# CALDERÓN'S INVERSE PROBLEM FOR ANISOTROPIC CONDUCTIVITY IN THE PLANE

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**Abstract:** We study inverse conductivity problem for an anisotropic conductivity $\sigma \in L^\infty$ in bounded and unbounded domains. Also, we give applications of the results in the case when Dirichlet-to-Neumann and Neumann-to-Dirichlet maps are given only on a part of the boundary.

## 1. INTRODUCTION

Let us consider the anisotropic conductivity equation in two dimensions

$$\nabla\cdot \sigma\nabla u = \sum_{j,k=1}^{2} \frac{\partial}{\partial x^j}\sigma^{jk}(x)\frac{\partial}{\partial x^k}u = 0 \text{ in } \Omega, \qquad u|_{\partial\Omega} = \phi. \tag{1}$$

Here $\Omega \subset \mathbb{R}^2$ is a simply connected domain. The conductivity $\sigma = [\sigma^{jk}]_{j,k=1}^2$ is a symmetric, positive definite matrix function, and $\phi \in H^{1/2}(\partial\Omega)$ is the prescribed voltage on the boundary. Then it is well known that equation (1) has a unique solution $u \in H^1(\Omega)$.

In the case when $\sigma$ and $\partial\Omega$ are smooth, we can define the voltage-to-current (or Dirichlet-to-Neumann) map by

$$\Lambda_\sigma(\phi) = Bu|_{\partial\Omega} \tag{2}$$

where

$$Bu = \nu\cdot \sigma\nabla u, \tag{3}$$

$u \in H^1(\Omega)$ is the solution of (1), and $\nu$ is the unit normal vector of $\partial\Omega$. Applying the divergence theorem, we have

$$Q_{\sigma,\Omega}(\phi) := \int_\Omega \sum_{j,k=1}^{2} \sigma^{jk}(x)\frac{\partial u}{\partial x^j}\frac{\partial u}{\partial x^k}dx = \int_{\partial\Omega} \Lambda_\sigma(\phi)\phi\, dS, \tag{4}$$

where $dS$ denotes the arc lenght on $\partial\Omega$. The quantity $Q_{\sigma,\Omega}(\phi)$ represents the power needed to maintain the potential $\phi$ on $\partial\Omega$. By symmetry of $\Lambda_\sigma$, knowing $Q_{\sigma,\Omega}$ is equivalent with knowing $\Lambda_\sigma$. For general $\Omega$ and $\sigma \in L^\infty(\Omega)$, the trace $u|_{\partial\Omega}$ is defined as the equivalence class of $u$ in $H^1(\Omega)/H^1_0(\Omega)$ (see [6]) and formula (4) is used to define the map $\Lambda_\sigma$.

If $F:\Omega\to\Omega$, $\quad F(x)=(F^1(x),F^2(x))$, is a diffeomorphism with $F|_{\partial\Omega}=\text{Identity}$, then by making the change of variables $y=F(x)$ and setting $v=u\circ F^{-1}$ in the first integral in (4), we obtain

$$\nabla\cdot(F_*\sigma)\nabla v=0\quad\text{in }\Omega,$$

where

$$(F_*\sigma)^{jk}(y)=\frac{1}{\det[\frac{\partial F^j}{\partial x^k}(x)]}\sum_{p,q=1}^{2}\frac{\partial F^j}{\partial x^p}(x)\,\frac{\partial F^k}{\partial x^q}(x)\sigma^{pq}(x)\bigg|_{x=F^{-1}(y)},\tag{5}$$

or

$$F_*\sigma(y)=\frac{1}{J_F(x)}DF(x)\,\sigma(x)\,DF(x)^t\bigg|_{x=F^{-1}(y)},\tag{6}$$

is the push-forward of the conductivity $\sigma$ by $F$. Moreover, since $F$ is identity at $\partial\Omega$, we obtain from (4) that

$$\Lambda_{F_*\sigma}=\Lambda_\sigma.$$

Thus, the change of coordinates shows that there is a large class of conductivities which give rise to the same electrical measurements at the boundary.

We consider here the converse question, that if we have two conductivities which have the same Dirichlet-to-Neumann map, is it the case that each of them can be obtained by pushing forward the other.

In applied terms, this inverse problem to determine $\sigma$ (or its properties) from $\Lambda_\sigma$ is also known as *Electrical Impedance Tomography.* It has been proposed as a valuable diagnostic, see [11].

In the case where $\sigma^{jk}(x)=\sigma(x)\delta^{jk}$, $\sigma(x)\in\mathbb{R}_+$, the metric is said to be isotropic. In 1980 it was proposed by A. Calderón [9] that in the isotropic case any bounded conductivity $\sigma(x)$ might be determined solely from the boundary measurements, i.e., from $\Lambda_\sigma$. Recently this has been confirmed in the two dimensional case (c.f. [6]). In the case when isotropic $\sigma$ is smoother than just a $L^\infty$-function, the same conclusion is known to hold also in higher dimensions.

The first global uniqueness result was obtained for a $C^\infty$–smooth conductivity in dimension $n\geq 3$ by J. Sylvester and G. Uhlmann in 1987 [36]. In dimension two A. Nachman [29] produced in 1995 a uniqueness result for conductivities with two derivatives. The corresponding algorithm has been successfully implemented and proven to work efficiently even with real data [33, 27]. The reduction of regularity assumptions has since been under active study. In dimension two the optimal $L^\infty$-regularity was obtained in [6]. In dimension $n\geq 3$ the uniqueness has presently been shown for isotropic conductivities $\sigma\in W^{3/2,\infty}(\Omega)$ in [31] and for globally $C^{1+\varepsilon}$–smooth isotropic conductivities having only co-normal singularities in [13].

Also, the stability of reconstructions of the inverse conductivity problem have been extensively studied. For these results, see [2, 3, 4] where stability results are based on reconstruction techniques of [8] in dimension two and those of [28] in dimensions $n \geq 3$.

In anisotropic case, where $\sigma$ is a matrix function and the problem is to recover the conductivity $\sigma$ up to the action of a class of diffeomorphisms, much less is known. In dimensions $n \geq 3$ it is generally known only that piecewise analytic conductivities can be constructed (see [18, 19]). For Riemannian manifolds this kind of technique has been generalized in [22, 20, 21]. In dimension $n = 2$ the inverse problem has been considered by J. Sylvester [35] for $C^3$ and Z. Sun and G. Uhlmann [34] for $W^{1,p}$-conductivities. The idea of [35] and [34] is that under quasiconformal change of coordinates (cf. [1, 17]) any anisotropic conductivity can be changed to isotropic one, see also section 3 below. The purpose of this paper is to carry this technique over to the $L^\infty$–smooth case and then use the result of [6] to obtain uniqueness up to the group of diffeomorphisms.

