Parametrizing Response Matrices

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Abstract

We give parametrizations for general spoked wheel graphs and generalize results given in McCormick's paper for spoked wheels.

Contents

In Megan McCormick's 2005 REU Paper, she discusses parametrization for certain classes of response matrices, e.g. for lattices and circles/spokes graphs. However, the permutations did not take into consideration the so-called "Right Sign" properties of the subdeterminants. In this paper we will outline parametrizations for certain classes of graphs taking the Right Sign properties into consideration.

1 Introduction

Definition A "spoked wheel" graph is given as below (on the next page as is)

A spoked wheel with n boundary vertices will be called an n-spoked wheel.

Claim The response matrix of an n-spoked wheel should be parametrized with 3n elements, as there as 3n edges in an n-spoked wheel.

Spoked Wheel.png

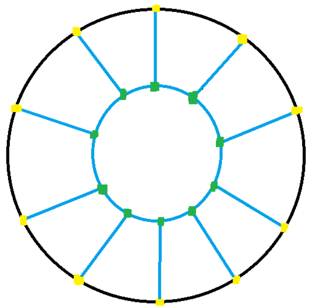


Figure 1

We will number the boundary vertices in a linear order, counterclockwise starting with 1 and ending with n. We will call boundary vertices i and j neighbors if i=j+1, j=i+1, or $\{i, j\} = \{1, n\}$.

Recall that as given in [1], the response matrix is computed from the Kirchhoff matrix of the graph (taking negatives of the conductivites for off-diagonal entries) by taking the Schur complement with respect to the interior-to-interior edges. Furthermore, a given subdeterminant is zero if and only if the corresponding connection between those vertices doesn't exist. The basic computation for additional entries out of those given will consist of using that property, while at the same time making sure that the connections that DO exist will not correspond to zero determinants. Curtis and Morrow give the following theorem:

Theorem 1.1 *Suppose $\Gamma = (G, \gamma)$ is a connected resistor network with boundary. Let $P = (p_1, p_2, \cdots p_k)$ and $Q = (q_1, q_2, \cdots q_k)$ be disjoint sequences of boundary nodes. Then*

$$det\Lambda(P; Q) \times detK(I; I) = (-1)^k \Sigma_{\tau \in S_k} sgn(\tau)\{\Sigma_{\alpha, \tau_\alpha = \tau} \Pi_{e \in E_\alpha}(\gamma(e) D_\alpha)\}$$

The proof can be found in [1], Section 3.7. For the case of circular planar graphs, it can be shown that there is only 1 permutation τ of the boundary nodes P and Q which lead to a connection existing, i.e. that of the circular pair.

2 Parametrizing Spoked Wheels

In McCormick's paper, she gives a detailed parametrization of the (4n+1) spoked wheel. However, there are no specifics given for spoked wheels of other equivalency class mod 4. The class of (4n+2) spoked wheels proves to be the easiest to describe, although the case for a (4n)-spoked wheel is effectively the same, and through this inspiration the case of a (4n+3) spoked wheel also comes out. There are several preliminary results that are needed, taked as read from McCormick's paper.

Lemma 2.1 *A two-connection exists for every circular pair on a wheel graph.*

Proof Suppose the circular pairs are $\{i_1, i_2\}$ and $\{j_1, j_2\}$, where the connections would be between i_1 and j_1 and then i_2 and j_2. Then the path from i_1 to j_1 can proceed clockwise, and the path from i_2 to j_2 counterclockwise, as shown. They clearly don't intersect, so a 2 connection will always exist.

Theorem 2.2 *Suppose* $\Gamma = (G, \gamma)$ *is a circular planar resistor network and* $(P; Q) = (p_1, p_2, \cdots p_k; q_1, q_2, \cdots q_k)$ *is a circular pair of sequences of boundary nodes. Then*
(a) If $(P; Q)$ *are not connected through G, then* $det(\Lambda_{(P;Q)}) = 0$
(b) If $(P; Q)$ *is connected through G, then* $(-1)^k (det\Lambda_{(P;Q)}) > 0$.

Lemma 2.3 *Let* $(P; Q)$ *be a circular pair on a spoked wheel graph. If* $det(\Lambda_{[p_1,p_2,p_3],[q_1,q_2,q_3]}) = 0$, *then one of the entries in the 3 by 3 submatrix can be determined in terms of the other eight.*

Lemma 2.4 *Let* $(P; Q) = (p_1, p_2, p_3; q_1, q_2, q_3)$ *be a circular pair of seqiences of boundary nodes on a generic spoked wheel graph. If none of the nodes in P are neighbors of any nodes in Q, there is no 3-connection between P and Q.*

We first need to ascertain that spoked wheel graphs are recoverable. To determine the connectivities from the response matrix we have to have as many independent entries as edges. Given an n-spoked wheel, there are 3n edges and $\binom{n}{2}$ entries in the response matrix. So the wheel will only be recoverable if $3n \leq \binom{n}{2}$, i.e. if $n \geq 7$. Therefore we should be able to give parametrizations for all spoked wheels with at least 7 spokes.

