THE GROTHENDIECK GROUP OF A QUANTUM PROJECTIVE SPACE BUNDLE

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ABSTRACT. We compute the Grothendieck group of non-commutative analogues of projective space bundles. Our results specialize to give the Grothendieck groups of non-commutative analogues of projective spaces, and specialize to recover the Grothendieck group of a usual projective space bundle over a regular noetherian separated scheme. As an application, we develop an intersection theory for quantum ruled surfaces.

0. INTRODUCTION

A basic result in the K-theory of algebraic varieties is the computation of the K-groups for projective space bundles [2], [5], [11]. In this paper we compute K_0 for non-commutative analogues of projective space bundles.

Our motivation comes from the problem of classifying non-commutative surfaces. This program is in its early stages but, in analogy with the commutative case, the non-commutative analogues of ruled surfaces appear to play a central role [1], [10], [15]. An intersection theory will be an essential ingredient in the study of non-commutative surfaces. One may develop an intersection theory by defining an intersection multiplicity as a bilinear $\mathbb{Z}$-valued form, the Euler form, on the Grothendieck group of the surface (cf. [4] and [6]). One of our main results yields a formula (6-1) for the Grothendieck group for quantum ruled surfaces, and we use this to show that the associated intersection theory gives natural analogues of the commutative results: if X is a smooth (commutative!) projective curve and $f : \mathbb{P}(\mathcal{E}) \to X$ a quantum ruled surface over X, then fibers do not meet and a section meets a fiber exactly once (Theorem 6.8).

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1. TERMINOLOGY AND NOTATION

The utility of co-monads in non-commutative algebraic geometry was first realized by Rosenberg [12]. Later Van den Bergh [14] developed very effective methods based on this idea and replaced the language of co-monads by terminology and notation that is closer to the classical language of algebra and algebraic geometry. We now recall some of his language [14].

1.1. **Bimodules** [14, Sect. 3.1]**.** Let L and K be abelian categories. The category BIMOD(K, L) of weak K-L-bimodules is the opposite of the category of left exact functors L → K. If $\mathcal{F} = {}_K\mathcal{F}_L$ is a weak K-L-bimodule, we write $\mathcal{H}om_L(\mathcal{F}, -)$ for the corresponding left exact functor. We call $\mathcal{F}$ a bimodule if $\mathcal{H}om_L(\mathcal{F}, -)$ has a left adjoint (which we denote by $- \otimes_K \mathcal{F}$).

The Yoneda embedding K → BIMOD(Ab, K) realizes objects in K as bimodules and every Ab-K-bimodule is isomorphic to an object of K in this sense.

The composition of left exact functors is left exact. The composition of the weak bimoduless (i.e., left exact functors) ${}_K\mathcal{F}_L$ and ${}_L\mathcal{G}_M$ is denoted by $\mathcal{F} \otimes_L \mathcal{G}$. This composition is a weak K-M-bimodule and $\mathcal{H}om_M(\mathcal{F} \otimes_L \mathcal{G}, -) = \mathcal{H}om_L(\mathcal{F}, \mathcal{H}om_M(\mathcal{G}, -))$. Thus BIMOD(K, K) is a monoidal category with identity object o_K defined by $\mathcal{H}om_K(o_K, -) = \mathrm{Id}_K$.

1.2. **Algebras** [14, Sect. 3.1]**.** An algebra object in BIMOD(K, K) is called a weak K-algebra: it is a triple (A, μ, η), where $A \in$ BIMOD(K, K), and $\mu : A \otimes_K A \to A$ and $\eta : o_K \to A$ are morphisms of weak bimodules such that the obvious diagrams commute. If $\mathcal{H}om_K(A, -)$ has a left adjoint we drop the adjective "weak" and simply call A a K-algebra. In this paper we are interested in algebras, not weak algebras.

An A-module is an object M in K together with a morphism $M \otimes_K A \to M$ in BIMOD(Ab, K) making the obvious diagrams commute. If R is a commutative ring and K = Mod$R$, then R-algebras in the usual sense give K-algebras, and the new notion of a module coincides with the old one.

Thus, the algebras in this paper are endo-functors with additional structure, modules are objects in categories on which the functors act, morphisms between algebras are natural transformations, et cetera. Van den Bergh's language closely parallels the classical language of rings and modules but one must be vigilant. Sometimes the proofs and statements of results that are trivial for ordinary algebras and modules must be modified in subtle ways to obtain the appropriate analogue for these "functor algebras". For example, A itself is not an A-module, but it is an A-A-bimodule.

1.3. **Quasi-schemes** [14, Sect. 3.5]**.** A quasi-scheme X is a Grothendieck category. We will use the symbol X when we think of the quasi-scheme as a geometric object and Mod$X$ when we think of it as a category. Objects in Mod$X$ will be called X-modules. We write mod$X$ for the full subcategory of noetherian X-modules. We say that X is noetherian if mod$X$ generates Mod$X$. Throughout X will denote a noetherian quasi-scheme.

We use the word "noetherian" in the same way as Van den Bergh [14, page 27] but it is not compatible with the usual notion of a noetherian scheme. If X is a noetherian scheme, it need not be the case that coh$X$ generates Qcoh$X$.

In this paper X will play the role of the base space for a quantum projective space bundle.

We write o_X for the identity functor on Mod$X$. It is an algebra in the sense of section 1.2 and in some sense plays a role similar to the structure sheaf or, more precisely, $- \otimes \mathcal{O}_X$.

1.4. **Graded X-modules** [14, Sect 3.2]**.** A graded X-module M is a sequence of objects $M = (M_n)_{n \in \mathbb{Z}}$ in Mod$X$. We call M_n the degree n component and, abusing notation, write $M = \oplus M_n$. We define the category GrMod$X$ by declaring

a morphism $f : M \to N$ to be a sequence of X-module maps $f_n : M_n \to N_n$. We define $M_{>n}$ by declaring that $(M_{>n})_j = 0$ if $j \leq n$ and $(M_{>n})_j = M_j$ if $j > n$.

1.5. Graded X-algebras.

Graded X-algebras $A = \oplus_{i \in \mathbb{Z}} A_i$ are defined by combining the ideas in sections 1.2 and 1.4 [14, page 19]. The degree zero component A_0 of a graded algebra A inherits an X-algebra structure from A and can also be given the structure of a graded X-algebra concentrated in degree zero. There is a graded X-algebra monomorphism $A_0 \to A$ and an augmentation map of graded X-algebras $A \to A_0$. The composition $A_0 \to A \to A_0$ is the identity. A graded algebra A is connected if the structure map $o_X \to A_0$ is an isomorphism.

A graded A-module is a graded X-module with additional structure [14, Sect. 3.2]. We write $\mathsf{GrMod}A$ for the category of graded A-modules and $\mathsf{grmod}A$ for the full subcategory of noetherian modules.

The identity functor on $\mathsf{GrMod}X$ can be given the structure of a graded X-algebra. It is concentrated in degree zero, and its degree zero component is isomorphic to o_X. Thus, we write o_X for both the X-algebra associated to $\mathrm{Id}_{\mathsf{Mod}X}$ and for the graded X-algebra associated to $\mathrm{Id}_{\mathsf{GrMod}X}$.

1.6. The quasi-scheme $\mathsf{Proj}_{\mathrm{nc}}A$.

Let A be a connected graded X-algebra. We define $\mathsf{Tors}A$ to be the full subcategory of $\mathsf{GrMod}A$ consisting of direct limits of graded A-modules that are annihilated by $A_{\geq n}$ for some $n \gg 0$. The full subcategory of noetherian objects in $\mathsf{Tors}A$ is denoted by $\mathsf{tors}A$.

The projective quasi-scheme associated to A is denoted by $\mathsf{Proj}_{\mathrm{nc}}A$ and defined by declaring that

$$\mathsf{Mod}(\mathsf{Proj}_{\mathrm{nc}}A) := \mathsf{GrMod}A/\mathsf{Tors}A.$$

We write $\pi : \mathsf{GrMod}A \to \mathsf{Proj}_{\mathrm{nc}}A$ for the quotient functor and ω for its right adjoint.

1.7. $\mathcal{O}_X$-bimodule algebras [13].

Let X and Y be noetherian schemes over a field k. We write $\mathrm{SSupp}\,\mathcal{M}$ for the scheme-theoretic support of a quasi-coherent $\mathcal{O}_{X \times Y}$-module $\mathcal{M}$.

A coherent $\mathcal{O}_X$-$\mathcal{O}_Y$ bimodule is a coherent $\mathcal{O}_{X \times Y}$-module $\mathcal{M}$ such that the projections $pr_1 : \mathrm{SSupp}\,\mathcal{M} \to X$ and $pr_2 : \mathrm{SSupp}\,\mathcal{M} \to Y$ are finite. A morphism of coherent $\mathcal{O}_X$-$\mathcal{O}_Y$ bimodules is a morphism of $\mathcal{O}_{X \times Y}$-modules. The category of coherent $\mathcal{O}_X$-$\mathcal{O}_Y$ bimodules is denoted by $\mathsf{Bimod}(\mathcal{O}_X, \mathcal{O}_Y)$.

An $\mathcal{O}_X$-bimodule algebra is an algebra object in the category of $\mathcal{O}_X$-$\mathcal{O}_X$-bimodules. Graded $\mathcal{O}_X$-bimodule algebras are then defined in an obvious way. A graded $\mathcal{O}_X$-bimodule algebra $\mathcal{B}$ is analogous to a sheaf of $\mathcal{O}_X$-algebras, but it is actually a sheaf on $X \times X$ and local sections of $\mathcal{B}$ do not have an algebra structure; its global sections do, and they typically form a non-commutative graded algebra.

1.8. Quantum $\mathbb{P}^n$-bundles (see section 5 for details).

The non-commutative analogue of a $\mathbb{P}^n$-bundle over a scheme X will be defined in the following way. First, take a coherent $\mathcal{O}_{X \times X}$-module $\mathcal{E}$ such that each $pr_{i*}\mathcal{E}$ is locally free of finite rank, then use $\mathcal{E}$ to construct a graded $\mathcal{O}_X$-bimodule algebra $\mathcal{B}$ that is analogous to the symmetric algebra of a locally free $\mathcal{O}_X$-module. The quasi-scheme $\mathbb{P}(\mathcal{E})$ is defined implicitly by declaring that its module category is the quotient category $\mathsf{Mod}\mathbb{P}(\mathcal{E}) := \mathsf{Proj}_{\mathrm{nc}}\mathcal{B} := \mathsf{GrMod}\mathcal{B}/\mathsf{Tors}\mathcal{B}$, where $\mathsf{Tors}\mathcal{B}$ consists of the direct limits of graded modules that are annihilated by a power of the augmentation ideal. This definition is due to Van den Bergh [15] (see also the thesis of David Patrick [10]). If $\mathcal{E}$ is a locally free $\mathcal{O}_X$-module where X is identified with the diagonal in $X \times X$, then

$\mathcal{B}$ is the symmetric algebra $S(\mathcal{E})$ and $\mathsf{Mod}\mathbb{P}(\mathcal{E})$ is $\mathsf{Qcoh}(\mathsf{Proj}\mathcal{B})$, the quasi-coherent sheaves on the usual projective space bundle associated to $\mathcal{E}$.

