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Partial Dirichlet–Neumann Boundary Value Problems on finite Networks

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We aim here at studying *partial Dirichlet boundary value problems on finite networks*. **Given two disjoint subset, A and B, of the vertex boundary of a set, F, a partial boundary value problem consists on finding a harmonic function with prescribed values on $\delta(F)\setminus B$ and with prescribed values of its normal derivative on A. We study existence and uniqueness of the solutions.**

1. Preliminaries

Throughout the paper, $\Gamma=(V,E)$ denotes a simple, finite and connected graph without loops, with vertex set V and edge set E. Two different vertices, $x,y\in V$, are called *adjacent*, which will be represented by $x\sim y$, if $\{x,y\}\in E$.

Given a vertex subset $F\subset V$, we denote by F^c its complementary in V and we call *boundary* and *closure of* F, the sets $\delta(F)=\{x\in V: x\sim y \text{ for some } y\in F\}$ and $\bar F=F\cup\delta(F)$, respectively. If $F\subset V$ is a proper subset, we say that F is *connected* if for any $x,y\in V$ there exists a path joined x and y whose vertices are all in F. It is easy to prove that $\bar F$ is connected when F is.

The sets of functions and non-negative functions on V are denoted by $\mathcal{C}(V)$ and $\mathcal{C}^+(V)$ respectively. If $u\in\mathcal{C}(V)$, its *support* is given by $\mathsf{supp}(u)=\{x\in V: u(x)\neq 0\}$. Moreover, if F is a non empty subset of V, its characteristic function is denoted by $\chi_{_F}$ and we can consider the sets $\mathcal{C}(F)=\{u\in\mathcal{C}(V):\mathsf{supp}(u)\subset F\}$ and $\mathcal{C}^+(F)=\mathcal{C}(F)\cap\mathcal{C}^+(V)$. We call *weight on* F any function $\sigma\in\mathcal{C}^+(F)$ such that $\mathsf{supp}(\sigma)=F$. The set of weights on F is denoted by $\Omega(F)$.

Given a weight $\nu\in\Omega(V)$, for any $u\in\mathcal{C}(F)$, we denote by $\displaystyle\int_F u(x)\,d\nu(x)$ or simply by $\displaystyle\int_F u\,d\nu$ the value $\displaystyle\sum_{x\in F}u(x)\nu(x)$. When $\nu(x)=1$ for any $x\in V$, we denote $\displaystyle\int_F u\,d\nu$ simply by $\displaystyle\int_F u\,dx$.

We call *conductance on* Γ a function $c\colon V\times V\longrightarrow \mathbb{R}^+$ such that $c(x,y)>0$ iff $x\sim y$. We call *network* any triple (Γ,c,ν), where c is a conductance on Γ and $\nu\in\Omega(V)$. In what follows we consider fixed the network (Γ,c,ν) and we refer to it simply by Γ. The function $\kappa\in\mathcal{C}^+(V)$ defined as $\kappa(x)=\displaystyle\int_V c(x,y)\,dy$ for any $x\in V$ is called the *degree of* Γ that clearly satisfies that $\mathsf{supp}(\kappa)=V$. In addition, for any proper subset $F\subset V$ we call *the boundary degree of* F the function $\kappa_{_F}\in\mathcal{C}\big(\delta(F)\big)$ defined as $\kappa_{_F}(x)=\displaystyle\int_F c(x,y)\,dy$ for any $x\in\delta(F)$, that clearly satisfies that $\mathsf{supp}(\kappa_{_F})=\delta(F)$.

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The *combinatorial Laplacian* or simply the *Laplacian* of Γ is the linear operator $\mathcal{L}:\mathcal{C}(V)\longrightarrow\mathcal{C}(V)$ that assigns to each $u\in\mathcal{C}(V)$ the function

$$\mathcal{L}(u)(x)=\frac{1}{\nu(x)}\int_V c(x,y)\big(u(x)-u(y)\big)\,dy,\quad x\in V. \tag{1}$$

Given $q\in\mathcal{C}(V)$ the *Schrödinger operator* on Γ with *potential* q is the linear operator $\mathcal{L}_q:\mathcal{C}(V)\longrightarrow\mathcal{C}(V)$ that assigns to each $u\in\mathcal{C}(V)$ the function $\mathcal{L}_q(u)=\mathcal{L}(u)+qu$.