The advantage of the reduction of the smoothness assumptions up to $L^\infty$ does not lie solely on the fact that many conductivities have jump-type singularities but it also allows us to consider much more complicated singular structures such as porous rocks [10]. Moreover it is important that this approach enables us to consider general diffeomorphisms. Thus anisotropic inverse problems in half-space or exterior domains can be solved simultaneously. This will be considered in Section 2.

If $\Omega \subset \mathbb{R}^2$ is a bounded domain, it is convenient to consider the class of matrix functions $\sigma = [\sigma^{jk}]$ such that

$$[\sigma^{ij}] \in L^\infty(\Omega; \mathbb{R}^{2\times 2}), \quad [\sigma^{ij}]^t = [\sigma^{ij}], \quad C_0^{-1} I \leq [\sigma^{ij}] \leq C_0 I \tag{7}$$

where $C_0 > 0$. In sequel, the minimal possible value of $C_0$ is denoted by $C_0(\sigma)$. We use the notation

$$\Sigma(\Omega) = \{\sigma \in L^\infty(\Omega; \mathbb{R}^{2\times 2}) \quad | \quad C_0(\sigma) < \infty\}.$$

Note that it is necessary to require $C_0(\sigma) < \infty$ as otherwise there would be counterexamples showing that even the equivalence class of the conductivity can not be recovered [14, 15].

Our main goal in this paper is to show that an anisotropic $L^\infty$–conductivity can be determined up to a $W^{1,2}$-diffeomorphism:

**Theorem 1.** *Let $\Omega \subset \mathbb{R}^2$ be a simply connected bounded domain and $\sigma \in L^\infty(\Omega; \mathbb{R}^{2\times 2})$. Suppose that the assumptions (7) are valid. Then the Dirichlet-to-Neumann map $\Lambda_\sigma$ determines the equivalence class*

$$E_\sigma = \{\sigma_1 \in \Sigma(\Omega) \quad | \quad \sigma_1 = F_*\sigma,\ F : \Omega \to \Omega \text{ is } W^{1,2}\text{-diffeomorphism and } F|_{\partial\Omega} = I\}.$$

We prove this result in Section 3.

Finally, note that the $W^{1,2}$–diffeomorphisms $F$ preserving the class $\Sigma(\Omega)$ are precisely the quasiconformal mappings. Namely, if $\sigma_0 \in \Sigma(\Omega)$ and $\sigma_1 = F_*(\sigma_0) \in$

$\Sigma(F(\Omega))$ then

$$\frac{1}{C_0}||DF(x)||^2 I \leq DF(x)\,\sigma_0(x)\,DF(x)^t \leq C_1 J_F(x) I \tag{8}$$

where $I = [\delta^{ij}]$ and we obtain

$$||DF(x)||^2 \leq K J_F(x), \quad \text{for a.e.} \quad x \in \Omega \tag{9}$$

where $K = C_1 C_0 < \infty$. Conversely, if (9) holds and $F$ is $W^{1,2}_{loc}$-homeomorphism then $F_*\sigma \in \Sigma(F(\Omega))$ whenever $\sigma \in \Sigma(\Omega)$. Furhtermore, recall that a map $F : \Omega \to \widetilde{\Omega}$ is quasiregular if $F \in W^{1,2}_{loc}(\Omega)$ and the condition (9) holds. Moreover, a map $F$ is quasiconformal if it is quasiregular and a $W^{1,2}$–homeomorphism.

## 2. Consequences and applications of Theorem 1

Here we consider applications of the diffeomorphism-technique to various inverse problem. The formulated results, Theorems 2.1–2.3 are proven in Section 4.

2.1. **Inverse Problem in the Half Space.** Inverse problem in half space is of crucial importance in geophysical prospecting, seismological imaging, non-destructive testing etc. For instance, the imaging of soil was the original motivation of Calderón's seminal paper [9]. As we can use a diffeomorphism to map the open half space to the unit disc, we can apply the previous result for the half space case. One should observe that in this deformation even infinitely smooth conductivities can become non-smooth at the boundary (e.g. conductivity oscillating near infinity produces a non-Lipschitz conductivity in push-forward) and thus the low-regularity result [6] is essential for the problem.

Thus, for $\sigma \in \Sigma(\mathbb{R}^2_-)$ let us consider the problem

$$\nabla\cdot \sigma\nabla u = 0 \text{ in } \mathbb{R}^2_- = \{(x^1,x^2)\,|\,x^2<0\}, \tag{10}$$

$$u|_{\partial\mathbb{R}^2_-} = \phi, \tag{11}$$

$$u \in L^\infty(\mathbb{R}^2_-). \tag{12}$$

Notice that here the radiation condition at infinity (12) is quite simple. We assume just that the potential $u$ does not blow up at infinity. The equation (10–12) is uniquely solvable and as before we can define

$$\Lambda_\sigma : H^{1/2}_{comp}(\partial\mathbb{R}^2_-) \to H^{-1/2}(\partial\mathbb{R}^2_-), \quad \phi \mapsto \nu\cdot \sigma\nabla u|_{\partial\mathbb{R}^2_-}.$$

**Theorem 2.1.** *The map $\Lambda_\sigma$ determines the equivalence class*

$$E_\sigma = \{\sigma_1 \in \Sigma(\mathbb{R}^2_-) \ \ | \ \ \sigma_1 = F_*\sigma,\ F:\mathbb{R}^2_- \to \mathbb{R}^2_- \text{ is } W^{1,2}\text{-diffeomorphism}, \ F|_{\partial\mathbb{R}^2_-} = I\}.$$

*Moreover, each orbit $E_\sigma$ contains at most one isotropic conductivity, and consequently if $\sigma$ is known to be isotropic, it is determined uniquely by $\Lambda_\sigma$.*

Note that the natural growth requirement $\lim_{|z|\to\infty}|F(z)|=\infty$ follows automatically from the above assumptions on $F$.

2.2. **Inverse Problem in the Exterior Domain.** An inverse problem similar to that of the half space can be considered in an exterior domain where one wants to find the conductivity in a complement of a bounded simply connected domain. This type of problem is encountered in cases where measurement devices are embedded to an unknown domain.