1.9. **Notation for Grothendieck groups.** Our goal is to compute the Grothendieck group of $\mathsf{mod}\mathbb{P}(\mathcal{E})$, the category of noetherian modules on the noetherian quasi-scheme $\mathbb{P}(\mathcal{E})$.

We want to use notation that is the same as, or at least compatible with, the classical notation so we adopt the following conventions:

(1) We will write $K_0'(\mathsf{C})$ for the Grothendieck group of an abelian category C with the following exceptions.
(2) If X is a noetherian scheme we will write $K_0'(X)$ for $K_0'(\mathsf{coh}X)$ and
(3) $K_0(X)$ for the Grothendieck group of the additive category of coherent locally free $\mathcal{O}_X$-modules.
(4) When X is a separated, regular, noetherian scheme, the natural map $K_0(X) \to K_0'(X)$ is an isomorphism so we identify these groups.
(5) If X is a noetherian quasischeme we will write $K_0'(X)$ for $K_0'(\mathsf{mod}X)$. In particular, we do this when $X = \mathbb{P}(\mathcal{E})$ and when $X = \mathsf{Proj}_{\mathsf{nc}}A$.
(6) Because every noetherian $\mathbb{P}(\mathcal{E})$-module has a finite resolution by modules that behave like locally free sheaves $\mathbb{P}(\mathcal{E})$ is regular in a suitable sense, and we therefore write $K_0(\mathbb{P}(\mathcal{E}))$ for $K_0'(\mathbb{P}(\mathcal{E}))$.
(7) When X is a separated, regular, noetherian scheme and A a flat, noetherian, connected graded X-algebra such that $\mathcal{T}or_q^A(-, o_X) = 0$ for $q \gg 0$ we will write $K_0(\mathsf{Proj}_{\mathsf{nc}}A)$ for $K_0'(\mathsf{Proj}_{\mathsf{nc}}A)$. The justification for doing this is explained prior to Theorem 4.7.

If M is in C, we write $[M]$ for its class in $K_0'(\mathsf{C})$.

1.10. **Computation of $K_0(\mathsf{Proj}_{\mathsf{nc}}A)$.** We were unable to compute $K_0'(\mathbb{P}(\mathcal{E}))$ by adapting the method that Quillen used in [11] to compute the K_0'-group of a $\mathbb{P}^n$-bundle. Instead we follow Manin's method [5].

The first step towards that is to associate to the $\mathcal{O}_X$-bimodule algebra $\mathcal{B}$, that is used to construct $\mathbb{P}(\mathcal{E})$ (see section 1.8), a graded X-algebra A, in the sense of section 1.5. We define A by defining the left exact functors

$$\mathcal{H}om_X(A_i, -) := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{B}_i, -), \qquad i \geq 0.$$

We show in section 5 that there is an equivalence of categories

$$\mathsf{Mod}\mathbb{P}(\mathcal{E}) \equiv \mathsf{GrMod}A/\mathsf{Tors}A$$

and therefore an isomorphism

$$K_0'(\mathsf{mod}\mathbb{P}(\mathcal{E})) \cong K_0'(\mathsf{grmod}A/\mathsf{tors}A) = K_0'(\mathsf{Proj}_{\mathsf{nc}}A).$$

In section 4 we compute $K_0'(\mathsf{Proj}_{\mathsf{nc}}A)$ for a somewhat larger class of graded X-algebras. We allow X to be any noetherian quasi-scheme, and A any flat, noetherian, connected, graded X-algebra such that $\underline{\mathcal{T}or}_q^A(-, o_X) = 0$ for $q \gg 0$ (i.e., the "fibers" of $\mathsf{Proj}_{\mathsf{nc}}A \to X$ are "smooth"). We show that

$$K_0'(\mathsf{grmod}A) \cong K_0'(\mathsf{grmod}X) \cong K_0'(X)[T, T^{-1}]$$

and

$$K_0'(\mathsf{grmod}A/\mathsf{tors}A) \cong K_0'(X)[T, T^{-1}]/I$$

where I is the image of the map

$$\rho : K_0'(\mathsf{grmod}X) \to K_0'(\mathsf{grmod}X), \qquad \rho[V] := \sum (-1)^q [\underline{\mathcal{T}or}_q^A(V, o_X)],$$

where on the right hand side of the equation, V is viewed as a graded A-module via the augmentation map $A \to o_X$. Up to this point in the argument the structure sheaf of X plays no role: X is only assumed to be a quasi-scheme so does not have a structure sheaf.

1.11. **Computation of $K_0(\mathbb{P}(\mathcal{E}))$.** In section 5 we return to the main line of the argument by applying the computation of $K_0'(\mathsf{Proj}_{\mathsf{nc}}A)$ in section 4 to the special A that is defined in terms of $\mathcal{B}$, the $\mathcal{O}_X$-bimodule algebra used to coinstruct $\mathbb{P}(\mathcal{E})$. In particular, we now assume that X is a separated, regular, noetherian scheme.

We also write $K_0(\mathsf{Proj}_{\mathsf{nc}}A)$ for $K_0'(\mathsf{grmod}A/\mathsf{tors}A)$ (the remarks prior to Theorem 4.7 justify this notational change).

A key step is to show that the ideal I mentioned in section 1.10 is principal: we show that

$$\rho[V] = [V].\rho[\mathcal{O}_X]$$

for all $V \in \mathsf{grmod}X$, where on the right hand side we use the usual product in $K_0(X)[T, T^{-1}]$. Therefore

$$K_0'(\mathbb{P}(\mathcal{E})) \cong K_0'(\mathsf{Proj}_{\mathsf{nc}}A) \cong \frac{K_0(X)[T, T^{-1}]}{(\rho[\mathcal{O}_X])}.$$

The construction of $\mathcal{B}$ is such that its "trivial module", namely $\mathcal{O}_X$, has a Koszul-like resolution (Definition 5.6). Now $\rho[\mathcal{O}_X]$ can be read off from this resolution, and is equal to the inverse of the Hilbert series of $\mathcal{B}$, $\sum [\mathcal{B}_n]T^n \in K_0(X)[[T]]$ where $\mathcal{B}_n$ is the degree n component of $\mathcal{B}$, and $[\mathcal{B}_n]$ denotes the class of $\mathcal{B}_n$ in $K_0(X)$ when it is viewed as a right $\mathcal{O}_X$-module; to be precise, $[\mathcal{B}_n] = [pr_{2*}\mathcal{B}_n]$ where $pr_2 : X \times X \to X$ is the projection onto the second factor.

1.12. **Quantum ruled surfaces.** In section 6 we further specialize to the case where X is a smooth projective curve and $pr_{1*}\mathcal{E}$ and $pr_{2*}\mathcal{E}$ are locally free of rank two. We then call $\mathbb{P}(\mathcal{E})$ a quantum ruled surface. We show there is a well-behaved intersection theory on $K_0(\mathbb{P}(\mathcal{E}))$. We define a fiber and a section as appropriate elements of $K_0(\mathbb{P}(\mathcal{E}))$. As an explicit application, we show that "fibers do not meet" and "a fiber and a section meet exactly once", as in the commutative case.

2. Connected graded algebras over a quasi-scheme

From now on we will assume that A is connected. The unit and augmentation maps, $\eta : o_X \to A$ and $\varepsilon : A \to o_X$ are maps of graded X-algebras and $\varepsilon\eta = \mathrm{Id}_{o_X}$. Associated to η and ε are various functors [14, (3.8), (3.9), (3.10)]. For example, associated to η is the forgetful functor

$$(-)_X : \mathsf{GrMod}A \to \mathsf{GrMod}X, \tag{2-1}$$

its left adjoint $- \otimes_X A$, and its right adjoint $\underline{\mathcal{H}om}_X(A, -)$. Associated to ε is the functor $(-)_A : \mathsf{GrMod}X \to \mathsf{GrMod}A$, its left adjoint $- \otimes_A o_X$, and its right adjoint $\underline{\mathcal{H}om}_A(o_X, -)$. The functors $(-)_X$ and $(-)_A$ are exact because they have both left and right adjoints. Therefore their right adjoints $\underline{\mathcal{H}om}_X(A, -)$ and $\underline{\mathcal{H}om}_A(o_X, -)$ preserve injectives. We observe for later use that if $- \otimes_X A$ is exact, then $(-)_X$ preserves injectives. The functor $(-)_A$ is fully faithful so we often view $\mathsf{GrMod}X$ as a full subcategory of $\mathsf{GrMod}A$.

The functors $\underline{\mathcal{H}om}_A(o_X, -)$ and $- \otimes_A o_X$ associated to the algebra map $A \to o_X$ may be composed with the functor $(-)_A$ to obtain an adjoint pair of functors sending $\mathsf{GrMod}A$ to $\mathsf{GrMod}A$, thus endowing o_X with the structure of a graded A-A-bimodule. With this point of view, the appropriate notation for these two functors is still $\underline{\mathcal{H}om}_A(o_X, -)$ and $- \otimes_A o_X$. The relation between these functors is set out below, where on the left hand side of each equality o_X is viewed as an A-A-bimodule, and on the right hand side it is viewed as a graded X-algebra:

$$- \otimes_A o_X = (- \otimes_A o_X)_A,$$
$$(- \otimes_A o_X)_X = - \otimes_A o_X,$$
$$\underline{\mathcal{H}om}_A(o_X, -) = \underline{\mathcal{H}om}_A(o_X, -)_A,$$
$$\underline{\mathcal{H}om}_A(o_X, -)_X = \underline{\mathcal{H}om}_A(o_X, -).$$

To illustrate how these functors are used we prove the following basic result.