If F is a proper subset of V, for each $u\in\mathcal{C}(\bar F)$ we define the *normal derivative of u on F* as the function in $\mathcal{C}(\delta(F))$ given by

$$\left(\frac{\partial u}{\partial \mathsf{n}_F}\right)(x)=\frac{1}{\nu(x)}\int_F c(x,y)\big(u(x)-u(y)\big)\,dy,\quad \text{for any } x\in\delta(F). \tag{2}$$

The *normal derivative on F* is the operator $\dfrac{\partial}{\partial\mathsf{n}_F}:\mathcal{C}(\bar F)\longrightarrow\mathcal{C}(\delta(F))$ that to any $u\in\mathcal{C}(\bar F)$ assigns its normal derivative on F.

The relation between the values of the Schrödinger operator with potential q on F and the values of the normal derivative at $\delta(F)$ is given by the *First Green Identity*, proved in [1]

$$\int_F v\,\mathcal{L}_q(u)\,d\nu=\frac{1}{2}\int_{\bar F}\int_{\bar F}c_F(x,y)(u(x)-u(y))(v(x)-v(y))\,dxdy+\int_F quv\,d\nu-\int_{\delta(F)}v\,\frac{\partial u}{\partial\mathsf{n}_F}\,d\nu,$$

where $u,v\in\mathcal{C}(\bar F)$ and $c_F=c\cdot\chi_{(\bar F\times\bar F)\setminus(\delta(F)\times\delta(F))}$. A direct consequence of the above identity is the so–called *Second Green Identity*

$$\int_F\Big(v\,\mathcal{L}_q(u)-u\,\mathcal{L}_q(v)\Big)\,d\nu=\int_{\delta(F)}\left(u\,\frac{\partial v}{\partial\mathsf{n}_F}-v\,\frac{\partial u}{\partial\mathsf{n}_F}\right)d\nu,\quad\text{for all}\quad u,v\in\mathcal{C}(\bar F).$$

2. Partial Dirichlet–Neumann Boundary Value Problems

Through this section we fixed $F\subset V$ a proper and connected subset and $A,B\subset\delta(F)$ non-empty subsets such that $A\cap B=\emptyset$. Moreover we denote by R the set $R=\delta(F)\setminus(A\cup B)$, so $\delta(F)=A\cup B\cup R$ is a partition of $\delta(F)$. We remark that R can be an empty set. Our aim is to study self-adjoint boundary value problems associated with the Schrödinger operator with potential $q\in\mathcal{C}(\bar F)$.

Definition 2.1. *For any $h\in\mathcal{C}(F)$, $f\in\mathcal{C}(A\cup R)$ and $g\in\mathcal{C}(A)$, the partial Dirichlet–Neumann boundary value problem on F with data h,f,g consists in finding $u\in\mathcal{C}(\bar F)$ such that*

$$\mathcal{L}_q(u)=h\quad\text{on}\quad F,\qquad\frac{\partial u}{\partial\mathsf{n}_F}=g\quad\text{on}\quad A\qquad\text{and}\qquad u=f\quad\text{on}\quad A\cup R. \tag{3}$$

Moreover, the homogeneous partial Dirichlet–Neumann boundary value problem on F consists in finding $u\in\mathcal{C}(\bar F)$ such that

$$\mathcal{L}_q(u)=0\quad\text{on}\quad F,\qquad\frac{\partial u}{\partial\mathsf{n}_F}=u=0\quad\text{on}\quad A\qquad\text{and}\qquad u=0\quad\text{on}\quad R. \tag{4}$$

Notice that the above boundary condition fix the values of u and of its normal derivative on A and we do not have any requirements on the values of u or $\dfrac{\partial u}{\partial\mathsf{n}_F}$ on B. It is clear that the set of solutions of the homogeneous boundary value problem is a vector subspace of $\mathcal{C}(F\cup B)$ that we will denote by $\mathcal{V}_B$. Moreover if Problem (3) has solutions and u is a particular one, then $u+\mathcal{V}_B$ describes the set of all its solutions.