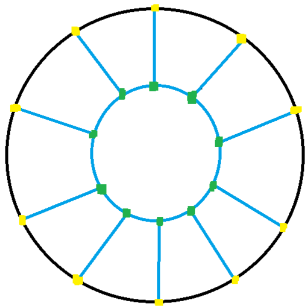
Let us first examine the (4n+2) case. After we analyze this case, the case for general wheels with an even number of vertices becomes obvious, as the techniques used here generalize elsewhere, also to wheels with an odd number of spokes.

A (4n+2) spoked wheel has (4n+2) boundary vertices, hence the response matrix is a (4n+2) square matrix. Split the matrix into four equally sized square blocks:

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{D} \end{pmatrix}$$

We give the parametrization for a 10-spoked graph.

Provides the requisite figure, with yellow vertices representing the boundary, green vertices representing the interior, black edges representing boundary-to-boundary edges, and blue edges representing interior-to-interior or boundary-to-interior edges. Examining intuitively, it is clear that any 2-connection exists, as one path can proceed along one "direction" (counterclockwise) and the other set will be "clockwise," i.e. suppose we



Spoked Wheel.png

Figure 2

have sets $P = (p_1, p_2)$ and $Q = (q_1, q_2)$ of circular pairs with paths from p_1 to q_1 and p_2 to q_2, then we can move into the inner ring and then proceed on disjoint paths.

With 3-connections, the same holds to a certain degree. If two vertices are neighbors, there is no disruption and the 2-connection will exist if we simply move all edges in the pathway into boundary-interior or interior-to-interior edges. If no two vertices are not neighbors, then conversely no 3-connection exists.

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\end{pmatrix}$$

Where the check marks are given and the question marks are to be computed from the existing check marks. We compute the entries as follows:

Under the knowledge that there is no 3 connection on a spoked wheel unless two vertices are neighbors and also that every 2-connection exists on a spoked wheel, we can compute each entry of the matrix from a 3 by 3 determinant that is 0 if we know the other 8 entries.

$$\begin{pmatrix} \Sigma & 41 & 40 & 34 & 26 & 17 & 13 & 1 & 2 & 3 \\ 41 & \Sigma & 45 & 35 & 27 & 18 & 14 & 4 & 5 & 6 \\ 40 & 45 & \Sigma & 36 & 28 & 19 & 15 & 7 & 8 & 9 \\ 34 & 35 & 36 & \Sigma & 29 & 20 & 16 & 12 & 10 & 11 \\ 26 & 27 & 28 & 29 & \Sigma & 25 & 24 & 23 & 22 & 21 \\ 17 & 18 & 19 & 20 & 25 & \Sigma & 33 & 32 & 31 & 30 \\ 13 & 14 & 15 & 16 & 24 & 33 & \Sigma & 42 & 43 & 37 \\ 1 & 4 & 7 & 12 & 23 & 32 & 42 & \Sigma & 44 & 38 \\ 2 & 5 & 8 & 10 & 22 & 31 & 43 & 44 & \Sigma & 39 \\ 3 & 6 & 9 & 11 & 21 & 30 & 37 & 38 & 39 & \Sigma \end{pmatrix}$$

The entries are determined as follows

- 1-9: given as parameters
- 10-11: given as parameters
- 12: computed by taking the subdeterminant $\Lambda_{[2,3,4],[8,9,10]}$ to be 0
- 13-14: given as parameters
- 15: computed by taking the subdeterminant $\Lambda_{[1,2,3],[7,8,9]}$ to be 0
- 16: computed by taking the subdeterminant $\Lambda_{[2,3,4],[7,8,9]}$ to be 0
- 17-18: given as parameter
- 19: computed by taking the subdeterminant $\Lambda_{[1,2,3],[6,7,8]}$ to be 0
- 20: computed by taking the subdeterminant $\Lambda_{[2,3,4],[6,7,8]}$ to be 0
- 21-22: given as parameter
- 23: computed by taking the subdeterminant $\Lambda_{[3,4,5],[8,9,10]}$ to be 0
- 24: computed by taking the subdeterminant $\Lambda_{[3,4,5],[7,8,9]}$ to be 0
- 25-27: given as parameter
- 28: computed from taking subdeterminant $\Lambda_{[1,2,3],[5,6,7]}$ to be 0
- 29: given as parameter
- 30-31: given as parameter
- 32: computed from taking subdeterminant $\Lambda_{[4,5,6],[8,9,10]}$ to be 0
- 33: given as parameter
- 34: computed from taking subdeterminant $\Lambda_{[4,5,6],[9,10,1]}$ to be 0
- 35: computed from taking subdeterminant $\Lambda_{[4,5,6],[10,1,2]}$ to be 0
- 36: given as parameter
- 37: computed from taking subdeterminant $\Lambda_{[5,6,7],[10,1,2]}$ to be 0
- 38: computed from taking subdeterminant $\Lambda_{[6,7,8],[10,1,2]}$ to be 0
- 39: given as parameter
- 40: computed from taking subdeterminant $\Lambda_{[3,4,5],[9,10,1]}$ to be 0
- 41: given as parameter
- 42: given as parameter
- 43: computed from taking subdeterminant $\Lambda_{[9,10,1],[5,6,7]}$ to be 0
- 44: given as parameter
- 45: given as parameter