Lemma 2.1. *Let A be a connected graded X-algebra. Let $M \in \mathsf{GrMod}A$, and suppose that $M_i = 0$ for $i < n$. Then the map $M = M \otimes_A A \to M \otimes_A o_X$ of graded A-modules induces an isomorphism $M_n \cong (M \otimes_A o_X)_n$ of X-modules. The kernel of the epimorphism $M \to M \otimes_A o_X$ is contained in $M_{>n}$.*

Proof. The A-module structure on M is determined by a morphism $h : M \otimes_X A \to M$, and $M \otimes_A o_X$ is, by definition, the coequalizer in $\mathsf{GrMod}A$ of

$$
\begin{array}{ccc}
M \otimes_X A & \xrightarrow{\ \ h\ \ } & M \\
{\scriptstyle 1 \otimes \varepsilon}\Big\downarrow & & \\
M \otimes_X o_X \cong M. & &
\end{array}
\qquad\qquad (2\text{-}2)
$$

Let $\theta_i : \mathsf{Mod}X \to \mathsf{GrMod}X$ be defined by $\theta_i(V)_j$ equals V if $i = j$ and 0 if $i \neq j$. If $M_{<n} = 0$, then for every injective $I \in \mathsf{Mod}X$,

$$
\begin{aligned}
\mathrm{Hom}_X((M \otimes_X A)_n, I) &= \mathrm{Hom}_{\mathsf{GrMod}X}(M \otimes_X A, \theta_n(I)) \\
&= \mathrm{Hom}_{\mathsf{GrMod}X}(M, \underline{\mathcal{H}om}_X(A, \theta_n(I))) \\
&= \prod_i \mathrm{Hom}_X(M_i, \underline{\mathcal{H}om}_X(A, \theta_n(I))_i) \\
&= \prod_i \mathrm{Hom}_X(M_i, \mathcal{H}om_X(A_{n-i}, I)) \\
&= \mathrm{Hom}_X(M_n, \mathcal{H}om_X(A_0, I)) \\
&= \mathrm{Hom}_X(M_n, I)
\end{aligned}
$$

so $(M \otimes_X A)_n \cong M_n$. So the degree n part of the diagram (2-2) consists of two identity maps, whence $(M \otimes_A o_X)_n \cong M_n$. $\qquad\square$

We call $- \otimes_X A : \mathsf{GrMod}X \to \mathsf{GrMod}A$ the induction functor. Let $V \in \mathsf{GrMod}X$. The structure map $h : V \otimes_X A \otimes_X A \to V \otimes_X A$ making $V \otimes_X A$ a graded A-module is $V \otimes \mu$, where $\mu : A \otimes_X A \to A$ is the multiplication map. A module $M \in \mathsf{GrMod}A$ is induced if $M \cong V \otimes_X A$ for some $V \in \mathsf{GrMod}X$.

We say that A is noetherian if X is noetherian and the induction functor $- \otimes_X A : \mathsf{GrMod}X \to \mathsf{GrMod}A$ preserves noetherian modules [14, page 27]. If A is noetherian and $M \in \mathsf{grmod}A$, then each M_i is a noetherian X-module.

Lemma 2.2. *Let A be an $\mathbb{N}$-graded X-algebra.*

(1) *Every $M \in \mathsf{GrMod}A$ is a quotient of an induced module.*

(2) *If X is noetherian, and A is noetherian, then every $M \in \mathsf{grmod}A$ is a quotient of an induced module $V \otimes_X A$ with V a noetherian graded X-module.*

Proof. (1) Let $M \in \mathsf{GrMod}A$, and let $h : M \otimes_X A \to M$ be the structure map. Forgetting the A-structure on M, we can form the induced module $M \otimes_X A$. Since M is an A-module $h \circ (M \otimes \mu) = h \circ (h \otimes A)$, from which it follows that h is a map of graded A-modules. To show h is an epimorphism in $\mathsf{GrMod}A$ it suffices to show it is an epimorphism in $\mathsf{GrMod}X$. Suppose that $u : M \to N$ is a map of graded X-modules such that $u \circ h = 0$. Since (M, h) is an A-module, $h \circ (M \otimes \eta) = \mathrm{id}_M$; composing this equality on the left with u we see that $u = 0$, whence h is epic.

(2) By (1) there is an epimorphism $\oplus_i(M_i \otimes_X A) \to M$ where $M_i \in \mathsf{grmod}X$ is concentrated in degree i. Since M is noetherian, there is a finite direct sum that still maps onto M, i.e.,

$$(\oplus_{i=a}^b M_i) \otimes_X A \cong \oplus_{i=a}^b (M_i \otimes_X A) \to M$$

is epic for some $a \leq b$. By an earlier remark, each M_i is a noetherian X-module, so $\oplus_{i=a}^b M_i \in \mathsf{grmod}X$. $\qquad\square$

We will prove a version of Nakayama's Lemma. Recall that if k is a field, and A is a connected k-algebra, then a finitely generated graded A-module M is zero if and only if $M \otimes_A k = 0$.

The augmentation ideal of A is $\mathfrak{m} := \ker(\varepsilon : A \to o_X)$. Since $(-)_A : \mathsf{GrMod}X \to \mathsf{GrMod}A$ has a left and a right adjoint, $\mathsf{GrMod}X$ is a closed subcategory [14, page 20] of $\mathsf{GrMod}A$, and hence $\mathfrak{m}$ is an ideal, not just a weak ideal, in A [14, Section 3.4]. If $M \in \mathsf{GrMod}A$, we write $M\mathfrak{m}$ for the image of the composition $M \otimes_A \mathfrak{m} \to M \otimes_A A \cong M$.

Lemma 2.3 (Nakayama's Lemma). *Let A be a connected X-algebra, and $M \in \mathsf{grmod}A$. Then the following are equivalent:*

(1) $M = 0$;

(2) $M\mathfrak{m} = M$;

(3) $M \otimes_A o_X = 0$.

Proof. Clearly that (1) implies both (2) and (3).

The exact sequence $0 \to \mathfrak{m} \to A \to o_X \to 0$ in $\mathsf{BiGr}(A, A)$ induces an exact sequence

$$M \otimes_A \mathfrak{m} \to M \otimes_A A \cong M \to M \otimes_A o_X \to 0$$

in $\mathsf{GrMod}A$, so $M\mathfrak{m} = \ker(M \to M \otimes_A o_X)$. Hence (2) and (3) are equivalent. Because M is noetherian, the chain of submodules $\ldots \subset M_{\geq n} \subset M_{\geq n-1} \subset \ldots$ eventually stops. Hence, if $M \neq 0$, there exists n such that $M = M_{\geq n}$ and $M_n \neq 0$. It then follows from Lemma 2.1 that $M \otimes_A o_X \neq 0$. Therefore (3) implies (1). $\quad\square$

We now wish to consider o_X as an A-A-bimodule and examine $\underline{\mathcal{T}or}_q^A(M, o_X)$ for $M \in \mathsf{GrMod}A$. The bifunctors $\mathcal{T}or$ are defined in [14, Section 3.1]. Because M is a $\mathbb{Z}$-A-bimodule the general definition in [14] admits a small simplification.

Lemma 2.4. *Let* D *and* E *be Grothendieck categories. Let* $\mathcal{M} \in \mathsf{D}$ *and* $\mathcal{N} \in$ $\mathsf{BiMod}(\mathsf{D}, \mathsf{E})$. *Then* $\mathcal{T}or_q^{\mathsf{D}}(\mathcal{M}, \mathcal{N})$ *is in* E, *and is uniquely determined by the requirement that there is a functorial isomorphism*

$$\mathrm{Hom}_{\mathsf{E}}(\mathcal{T}or_q^{\mathsf{D}}(\mathcal{M}, \mathcal{N}), I) \cong \mathrm{Ext}_{\mathsf{D}}^q(\mathcal{M}, \mathcal{H}om_{\mathsf{E}}(\mathcal{N}, I))$$

for all injectives I *in* E.

Proof. If $\mathcal{M} \in \mathsf{D}$ is viewed as a $\mathbb{Z}$-D-bimodule, then $\mathcal{T}or_q^{\mathsf{D}}(\mathcal{M}, \mathcal{N})$ is defined to be the $\mathbb{Z}$-E-bimodule such that $\mathcal{H}om_{\mathsf{E}}(\mathcal{T}or_q^{\mathsf{D}}(\mathcal{M}, \mathcal{N}), I) = \mathcal{E}xt_{\mathsf{D}}^q(\mathcal{M}, \mathcal{H}om_{\mathsf{E}}(\mathcal{N}, I))$ for all injective I [14, page 12]. However, if $\mathcal{F} \in \mathcal{D}$, then $\mathcal{E}xt_{\mathsf{D}}^q(\mathcal{M}, \mathcal{F}) = \mathrm{Ext}_{\mathsf{D}}^q(\mathcal{M}, \mathcal{F})$, where $\mathrm{Ext}_{D}^q(-, \mathcal{F})$ is the usual q^{th} right derived functor of $\mathrm{Hom}_{\mathsf{D}}(-, \mathcal{F})$. This is a consequence of [14, 3.1.2(4)], and the fact that the inclusion $\mathsf{D} \to \mathsf{BIMOD}(\mathsf{Ab}, \mathsf{D})$ is exact and preserves injectives.

Let InjE be the full subcategory of injectives in E. There is a unique left exact functor $\bar{F} : \mathsf{E} \to \mathsf{Ab}$ extending the functor $F : \mathsf{InjE} \to \mathsf{Ab}$ defined by

$$F(I) := \mathrm{Ext}_{\mathsf{D}}^i(\mathcal{M}, \mathcal{H}om_{\mathsf{E}}(\mathcal{N}, I)).$$

We will show that $\bar{F}$ is representable. This is sufficient to prove the lemma.

