Recovering Negative Conductivities

Cynthia Wu

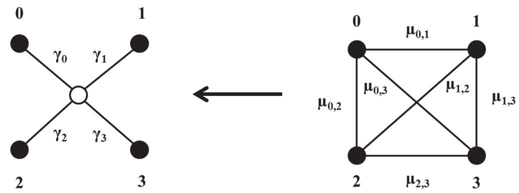
University of Washington Mathematics REU 2012

Abstract. We consider resistor networks composed of 4-stars with the possibility of several negative conductivities. Using equation 2 in [1] to recover conductivities from the conductivities of edges in the R-Multigraph poses problems due to the determination of the α's. A method of determining these α's is given along with a comparison of equation 2 in [1] to the general formula given in [2].

Contents

1 The General Formula

Suppose we wish to find the conductivities of the edges in a resistor network (represented by γ_i's) composed of 4-stars. After applying a Star-K transformation and obtaining the conductivities of the edges in the respective R-Multigraph (represented as μ_{ij}'s), one can use the general formula in [2] to acquire the conductivities of the edges in the original resistor network by considering each single quadrilateral in the R-Multigraph.



A 4-star and its respective R-Multigraph, a quadrilateral

According to the general formula,

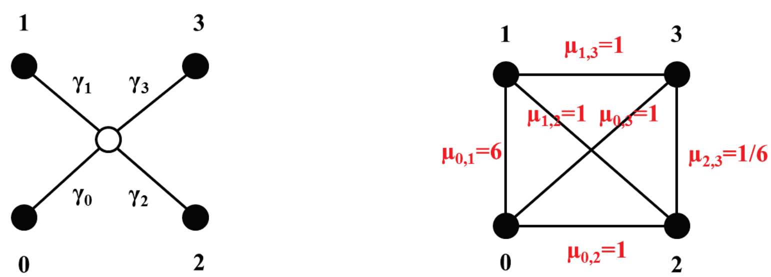
$$\gamma_i = -\frac{\det\begin{bmatrix}\mu_{i,i} & \mu_{i,j}\\ \mu_{i,k} & \mu_{j,k}\end{bmatrix}}{\mu_{j,k}} \tag{1}$$

where

$$\mu_{i,i} = -(\mu_{i,j} + \mu_{i,k} + \mu_{i,l})$$

Note that i, j, k, and l are vertices in the same quadrilateral.

As an example, we will use quadrilaterals 1 and 3 from the Pseudo 2 to 1 graph in [3].



The 4-star and Quadrilateral 1 from the Pseudo 2 to 1 graph

By (1),

$$\mu_{0,0} = -(\mu_{0,1} + \mu_{0,2} + \mu_{0,3}) = -(6+1+1) = -8$$

So

$$\gamma_0 = -\frac{\det\begin{bmatrix}\mu_{0,0} & \mu_{0,1}\\ \mu_{0,2} & \mu_{1,2}\end{bmatrix}}{\mu_{1,2}} = -\frac{\det\begin{bmatrix}-8 & 6\\ 1 & 1\end{bmatrix}}{1} = 14$$

Similarly,

$$\mu_{1,1} = -(\mu_{1,3} + \mu_{1,2} + \mu_{0,1}) = -(1 + 1 + 6) = -8$$

$$\gamma_1 = -\frac{\det \begin{bmatrix} \mu_{1,1} & \mu_{1,3} \\ \mu_{0,1} & \mu_{0,3} \end{bmatrix}}{\mu_{0,3}} = -\frac{\det \begin{bmatrix} -8 & 1 \\ 6 & 1 \end{bmatrix}}{1} = 14$$

$$\mu_{2,2} = -(\mu_{0,2} + \mu_{1,2} + \mu_{2,3}) = -(1 + 1 + \frac{1}{6}) = -\frac{13}{6}$$