In the case of $S=\mathbb{R}^2\setminus\overline{D}$, where $D$ is a bounded Jordan domain, we consider the problem

$$\nabla\cdot\sigma\nabla u = 0 \quad \text{in } S, \tag{13}$$

$$u|_{\partial S} = \phi\in H^{1/2}(\partial S), \tag{14}$$

$$u \in L^\infty(S). \tag{15}$$

Again, the radiation condition (15) of infinity is only that the solution is uniformly bounded. For this equation we define

$$\Lambda_\sigma : H^{1/2}(\partial S)\to H^{-1/2}(\partial S),\quad \phi\mapsto \nu\cdot\sigma\nabla u|_{\partial S}.$$

Surprisingly, the result is different from the half-space case. The reason for this is the phenomenon that the group of diffeomorphisms preserving the data does not fix the point of the infinity. More precisely, there are two points $x_0,x_1\in S\cup\{\infty\}$ such that $F(x_0)=\infty$, $F^{-1}(x_1)=\infty$, and $F:S\setminus\{x_0\}\to S\setminus\{x_1\}$. In particular, this means that the uniqueness does not hold up to diffeomorphisms mapping the exterior domain to itself.

For convenience, we compactify $S$ by adding one infinity point, denote $\overline{S}=S\cup\{\infty\}$, and define $\sigma(\infty)=1$. We say that $F:\overline{S}\to\overline{S}$ is a $W^{1,2}$-diffeomorphism if $F$ is homeomorphism and a $W^{1,2}$-diffeomorphism in spherical metric [1].

**Theorem 2.2.** *Let $\sigma\in\Sigma(S)$. Then the map $\Lambda_\sigma$ determines the equivalence class*

$$E_{\sigma,S}=\{\sigma_1\in\Sigma(S)\ \mid\ \sigma_1=F_*\sigma,\ F:\overline{S}\to\overline{S}\ \textit{is a } W^{1,2}\textit{-diffeomorphism},\ F|_{\partial S}=I\,\}.$$

*Moreover, if $\sigma$ is known to be isotropic, it is determined uniquely by $\Lambda_\sigma$.*

2.3. **Data on Part of the Boundary.** In many inverse problems data is measured only on a part of the boundary. For the conductivity equation in dimensions $n\geq 3$ it has been shown that if the measurements are done on a part of the boundary, then the integrals of the unknown conductivity over certain 2-planes can be determined [12]. In one-dimensional inverse problems partial data is often considered with two different boundary conditions, see e.g. [24, 25]. For instance, in the inverse spectral problem for a one-dimensional Schrödinger operator, it is known that measuring spectra corresponding to two different boundary conditions determine the potential uniquely. Here we consider similar results for the 2-dimensional conductivity equation assuming that we know measurements on part of the boundary for two different boundary conditions.

Let us consider the conductivity equation with the Dirichlet boundary condition

$$\begin{aligned} \nabla\cdot \sigma\nabla u &= 0 \text{ in } \Omega, \\ u|_{\partial\Omega} &= \phi \end{aligned} \tag{16}$$

and with the Neumann boundary condition

$$\begin{aligned} \nabla\cdot \sigma\nabla v &= 0 \text{ in } \Omega, \\ \nu\cdot \sigma\nabla v|_{\partial\Omega} &= \psi, \end{aligned} \tag{17}$$

normalized by $\int_{\partial\Omega} v\, dS = 0$. Let $\Gamma \subset \partial\Omega$ be open. We denote by $H^s_0(\Gamma)$ the space of functions $f \in H^s(\partial\Omega)$ that are supported on $\Gamma$ and by $H^s(\Gamma)$ the space of restrictions $f|_\Gamma$ of $f \in H^s(\partial\Omega)$. We define the Dirichlet-to-Neumann map $\Lambda_\Gamma$ and Neumann-to-Dirichlet map $\Sigma_\Gamma$ by

$$\Lambda_\Gamma : H^{1/2}_0(\Gamma) \to H^{-1/2}(\Gamma), \quad \phi \mapsto (\nu\cdot \sigma\nabla u)|_\Gamma,$$
$$\Sigma_\Gamma : H^{-1/2}_0(\Gamma) \to H^{1/2}(\Gamma), \quad \psi \mapsto v|_\Gamma.$$

**Theorem 2.3.** *Let $\Gamma \subset \partial\Omega$ be open. Then knowing $\partial\Omega$ and both of the maps $\Lambda_\Gamma$ and $\Sigma_\Gamma$ determine the equivalence class*

$$E_{\sigma,\Gamma} = \{\sigma_1 \in \Sigma(\Omega) \ \ | \ \ \sigma_1 = F_*\sigma,\ F : \Omega \to \Omega \text{ is a } W^{1,2}\text{-diffeomorphism}, \ F|_\Gamma = I\}.$$

*Moreover, if $\sigma$ is known to be isotropic, it is determined uniquely by $\Lambda_\Gamma$ and $\Sigma_\Gamma$.*

## 3. PROOF OF THEOREM 1

3.1. **Preliminary Considerations.** In the following we identify $\mathbb{R}^2$ and $\mathbb{C}$ by the map $(x^1, x^2) \mapsto x^1 + ix^2$ and denote $z = x^1 + ix^2$. We use the standard notations

$$\partial_z = \frac{1}{2}(\partial_1 - i\partial_2), \quad \partial_{\overline{z}} = \frac{1}{2}(\partial_1 + i\partial_2),$$

where $\partial_j = \partial/\partial x^j$. Below we consider $\sigma : \Omega \to \mathbb{R}^{2\times 2}$ to be extended as a function $\sigma : \mathbb{C} \to \mathbb{R}^{2\times 2}$ by defining $\sigma(z) = I$ for $z \in \mathbb{C}\setminus\Omega$. In following, we denote $C_0 = C_0(\sigma)$. For the conductivity $\sigma = \sigma^{jk}$ we define the corresponding Beltrami coefficient (see [35, 6, 17])

$$\mu_1(z) = \frac{-\sigma^{11}(z) + \sigma^{22}(z) - 2i\,\sigma^{12}(z)}{\sigma^{11}(z) + \sigma^{22}(z) + 2\sqrt{\det(\sigma(z))}}. \tag{18}$$

The coefficient $\mu_1(z)$ satisfies $|\mu_1(z)| \le \kappa < 1$ and is compactly supported.

Next we introduce a $W^{1,2}$-diffeomorphism (not necessarily preserving the boundary) that transforms the conductivity to an isotropic one.