Since $\bar{F}$ is left exact, it is enough to show that it commutes with products because then it will have a left adjoint and that left adjoint evaluated at $\mathbb{Z}$ will give the representing object. Let $\{W_j\}$ be a family of objects in E, and for each j take an injective resolution $0 \to W_j \to J_j^\bullet$. Since the product functor $\prod$ is left exact and exact on injectives, and the product of injectives is injective, $0 \to \prod W_j \to \prod J_j^\bullet$ is an injective resolution of $\prod W_j$. Therefore $\bar{F}(\prod W_j) = \mathrm{Ker}(F(\prod J_j^0) \to F(\prod J_j^1))$. Since $\mathcal{H}om_{\mathsf{E}}(\mathcal{N}, -)$ and $\mathrm{Ext}_{\mathsf{D}}^i(\mathcal{M}, -)$ both commute with products, so does F. Hence

$$\bar{F}(\prod W_j) = \mathrm{Ker}(\prod F(J_j^0) \to \prod F(J_j^1))$$
$$= \prod \mathrm{Ker}(F(J_j^0) \to F(J_j^1))$$
$$= \prod \bar{F}(W_j).$$

That is, $\bar{F}$ commutes with products. $\qquad\square$

There is a graded version of this lemma that may be applied to the present situation. Thus, if $M \in \mathsf{GrMod}A$, then $\underline{\mathcal{T}or}_q^A(M, o_X) \in \mathsf{GrMod}A$ is determined by the requirement that

$$\mathrm{Hom}_{\mathsf{GrMod}A}(\underline{\mathcal{T}or}_q^A(M, o_X), I) = \mathrm{Ext}_{\mathsf{GrMod}A}^q(M, \underline{\mathcal{H}om}_A(o_X, I)) \qquad (2\text{-}3)$$

for all injectives $I \in \mathsf{GrMod}A$.

We may also view o_X as an A-o_X-bimodule, and examine $\underline{\mathcal{T}or}_q^A(M, o_X)$ in that context. As remarked in [14, page 18], we have

$$\underline{\mathcal{T}or}_q^A(M, o_X) = \underline{\mathcal{T}or}_q^A(M, o_X)_X,$$

where on the left o_X is an A-o_X-bimodule and on the right it is an A-A-bimodule.

There is a notion of minimal resolution in $\mathsf{grmod}A$.

Definition 2.5. A complex $\{P_\bullet, d_n\}$ in $\mathsf{grmod}A$ is minimal if $\mathrm{Im}\, d_n \subseteq P_n\mathfrak{m}$ for all n.

By [14, 3.1.5], $\underline{\mathcal{T}or}_\bullet^A(-, o_X)$ is a δ-functor.

Lemma 2.6. *Let A be a connected noetherian X-algebra, and $M \in \mathsf{grmod}A$. If induced modules are acyclic for $\underline{\mathcal{T}or}^A_\bullet(-, o_X)$, then $\underline{\mathcal{T}or}^A_i(M, o_X)_X \in \mathsf{grmod}X$. In particular, if M admits a minimal resolution of the form $P_\bullet = V_\bullet \otimes_X A \to M$, there is an isomorphism of graded X-modules*

$$V_i \cong \underline{\mathcal{T}or}^A_i(M, o_X)_X.$$

Proof. By Lemma 2.2, there is an epimorphism $V_0 \otimes_X A \to M$ for some $V_0 \in \mathsf{grmod}X$. Repeatedly applying this process to the kernels of the epimorphisms, we can construct a noetherian induced resolution $P_\bullet = V_\bullet \otimes_X A \to M$. Since $P_i \otimes_A o_X = V_i \otimes_X A \otimes_A o_X \cong V_i \in \mathsf{grmod}X$, by the acyclicity hypothesis, $\underline{\mathcal{T}or}^A_i(M, o_X)_X = H_i(P_\bullet \otimes_A o_X) = H_i(V_\bullet) \in \mathsf{grmod}X$. If the resolution $P_\bullet = V_\bullet \otimes_X A \to M$ is minimal, then the differentials in $P_\bullet \otimes_A o_X \cong V_\bullet$ are all zero, hence $\underline{\mathcal{T}or}^A_i(M, o_X)_X \cong V_i$. □

Warning. Graded modules over a connected algebra in our sense need not have minimal resolutions. For example, if A is the polynomial ring $k[x, y]$ with $\deg x = 0$ and $\deg y = 1$, then A is a connected $\mathbb{A}^1$-algebra, but $A/(x)$ does not have a minimal resolution in the sense just defined. If A_0 is semi-simple, then every left bounded graded A-module has a minimal resolution.

For our applications to a quantum projective space bundle $\mathbb{P}(\mathcal{E}) \to X$ over a (commutative!) scheme X, we will only need that one very special module has a minimal resolution, namely $\mathcal{O}_X$, and that is guaranteed by the very definition of $\mathbb{P}(\mathcal{E})$.

3. Flat connected graded X-algebras

We retain the notation from the previous section. Thus X is a noetherian quasi-scheme and A is a noetherian, connected, graded X-algebra.

Definition 3.1. We say that A is a **flat X-algebra** if $- \otimes_X A_i$ is exact for all i.

Throughout this section we assume that A is a flat X-algebra.

Thus A is analogous to a locally free sheaf of graded algebras on a scheme X.

Lemma 3.3 shows that the flatness of A implies that induced modules are acyclic for $\underline{\mathcal{T}or}^A_\bullet(-, o_X)$. This is used to characterize those $M \in \mathsf{grmod}A$ for which $\underline{\mathcal{T}or}^A_1(M, o_X) = 0$.

Lemma 3.2. *Suppose that A is a flat, noetherian, connected, graded X-algebra. If $V \in \mathsf{GrMod}X$ and $M \in \mathsf{GrMod}A$, then there is a functorial isomorphism*

$$\operatorname{Ext}^q_{\mathsf{GrMod}A}(V \otimes_X A, M) \cong \operatorname{Ext}^q_{\mathsf{GrMod}X}(V, M_X).$$

Proof. If $0 \to M \to I^\bullet$ is an injective resolution in $\mathsf{GrMod}A$, then $0 \to M_X \to I^\bullet_X$ is an injective resolution in $\mathsf{GrMod}X$. Thus

$$
\begin{aligned}
\operatorname{Ext}^q_{\mathsf{GrMod}A}(V \otimes_X A, M) &= H^q(\operatorname{Hom}_{\mathsf{GrMod}A}(V \otimes_X A, I^\bullet)) \\
&= H^q(\operatorname{Hom}_{\mathsf{GrMod}X}(V, I^\bullet_X)) \\
&= \operatorname{Ext}^q_{\mathsf{GrMod}X}(V, M_X).
\end{aligned}
$$

□

Lemma 3.3. *Suppose that A is a flat, noetherian, connected, graded X-algebra. Then induced modules are acyclic for $\underline{\mathcal{T}or}^A_\bullet(-, o_X)$.*

Proof. Let $I \in \mathsf{GrMod}A$ be injective. By the definition of $\underline{\mathcal{T}or}$,

$$\operatorname{Hom}_{\mathsf{GrMod}A}(\underline{\mathcal{T}or}^A_q(V \otimes_X A, o_X), I) = \operatorname{Ext}^q_{\mathsf{GrMod}A}(V \otimes_X A, \underline{\mathcal{H}om}_A(o_X, I)).$$

By Lemma 3.2, this is isomorphic to $\operatorname{Ext}^q_{\mathsf{GrMod}X}(V, \underline{\mathcal{H}om}_A(o_X, I)_X)$. In the proof so far o_X has been viewed as a graded A-A-bimodule. But $\underline{\mathcal{H}om}_A(o_X, I)_X = \underline{\mathcal{H}om}_A(o_X, I)$, where on the right hand side o_X is viewed as a graded X-algebra. Since $\underline{\mathcal{H}om}_A(o_X, -)$ is right adjoint to the exact functor $(-)_A$ it preserves injectives. Hence $\underline{\mathcal{H}om}_A(o_X, I)$ is injective and, if $q \geq 1$, $\underline{\mathcal{T}or}^A_q(V \otimes_X A, o_X) = 0$. $\square$

If k is a field, A a connected k-algebra, and M a finitely generated graded A-module, then $\operatorname{Tor}^A_1(M, k) = 0$ if and only if M is a free A-module. The next result is an analogue of this.

Terminology: Given a sequence of subobjects $\cdots \subset M_1 \subset M_2 \subset \cdots$ in an abelian category, we call the subquotients M_i/M_{i-1} the slices of this sequence.

Lemma 3.4. *Let A be a flat, noetherian, connected, graded X-algebra. For $M \in \mathsf{grmod}A$, $\underline{\mathcal{T}or}^A_1(M, o_X) = 0$ if and only if M has a finite filtration by graded A-submodules such that each slice is an induced module.*

Proof. ($\Leftarrow$) By the long exact sequence for $\underline{\mathcal{T}or}^A_\bullet(-, o_X)$, it suffices to show that $\underline{\mathcal{T}or}^A_1(V \otimes_X A, o_X)$ is zero. But this is true by Lemma 3.3.

($\Rightarrow$) By Lemma 2.2, there are integers a and b for which the natural map

$$\oplus^b_{i=a} M_i \otimes_X A \to M$$

is an epimorphism. We will prove the result by induction on $b - a$.

Suppose that $b - a = 0$. Then there are exact sequences

$$0 \to K \to M_a \otimes_X A \to M \to 0$$

and

$$0 = \underline{\mathcal{T}or}^A_1(M, o_X) \to K \otimes_A o_X \to M_a \otimes_X A \otimes_A o_X \xrightarrow{\varphi} M \otimes_A o_X \to 0$$

of graded X-modules. But $M_a \otimes_X A \otimes_A o_X \cong M_a$ because $- \otimes_X A \otimes_A o_X$ is equivalent to the identity functor, and $M \otimes_A o_X \cong M_a$ by Lemma 2.1, so φ is monic. Thus $K \otimes_A o_X = 0$. Since M is noetherian, M_a is a noetherian X-module. Since A is noetherian, $M_a \otimes_X A$ is a noetherian A-module, whence K is a noetherian A-module. Hence by Nakayama's Lemma, $K = 0$. Thus $M \cong M_a \otimes_X A$ is induced.

Now suppose that $b - a \geq 1$. Write $C := \operatorname{coker}(M_a \otimes_X A \to M)$ and $K := \operatorname{im}(M_a \otimes_X A \to M)$. Hence there are exact sequences

$$0 \to K \to M \to C \to 0 \tag{3-1}$$

and

$$0 = \underline{\mathcal{T}or}^A_1(M, o_X) \to \underline{\mathcal{T}or}^A_1(C, o_X) \to K \otimes_A o_X \to M \otimes_A o_X \to C \otimes_A o_X \to 0.$$

However, $M_a \otimes_X A \otimes_A o_X \to M \otimes_A o_X$ is monic, so $K \otimes_A o_X \to M \otimes_A o_X$ is monic. Therefore $\underline{\mathcal{T}or}^A_1(C, o_X) = 0$.

The top and right arrows in the diagram

$$
\begin{array}{ccc}
\bigoplus^b_{i=a} M_i \otimes_X A & \longrightarrow & M \\
\downarrow & & \downarrow \\
\bigoplus^b_{i=a} C_i \otimes_X A & \longrightarrow & C
\end{array}
$$

are epimorphisms, so the bottom arrow is epic too. However $C_a = 0$, so the induction hypothesis implies that C has a finite filtration with each slice an induced module.