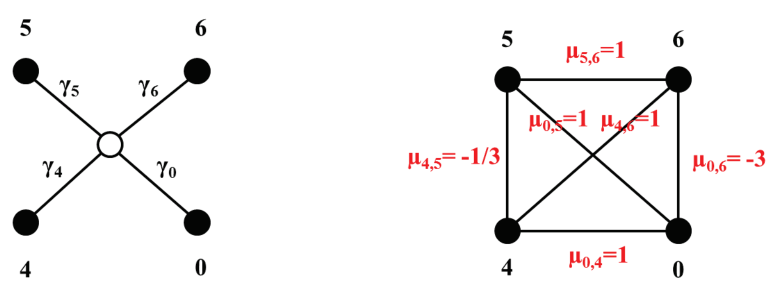
$$\gamma_2 = -\frac{\det \begin{bmatrix} \mu_{2,2} & \mu_{2,3} \\ \mu_{0,2} & \mu_{0,3} \end{bmatrix}}{\mu_{0,3}} == -\frac{\det \begin{bmatrix} -\frac{13}{6} & \frac{1}{6} \\ 1 & 1 \end{bmatrix}}{1} = \frac{7}{3}$$

$$\mu_{3,3} = -(\mu_{0,3} + \mu_{1,3} + \mu_{2,3}) = -(1 + 1 + \frac{1}{6}) = -\frac{13}{6}$$

$$\gamma_3 = -\frac{\det \begin{bmatrix} \mu_{3,3} & \mu_{1,3} \\ \mu_{2,3} & \mu_{1,2} \end{bmatrix}}{\mu_{1,2}} = -\frac{\det \begin{bmatrix} -\frac{13}{6} & 1 \\ \frac{1}{6} & 1 \end{bmatrix}}{1} = \frac{7}{3}$$

Thus, we have found all the conductivities of the edges in the 4-star corresponding to quadrilateral 1 using the general formula. Note that all conductivities are positive in this example.

Now consider quadrilateral 3 from the Pseudo 2 to 1 graph.



The 4-star and Quadrilateral 3 from the Pseudo 2 to 1 graph

By (1),

$$\mu_{0,0} = -(\mu_{0,6} + \mu_{0,5} + \mu_{0,4}) = -(-3 + 1 + 1) = 1$$

So

$$\gamma_0 = -\frac{\det\begin{bmatrix}\mu_{0,0} & \mu_{0,6}\\ \mu_{0,4} & \mu_{4,6}\end{bmatrix}}{\mu_{4,6}} = -\frac{\det\begin{bmatrix}1 & -3\\ 1 & 1\end{bmatrix}}{1} = -4$$

Similarly,

$$\mu_{4,4} = -(\mu_{4,5} + \mu_{4,6} + \mu_{4,0}) = -(-\frac{1}{3} + 1 + 1) = -\frac{5}{3}$$

$$\gamma_4 = -\frac{\det\begin{bmatrix}\mu_{4,4} & \mu_{4,5}\\ \mu_{4,0} & \mu_{0,5}\end{bmatrix}}{\mu_{0,5}} = -\frac{\det\begin{bmatrix}-\frac{5}{3} & -\frac{1}{3}\\ 1 & 1\end{bmatrix}}{1} = \frac{4}{3}$$

$$\mu_{5,5} = -(\mu_{5,6} + \mu_{5,0} + \mu_{5,4}) = -(1 + 1 - \frac{1}{3}) = -\frac{5}{3}$$

$$\gamma_5 = -\frac{\det\begin{bmatrix}\mu_{5,5} & \mu_{5,6}\\ \mu_{5,4} & \mu_{4,6}\end{bmatrix}}{\mu_{4,6}} == -\frac{\det\begin{bmatrix}-\frac{5}{3} & 1\\ -\frac{1}{3} & 1\end{bmatrix}}{1} = \frac{4}{3}$$

$$\mu_{6,6} = -(\mu_{6,5} + \mu_{6,4} + \mu_{6,0}) = -(1 + 1 - 3) = 1$$

$$\gamma_6 = -\frac{\det\begin{bmatrix}\mu_{6,6} & \mu_{6,5}\\ \mu_{6,0} & \mu_{5,0}\end{bmatrix}}{\mu_{5,0}} = -\frac{\det\begin{bmatrix}1 & 1\\ -3 & 1\end{bmatrix}}{1} = -4$$

Thus, we have found all the conductivities of the edges in the 4-star corresponding to quadrilateral 3 using the general formula.

It is important to note that the general formula can be used regardless of whether or not the conductivities of the edges in the R-Multigraph are positive or negative.

2 The Alternate Formula

An alternate formula can be used in obtaining the conductivities of edges in the resistor network from the conductivities of edges in the R-Multigraph.