**Lemma 3.1.** *There is a quasiconformal homeomorphism $F:\mathbb{C}\to\mathbb{C}$ such that*

$$F(z) = z + \mathcal{O}(\frac{1}{z}) \quad \text{as } |z|\to\infty \tag{19}$$

*and such that $F\in W^{1,p}_{loc}(\mathbb{C};\mathbb{C})$, $2<p<p(C_0)=\frac{2C_0}{C_0-1}$ for which*

$$(F_*\sigma)(z) = \widetilde{\sigma}(z) := \det(\sigma(F^{-1}(z)))^{\frac{1}{2}}. \tag{20}$$

**Proof.** The proof can be found from [35] for $C^3$-smooth conductivities, see also [17]. Because of varying sign conventions, we sketch here the proof for readers convenience. We need to find a quasiconformal map $F$ such that

$$DF\,\sigma\,DF^t = \sqrt{\det(\sigma)}\,J_F I \tag{21}$$

where $J_F=\det(DF)$ is the Jacobian of $F$. Denoting by $G=[g_{ij}]_{i,j=1}^2$ the inverse of the matrix $\sigma/\sqrt{\det(\sigma)}$ we see that the claim is equivalent to proving the following:

For any symmetric matrix $G$ with $\det(G)=1$ and $\frac{1}{K}I\le G\le KI$ there exists a quasiconformal map $F$ such that

$$J_F G = DF^t DF. \tag{22}$$

Next, the non-linear equation (22) can be replaced in complex notation by a linear one. Indeed, if $F=u+iv$ then (22) is equivalent to

$$\nabla v = JG^{-1}\nabla u, \quad \text{where } J=\begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix}. \tag{23}$$

This follows readily from the identity

$$DF^t J = \det(DF)\,J\,(DF)^{-1} = JG^{-1}DF^t$$

where the latter equality uses (22). The matrix $J$ corresponds to the multiplication with the imaginary unit $i$ in complex notation. Denoting by $R=\begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}$ (the matrix corresponding to complex conjugation) we see that (23) is equivalent to

$$\nabla u + J\nabla v = (G-1)(G+1)^{-1}(\nabla u - J\nabla v). \tag{24}$$

But, $\nabla u + J\nabla v = 2\partial_{\overline{z}}F$ and $R(\nabla u - J\nabla v) = 2\partial_z F$ in complex notation and hence (24) becomes

$$\partial_{\overline{z}}F = \mu_1(z)\partial_z F \tag{25}$$

where

$$\mu_1 = (G-1)(G+1)^{-1}R = (\sqrt{\det\sigma}I-\sigma)(\sqrt{\det\sigma}I+\sigma)^{-1}R.$$

A direct calculation gives

$$\mu_1 = \frac{1}{2+g_{11}+g_{22}}\begin{pmatrix} g_{11}-g_{22} & -2g_{12}\\ 2g_{12} & g_{11}-g_{22}\end{pmatrix}$$

which shows that the matrix $\mu_1 = (G-1)(G+1)^{-1}R$ corresponds to a multiplication operator (in complex notation) by the function

$$\mu_1(z) = \frac{g_{11}(z) - g_{22}(z) + 2ig_{12}(z)}{2 + g_{11}(z) + g_{22}(z)}.$$

This gives (18) since $G^{-1} = \sigma/\sqrt{\det(\sigma)}$. Since $|\mu_1(z)| \leq \kappa < 1$ for every $z \in \mathbb{C}$ it is well known by [1, Thm. V.1, V.2] that the equation (25) with asymptotics

$$F(z) = z + \mathcal{O}(\frac{1}{z}), \quad \text{as } z \to \infty$$

has a unique (quasiconformal) solution $F$. The fact that $F \in W^{1,p}_{loc}(\mathbb{C};\mathbb{C})$, $2 < p < \frac{2C_0}{C_0-1}$ follows from [5]. $\square$

In this section we denote by $F = F_\sigma$ the diffeomorphism determined by Lemma 3.1. We also denote $\widetilde\Omega = F(\Omega)$ where $F$ is as in Lemma 3.1. Note that (19) implies also that

$$F^{-1}(z) = z + \mathcal{O}(\frac{1}{|z|}) \quad \text{as } |z| \to \infty. \tag{26}$$

Later we will use the obvious fact that the knowledge of map $\Lambda_\sigma$ is equivalent to the knowledge of the Cauchy data pairs

$$C_\sigma = \{(u|_{\partial\Omega}, \nu\cdot \sigma\nabla u|_{\partial\Omega}) \mid u \in H^1(\Omega),\ \nabla\cdot \sigma\nabla u = 0\}.$$

In addition to the anisotropic conductivity equation (1) we consider the corresponding conductivity equation with isotropic conductivity. For these considerations, we observe that if $u$ satisfies equation (1) and $\widetilde\sigma$ is as in (20) then the function

$$w(x) = u(F^{-1}(x)) \in H^1(\widetilde\Omega)$$

satisfies the isotropic conductivity equation

$$\begin{aligned} &\nabla\cdot \widetilde\sigma\nabla w = 0 \quad \text{in } \widetilde\Omega, \\ &w|_{\partial\widetilde\Omega} = \phi\circ F^{-1}. \end{aligned} \tag{27}$$

Thus, $\widetilde\sigma$ can be considered as a scalar, isotropic $L^\infty$–smooth conductivity $\widetilde\sigma I$. We continue also the function $\widetilde\sigma : \widetilde\Omega \to \mathbb{R}_+$ to a function $\widetilde\sigma : \mathbb{C} \to \mathbb{R}_+$ by defining $\widetilde\sigma(x) = 1$ for $x \in \mathbb{C} \setminus \widetilde\Omega$.

3.2. **Conjugate Functions.** While solving the isotropic inverse problem in [6], the interplay of the scalar conductivities $\sigma(x)$ and $\frac{1}{\sigma(x)}$ played a crucial role. Motivated by this, we define

$$\widehat\sigma^{jk}(x) = \frac{1}{\det(\sigma(x))}\sigma^{jk}(x).$$

Note that for a isotropic conductivity $\widehat\sigma = 1/\sigma$.