In addition, if u is a solution of Problem (3), then for any $x\in A$ we get that

$$\int_F c(x,y)u(y)dy=f(x)\kappa_F(x)-\nu(x)g(x).$$

Therefore, if u is a solution of Problem (4), then for any $x \in A$ we get that

$$\int_F c(x,y)u(y)dy = 0.$$

Let us define the adjoint of problem (4).

Definition 2.2. *The adjoint problem of the partial Dirichlet–Neumann boundary value problem on F (4) is given by*

$$\mathcal{L}_q(v) = 0 \quad on \quad F, \qquad \frac{\partial v}{\partial \mathsf{n}_\mathsf{F}} = v = 0 \quad on \quad B \qquad and \qquad v = 0 \quad on \quad R. \tag{5}$$

The subspace of solution of the above problem will be denoted by $\mathcal{V}_{_A}$. It is clear that $\mathcal{V}_{_A} \subset \mathcal{C}(F \cup A)$.

The Second Green Identity leads to the following result.

Proposition 2.3. *Problems (4) and (5) are mutually adjoint; that is*

$$\int_F v\mathcal{L}_q(u)d\nu = \int_F u\mathcal{L}_q(v)d\nu,$$

for any $u, v \in \mathcal{C}(\bar{F})$ such that $\dfrac{\partial u}{\partial \mathsf{n}_\mathsf{F}} = u = 0$ on A, $\dfrac{\partial v}{\partial \mathsf{n}_\mathsf{F}} = v = 0$ on B and $u = v = 0$ on R.

Proposition 2.4 (Fredholm Alternative)**.** *Given $h \in \mathcal{C}(F)$, $f \in \mathcal{C}(A \cup R)$, $g \in \mathcal{C}(A)$, the boundary value problem*

$$\mathcal{L}_q(u) = h, \quad on \quad F, \qquad \frac{\partial u}{\partial \mathsf{n}_\mathsf{F}} = g \quad on \quad A \quad and \quad u = f \quad on \quad A \cup R$$

has solution iff

$$\int_F hv\, d\nu + \int_P gv\, d\nu = \int_{A\cup R} f\, \frac{\partial v}{\partial \mathsf{n}_\mathsf{F}}\, d\nu, \qquad for\ each\ v \in \mathcal{V}_{_A}.$$

In addition, when the above condition holds, then there exists a unique solution of the boundary value problem in $\mathcal{V}_{_B}^{\perp}$, i.e. a unique solution u, such that

$$\int_{F\cup B} uz\, d\nu = 0, \quad for\ any\ z \in \mathcal{V}_{_B}.$$

Proof. First observe that problem (3) is equivalent to the boundary value problem

$$\mathcal{L}_q(u) = h - \mathcal{L}(f) \quad \text{on} \quad F, \quad \frac{\partial u}{\partial \mathsf{n}_\mathsf{F}} = g - f\frac{\kappa_F}{\nu}{}^{1} \quad \text{on} \quad A \quad \text{and} \quad u = 0, \quad \text{on} \quad A \cup R$$

in the sense that u is a solution of the this problem iff $u + f$ is a solution of (3).

Consider now the linear operators $\mathcal{F}\colon \mathcal{C}(F \cup B) \longrightarrow \mathcal{C}(F \cup A)$ and $\mathcal{F}^*\colon \mathcal{C}(F \cup A) \longrightarrow \mathcal{C}(F \cup B)$ defined as

$$\mathcal{F}(u) = \begin{cases} \mathcal{L}_q(u), & \text{on} \quad F, \\ \dfrac{\partial u}{\partial \mathsf{n}_\mathsf{F}}, & \text{on} \quad A, \end{cases} \qquad \text{and} \qquad \mathcal{F}^*(v) = \begin{cases} \mathcal{L}_q(v), & \text{on} \quad F, \\ \dfrac{\partial v}{\partial \mathsf{n}_\mathsf{F}}, & \text{on} \quad B, \end{cases}$$

respectively. Then, for any $u \in \mathcal{C}(F \cup B)$ and $v \in \mathcal{C}(F \cup A)$ it is verified that

$$\begin{aligned}\int_{F\cup A} v\mathcal{F}(u)\, d\nu &= \int_F v\mathcal{L}_q(u)\, d\nu + \int_{\delta(F)} v\frac{\partial u}{\partial \mathsf{n}_\mathsf{F}}\, d\nu \\ &= \int_F u\mathcal{L}_q(v)\, d\nu + \int_{\delta(F)} u\frac{\partial v}{\partial \mathsf{n}_\mathsf{F}}\, d\nu = \int_{F\cup B} u\mathcal{F}^*(v)\, d\nu.\end{aligned}$$