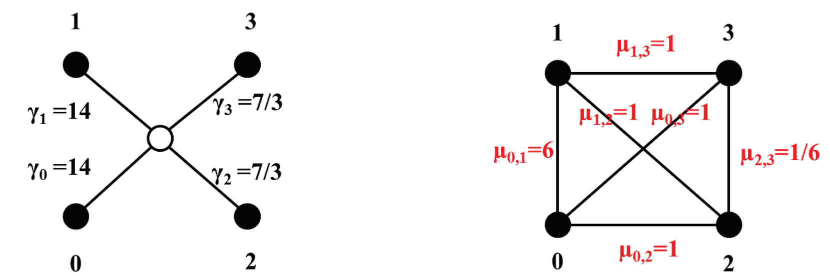
$$\gamma_i = \alpha_i \sum_m \alpha_m \tag{2}$$

where

$$\alpha_i = \sqrt{\frac{\mu_{i,j}\mu_{i,k}}{\mu_{j,k}}}$$

We will use (2) to recalculate the conductivities of edges in quadrilateral 1 and quadrilateral 3 of the Pseudo 2 to 1 graph and compare our results with the ones obtained from using (1).

Here are our results from using (1).



The 4-star with conductivities obtained from using (1) and Quadrilateral 1 from the Pseudo 2 to 1 graph

Using (2),

$\alpha_0 = \sqrt{\frac{\mu_{0,1}\mu_{0,2}}{\mu_{1,2}}} = \sqrt{6}$

$\alpha_1 = \sqrt{\frac{\mu_{1,3}\mu_{0,1}}{\mu_{0,3}}} = \sqrt{6}$

$\alpha_2 = \sqrt{\frac{\mu_{2,3}\mu_{0,2}}{\mu_{0,3}}} = \sqrt{\frac{1}{6}}$

$\alpha_3 = \sqrt{\frac{\mu_{1,3}\mu_{2,3}}{\mu_{1,2}}} = \sqrt{\frac{1}{6}}$

Since $\sum_m \alpha_m$ is the sum of all the α_m's in the quadrilateral,

$$\sum_m \alpha_m = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = \frac{14}{\sqrt{6}}$$

Thus, we have the following conductivities for the edges in the 4-star corresponding to quadrilateral 1 using (2).

$\gamma_0 = \alpha_0 \sum_m \alpha_m = \sqrt{6}(\frac{14}{\sqrt{6}}) = 14$

$\gamma_1 = \alpha_1 \sum_m \alpha_m = \sqrt{6}(\frac{14}{\sqrt{6}}) = 14$

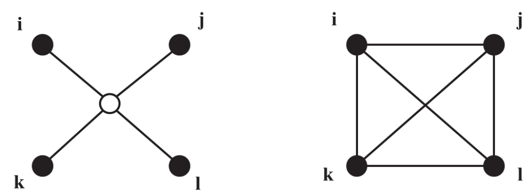
$\gamma_2 = \alpha_2 \sum_m \alpha_m = \sqrt{\frac{1}{6}}(\frac{14}{\sqrt{6}}) = \frac{7}{3}$

$\gamma_3 = \alpha_3 \sum_m \alpha_m = \sqrt{\frac{1}{6}}(\frac{14}{\sqrt{6}}) = \frac{7}{3}$

Thus, we obtain the same conductivities for quadrilateral 1 using (1) or (2). In general,when conductivities are positive in the R-Multigraph, (1) and (2) will always yield the same conductivities for the resistor network.

Theorem 1. Assuming positive conductivities, (1) and (2) produce the same results.

Proof. Suppose we are given the following 4-star and corresponding R-Multigraph. Assume that the conductivities of edges in the R-Multigraph are positive.



4-star and corresponding R-Multigraph

Without loss of generality, we will show that the conductivity, γ_i, will be the same regardless of which equation we use.