Let now $F$ be the quasiconformal map defined in Lemma 3.1 and $\widetilde{\sigma} = F_*\sigma$ as in (20). We say that $\widehat{w} \in H^1(\widetilde{\Omega})$ is a $\widetilde{\sigma}$-harmonic conjugate of $w$ if

$$\begin{aligned} \partial_1 \widehat{w}(z) &= -\widetilde{\sigma}(z)\partial_2 w(z), \\ \partial_2 \widehat{w}(z) &= \widetilde{\sigma}(z)\partial_1 w(z) \end{aligned} \tag{28}$$

for $z = x^1 + ix^2 \in \mathbb{C}$. Using $\widehat{w}$ we define the function $\widehat{u}$ that we call the $\sigma$-harmonic conjugate of $u$,

$$\widehat{u}(x) = \widehat{w}(F(x)).$$

To find the equation governing $\widehat{u}$, it easily follows that (c.f. [6])

$$\nabla\cdot \frac{1}{\widetilde{\sigma}}\nabla \widehat{w} = 0 \quad \text{ in } \widetilde{\Omega}, \tag{29}$$

and by changing coordinates to $y = F(x)$ we see that $1/\widetilde{\sigma} = F_*\widehat{\sigma}$. These facts imply

$$\nabla\cdot \widehat{\sigma}\nabla \widehat{u} = 0 \quad \text{ in } \Omega. \tag{30}$$

Thus $\widehat{u}$ is the $\widehat{\sigma}$-harmonic conjugate function of $u$ and we have

$$\nabla \widehat{u} = J\sigma\nabla u, \quad \nabla u = J\widehat{\sigma}\nabla\widehat{u}. \tag{31}$$

Since $u$ is a solution of the conductivity equation if and only if $u + c$, $c \in \mathbb{C}$, is solution, we see from (31) that the Cauchy data pairs $C_\sigma$ determine the pairs $C_{\widehat{\sigma}}$ and vice versa. Thus we get, almost free, that $\Lambda_\sigma$ determines $\Lambda_{\widehat{\sigma}}$, too.

Let us next consider the function

$$f(z) = w(z) + i\widehat{w}(z). \tag{32}$$

By [6], it satisfies the pseudo-analytic equation of second type,

$$\partial_{\overline{z}} f = \widetilde{\mu}_2\, \overline{\partial_z f} \tag{33}$$

where

$$\widetilde{\mu}_2(z) = \frac{1-\widetilde{\sigma}(z)}{1+\widetilde{\sigma}(z)}, \quad |\widetilde{\mu}_2(z)| \le \frac{C_0 - 1}{C_0 + 1} < 1. \tag{34}$$

Using this Beltrami coefficient, we define $\mu_2 = \widetilde{\mu}_2 \circ F$.

We will need the following:

**Lemma 3.2.** *Let $g = f \circ F$ where $F : \Omega \to \widetilde{\Omega}$ is a quasiconformal homeomorphism and $f$ is a quasiregular map satisfying*

$$\partial_{\overline{z}} f = \widetilde{\mu}_2 \overline{\partial_z f} \quad \text{ and } \quad \partial_{\overline{z}} F = \mu_1 \partial_z F, \tag{35}$$

*where $\widetilde{\mu}_2 = \mu_2 \circ F^{-1}$ and $\mu_1$ satisfies $|\mu_j| \le \kappa < 1$ and $\mu_2$ is real. Then $g$ is quasiregular and satisfies*

$$\partial_{\overline{z}} g = \nu_1 \partial_z g + \nu_2 \overline{\partial_z g}, \tag{36}$$

*where*

$$\nu_1=\mu_1\frac{1-\mu_2^2}{1-|\mu_1|^2\mu_2^2},\qquad \textit{and}\quad \nu_2=\mu_2\frac{1-|\mu_1|^2}{1-|\mu_1|^2\mu_2^2}. \tag{37}$$

*Conversely, if $g$ satisfies (36) with $\nu_2$ real and $|\nu_1|+|\nu_2|\le\kappa'<1$ then there exists unique $\mu_1$ and $\mu_2$ such that (37) holds and $f=g\circ F^{-1}$ satisfies (35).*

**Proof.** We apply the chain rule

$$\begin{aligned}\partial(f\circ F) &= (\partial f)\circ F\cdot\partial F+(\overline{\partial}f)\circ F\cdot\overline{\overline{\partial}F},\\ \overline{\partial}(f\circ F) &= (\partial f)\circ F\cdot\overline{\partial}F+(\overline{\partial}f)\circ F\cdot\overline{\partial F},\end{aligned}$$

and obtain

$$\nu_1\partial_z g+\nu_2\overline{\partial_z g}=\partial f\circ F\cdot\partial F\cdot(\nu_1+\nu_2\mu_1\mu_2)+\overline{\partial f\circ F}\cdot\overline{\partial}F\cdot(\nu_2+\nu_1\overline{\mu}_1\mu_2)$$

and

$$\partial_{\overline{z}}g=\mu_1\cdot\partial f\circ F\cdot\partial F+\mu_2\cdot\overline{\partial f\circ F}\cdot\overline{\partial}F.$$

Hence, if $\mu_1,\mu_2,\nu_1$, and $\nu_2$ are related so that

$$\mu_1=\nu_1+\nu_1\mu_2,\quad \mu_2=\nu_2+\overline{\nu}_1\mu_2,$$

we see that (36) and (37) are satisfied.

It is not difficult to see that for each $\nu_1$ and $\nu_2$ (37) has a unique solution $\mu_1,\mu_2$ with $|\mu_j|\le\kappa'<1$, $j=1,2$. Again, the general theory of quasiregular maps [1] implies that (35) has a solution and the factorization $g=f\circ F$ holds. □

Note that (37) implies that

$$|\nu_1|+|\nu_2|=\frac{|\mu_1|+|\mu_2|}{1+|\mu_1|\,|\mu_2|}\le\frac{2\kappa}{1+\kappa^2}<1. \tag{38}$$

Lemma 3.2 has the following important corollary, that is the main goal of this subsection.

**Corollary 3.3.** *If $u\in H^1(\Omega)$ is a real solution of the conductivity equation (1), there exists $\widehat{u}\in H^1(\Omega)$, unique up to a constant, such that $g=u+i\widehat{u}$ satisfies (36) where*

$$\nu_1=\frac{\sigma^{22}-\sigma^{11}-2i\sigma^{12}}{1+\operatorname{tr}\sigma+\det(\sigma)},\qquad \textit{and}\quad \nu_2=\frac{1-\det(\sigma)}{1+\operatorname{tr}\sigma+\det(\sigma)}. \tag{39}$$

*Conversely, if $\nu_1$ and $\nu_2$, $|\nu_1|+|\nu_2|\le\kappa'<1$ are as in Lemma 3.2 then there are unique $\sigma$ and $\widehat{\sigma}$ such that for any solution $g$ of (36) $u=Re\,g$ and $\widehat{u}=Im\,g$ satisfy the conductivity equations*

$$\nabla\cdot\sigma\nabla u=0,\quad \textit{and}\quad \nabla\cdot\widehat{\sigma}\nabla\widehat{u}=0. \tag{40}$$

**Proof.** Since $g = f \circ F$ where $f = w + i\widehat{w}$ according to (32), we obtain immediately the existence of $\widehat{u} = \widehat{w} \circ F$. Thus we need only to calculate $\nu_1$ and $\nu_2$ in terms of $\sigma$. Note that by (18),

$$|\mu_1|^2 = \frac{\mathrm{tr}\,(\sigma) - 2\det(\sigma)^{1/2}}{\mathrm{tr}\,(\sigma) + 2\det(\sigma)^{1/2}}. \tag{41}$$

We recall that

$$\mu_2 = \frac{1 - \det(\sigma)^{1/2}}{1 + \det(\sigma)^{1/2}} \tag{42}$$

and thus

$$1 - |\mu_1|^2\mu_2^2 = \frac{4(\det(\sigma)^{1/2}\mathrm{tr}\,(\sigma) + (1 + \det(\sigma))\det(\sigma)^{1/2})}{(1 + \det(\sigma)^{1/2})^2(\mathrm{tr}\,(\sigma) + 2\det(\sigma)^{1/2})}$$

which readily yields (39) from (37).