Since C has a filtration by induced modules, $\underline{\mathcal{T}or}_2^A(C, o_X) = 0$. Therefore $\underline{\mathcal{T}or}_1^A(K, o_X) = 0$. Since the morphism $K_a \otimes_X A \to K$ is epic the proof of the case $b = a$ shows that K is induced. It follows from the exact sequence (3-1) that M has a filtration of the required form. $\square$

4. Grothendieck groups

We retain the notation from the previous section. Thus X is a noetherian quasi-scheme and A is a flat, noetherian, connected, graded X-algebra.

The degree shift functors $M \mapsto M(-1)$ induce automorphisms of $K_0'(\mathsf{grmod}X)$ and $K_0'(\mathsf{grmod}A)$. Let $\mathbb{Z}[T, T^{-1}]$ be the ring of Laurent polynomials and make $K_0'(\mathsf{grmod}X)$ and $K_0'(\mathsf{grmod}A)$ into $\mathbb{Z}[T, T^{-1}]$-modules by defining

$$[M].T := [M(-1)].$$

Proposition 4.1. (cf. [11, Theorem 6, page 110]) *Let A be a flat, noetherian, connected, graded X-algebra. Suppose $\underline{\mathcal{T}or}_q^A(M, o_X) = 0$ for $q \gg 0$ for all $M \in \mathsf{grmod}A$. Then there is an isomorphism*

$$\theta : K_0'(\mathsf{grmod}X) \xrightarrow{\;\sim\;} K_0'(\mathsf{grmod}A), \quad \theta[V] := [V \otimes_X A],$$

of $\mathbb{Z}[T, T^{-1}]$-modules whose inverse is the map

$$\rho : K_0'(\mathsf{grmod}A) \xrightarrow{\;\sim\;} K_0'(\mathsf{grmod}X), \; \rho[M] := \sum (-1)^i [\underline{\mathcal{T}or}_i^A(M, o_X)]. \tag{4-1}$$

Proof. Since $- \otimes_X A$ is exact it induces a group homomorphism θ. It is clear that θ is a $\mathbb{Z}[T, T^{-1}]$-module homomorphism.

To see that ρ is well-defined first observe that $\underline{\mathcal{T}or}_i^A(M, o_X) \in \mathsf{grmod}X$ by Lemmas 2.6 and 3.3, and the sum (4-1) is finite by hypothesis. The long exact sequence for $\underline{\mathcal{T}or}_\bullet^A(-, o_X)$ then ensures that ρ is defined on $K_0'(\mathsf{grmod}A)$.

Now we show that θ is surjective. Let $M \in \mathsf{grmod}A$. By Lemma 3.4, there is an epimorphism $L := V \otimes_X A \to M$ for some $V \in \mathsf{grmod}X$. By Lemma 3.3, $\underline{\mathcal{T}or}_j^A(L, o_X) = 0$ for all $j \geq 1$. By induction, there is a resolution of graded A-modules

$$\cdots \to L_n \to \cdots \to L_0 \to M \to 0$$

with each L_i induced from a noetherian graded X-module. Consider the truncation

$$0 \to N \to L_{d-1} \to \cdots \to L_0 \to M \to 0.$$

Since $\underline{\mathcal{T}or}_i^A(L_j, o_X) = 0$ for $0 \leq j \leq d - 1$ and $i \geq 1$, dimension shifting gives

$$\underline{\mathcal{T}or}_1^A(N, o_X) \cong \underline{\mathcal{T}or}_{d+1}^A(M, o_X) = 0.$$

By Lemma 3.4, N has a filtration with slices being induced modules. Each L_j is an induced module so, in $K_0'(\mathsf{grmod}A)$, $[M]$ is a linear combination of induced modules. This proves the surjectivity.

If $V \in \mathsf{grmod}X$, then Lemma 3.3 shows that

$$(\rho \circ \theta)[V] = \sum (-1)^i [\underline{\mathcal{T}or}_i^A(V \otimes_X A, o_X)] = [V \otimes_X A \otimes_A o_X] = [V],$$

so $\rho\theta = \mathrm{id}_{K_0'(\mathsf{grmod}X)}$. Therefore θ is injective and ρ is its inverse. $\square$

Remark 4.2. If M is a noetherian graded X-module, then its degree i homogeneous component, M_i, is zero for $i \ll 0$ and for $i \gg 0$. Hence there is an isomorphism of $\mathbb{Z}[T, T^{-1}]$-modules

$$K_0'(\mathsf{grmod}X) \xrightarrow{\sim} K_0'(X)[T, T^{-1}] := K_0'(X) \otimes_{\mathbb{Z}} \mathbb{Z}[T, T^{-1}].$$

$$[M] \longmapsto \sum_i [M_i]T^i.$$

From now on we will make the identification

$$K_0'(\mathsf{grmod}X) \equiv K_0'(X)[T, T^{-1}]$$

without further comment.

Corollary 4.3. *Under the hypotheses of Proposition 4.1,*

$$K_0'(\mathsf{grmod}A) \cong K_0'(X)[T, T^{-1}].$$

Proof. Combine Proposition 4.1 and Remark 4.2. $\square$

We now turn to the computation of

$$K_0'(\mathsf{Proj}_{\mathsf{nc}}A) = K_0'(\mathsf{grmod}A/\mathsf{tors}A).$$

The degree shift functor on graded A-modules preserves $\mathsf{Tors}A$, so induces an auto-equivalence of $\mathsf{Proj}_{\mathsf{nc}}A$ that sends noetherian objects to noetherian objects, and hence induces an automorphism of $K_0'(\mathsf{Proj}_{\mathsf{nc}}A)$. We make $K_0'(\mathsf{Proj}_{\mathsf{nc}}A)$ a $\mathbb{Z}[T, T^{-1}]$-module by defining

$$[\mathcal{M}].T := [\mathcal{M}(-1)].$$

A map f of quasi-schemes is an adjoint pair of functors (f^*, f_*) between their module categories [14, Section 3.5]. We define a map

$$f : \mathsf{Proj}_{\mathsf{nc}}A \to X$$

of quasi-schemes as follows. If $\mathcal{M} \in \mathsf{Proj}_{\mathsf{nc}}A$, we define $f_*\mathcal{M} = (\omega\mathcal{M})_0$, the degree zero component. If $V \in \mathsf{Mod}X$, we define $f^*V = \pi(V \otimes_X A)$.

If A has the properties specified in sections 5 and 6 we think of $\mathsf{Proj}_{\mathsf{nc}}A$ as a non-commutative analogue of a projective space bundle over X with structure map $f : \mathsf{Proj}_{\mathsf{nc}}A \to X$.

Theorem 4.4. *Retain the hypotheses of Proposition 4.1. The group homomorphism*

$$K_0'(X)[T, T^{-1}] \to K_0'(\mathsf{Proj}_{\mathsf{nc}}A), \qquad [\mathcal{F}]T^i \mapsto [f^*\mathcal{F}(-i)],$$

induces a $\mathbb{Z}[T, T^{-1}]$-module isomorphism

$$\frac{K_0'(X)[T, T^{-1}]}{I} \cong K_0'(\mathsf{Proj}_{\mathsf{nc}}A),$$

where I is the image in $K_0'(\mathsf{grmod}X) \cong K_0'(X)[T, T^{-1}]$ of the restriction to $K_0'(\mathsf{tors}A)$ of the map ρ in (4-1).

Proof. The localization sequence for K-theory gives an exact sequence

$$K_0'(\mathsf{tors}A) \xrightarrow{\psi} K_0'(\mathsf{grmod}A) \longrightarrow K_0'(\mathsf{Proj}_{\mathsf{nc}}A) \longrightarrow 0.$$

Using the action of the degree shift functor, this is a sequence of $\mathbb{Z}[T, T^{-1}]$-modules. If $M \in \mathsf{tors}A$, then $M\mathfrak{m}^i = 0$ for $i \gg 0$, so objects of $\mathsf{tors}A$ have finite filtrations

with slices belonging to $\mathsf{grmod}X$. By Dévissage, the inclusion $\mathsf{grmod}X \to \mathsf{tors}A$ induces an isomorphism $K_0'(\mathsf{grmod}X) \to K_0'(\mathsf{tors}A)$ which gives the left-most vertical isomorphisms in the diagram

$$
\begin{array}{ccccccc}
K_0'(\mathsf{tors}A) & \xrightarrow{\ \psi\ } & K_0'(\mathsf{grmod}A) & \longrightarrow & K_0'(\mathsf{Proj_{nc}}A) & \longrightarrow & 0 \\[4pt]
\cong \big\uparrow & & \cong \big\downarrow \rho & & & & \\[4pt]
K_0'(\mathsf{grmod}X) & & K_0'(\mathsf{grmod}X) & & & & \\[4pt]
\cong \big\uparrow & & \big\downarrow \cong & & & & \\[4pt]
K_0'(X)[T,T^{-1}] & & K_0'(X)[T,T^{-1}], & & & &
\end{array}
$$

But ρ is an isomorphism, so $K_0'(\mathsf{Proj_{nc}}A) \cong \mathrm{coker}(\rho \circ \psi) \cong K_0'(X)[T,T^{-1}]/I$. $\square$

For the rest of this section, we suppose that X is a noetherian scheme in the usual sense.

As usual, $\mathcal{O}_X$ denotes its structure sheaf. We define $\mathsf{Mod}X$ to be the category of quasi-coherent $\mathcal{O}_X$-modules. Thus $\mathcal{O}_X$ is in $\mathsf{grmod}X$, and hence in $\mathsf{grmod}A$ via the map $A \to o_X$ of graded X-algebras and the associated fully faithful functor $(-)_A : \mathsf{GrMod}X \to \mathsf{GrMod}A$. Thus $\mathcal{O}_X$ is a torsion A-module concentrated in degree zero.

Proposition 4.5. *Retain the hypotheses of Proposition 4.1 and let ρ be the map defined by (4-1). Further, suppose that X is a separated regular noetherian scheme. Let $\mathcal{F} \in \mathsf{grmod}X$, and view $\mathcal{F}_A = \mathcal{F}$ as a graded A-module annihilated by $A_{>0}$. Then*

$$\rho[\mathcal{F}_A] = [\mathcal{F}].\rho[\mathcal{O}_X]$$

where the product on the right hand side is the usual product in $K_0'(X)$ extended to $K_0'(X)[T,T^{-1}]$ in the natural way.