According to (1),

$$\gamma_i = -\frac{\det \begin{bmatrix} \mu_{i,i} & \mu_{i,j} \\ \mu_{i,k} & \mu_{j,k} \end{bmatrix}}{\mu_{j,k}} = -\frac{\det \begin{bmatrix} -(\mu_{i,j}+\mu_{i,k}+\mu_{i,l}) & \mu_{i,j} \\ \mu_{i,k} & \mu_{j,k} \end{bmatrix}}{\mu_{j,k}} = \mu_{ij}+\mu_{il}+\mu_{ik}+\frac{\mu_{ik}\mu_{ij}}{\mu_{jk}}$$

According to (2),

$$\alpha_i = \sqrt{\frac{\mu_{i,j}\mu_{i,k}}{\mu_{j,k}}}$$

$$\alpha_j = \sqrt{\frac{\mu_{j,i}\mu_{j,l}}{\mu_{i,l}}}$$

$$\alpha_k = \sqrt{\frac{\mu_{k,i}\mu_{k,l}}{\mu_{i,l}}}$$

$$\alpha_l = \sqrt{\frac{\mu_{l,j}\mu_{l,k}}{\mu_{j,k}}}$$

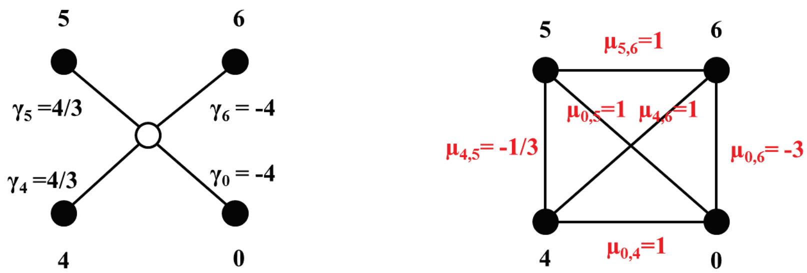
Note that $\frac{\sqrt{\mu_{i,k}\mu_{j,l}}}{\sqrt{\mu_{j,k}\mu_{i,l}}} = 1$ and $\frac{\sqrt{\mu_{i,j}\mu_{k,l}}}{\sqrt{\mu_{j,k}\mu_{i,l}}} = 1$ by the quadrilateral rule $\mu_{i,k}\mu_{j,l} = \mu_{i,j}\mu_{k,l} = \mu_{i,l}\mu_{j,k}$ (see [1]). Also note that by the quadrilateral rule, we will replace $\mu_{i,j}\mu_{l,k}$ with $\mu_{i,l}\mu_{j,k}$ and $\mu_{i,k}\mu_{l,j}$ with $\mu_{i,l}\mu_{j,k}$. So, by (2),

$$
\begin{aligned}
\gamma_i &= \alpha_i \sum_m \alpha_m \\
&= \sqrt{\frac{\mu_{i,j}\mu_{i,k}}{\mu_{j,k}}}\left(\sqrt{\frac{\mu_{i,j}\mu_{i,k}}{\mu_{j,k}}} + \sqrt{\frac{\mu_{j,i}\mu_{j,l}}{\mu_{i,l}}} + \sqrt{\frac{\mu_{k,i}\mu_{k,l}}{\mu_{i,l}}} + \sqrt{\frac{\mu_{l,j}\mu_{l,k}}{\mu_{j,k}}}\right) \\
&= \frac{\mu_{i,j}\mu_{i,k}}{\mu_{j,k}} + \frac{\sqrt{\mu_{i,j}}\sqrt{\mu_{i,k}}}{\sqrt{\mu_{j,k}}} * \frac{\sqrt{\mu_{i,j}}\sqrt{\mu_{j,l}}}{\sqrt{\mu_{i,l}}} + \frac{\sqrt{\mu_{i,j}}\sqrt{\mu_{i,k}}}{\sqrt{\mu_{j,k}}} * \frac{\sqrt{\mu_{k,i}}\sqrt{\mu_{k,l}}}{\sqrt{\mu_{i,l}}} + \frac{\sqrt{\mu_{i,j}}\sqrt{\mu_{i,k}}}{\sqrt{\mu_{j,k}}} * \frac{\sqrt{\mu_{l,j}}\sqrt{\mu_{l,k}}}{\sqrt{\mu_{j,k}}} \\
&= \frac{\mu_{i,j}\mu_{i,k}}{\mu_{j,k}} + \frac{\mu_{i,j}\sqrt{\mu_{i,k}\mu_{j,l}}}{\sqrt{\mu_{j,k}\mu_{i,l}}} + \frac{\mu_{i,k}\sqrt{\mu_{i,j}\mu_{k,l}}}{\sqrt{\mu_{j,k}\mu_{i,l}}} + \frac{\sqrt{\mu_{i,j}\mu_{l,k}\mu_{i,k}\mu_{l,j}}}{\mu_{j,k}} \\
&= \frac{\mu_{i,j}\mu_{i,k}}{\mu_{j,k}} + \mu_{i,j} + \mu_{i,k} + \frac{\sqrt{(\mu_{i,j}\mu_{l,k})(\mu_{i,k}\mu_{l,j})}}{\mu_{j,k}} \\
&= \frac{\mu_{i,j}\mu_{i,k}}{\mu_{j,k}} + \mu_{i,j} + \mu_{i,k} + \frac{\sqrt{(\mu_{i,l}\mu_{j,k})(\mu_{i,l}\mu_{j,k})}}{\mu_{j,k}} \\
&= \frac{\mu_{i,j}\mu_{i,k}}{\mu_{j,k}} + \mu_{i,j} + \mu_{i,k} + \frac{\sqrt{\mu_{i,l}^2\mu_{j,k}^2}}{\mu_{j,k}} \\
&= \frac{\mu_{i,j}\mu_{i,k}}{\mu_{j,k}} + \mu_{i,j} + \mu_{i,k} + \frac{\sqrt{\mu_{i,l}^2}\sqrt{\mu_{j,k}^2}}{\mu_{j,k}} \\
&= \frac{\mu_{i,j}\mu_{i,k}}{\mu_{j,k}} + \mu_{i,j} + \mu_{i,k} + \frac{\mu_{i,l}\mu_{j,k}}{\mu_{j,k}} \\
&= \frac{\mu_{i,j}\mu_{i,k}}{\mu_{j,k}} + \mu_{i,j} + \mu_{i,k} + \mu_{i,l}
\end{aligned}
$$