Note that since $\nu_1$ and $\nu_2$ uniquely determine $\mu_1$ and $\mu_2$, they by (41) and (42) also determine $\det(\sigma)$ and $\mathrm{tr}(\sigma)$. After observing this, it is clear from (39) that $\sigma$ is uniquely determined by $\nu_1$ and $\nu_2$. $\square$

Now one can write equations (31) in more explicit form

$$\tau \cdot \nabla \widehat{u}|_{\partial\Omega} = \Lambda_\sigma(u|_{\partial\Omega}) \tag{43}$$

where $\tau = (-\nu_2, \nu_1)$ is a unit tangent vector of $\partial\Omega$. As $\mathrm{Re}\, g|_{\partial\Omega} = u|_{\partial\Omega}$ and $\mathrm{Im}\, g|_{\partial\Omega} = \widehat{u}|_{\partial\Omega}$, we see that $\Lambda_\sigma$ determines the $\sigma$-Hilbert transform $\mathcal{H}_\sigma$ defined by

$$\begin{aligned} \mathcal{H}_\sigma \quad &: \quad H^{1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)/\mathbb{C}, \\ &\mathrm{Re}\, g|_{\partial\Omega} \mapsto \mathrm{Im}\, g|_{\partial\Omega} + \mathbb{C}. \end{aligned} \tag{44}$$

Put yet in another terms, for $u, \widehat{u} \in H^{1/2}(\partial\Omega)$, $\widehat{u} = \mathcal{H}_\sigma u$ if and only if the map $g(\xi) = (u + i\widehat{u})(\xi)$, $\xi \in \partial\Omega$, extends to $\Omega$ so that (36) is satisfied.

Summarizing the previous results, we have

**Lemma 3.4.** *The Dirichlet-to-Neumann map $\Lambda_\sigma$ determines the maps $\Lambda_{\widehat{\sigma}}$ and $\mathcal{H}_\sigma$.*

3.3. **Solutions of Complex Geometrical Optics.** Next we consider exponentially growing solutions, i.e., solutions of complex geometrical optics originated by Calderón for linearized inverse problems and by Sylvester and Uhlmann for nonlinear inverse problems. In our case, we seek solutions $G(z, k)$, $z \in \mathbb{C} \setminus \Omega$, $k \in \mathbb{C}$ satisfying

$$\partial_{\overline{z}} G(z, k) = 0 \quad \text{for } z \in \mathbb{C} \setminus \overline{\Omega}, \tag{45}$$

$$G(z, k) = e^{ikz}(1 + \mathcal{O}_k(\frac{1}{z})), \tag{46}$$

$$\mathrm{Im}\, G(z, k)|_{z \in \partial\Omega} = \mathcal{H}_\sigma(\mathrm{Re}\, G(z, k)|_{z \in \partial\Omega}). \tag{47}$$

Here $\mathcal{O}_k(h(z))$ means a function of $(z, k)$ that satisfies $|\mathcal{O}_k(h(z))| \leq C(k)|h(z)|$ for all $z$ with some constant $C(k)$ depending only on $k \in \mathbb{C}$. For the conductivity $\widetilde{\sigma}$ we consider the corresponding exponentially growing solutions $W(z,k)$, $z \in \mathbb{C}$, $k \in \mathbb{C}$ where

$$\partial_{\overline{z}} W(z,k) = \widetilde{\mu}_2(z)\overline{\partial_z W(z,k)}, \quad \text{for } z \in \mathbb{C}, \tag{48}$$

$$W(z,k) = e^{ikz}(1 + \mathcal{O}_k(\frac{1}{z})). \tag{49}$$

Note that in this stage, $z \mapsto G(z,k)$ is defined only in the exterior domain $\mathbb{C} \setminus \Omega$ but $z \mapsto W(z,k)$ in the whole complex plane. These two solutions are closely related:

**Lemma 3.5.** *For all $k \in \mathbb{C}$ we have:*

*i. The system (48) has a unique solution $W(z,k)$ in $\mathbb{C}$.*

*ii. The system (45–47) has a unique solution $G(z,k)$ in $\mathbb{C} \setminus \Omega$.*

*iii. For $z \in \mathbb{C} \setminus \Omega$ we have*

$$G(z,k) = W(F(z),k). \tag{50}$$

**Proof.** For the claim i. we refer to [6, Theorem 4.2].

Next we consider ii. and iii. simultaneously. Assume that $G(z,k)$ is a solution of (45–47). By Lemma 3.2 and boundary condition (47) we see that equation

$$\overline{\partial} h(z,k) = \nu_1 \partial_z h + \nu_2 \overline{\partial_z h}, \quad \text{in } \Omega,$$
$$h(\cdot\,,k)|_{\partial\Omega} = G(\cdot\,,k)|_{\partial\Omega}$$

has a unique solution where $\nu_1$ and $\nu_2$ are given in (39).

Let

$$H(z,k) = \begin{cases} G(z,k) & \text{for } z \in \mathbb{C} \setminus \Omega \\ h(z,k) & \text{for } z \in \Omega \end{cases} \tag{51}$$

and $\widetilde{H}(z,k) = H(F^{-1}(z),k)$. Then $\widetilde{H}(z,k)$ satisfies equations

$$\partial_{\overline{z}} \widetilde{H}(z,k) = 0, \quad \text{for } z \in \mathbb{C} \setminus \widetilde{\Omega},$$
$$\partial_{\overline{z}} \widetilde{H}(z,k) = \widetilde{\mu}_2(z)\overline{\partial_z \widetilde{H}(z,k)}, \quad \text{for } z \in \widetilde{\Omega},$$

and traces from both sides of $\partial\widetilde{\Omega}$ coincide. Thus $\widetilde{H}(z,k)$ satisfies equation in (48). Now (26) and (46) yield that

$$\begin{aligned} \widetilde{H}(z,k) &= H(F^{-1}(z),k) \\ &= \exp(ikF^{-1}(z))(1 + \mathcal{O}_k(\frac{1}{1+|F^{-1}(z)|})) \\ &= \exp(ikz)(1 + \mathcal{O}_k(\frac{1}{1+|z|})) \end{aligned} \tag{52}$$

showing that $\widetilde{H}$ satisfies (48–49). Thus by i., $\widetilde{H}(z,k) = W(z,k)$. This proves both ii. and iii. □

3.4. **Proof of Theorem 1.** As $G(z,k)$ is the unique solution of (45–47) and the operator appearing in boundary condition (47) is known, Lemmata 3.4 and 3.5 imply the following:

**Lemma 3.6.** *The Dirichlet-to-Neumann map $\Lambda_\sigma$ determines $G(z,k)$, $z \in \mathbb{C} \setminus \Omega$, $k \in \mathbb{C}$.*

Next we use this to find the diffeomorphism $F_\sigma$ outside $\Omega$.