In the general case, we parameterize

- all entries $a_{k,k+1}$
- top 2 rows and rightmost 2 columns of **B**
- $a_{3,4n}$
- $a_{2n+2,4n+1}$ and $a_{2n+2,4n+2}$
- $a_{1,2n+1}$ and $a_{2,2n+1}$

which will in turn give us everything else. The general idea will be that we propagate up columns, with 3 by 3 subdeterminant known to be 0, with 8 entries given, to compute the last entry. For example, we propagate from the bottom up in the fourth column to compute one entry, then in the fourth-from-bottom row, which then allows us to compute another entry, and so forth. As we propagate through more columns, we obtain more entries, eventually filling up the entirety of blocks **B**. We can equivalently compute the leftmost column of of **D** and the lowermost column of **A**.

Theorem 2.5 *The above parametrization completely determines every entry of a response matrix for a (4n+2)-spoked graph.*

Proof We can trivially compute the bottom two rows of the matrix, using the subdeterminant $\Lambda_{[1,2,4n+2],[2n+1,2n+2,2n+3]}$ to compute entry $a_{4n+2,2n+3}$ and similarly using $\Lambda_{[1,2,4n+2],[2n+1,2n+2,2n+3]}$ to compute entry $a_{4n+1,2n+3}$, as we know eight entries in each subdeterminant. Similarly, we can propagate across the bottom two rows, as well as then across the top two rows, as well as the two leftmost and rightmost columns.

Then we can propagate across the third row, for example using $\Lambda_{[2,3,4],[4n,4n+1,4n+2]}$ to compute $a_{4,4n}$, and so forth, to compute entries in the 4n-row, and then in the 4n-th column, the third row and third column. We simply propagate up, although notably avoiding the entries of form $a_{k,k+1}$ and $a_{k+1,k}$ as they are equivalent to neighboring nodes/connections, and any 3 by 3 determinant containing them will be nonzero.

The idea in parametrizing the rest of blocks **A** and **D** is as follows: As we see, the two leftmost columns in the parametrization on the upper triangular part of the matrix will coincide with the rightmost two columns of the lower triangular part, which will allow us to propagate over and compute all entries in the bottom two rows of the lower triangular part not of form $a_{k+1,k}$, and in turn to propagate up. Similarly, the two lowermost rows in the upper triangular part will coincide with the two uppermost rows of the lower triangular part, whence we can then propagate to the left.

In fact the same parametrization holds for 4n-spoked graphs. Divide the response into for $2n \times 2n$ blocks as below

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{D} \end{pmatrix}$$

We parametrize the following entries, examining only the upper triangular part and noting symmetry for the lower triangular part.

- all entries of form $a_{k,k+1}$
- top 2 rows and rightmost 2 columns of $\mathbf{B}$
- $a_{3,4n-2}$
- $a_{2n+1,4n-1}$ and $a_{2n+1,4n}$
- $a_{1,2n}$ and $a_{2,2n}$

Similarly, in the case of a 4n+3-spoked wheel, the parametrization is the same as in given in McCormick's paper, i.e. say in the 11-spoked case

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\end{array}$$

which is equivalent to the (4n+1) spoked wheel case. The specific entries are, for a 2k+1 spoked wheel, we must first divide the matrix into blocks

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{D} \\ \mathbf{B}^T & \mathbf{C} & \mathbf{E} \\ \mathbf{D}^T & \mathbf{E}^T & \mathbf{F} \end{pmatrix}$$

where $\mathbf{A}$, $\mathbf{F}$, $\mathbf{D}$ are $k \times k$ blocks. $\mathbf{B}$ is $1 \times k$, $\mathbf{E}$ is $k \times 1$, and $\mathbf{C}$ is simply a single entry. We examine only the entries in the upper triangular case, as normal, and use symmetry to determine the entries in the lower triangular half.