Thus, the conductivity, γ_i, will be the same regardless of which equation we use. □

Let's compare the results for quadrilateral 3 using (1) and (2). Note that we are now considering negative conductivities.

Here are our results from the general formula.



The 4-star with conductivities obtained using (1) and Quadrilateral 3 from the Pseudo 2 to 1 graph

Using (2),

$\alpha_0 = \sqrt{\frac{\mu_{0,6}\mu_{0,5}}{\mu_{5,6}}} = \sqrt{-3} = i\sqrt{3}$

For now, we will only choose the positive roots for our α's.

$\alpha_4 = \sqrt{\frac{\mu_{4,5}\mu_{4,6}}{\mu_{5,6}}} = \sqrt{-\frac{1}{3}} = i\sqrt{\frac{1}{3}}$

$\alpha_5 = \sqrt{\frac{\mu_{5,6}\mu_{0,5}}{\mu_{0,6}}} = \sqrt{\frac{1}{-3}} = i\sqrt{\frac{1}{3}}$

$\alpha_6 = \sqrt{\frac{\mu_{5,6}\mu_{4,6}}{\mu_{4,5}}} = \sqrt{-3} = i\sqrt{3}$

Since $\sum_m \alpha_m$ is the sum of all the α_m's in the quadrilateral,

$$\sum_m \alpha_m = \alpha_0 + \alpha_4 + \alpha_5 + \alpha_6 = \frac{8i}{\sqrt{3}}$$

Thus, we have the following conductivities for the edges in the 4-star corresponding to quadrilateral 3 using (2). However, these conductivities are incorrect.

$\gamma_0 = \alpha_0 \sum_m \alpha_m = i\sqrt{3}(\frac{8i}{\sqrt{3}}) = -8$

$\gamma_4 = \alpha_4 \sum_m \alpha_m = i\sqrt{\frac{1}{3}}(\frac{8i}{\sqrt{3}}) = \frac{-8}{3}$

$\gamma_5 = \alpha_5 \sum_m \alpha_m = i\sqrt{\frac{1}{3}}(\frac{8i}{\sqrt{3}}) = \frac{-8}{3}$

$\gamma_6 = \alpha_6 \sum_m \alpha_m = i\sqrt{3}(\frac{8i}{\sqrt{3}}) = -8$

A different choice of positive and negative roots for the α's produces correct conductivities.