**Lemma 3.7.** *The Dirichlet-to-Neumann map $\Lambda_\sigma$ determines the values the restriction $F_\sigma|_{\mathbb{C}\setminus\Omega}$.*

**Proof.** By (50), $G(z,k) = W(F(z),k)$, where $W(z,k)$ is the exponentially growing solution corresponding to the isotropic conductivity $\widetilde{\sigma}$. Thus by applying the subexponential growth results for such solutions, [6, Lemma 7.1 and Thm. 7.2], we have representation

$$W(z,k) = \exp(ik\varphi(z,k)) \tag{53}$$

where

$$\lim_{k\to\infty} \sup_{z\in\mathbb{C}} |\varphi(z,k) - z| = 0. \tag{54}$$

As $F(z) = z + \mathcal{O}(1/z)$, and $G(z,k) = W(F(z),k)$ we have

$$\lim_{k\to\infty} \frac{\log G(z,k)}{ik} = \lim_{k\to\infty} \varphi(F(z),k) = F(z). \tag{55}$$

By Lemma 3.6 we know the values of limit (55) for any $z \in \mathbb{C}\setminus\Omega$. Thus the claim is proven. □

We are ready to prove Theorem 1.

**Proof.** As we know $F|_{\mathbb{C}\setminus\Omega} \in W^{1,p}$, $2 < p < p(C_0)$, we in particular know $\widetilde{\Omega} = \mathbb{C} \setminus (F(\mathbb{C}\setminus\Omega))$. When $u$ is the solution of conductivity equation (1) with Dirichlet boundary value $\phi$ and $w$ is the solution of (27) with Dirichlet boundary value $\widetilde{\phi} = \phi \circ h$, where $h = F^{-1}|_{\partial\widetilde{\Omega}}$ we see that

$$\int_{\partial\widetilde{\Omega}} \widetilde{\phi}\Lambda_{\widetilde{\sigma}}\widetilde{\phi}\, dS = Q_{\widetilde{\sigma},\widetilde{\Omega}}(w) = Q_{\sigma,\Omega}(u) = \int_{\partial\Omega} \phi\Lambda_\sigma\phi\, dS. \tag{56}$$

Here, the second identity is justified by the fact that $F$ is quasiconformal and hence satisfies (9). Since $\Lambda_\sigma$ and $\Lambda_{\widetilde{\sigma}}$ are symmetric, this implies

$$\int_{\partial\widetilde{\Omega}} \widetilde{\psi}\Lambda_{\widetilde{\sigma}}\widetilde{\phi}\, dS = \int_{\partial\Omega} \psi\Lambda_\sigma\phi\, dS \tag{57}$$

for any $\widetilde{\psi}, \widetilde{\phi} \in H^{1/2}(\partial\widetilde{\Omega})$ and $\psi, \phi \in H^{1/2}(\partial\Omega)$ are related by $\widetilde{\psi} = \psi\circ h$ and $\widetilde{\psi} = \psi\circ h$. Note that $\phi \in H^{1/2}(\partial\Omega)$ if and only if $\widetilde{\phi} = \phi \circ h \in H^{1/2}(\partial\widetilde{\Omega})$. To see this, extend

$\phi$ to a $H^1(\Omega)$ function and after that define $\widetilde\phi$ in the interior of $\widetilde\Omega$ by $\widetilde\phi = \phi\circ F^{-1}$. Now

$$||\nabla\phi||^2_{L^2(\Omega)} \sim \int_\Omega \nabla\phi\cdot\sigma\overline{\nabla\phi}\,dx \sim \int_{\widetilde\Omega} \nabla\widetilde\phi\cdot\widetilde\sigma\overline{\nabla\widetilde\phi}\,dx \sim ||\nabla\widetilde\phi||^2_{L^2(\widetilde\Omega)}$$

and hence

$$||\phi||^2_{H^{1/2}(\partial\Omega)} \sim ||\phi||^2_{H^1(\Omega)} \sim ||\widetilde\phi||^2_{H^1(\widetilde\Omega)} \sim ||\widetilde\phi||^2_{H^{1/2}(\partial\widetilde\Omega)}.$$

As we know $F|_{\mathbb{C}\setminus\Omega}$ and $\Lambda_\sigma$, we can find $\Lambda_{\widetilde\sigma}$ using formula (57). By [6], the map $\Lambda_{\widetilde\sigma}$ determines uniquely the conductivity $\widetilde\sigma$ on $\widetilde\Omega$ in a constructive manner.

Knowing $\Omega$, $\widetilde\Omega$, and the boundary value $f = F|_{\partial\Omega}$ of the map $F:\Omega\to\widetilde\Omega$, we next construct a sufficiently smooth diffeomorphism $H:\widetilde\Omega\to\Omega$. First, by the Riemann mapping theorem we can map $\Omega$ and $\widetilde\Omega$ to the unit disc $\mathbb{D}$ by the conformal maps $R$ and $\widetilde R$, respectively. Now

$$G = \widetilde R\circ F\circ R^{-1} : \mathbb{D}\to\mathbb{D}$$

is a quasiconformal map and since we know $R$ and $\widetilde R$, we know the function $g = G|_{\partial\mathbb{D}}$ mapping $\partial\mathbb{D}$ onto itself. The map $g$ is quasisymmetric (cf. [1]) and by Ahlfors-Beurling extension theorem [1, Thm. IV.2] it has a quasiconformal extension $\mathcal{A}B(g)$ mapping $\overline{\mathbb{D}}$ onto itself. Note that one can obtain $\mathcal{A}B(g)$ from $g$ constructively by an explicit formula [1, p. 69]. Thus we may find a quasiconformal diffeomorphism $H = R^{-1}\circ[\mathcal{A}B(g)]^{-1}\circ\widetilde R$, $H:\widetilde\Omega\to\Omega$ satisfying $H|_{\partial\widetilde\Omega} = F^{-1}|_{\partial\widetilde\Omega}$.