Proof. The hypothesis on X ensures that $\mathcal{F}$ has a finite resolution by locally free $\mathcal{O}_X$-modules, say $\mathcal{W}_\bullet \to \mathcal{F}$. The ring structure on $K_0'(X)[T,T^{-1}]$ is defined by

$$[\mathcal{F}].[\mathcal{G}] = \sum (-1)^i [\underline{\mathrm{Tor}}_i^X(\mathcal{F},\mathcal{G})]$$

where $\underline{\mathrm{Tor}}^X$ denotes the sheaf Tor of graded X-modules.

By definition of $\rho[\mathcal{O}_X]$, we have

$$
\begin{aligned}
[\mathcal{F}].\rho[\mathcal{O}_X] &= \sum_q (-1)^q [\mathcal{F}].[\underline{\mathcal{T}or}_q^A(\mathcal{O}_X, o_X)] \\
&= \sum_q (-1)^q \sum_p (-1)^p [\underline{\mathrm{Tor}}_p^X(\mathcal{F}, \underline{\mathcal{T}or}_q^A(\mathcal{O}_X, o_X))],
\end{aligned}
$$

Hence, to prove the proposition, we must show that

$$\sum_n (-1)^n [\underline{\mathcal{T}or}_n^A(\mathcal{F}, o_X)] = \sum_{p,q} (-1)^{p+q} [\underline{\mathrm{Tor}}_p^X(\mathcal{F}, \underline{\mathcal{T}or}_q^A(\mathcal{O}_X, o_X))].$$

This will follow in the standard way if we can prove the existence of a convergent spectral sequence

$$\underline{\mathrm{Tor}}_p^X(\mathcal{F}, \underline{\mathcal{T}or}_q^A(\mathcal{O}_X, o_X)) \Rightarrow \underline{\mathcal{T}or}_{p+q}^A(\mathcal{F}, o_X). \tag{4-2}$$

This is a mild variation of the usual spectral sequence—the details follow.

By Lemma 2.2, there is a resolution of $\mathcal{O}_X$ in $\mathsf{grmod}A$ by noetherian induced modules. Since every $\mathcal{O}_X$-module is a quotient of a locally free $\mathcal{O}_X$-module there is a resolution $\mathcal{V}_\bullet \otimes_X A \to \mathcal{O}_X$ with each $\mathcal{V}_q$ a locally free graded $\mathcal{O}_X$-module. The spectral sequence we want will come from the bicomplex $\mathcal{W}_\bullet \otimes_X \mathcal{V}_\bullet$.

First we make a preliminary calculation. Applying the forgetful functor $(-)_X$, we can view $\mathcal{V}_\bullet \otimes_X A$ as an exact sequence in $\mathsf{GrMod}X$. Since $- \otimes_X A$ is an exact functor, each $\mathcal{V}_q \otimes_X A$ is a locally free $\mathcal{O}_X$-module. Hence $\underline{\mathrm{Tor}}_q^X(\mathcal{F}, \mathcal{O}_X) \cong H_q(\mathcal{F} \otimes_X \mathcal{V}_\bullet \otimes_X A)$. But $\underline{\mathrm{Tor}}_q^X(\mathcal{F}, \mathcal{O}_X) = \mathcal{T}or_q^X(\mathcal{F}, \mathcal{O}_X)$ is zero if $q \neq 0$, so we conclude that $\mathcal{F} \otimes_X \mathcal{V}_\bullet \otimes_X A \to \mathcal{F}$ is exact, whence it is a resolution of $\mathcal{F}$ in $\mathsf{grmod}A$ by induced modules. Applying $- \otimes_A o_X$ to this complex, it follows from Lemma 3.3 that $\underline{\mathcal{T}or}_q^A(\mathcal{F}, o_X) \cong H_q(\mathcal{F} \otimes_X \mathcal{V}_\bullet)$. Similarly, $H_q(\mathcal{V}_\bullet) \cong \underline{\mathcal{T}or}_q^A(\mathcal{O}_X, o_X)$.

Now we consider the bicomplex $\mathcal{W}_\bullet \otimes_X \mathcal{V}_\bullet$. Since $- \otimes_X \mathcal{V}_q$ is exact, taking homology along the rows gives zero everywhere except in the first column where the homology is $\mathcal{F} \otimes_X \mathcal{V}_\bullet$; taking homology down this column gives $\underline{\mathcal{T}or}_n^A(\mathcal{F}, o_X)$ by the previous paragraph. On the other hand, because $\mathcal{W}_p$ is locally free, when we take homology down the pth column we get $\mathcal{W}_p \otimes_X H_\bullet(\mathcal{V}_\bullet) \cong \mathcal{W}_p \otimes_X \underline{\mathcal{T}or}_\bullet^A(\mathcal{O}_X, o_X)$. If we now take the homology of $\mathcal{W}_\bullet \otimes_X \underline{\mathcal{T}or}_\bullet^A(\mathcal{O}_X, o_X)$ we get $\underline{\mathrm{Tor}}_p^X(\mathcal{F}, \underline{\mathcal{T}or}_q^A(\mathcal{O}_X, o_X))$. Thus, we obtain the desired spectral sequence. $\qquad\square$

Remark 4.6. A connected graded algebra over a field k has finite global dimension if and only if the trivial module k has finite projective dimension; equivalently, if and only if $\mathrm{Tor}_q^A(k, k) = 0$ for $q \gg 0$. The spectral sequence (4-2) yields an analogue of this result. If X is a separated regular noetherian scheme, then $\underline{\mathcal{T}or}_n^A(\mathcal{F}, o_X) = 0$ for all $\mathcal{F} \in \mathsf{mod}X$ and all $n \gg 0$ if and only if $\underline{\mathcal{T}or}_q^A(\mathcal{O}_X, o_X) = 0$ for $q \gg 0$.

The next result is obtained by combining Theorem 4.4 and Proposition 4.5. First, however, we make a change of notation.

When X is a separated, regular, noetherian scheme, every coherent $\mathcal{O}_X$-module has a finite resolution by locally free $\mathcal{O}_X$-modules, so the natural map $K_0(X) \to K_0'(X)$ from the Grothendieck group of locally free coherent $\mathcal{O}_X$-modules is an isomorphism. When X has these properties, and A is a flat, noetherian, connected, graded X-algebra such that $\underline{\mathcal{T}or}_q^A(-, o_X) = 0$ for $q \gg 0$, every noetherian graded A-module has a finite resolution by iterated extensions of modules of the form $\mathcal{V} \otimes_X A$ where $\mathcal{V}$ is a locally free coherent $\mathcal{O}_X$-module (Proposition 4.1). Since an extension by modules of the form $\mathcal{V} \otimes_X A$ is acyclic for $\underline{\mathcal{T}or}_\bullet^A(-, o_X)$ it behaves like a flat A-module. Although we do not have a notion of locally free for $\mathsf{Proj}_{\mathsf{nc}}A$, the image of a flat A-module in $\mathsf{Proj}_{\mathsf{nc}}A$ behaves like a locally free module. So every noetherian module over $\mathsf{Proj}_{\mathsf{nc}}A$ has a finite resolution by modules that behave like locally free modules. We therefore write $K_0(\mathsf{Proj}_{\mathsf{nc}}A)$ for $K_0'(\mathsf{Proj}_{\mathsf{nc}}A)$ in this situation.

Theorem 4.7. *Let X be a separated regular noetherian scheme and A a flat, noetherian, connected, graded X-algebra in the sense of section 1.2. Suppose that $\underline{\mathcal{T}or}_q^A(M, o_X) = 0$ for $q \gg 0$ for all $M \in \mathsf{grmod}A$. Then the group homomorphism*

$$K_0(X)[T, T^{-1}] \longrightarrow K_0(\mathsf{Proj}_{\mathsf{nc}}A), \qquad [\mathcal{F}]T^i \longmapsto [f^*\mathcal{F}(-i)],$$

induces a $\mathbb{Z}[T, T^{-1}]$-module isomorphism

$$K_0(\mathsf{Proj}_{\mathsf{nc}}A) \cong \frac{K_0(X)[T, T^{-1}]}{(F)}$$

where F is the polynomial $\rho[\mathcal{O}_X] = \sum_q (-1)^q [\underline{\mathcal{T}or}_q^A(\mathcal{O}_X, o_X)]$.

Proof. With the identification $K_0'(\mathsf{grmod}X) \equiv K_0(X)[T, T^{-1}]$, the large diagram in the proof of Theorem 4.4 becomes

$$
\begin{array}{ccccccc}
K_0'(\mathsf{tors}A) & \xrightarrow{\ \psi\ } & K_0'(\mathsf{grmod}A) & \longrightarrow & K_0(\mathsf{Proj}_{\mathsf{nc}}A) & \longrightarrow & 0 \\
\cong \downarrow & & \cong \downarrow \rho & & & & \\
K_0(X)[T, T^{-1}] & \xrightarrow[\ \bar{\psi}\ ]{} & K_0(X)[T, T^{-1}] & & & &
\end{array}
$$

and Proposition 4.5 shows that the map $\bar{\psi}$ is multiplication by $\rho[\mathcal{O}_X]$. The result follows. $\qquad\square$

Hilbert series. We say that $M \in \mathsf{GrMod}X$ is bounded below if $M_i = 0$ for $i \ll 0$. In particular, since A is a noetherian graded X-algebra such that $A_i = 0$ for $i < 0$, every $M \in \mathsf{grmod}A$ is bounded below. Hence the forgetful functor $(-)_X : \mathsf{grmod}A \to \mathsf{GrMod}X$ induces a well-defined map

$$H : \mathsf{grmod}A \longrightarrow K_0'(X)((T)),$$
$$M \longmapsto H(M) := \sum_i [M_i]T^i$$

taking values in the the ring of formal Laurent series. Since $(-)_X$ is exact, H induces a homomorphism

$$K_0'(\mathsf{grmod}A) \longrightarrow K_0'(X)((T))$$

of $\mathbb{Z}[T, T^{-1}]$-modules. We call $H(M)$ the Hilbert series of M.

Now suppose the hypotheses of Theorem 4.7 hold. Evaluating $- \otimes_X A$ at $\mathcal{O}_X$ gives $\mathcal{O}_X \otimes_X A \in \mathsf{grmod}A$ with degree i component $\mathcal{O}_X \otimes_X A_i \in \mathsf{mod}X$, so

$$H(\mathcal{O}_X \otimes_A A) = \sum_{i=0}^{\infty} [\mathcal{O}_X \otimes_X A_i]T^i \in K_0(X)[[T]].$$

Since A is connected the leading coefficient of this is $[\mathcal{O}_X \otimes_X o_X] = [\mathcal{O}_X]$ which is the identity element in the ring $K_0(X)[[T]]$. Therefore $H(\mathcal{O}_X \otimes_X A)$ is a unit in $K_0(X)[[T]]$.