$\alpha_0 = \sqrt{\frac{\mu_{0,6}\mu_{0,5}}{\mu_{5,6}}} = \sqrt{-3} = -i\sqrt{3}$

$\alpha_4 = \sqrt{\frac{\mu_{4,5}\mu_{4,6}}{\mu_{5,6}}} = \sqrt{-\frac{1}{3}} = i\sqrt{\frac{1}{3}}$

$\alpha_5 = \sqrt{\frac{\mu_{5,6}\mu_{0,5}}{\mu_{0,6}}} = \sqrt{\frac{1}{-3}} = i\sqrt{\frac{1}{3}}$

$\alpha_6 = \sqrt{\frac{\mu_{5,6}\mu_{4,6}}{\mu_{4,5}}} = \sqrt{-3} = -i\sqrt{3}$

Since $\sum_m \alpha_m$ is the sum of all the α_m's in the quadrilateral,

$$\sum_m \alpha_m = \alpha_0 + \alpha_4 + \alpha_5 + \alpha_6 = \frac{-4i}{\sqrt{3}}$$

Thus, we have the following conductivities for the edges in the 4-star corresponding to quadrilateral 3 using (2).

$\gamma_0 = \alpha_0 \sum_m \alpha_m = -i\sqrt{3}(\frac{-4i}{\sqrt{3}}) = -4$

$\gamma_4 = \alpha_4 \sum_m \alpha_m = i\sqrt{\frac{1}{3}}(\frac{-4i}{\sqrt{3}}) = \frac{4}{3}$

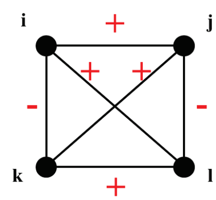
$\gamma_5 = \alpha_5 \sum_m \alpha_m = i\sqrt{\frac{1}{3}}(\frac{-4i}{\sqrt{3}}) = \frac{4}{3}$

$\gamma_6 = \alpha_6 \sum_m \alpha_m = -i\sqrt{3}(\frac{-4i}{\sqrt{3}}) = -4$

These conductivities match with the ones produced by (1). But how does one know how to choose the α's in such a way that using (2) produces the correct conductivities?

3 The Determination of α's in the Alternate Formula

Suppose we are given the R-Multigraph of some 4-star. In determining the α's for (2), we only care about the signs of the conductivities in the R-Multigraph. Consider the following example.



R-Multigraph with signs of conductivities

From [2], we know that

$$\mu_{ij} = \frac{\gamma_i \gamma_j}{\sigma} \tag{3}$$

where

$$\sigma = \gamma_i + \gamma_j + \gamma_k + \gamma_l$$

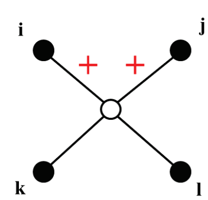
σ is the sum of the conductivities in the 4-star. If $\sigma = 0$, this would imply that the submatrix C in the Kirchhoff matrix is the zero matrix. However, zero matrices do not have inverses which would make the response matrix undefined. Thus, $\sigma > 0$ or $\sigma < 0$.

Case I: $\sigma > 0$

Since

$$\mu_{ij} = \frac{\gamma_i \gamma_j}{\sigma}$$

by (3) and $\mu_{ij} > 0$ from the figure above and $\sigma > 0$ from our assumption, this implies that γ_i and γ_j must have the same sign. Without loss of generality, suppose both γ_i and γ_j are positive.

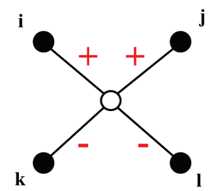


γ_i and γ_j are positive

Since

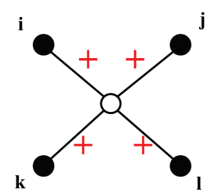
$$\mu_{kl} = \frac{\gamma_k \gamma_l}{\sigma}$$

by (3) and $\mu_{kl}>0$ from the figure above and $\sigma>0$ from our assumption, this implies that γ_k and γ_l must have the same sign. Let's first suppose γ_k and γ_l are negative.



γ_k and γ_l are negative

But this contradicts with the sign of $\mu_{il} = \frac{\gamma_i \gamma_l}{\sigma}$ because this would imply that a positive number is equal to a negative number. So let's suppose instead that γ_k and γ_l are positive.



γ_k and γ_l are now positive

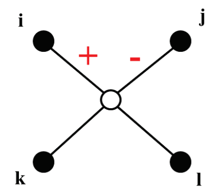
However, we still encounter a contradiction with the sign of $\mu_{ik} = \frac{\gamma_i \gamma_k}{\sigma}$ since this would imply that a negative number is a positive number. Thus, σ cannot be greater than 0.