Combining the above results, we can find $H_*\widetilde\sigma$ that is a representative of the equivalence class $E_\sigma$. □

In the above proof the Riemann mappings can not be found as explicitly as the Ahlfors-Beurling extension. However, there are numerical packages for approximative construction of Riemann mappings, see e.g. [26].

## 4. PROOFS OF CONSEQUENCES OF MAIN RESULT

Here we give proofs of Theorems 2.1–2.3.

**Proof of Theorem 2.1.** Let $F:\mathbb{R}^2_- = \mathbb{R}+i\mathbb{R}_-\to\mathbb{D}$ be the Möbius transform

$$F(z) = \frac{z+i}{z-i}.$$

Since this map is conformal, we see that $C_0(F_*\sigma) = C_0(\sigma)$. Let $\widetilde\sigma = F_*\sigma$ be the conductivity in $\mathbb{D}$. Then $\Lambda_{\widetilde\sigma}\phi$ is determined as in (57) for all $\phi\in C_0^\infty(\partial\mathbb{D}\setminus\{1\})$.

Since $\Lambda_{\widetilde\sigma}1 = 0$ and functions $\mathbb{C}\oplus C_0^\infty(\partial\mathbb{D}\setminus\{1\})$ are dense in the space $H^{1/2}(\partial\mathbb{D})$, we see that $\Lambda_\sigma$ determines the Dirichlet-to-Neumann map $\Lambda_{\widetilde\sigma}$ on $\partial\mathbb{D}$. Thus we can find the equivalence class of the conductivity on $\mathbb{D}$. Pushing these conductivities forward with $F^{-1}$ to $\mathbb{R}^2_-$, we obtain the claim. □

**Proof of Theorem 2.2.** Let $F : S \to \mathbb{D} \setminus \{0\}$ be the conformal map such that

$$\lim_{z\to\infty} F(z) = 0.$$

Again, since this map is conformal we have for $\widetilde{\sigma} = F_*\sigma$ the equality $C_0(\sigma) = C_0(F_*\sigma)$. Moreover, if $u$ is a solution of (13), we have that $w = u \circ F^{-1}$ is solution of

$$\begin{aligned} \nabla\cdot \widetilde{\sigma}\nabla w &= 0 \text{ in } \mathbb{D}\setminus\{0\}, \\ w|_{\partial\mathbb{D}} &= \phi\circ F^{-1}, \\ w &\in L^\infty(\mathbb{D}). \end{aligned} \tag{58}$$

Since set $\{0\}$ has capacitance zero in $\mathbb{D}$, we see that $w = W|_{\mathbb{D}\setminus\{0\}}$ where

$$\begin{aligned} \nabla\cdot \widetilde{\sigma}\nabla W &= 0 \text{ in } \mathbb{D}, \\ W|_{\partial\mathbb{D}} &= \phi\circ F^{-1}. \end{aligned} \tag{59}$$

Since $F$ can be constructed via the Riemann mapping theorem, we see that $\Lambda_\sigma$ determines $\Lambda_{\widetilde{\sigma}}$ on $\partial\mathbb{D}$ and thus the equivalence class $E_{\widetilde{\sigma}}$. When $\widetilde{F} : \mathbb{D} \to \mathbb{D}$ is a boundary preserving diffeomorphism, we see that $F^{-1}\circ\widetilde{F}\circ F$ defines a diffeomorphism $\overline{S} \to \overline{S}$. Since we have determined the conductivity $\widetilde{\sigma}$ up to a boundary preserving diffeomorphism, the claim follows easily. □

**Proof of Theorem 2.3.** Let $\mathbb{D}\subset\mathbb{C}$ be the unit disc and $\mathbb{D}_+ = \{z \in \mathbb{D} \mid \operatorname{Re} z > 0\}$. Let $F : \Omega \to \mathbb{D}_+$ be a Riemann mapping such that

$$\mathbb{D}_+ \subset \mathbb{R}\times\mathbb{R}_+, \quad F(\Gamma) = \partial\mathbb{D}_+ \setminus (\mathbb{R}\times\{0\}), \quad F(\partial\Omega\setminus\Gamma) = \partial\mathbb{D}_+ \cap (\mathbb{R}\times\{0\}).$$

Let $\eta : (x^1, x^2) \mapsto (x^1, -x^2)$ and define $\mathbb{D}_- = \eta(\mathbb{D}_+)$, and $\widetilde{\sigma} = F_*\sigma$. Let

$$\widehat{\sigma}(x) = \begin{cases} \sigma(x) & \text{for } x \in \mathbb{D}_+, \\ (\eta_*\sigma)(x) & \text{for } x \in \mathbb{D}_-. \end{cases}$$

Consider equation

$$\nabla\cdot \widehat{\sigma}\nabla w = 0 \text{ in } \mathbb{D}. \tag{60}$$

Using formula (57) we see that $F$ and $\Lambda_\Gamma$ determine the corresponding map $\Lambda_{F(\Gamma)}$ for $\widehat{\sigma}$. Similarly, we can find $\Sigma_{F(\Gamma)}$ for $\widehat{\sigma}$.

Then $\Lambda_{F(\Gamma)}$ determines the Cauchy data on the boundary for the solutions of (60) for which $w \in H^1(\mathbb{D})$, $w = -w\circ\eta$. On the other hand, $\Sigma_{F(\Gamma)}$ determines the Cauchy data on the boundary of the solutions of (60) for which $w \in H^1(\mathbb{D})$ and $w = w\circ\eta$. Now each solution $w$ of (60) can be written as a linear combination

$$w(x) = \frac{1}{2}(w(x) + w(\eta(x))) + \frac{1}{2}(w(x) - w(\eta(x))).$$

Thus the maps $\Lambda_{F(\Gamma)}$ and $\Sigma_{F(\Gamma)}$ together determine $C_{\widehat{\sigma}}$, and hence we can find $\widehat{\sigma}$ up to a diffeomorphism. We can choose a representative $\widehat{\sigma}_0$ of the equivalence class $E_{\widehat{\sigma}}$ such that $\widehat{\sigma}_0 = \widehat{\sigma}_0\circ\eta$. In fact, choosing a symmetric Ahlfors-Beurling extension in the construction given in the proof of Theorem 1, we obtain such a conductivity. Pushing the conductivity $\widehat{\sigma}_0$ from $\mathbb{D}_+$ to $\Omega$ with $F^{-1}$, we obtain the claim. □

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