Proposition 4.8. *Under the hypotheses of Theorem 4.7, if $\mathcal{O}_X$ has a minimal resolution in $\mathsf{grmod}A$ by noetherian induced modules, then*

$$\rho[\mathcal{O}_X] = H(\mathcal{O}_X \otimes_X A)^{-1}.$$

Proof. Let $\mathcal{V}_\bullet \otimes_X A \to \mathcal{O}_X$ be a minimal resolution in $\mathsf{grmod}A$ (Definition 2.5). It is necessarily finite. Associativity of tensor product for weak bimodules gives $\mathcal{V}_q \otimes_X A = \mathcal{V}_q \otimes_{\mathcal{O}_X} (\mathcal{O}_X \otimes_X A)$. Therefore, we have the following computation in $K_0'(\mathsf{grmod}X) \subset K_0(X)((T))$:

$$
\begin{aligned}
[\mathcal{O}_X] &= \sum (-1)^q [\mathcal{V}_q \otimes_X A] \\
&= \sum (-1)^q [\mathcal{V}_q].[\mathcal{O}_X \otimes A] \\
&= \sum (-1)^q [\underline{\mathcal{T}or}_q^A(\mathcal{O}_X, o_X)].H(\mathcal{O}_X \otimes A) \\
&= \rho[\mathcal{O}_X].H(\mathcal{O}_X \otimes A).
\end{aligned}
$$

This proves the propoosition. $\qquad\square$

Theorem 4.7 and Proposition 4.8 apply to ordinary algebras. If k is a field and A is a connected graded noetherian k-algebra of finite global dimension, then A is an algebra over $X = \operatorname{Spec} k$ such that $\underline{\operatorname{Tor}}_q^A(-, k) = 0$ for $q \gg 0$, and $- \otimes_k A_i$ is exact for every i. Since $\rho[k] = H(A)^{-1}$, where $H(A)$ is the usual Hilbert series of A, we get

$$K_0(\operatorname{Proj}_{\mathsf{nc}} A) \cong \mathbb{Z}[T]/(H(A)^{-1}),$$

thereby recovering [6, Theorem 2.4]. In particular, this computes the K_0 for various non-commutative analogues of $\mathbb{P}^n$.

5. Applications to quantum projective space bundles over a commutative scheme

Now we apply the results of the previous section to quantum projective space bundles over a commutative scheme.

We use the definitions in section 1.7. Thus in this section X, Y, and Z are noetherian schemes over a field k. Let pr_1 and pr_2 denote the canonical projections $pr_1 : X \times Y \to X$ and $pr_2 : X \times Y \to Y$. Let $\mathsf{Mod}X$ and $\mathsf{Mod}Y$ be the categories of quasi-coherent sheaves on X and Y respectively.

The tensor product of a coherent $\mathcal{O}_X$-$\mathcal{O}_Y$ bimodule $\mathcal{M}$ and a coherent $\mathcal{O}_Y$-$\mathcal{O}_Z$ bimodule $\mathcal{N}$ is the coherent $\mathcal{O}_X$-$\mathcal{O}_Z$ bimodule, defined by

$$\mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{N} := pr_{13*}(pr_{12}^* \mathcal{M} \otimes_{\mathcal{O}_{X \times Y \times Z}} pr_{23}^* \mathcal{N}). \tag{5-1}$$

We may tensor an $\mathcal{O}_X$-$\mathcal{O}_Y$ bimodule with an $\mathcal{O}_X$-module or $\mathcal{O}_Y$-module in the appropriate order. For example, if $\mathcal{F} \in \mathsf{Mod}X$ and $\mathcal{M}$ is an $\mathcal{O}_X$-$\mathcal{O}_Y$ bimodule, then

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{M} := pr_{2*}(pr_1^* \mathcal{F} \otimes_{\mathcal{O}_{X \times Y}} \mathcal{M}) \tag{5-2}$$

is in $\mathsf{Mod}Y$. This determines a functor $- \otimes_{\mathcal{O}_X} \mathcal{M} : \mathsf{Mod}X \to \mathsf{Mod}Y$. Formula (5-2) is a special case of (5-1) since $\mathcal{F}$ is naturally an $\mathcal{O}_{\operatorname{Spec} k}$-$\mathcal{O}_X$ bimodule.

Let $\mathcal{M}$ be a coherent $\mathcal{O}_X$-$\mathcal{O}_Y$ bimodule and let $\Gamma = \operatorname{SSupp} \mathcal{M}$ be its scheme-theoretic support. Since $pr_2 : \Gamma \to Y$ is finite, $pr_{2*} : \mathsf{Mod}\Gamma \to \mathsf{Mod}Y$ has a right adjoint $pr_2^! : \mathsf{Mod}Y \to \mathsf{Mod}\Gamma$. We define $\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{M}, -) : \mathsf{Mod}Y \to \mathsf{Mod}X$ by

$$\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{M}, \mathcal{G}) := \mathcal{H}om_{\mathcal{O}_Y}(pr_{2*}\mathcal{M}, \mathcal{G}) = pr_{1*}(\mathcal{H}om_{\mathcal{O}_{X \times Y}}(\mathcal{M}, pr_2^! \mathcal{G})) \tag{5-3}$$

for $\mathcal{G} \in \mathsf{Mod}Y$.

A special case of (5-2) arises when $\mathcal{F} = \mathcal{O}_X$. This amounts to viewing $\mathcal{M}$ as a right $\mathcal{O}_Y$-module. More precisely, the direct image is an exact functor $pr_{2*} : \mathsf{Bimod}(\mathcal{O}_X, \mathcal{O}_Y) \to \mathsf{Mod}Y$. Of course, pr_{2*} need not be exact on $\mathsf{Mod}X \times Y$, but its restriction to $\mathsf{Bimod}(\mathcal{O}_X, \mathcal{O}_Y)$ is because if $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is an exact sequence in $\mathsf{Bimod}(\mathcal{O}_X, \mathcal{O}_Y)$, then $\cup_{i=1}^3 \operatorname{Supp} \mathcal{F}_i$ is affine over Y.

Lemma 5.1. *The rule* $\mathcal{M} \longmapsto \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{M}, -)$ *determines a fully faithful functor* $\mathsf{Bimod}(\mathcal{O}_X, \mathcal{O}_Y) \longrightarrow \mathsf{BiMod}(X, Y)$ *that is compatible with the tensor products.*

Proof. Let $\mathcal{M}$ be a coherent $\mathcal{O}_X$-$\mathcal{O}_Y$ bimodule. First we show that the functor $- \otimes_{\mathcal{O}_X} \mathcal{M} : \mathsf{Mod}X \to \mathsf{Mod}Y$ defined by (5-2) is left adjoint to the functor $\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{M}, -) : \mathsf{Mod}Y \to \mathsf{Mod}X$ defined by (5-3).

Let $\Gamma = \mathrm{SSupp}\,\mathcal{M}$. Let pr_{2*} denote the restriction to $\mathsf{Mod}\Gamma$. Then

$$
\begin{aligned}
\mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{M}, \mathcal{G}) &= \mathrm{Hom}_{\mathcal{O}_Y}(pr_{2*}(pr_1^* \mathcal{F} \otimes_{\mathcal{O}_{X \times Y}} \mathcal{M}), \mathcal{G}) \\
&\cong \mathrm{Hom}_{\mathcal{O}_\Gamma}(pr_1^* \mathcal{F} \otimes_{\mathcal{O}_{X \times Y}} \mathcal{M}, pr_2^! \mathcal{G}) \\
&\cong \mathrm{Hom}_{\mathcal{O}_\Gamma}(pr_1^* \mathcal{F}, \mathcal{H}om_{\mathcal{O}_{X \times Y}}(\mathcal{M}, pr_2^! \mathcal{G})) \\
&\cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, pr_{1*} \mathcal{H}om_{\mathcal{O}_{X \times Y}}(\mathcal{M}, pr_2^! \mathcal{G})) \\
&= \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{M}, \mathcal{G})),
\end{aligned}
$$

which proves the adjointness claim.

We may view the left exact functor $\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{M}, -)$ as an object in the opposite of the category of left exact functors; as such it is an X-Y bimodule in the sense of section 1.1 that we will denote by $\mathcal{M}$.

The tensor product (5-1) of bimodules is associative [13, Proposition 2.5], so is compatible with the tensor product defined in section 1.1. A morphism of $\mathcal{O}_X$-$\mathcal{O}_Y$ bimodules $\mathcal{M} \to \mathcal{N}$ induces natural trasformations

$$
\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{N}, -) \to \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{M}, -)
$$

and

$$
- \otimes_{\mathcal{O}_X} \mathcal{M} \to - \otimes_{\mathcal{O}_X} \mathcal{N}.
$$

The functor is fully faithful by [15, Lemma 3.1.1]. $\qquad\square$

By symmetry $\mathcal{M} \otimes_{\mathcal{O}_Y} - : \mathsf{Mod}Y \to \mathsf{Mod}X$ has a right adjoint, so $\mathcal{M}$ also determines a Y-X bimodule.

Definition 5.2. A coherent $\mathcal{O}_X$-$\mathcal{O}_Y$ bimodule $\mathcal{E}$ is locally free if $pr_{1*}\mathcal{E}$ and $pr_{2*}\mathcal{E}$ are locally free on X and Y respectively. If $pr_{1*}\mathcal{E}$ and $pr_{2*}\mathcal{E}$ are locally free of the same rank r, then we say that $\mathcal{E}$ is locally free of rank r.

Let $\mathcal{M}$ be a coherent $\mathcal{O}_X$-$\mathcal{O}_Y$ bimodule. Then $pr_{1*}\mathcal{M}$ (resp. $pr_{2*}\mathcal{M}$) is locally free if and only if the functor $- \otimes_{\mathcal{O}_X} \mathcal{M}$ (resp. $\mathcal{M} \otimes_{\mathcal{O}_Y} -$) is exact. Let $\mathcal{E}$ be an $\mathcal{O}_X$-$\mathcal{O}_Y$ bimodule and $\mathcal{F}$ an $\mathcal{O}_Y$-$\mathcal{O}_Z$ bimodule. If $\mathcal{E}$ and $\mathcal{F}$ are locally free, so is $\mathcal{E} \otimes_{\mathcal{O}_Y} \mathcal{F}$.