Case II: $\sigma<0$

Since

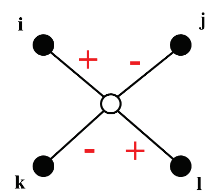
$$\mu_{ij} = \frac{\gamma_i \gamma_j}{\sigma}$$

and $\mu_{ij}>0$ from the figure above and $\sigma<0$ from our assumption, this implies that γ_i and γ_j must have opposite signs. Without loss of generality, suppose $\gamma_i>0$ and $\gamma_j<0$.



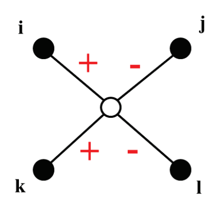
γ_i is positive and γ_j is negative

Since
$$\mu_{kl} = \frac{\gamma_k \gamma_l}{\sigma}$$
and $\mu_{kl}>0$ from the figure above and $\sigma<0$ from our assumption, this implies that γ_k and γ_l must have opposite signs. Let's first suppose $\gamma_k<0$ and $\gamma_l>0$.



γ_k is negative and γ_l is positive

But this contradicts with the sign of $\mu_{il} = \frac{\gamma_i \gamma_l}{\sigma}$ because this would imply that a positive number is equal to a negative number. So let's suppose instead that $\gamma_k>0$ and $\gamma_l<0$.



γ_k is positive and γ_l is negative

We check for any contradictions with the signs of the μ's using (3).

$$\mu_{ij} = \frac{\gamma_i \gamma_j}{\sigma} \qquad + = \frac{+-}{-}$$

$$\mu_{il} = \frac{\gamma_i \gamma_l}{\sigma} \qquad + = \frac{+-}{-}$$

$$\mu_{ik} = \frac{\gamma_i \gamma_k}{\sigma} \qquad - = \frac{++}{-}$$

$$\mu_{jk} = \frac{\gamma_j \gamma_k}{\sigma} \qquad + = \frac{-+}{-}$$

$$\mu_{jl} = \frac{\gamma_j \gamma_l}{\sigma} \qquad - = \frac{--}{-}$$

$$\mu_{kl} = \frac{\gamma_k \gamma_l}{\sigma} \qquad + = \frac{+-}{-}$$

There are no such contradictions so this pattern of signs for the conductivities in the 4-star works. Thus, γ_i and γ_k have opposite signs from γ_j and γ_l.

By (2),

$$\gamma_i = \alpha_i \sum_m \alpha_m$$

$$\gamma_j = \alpha_j \sum_m \alpha_m$$

$$\gamma_k = \alpha_k \sum_m \alpha_m$$

$$\gamma_l = \alpha_l \sum_m \alpha_m$$

Note that although $\sum_m \alpha_m$ may be a complex value, we will say that $\sum_m \alpha_m$ is negative if there is a negative sign in the sum. For example, we call $\sum_m \alpha_m = \frac{-4i}{\sqrt{3}}$ negative whereas we call $\sum_m \alpha_m = \frac{4i}{\sqrt{3}}$ positive.

Suppose $\sum_m \alpha_m$ is positive. In this case, if γ_i and γ_k are positive and γ_j and γ_l are negative, then α_i and α_k must be positive and α_j and α_l must be negative.

Suppose $\sum_m \alpha_m$ is negative. In this case, if γ_i and γ_k are positive and γ_j and γ_l are negative, then α_i and α_k must be negative and α_j and α_l must be positive.

Thus, in order for γ_i and γ_k to have opposite signs from γ_j and γ_l, this requires α_i and α_k to have opposite signs from α_j and α_l.

So we can choose positive roots for α_i and α_k and negative roots for α_j and α_l or we can choose negative roots for α_i and α_k and positive roots for α_j and α_l.

We can repeat this process given any R-Multigraph of some 4-star. As long as we know the signs of the conductivities of the edges in the R-Multigraph, we can determine the signs of the roots for our α's in (2) so that we can obtain the same results as (1).

References

[1] Kempton, *n-1 Graphs*, University of Washington Math REU (2011).

[2] Curtis, Morrow, *Inverse Problems for resistor Networks*, World Scientific, v.13, Series on Applied Mathematics, Singapore, (2000).

[3] Wu, *n to 1 Graphs*, University of Washington Math REU (2012).