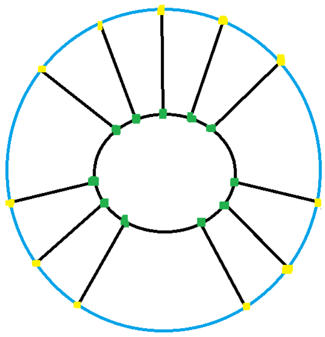
- top two rows and rightmost two columns of $\mathbf{D}$
- top two entries of $\mathbf{B}$
- entries $a_{1,k}$ and $a_{k+1,2k+1}$
- all entries of form $a_{k,k+1}$
- the entry $a_{3,2k-1}$

Theorem 2.6 *The above permutation will determine all entries of the response matrix.*

Proof The proof is essentially the same as McCormick's proof, as given in her paper. Note that the proof does not depend on the 4n+1-nature of the spoked wheel but only upon the fact that the wheel as an odd number of spokes.

3 Discussion

The differing parametrizations are pretty interesting. It does not seem like there is a very good reason for the parametrizations for even and odd spoked wheels to have different parameterizations. From a linear algebraic perspective it boils down to the fact that we don't want to completely determine a 3 by 3 complete subdeterminant simply based on given entries. My best explanation is follows.



Wheel.png

Figure 3

Suppose we have an 11-spoked wheel, such as (on the next page)

and take some collection $P = (p_1, p_2, p_3)$ and $Q = (q_1, q_2, q_3)$ forming a circular pair. In that case, we can see that the "spacing" between the vertices is critical. The entry $a_{2,5}$ is not given in the parametrization. Intuitively this can be placed to a difference in the number of vertices, between even and odd, as the placement (if the vertices are only 3 apart, i.e. vertices n and n+3), in particular between what we consider as vertices 2 and 5 (equivalent up to rotation).

We see that vertices 3 apart must be either the "inner" (those closest to each other) or in the middle. If the latter is true, then the "inner" vertices are neighbors, so a 3-connection exists in that case based on our logic with 2-connections. Thus all pairs of sets of circular pairs with non-existent connections must involve such vertices as the "inner" pair. In this case our examination switches to the other 2 vertices of each set P and Q, for which we determine that we have to go "inwards" as stated in the case of a 2-connection below.

As explained we can do the follows. In as follows

4 Further Research

• There are possible inroads to certain classes of annular planar graphs, such as layered wheels. McCormick gives us some insight into the $(4, 2)$ layered wheel case.

• General Lattices are also worth a look. There have been parametrizations given for certain classes, such as square lattices and $n \times (n + 1)$ lattices. The issue with giving parmetrizations to general lattices turns out to be the disparity between the number of vertices on each edge and the sizes of the blocks in the response that then arise. If we were to examine a general n by m lattice, then the response would be a square matrix of size $2(n + m)$ that

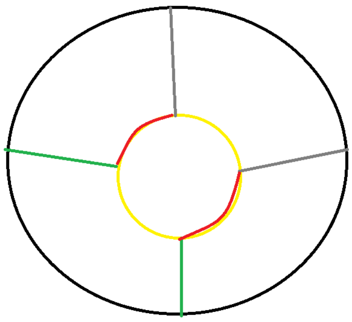


Figure 4

• Also certain classes of annular planar graphs, for the same reason. The particular class of $(2n, n)$ graphs could prove interesting, as with McCormick's presentation of the $(4, 2)$ class graphs. In the case of an annular planar graph, however, the determinantal connection formula is much more complicated, as we need to examine the respective permutation signs of any given pair of sets of equal numbers of vertices and since every connection of size less than n exists.

McCormick's paper gives the following permutation. There are eight boundary vertices, saying 1-4 on the outside circular boundary and 5-8 on the inner circular boundary. The permutation matrix is naturally an 8 by 8 matrix. There are no nonzero subdeterminants, in fact, so there are parametrizations, given by any 20 entries and using the fact that no 4-connection exists to compute one entry of a 4 by 4 determinant by knowing that no subdeterminant is 0 and then using the minor expansion to compute the last entry. In fact the other issue presented in this is that since no such determinants exist with determinant 0 that are also small enough to fit into the response matrix. This is because the annular planar graph, in this case, does not have any non-existent connections with the correct size, and therefore it will not have any subdeterminants whose determinants are 0.

Explanation is as follows: Suppose I have an arbitrary 4-connection. If I can prove that this exists, then all smaller connections must exist too by the above logic. If one set is on the inner boundary and the other on the outer the connection is obvious. If there is a direct correspondence it is also obvious. Suppose that there is no direct correspondence between the vertices of the 2 different sets, i.e. they do not exactly "match up" as according to the direct connections in the graph.

References

[1] Edward B. Curtis and James A. Morrow

Inverse Problems for Electrical Networks
World Scientific: New York, New York, 2011.

[2] Megan McCormick
Parametrizing Response Matrices
http://www.math.washington.edu/ reu/papers/2005/megan/parameterization5.pdf