[1] Recall that $\dfrac{\partial f}{\partial \mathsf{n}_\mathsf{F}} = f\dfrac{\kappa_F}{\nu}$, since $f \in \mathcal{C}(\delta(F))$.

Therefore the operators $\mathcal{F}$ and $\mathcal{F}^*$ are mutually adjoint with respect to the inner products induced in $\mathcal{C}(F\cup A)$ and $\mathcal{C}(F\cup B)$. By applying the classical Fredholm Alternative we obtain that $\mathrm{Img}\mathcal{F} = (\ker\mathcal{F}^*)^{\perp}$. Clearly, the subspace $\ker\mathcal{F}^*$ coincides with the space of solutions of the homogeneous problem (5) and moreover problem (3) has a solution iff the function $\tilde{h} \in \mathcal{C}(F\cup A)$ given by $\tilde{h} = h - \mathcal{L}(f)$ on F and $\tilde{h} = g - f\kappa_{_F}$ on A verifies that $\tilde{h} \in \mathrm{Img}\mathcal{F}$. Therefore, problem (3) has solution iff for any $v \in \mathcal{V}_{_A}$

$$\begin{aligned}0 = \int_{F\cup A} \tilde{h} v\, d\nu &= \int_F hv\, d\nu + \int_A gv\, d\nu - \int_F v\mathcal{L}(f)\, d\nu - \int_A v\, \frac{\partial f}{\partial \mathsf{n}_{\mathsf{F}}}\, d\nu \\ &= \int_F hv\, d\nu + \int_A gv\, d\nu - \int_F f\mathcal{L}(v)\, d\nu - \int_{\delta(F)} f\, \frac{\partial v}{\partial \mathsf{n}_{\mathsf{F}}}\, d\nu + \int_{R\cup B} v\, \frac{\partial f}{\partial \mathsf{n}_{\mathsf{F}}}\, d\nu \\ &= \int_F hv\, d\nu + \int_A gv\, d\nu - \int_{R\cup A} f\, \frac{\partial v}{\partial \mathsf{n}_{\mathsf{F}}}\, d\nu.\end{aligned}$$

Finally, the Fredholm Alternative also establishes that when the necessary and sufficient condition is attained there exists a unique $w \in (\ker\mathcal{F})^{\perp}$ such that $\mathcal{F}(w) = \tilde{h}$. Therefore, $u = w + f$ is the unique solution of problem (3) such that for any $z \in \ker\mathcal{F} = \mathcal{V}_{_B}$ verifies

$$\int_{F\cup B} uz\, d\nu = 0. \quad \square$$

Observe that Fredholm Alternative establishes the following formula

$$|A| - |B| = \mathrm{dim}\mathcal{V}_{_A} - \mathrm{dim}\mathcal{V}_{_B} \tag{6}$$

On the other hand, the existence of solution for any data is equivalent to be $\mathcal{V}_{_A} = \{0\}$, that is; iff $|B| - |A| = \mathrm{dim}\mathcal{V}_{_B} \geq 0$. Moreover, uniqueness of solutions is equivalent to be $|A| - |B| = \mathrm{dim}\mathcal{V}_{_A} \geq 0$. In particular, if $|A| = |B|$, the existence of solution of problem (3) for any data h, f and g is equivalent to the uniqueness of solution and hence it is equivalent to the fact that the homogeneous problem has $v = 0$ as its unique solution.

Next, we establish the variational formulation of the boundary value problem (3), that represents the discrete version of the weak formulation for boundary value problems.

Prior to describe the claimed formulation, we give some useful definitions. *The bilinear form* associated with the boundary value problem (3) is $\mathcal{B}\colon \mathcal{C}(\bar{F}) \times \mathcal{C}(\bar{F}) \longrightarrow \mathbb{R}$ given by

$$\mathcal{B}(u,v) = \int_F v\mathcal{L}_q(u)\, d\nu + \int_A v\frac{\partial u}{\partial \mathsf{n}_{\mathsf{F}}}\, d\nu. \tag{7}$$

Analogously, $\mathcal{B}^*\colon \mathcal{C}(\bar{F}) \times \mathcal{C}(\bar{F}) \longrightarrow \mathbb{R}$, *the bilinear form* associated with the adjoint boundary value problem (5) is given by

$$\mathcal{B}^*(u,v) = \int_F v\mathcal{L}_q(u)\, d\nu + \int_B v\frac{\partial u}{\partial \mathsf{n}_{\mathsf{F}}}\, d\nu. \tag{8}$$

Therefore, for all $u \in \mathcal{C}(F\cup B)$ and $v \in \mathcal{C}(F\cup A)$, it holds

$$\mathcal{B}^*(v,u) = \mathcal{B}(u,v)$$

Applying the First Green Identities, we obtain that

$$\begin{aligned}\mathcal{B}(u,v) = &\ \frac{1}{2}\int_{\bar{F}\times\bar{F}} c_{_F}(x,y)\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)\, dx\, dy + \int_F q\, u\, v\, d\nu - \int_{B\cup R} v\frac{\partial u}{\partial \mathsf{n}_F}\, d\nu, \\ \mathcal{B}^*(v,u) = &\ \frac{1}{2}\int_{\bar{F}\times\bar{F}} c_{_F}(x,y)\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)\, dx\, dy + \int_F q\, u\, v\, d\nu - \int_{A\cup R} u\frac{\partial v}{\partial \mathsf{n}_F}\, d\nu.\end{aligned} \tag{9}$$

Associated with any pair of functions $h \in \mathcal{C}(F)$ and $g \in \mathcal{C}(A)$ we define the linear functional $\ell\colon \mathcal{C}(\bar{F}) \longrightarrow \mathbb{R}$ as $\ell(v) = \displaystyle\int_F hv\, d\nu + \int_A gv\, d\nu$, whereas for any function $f \in \mathcal{C}(A\cup R)$ we consider the convex set $K_f = f + \mathcal{C}(F\cup B)$.

Proposition 2.5 (Variational Formulation). *Given $h \in \mathcal{C}(F)$, $g \in \mathcal{C}(A)$ and $f \in \mathcal{C}(A \cup R)$, then $u \in K_f$ is a solution of Problem (3) iff*

$$\mathcal{B}(u, v) = \ell(v), \quad \text{for any } \ v \in \mathcal{C}(F \cup A)$$

and in this case, the set $u + \Big\{w \in \mathcal{C}(F \cup B) : \mathcal{B}(w, v) = 0, \text{ for any } v \in \mathcal{C}(F \cup A)\Big\}$ describes all solutions of (3).

In particular, $\mathcal{V}_{_B} = \Big\{w \in \mathcal{C}(F \cup B) : \mathcal{B}(w, v) = 0, \text{ for any } v \in \mathcal{C}(F \cup A)\Big\}$ and, analogously $\mathcal{V}_{_A} = \Big\{w \in \mathcal{C}(F \cup A) : \mathcal{B}(v, w) = 0, \text{ for any } v \in \mathcal{C}(F \cup B)\Big\}$.

Proof. A function $u \in K_f$ satisfies that $\mathcal{B}(u, v) = \ell(v)$ for any $v \in \mathcal{C}(F \cup A)$ iff

$$\int_F v(\mathcal{L}_q(u) - h)\, d\nu + \int_A v \left(\frac{\partial u}{\partial \mathsf{n}_{_F}} - g\right) d\nu = 0.$$

Then, the first result follows by taking $v = \varepsilon_x$, $x \in F \cup A$. Finally, $\hat{u} \in K_f$ is another solution of (3) iff $\mathcal{B}(\hat{u}, v) = \ell(v)$ for any $v \in \mathcal{C}(F \cup A)$ and hence iff $\mathcal{B}(u - \hat{u}, v) = 0$ for any $v \in \mathcal{C}(F \cup A)$. ∎

3. Partial Dirichlet to Neumann map

In this section we study sufficient and necessary conditions so that $\mathcal{V}_{_B} = \{0\}$ and/or $\mathcal{V}_{_A} = \{0\}$. To do this we consider the Dirichlet problem

$$\mathcal{L}_q(u) = 0 \ \text{ on } \ F, \quad u = f \ \text{ on } \ \delta(F) \tag{10}$$

and we will assume that $\mathcal{L}_q$ is positive definite on $\mathcal{C}(F)$. We recall that this hypothesis assures the existence and uniqueness of solution for any data $f \in \mathcal{C}(\delta(F))$. We will denoted this solution by u_f. In fact, the following result holds.

Lemma 3.1 ([2], Proposition 4). *Given $q \in \mathcal{C}(F)$, the Schrödinger operator $\mathcal{L}_q$ is positive definite on $\mathcal{C}(F)$ iff there exists a weight $\sigma \in \Omega(F)$ such that $q \geq q_\sigma$ on F, where $q_\sigma = -\sigma^{-1}\mathcal{L}(\sigma)$.*

Let us consider the linear operator $\Lambda_{_{A,B}} : \mathcal{C}(A) \longrightarrow \mathcal{C}(B)$, defined for any $v \in \mathcal{C}(A)$ and any $x \in B$ by

$$\Lambda_{_{A,B}}(v) = \frac{\partial u_v}{\partial \mathsf{n}_{_F}} \chi_{_B}. \tag{11}$$

We define $\Lambda_{_{B,A}}$ in an analogous manner.

Proposition 3.2. $\Lambda^*_{_{A,B}} = \Lambda_{_{B,A}}$ *and, in addition,* $\mathsf{ker}\Lambda_{_{A,B}} = \mathcal{V}_{_A} \cdot \chi_{_A}$ *and* $\mathsf{ker}\Lambda_{_{B,A}} = \mathcal{V}_{_B} \cdot \chi_{_B}$.

Proof. Given $v \in \mathcal{C}(A)$ and $w \in \mathcal{C}(B)$, then from the Second Green Identity

$$\int_B w\Lambda_{_{A,B}}(v) d\nu = \int_B u_w \frac{\partial u_v}{\partial \mathsf{n}_{\mathsf{F}}} d\nu = \int_{\delta(F)} u_w \frac{\partial u_v}{\partial \mathsf{n}_{\mathsf{F}}} d\nu = \int_{\delta(F)} u_v \frac{\partial u_w}{\partial \mathsf{n}_{\mathsf{F}}} d\nu = \int_A u_v \frac{\partial u_w}{\partial \mathsf{n}_{\mathsf{F}}} d\nu = \int_A v\Lambda_{_{B,A}}(w) d\nu,$$

where we have taken into account that $\mathcal{L}_q(u_v) = \mathcal{L}_q(u_w) = 0$ on F.

Clearly, if $v \in \mathsf{ker}\Lambda_{_{A,B}}$, then $u_v \in \mathcal{V}_{_A}$ and $v = u_v \cdot \chi_{_A}$. Conversely, if $u \in \mathcal{V}_{_A}$ then $\mathcal{L}_q(u) = 0$ on F, $\dfrac{\partial u}{\partial \mathsf{n}_{\mathsf{F}}} = 0$ on B and $u = 0$ on $B \cup R$. Therefore, if we consider $v = u \cdot \chi_{_A}$, then $u = u_v$ and clearly $v \in \mathsf{ker}\Lambda_{_{A,B}}$. The equality for $\mathsf{ker}\Lambda_{_{B,A}}$ follows analogously. □

Corollary 3.3. *Problem (3) has solution for any data iff $\Lambda_{_{A,B}}$ is non-singular. Moreover, Problem (3) has uniqueness of solutions iff $\Lambda_{_{B,A}}$ is non-singular. In particular, when $|A| = |B|$, $\Lambda_{_{A,B}}$ is non-singular iff $\Lambda_{_{B,A}}$ is non-singular, and in this case Problem (3) has a unique solution for any data.*

Remark: If Γ is a circular planar resistor network and A, B is a circular pair of sequences of boundary nodes that are connected through Γ, then $\Lambda_{A,B}$ is an isomorphism, see [4, Theorem 4.2].

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