From now on we restrict to the case $X = Y$. An $\mathcal{O}_X$-$\mathcal{O}_X$ bimodule will be called $\mathcal{O}_X$-bimodule for short. Let Δ denote the diagonal in $X \times X$. Then $\mathcal{O}_\Delta$ is an $\mathcal{O}_X$-bimodule in an obvious way. We call it the trivial $\mathcal{O}_X$-bimodule. It is isomorphic to $\mathcal{O}_X$ on each side, and the $\mathcal{O}_X$-action is central.

Definition 5.3. A graded $\mathcal{O}_X$-bimodule algebra is an $\mathcal{O}_X$-bimodule $\mathcal{B}$ that is a direct sum $\mathcal{B} = \oplus_i \mathcal{B}_i$ of $\mathcal{O}_X$-bimodules equipped with maps of $\mathcal{O}_X$-bimodules $\mathcal{B}_i \otimes_{\mathcal{O}_X} \mathcal{B}_j \to \mathcal{B}_{i+j}$ (the multiplication) and $\mathcal{O}_\Delta \to \mathcal{B}_0$ (the unit) such that the usual diagrams commute. We say that $\mathcal{B}$ is connected if $\mathcal{B}$ is $\mathbb{N}$-graded and $\mathcal{B}_0 \cong \mathcal{O}_\Delta$ as $\mathcal{O}_X$-bimodules.

Definition 5.4. Let $\mathcal{B}$ be a graded $\mathcal{O}_X$-bimodule algebra. An $\mathcal{O}_X$-module $\mathcal{M}$ that is a direct sum $\mathcal{M} = \oplus_i \mathcal{M}_i$ of $\mathcal{O}_X$-modules is called a graded right $\mathcal{B}$-module if it is equipped with maps of $\mathcal{O}_X$-modules $\mathcal{M}_i \otimes_{\mathcal{O}_X} \mathcal{B}_j \to \mathcal{M}_{i+j}$ (the action) satisfying the usual compatibilities. The category of graded right $\mathcal{B}$-modules will be denoted by $\mathsf{GrMod}\mathcal{B}$.

A graded right $\mathcal{B}$-module is torsion if $\mathcal{M}_n = 0$ for $n \gg 0$. The subcategory consisting of direct limits of torsion modules is denoted by $\mathsf{Tors}\mathcal{B}$. The quantum projective scheme associated to $\mathcal{B}$ is

$$\mathsf{Proj}_{\mathsf{nc}}\mathcal{B} := \mathsf{GrMod}\mathcal{B}/\mathsf{Tors}\mathcal{B}.$$

We say that $\mathcal{B}$ is noetherian if $\mathsf{GrMod}\mathcal{B}$ is a locally noetherian Grothendieck category.

Definition 5.5. Let $\mathcal{B}$ be a graded $\mathcal{O}_X$-bimodule algebra. A graded $\mathcal{B}$-$\mathcal{O}_X$ bimodule is a direct sum of $\mathcal{O}_X$-bimodules, $\mathcal{M} = \oplus \mathcal{M}_i$, endowed with maps $\mathcal{B}_i \otimes_{\mathcal{O}_X} \mathcal{M}_j \to \mathcal{M}_{i+j}$ of $\mathcal{O}_X$-bimodules making the usual diagrams commute.

If $\mathcal{B}$ is connected then the augmentation map $\varepsilon : \mathcal{B} \to \mathcal{O}_\Delta$ makes $\mathcal{O}_\Delta$ a graded $\mathcal{B}$-$\mathcal{O}_X$-bimodule.

Definition 5.6. A quantum $\mathbb{P}^n$-bundle over a commutative scheme X is a quasi-scheme of the form $\mathsf{Proj}_{\mathsf{nc}}\mathcal{B}$ where $\mathcal{B}$ is a connected graded noetherian $\mathcal{O}_X$-bimodule algebra such that

- each $\mathcal{B}_i$ is a locally free coherent $\mathcal{O}_X$-bimodule;
- there is an exact sequence

$$0 \to \mathcal{B} \otimes_{\mathcal{O}_X} \mathcal{E}'_{n+1}(-n-1) \to \cdots \to \mathcal{B} \otimes_{\mathcal{O}_X} \mathcal{E}'_1(-1) \to \mathcal{B} \otimes_{\mathcal{O}_X} \mathcal{E}'_0 \to \mathcal{O}_\Delta \to 0 \tag{5-4}$$

  of graded $\mathcal{B}$-$\mathcal{O}_X$-bimodules in which each $\mathcal{E}'_i$ is a locally free $\mathcal{O}_X$-bimodule of rank $\binom{n+1}{i}$ concentrated in degree zero;
- there is an exact sequence

$$0 \to \mathcal{E}_{n+1} \otimes_{\mathcal{O}_X} \mathcal{B}(-n-1) \to \cdots \to \mathcal{E}_1 \otimes_{\mathcal{O}_X} \mathcal{B}(-1) \to \mathcal{E}_0 \otimes_{\mathcal{O}_X} \mathcal{B} \to \mathcal{O}_\Delta \to 0 \tag{5-5}$$

  of graded $\mathcal{O}_X$-$\mathcal{B}$-bimodules in which each $\mathcal{E}_i$ is a locally free $\mathcal{O}_X$-bimodule of rank $\binom{n+1}{i}$ concentrated in degree zero.

Usual projective space bundles are quantum $\mathbb{P}^n$-bundles: a locally free sheaf $\mathcal{E}$ on a scheme X may be viewed as a sheaf on Δ, the usual symmetric algebra $S(\mathcal{E})$ may play the role of $\mathcal{B}$, and there are exact sequences of the prescribed form with $\mathcal{E}_q = \mathcal{E}'_q = \wedge^q \mathcal{E}$, the qth exterior power.

Let $\mathcal{B}$ be a graded $\mathcal{O}_X$-bimodule algebra. If each $\mathcal{B}_i$ is a coherent $\mathcal{O}_X$-bimodule, then $\mathcal{B}$ determines a graded X-algebra A in the earlier sense by

$$\mathcal{H}om_X(A_i, -) := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{B}_i, -)$$

each of which has a left adjoint $- \otimes_X A_i = - \otimes_{\mathcal{O}_X} \mathcal{B}_i$. If $\mathcal{B}$ is connected, so is A. A graded right $\mathcal{B}$-module $\mathcal{M}$ can be thought of as a graded A-module because

$$\mathcal{M}_i \otimes_X A_j = \mathcal{M}_i \otimes_{\mathcal{O}_X} \mathcal{B}_j \to \mathcal{M}_{i+j}.$$

Conversely, a graded A module $\mathcal{N}$ can be thought of as a graded right $\mathcal{B}$-module because

$$\mathcal{N}_i \otimes_{\mathcal{O}_X} \mathcal{B}_j = \mathcal{N}_i \otimes_X A_j \to \mathcal{N}_{i+j}.$$

It follows that $\mathsf{GrMod}\mathcal{B} \cong \mathsf{GrMod}A$. Moreover, this equivalence preserves torsion objects, so

$$\mathsf{Proj}_{\mathsf{nc}}\mathcal{B} \cong \mathsf{Proj}_{\mathsf{nc}}A.$$

Lemma 5.7. *Let X be a scheme, and A a flat, noetherian, connected, graded X-algebra. If $\mathcal{E}$ is a locally free $\mathcal{O}_X$-bimodule and $M \in \mathsf{GrMod}A$, then $A \otimes_X \mathcal{E}$ is acyclic for $\underline{\mathcal{T}or}^A_\bullet(M, -)$.*

Proof. We can view $\mathcal{E}$ as a graded X-X-bimodule in the earlier sense, concentrated in degree zero. Composing left exact functors gives a graded A-X-bimodule $A \otimes_X \mathcal{E}$. By the graded version of Lemma 2.4, $\underline{\mathcal{T}or}^A_q(M, A \otimes_X \mathcal{E})$ is the graded A-module determined by the requirement that

$$\mathrm{Hom}_{\mathsf{GrMod}A}(\underline{\mathcal{T}or}^A_q(M, A \otimes_X \mathcal{E}), I) = \mathrm{Ext}^q_{\mathsf{GrMod}A}(M, \underline{\mathcal{H}om}_X(A \otimes_X \mathcal{E}, I))$$

for all injectives $I \in \mathsf{GrMod}X$. However, $\underline{\mathcal{H}om}_X(A \otimes_X \mathcal{E}, -)$ has an exact left adjoint $- \otimes_X A \otimes_{\mathcal{O}_X} \mathcal{E}$, so $\underline{\mathcal{H}om}_X(A \otimes_X \mathcal{E}, I)$ is injective. Therefore $\underline{\mathcal{T}or}^A_q(M, A \otimes_X \mathcal{E}) = 0$ if $q \geq 1$. $\qquad\square$

Theorem 5.8. *Let X be a separated regular noetherian scheme. If $\mathsf{Proj}_{\mathsf{nc}}\mathcal{B}$ is a quantum $\mathbb{P}^n$-bundle over X as in Definition 5.6, then the Hilbert series of $pr_{2*}\mathcal{B} = \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{B}$ is*

$$H(pr_{2*}\mathcal{B}) = \left(\sum_q (-1)^q [pr_{2*}\mathcal{E}_q] T^q \right)^{-1},$$

and

$$K_0(\mathsf{Proj}_{\mathsf{nc}}\mathcal{B}) \cong K_0(X)[T]/(H(pr_{2*}\mathcal{B})^{-1}).$$

Proof. Let A be the X-algebra determined by $\mathcal{B}$. Then $\mathsf{Proj}_{\mathsf{nc}}A = \mathsf{Proj}_{\mathsf{nc}}\mathcal{B}$. The result will follow from Theorem 4.7 and Proposition 4.8, but in order to apply those results we must check that A satisfies the hypotheses of Proposition 4.1.

First A is connected and noetherian because $\mathcal{B}$ is. Since each $\mathcal{B}_j$ is locally free, $- \otimes_X A$ is exact, and A is a flat X-algebra. The sequence (5-4) gives an exact sequence $A \otimes_X \mathcal{E}'_\bullet \to o_X \to 0$ in $\mathsf{BiGr}(A, X)$. By Lemma 5.7, we can compute $\underline{\mathcal{T}or}^A_q(M, o_X)$ as the homology of $M \otimes_A A \otimes_X \mathcal{E}'_\bullet$. But (5-4) is a bounded complex, so $\underline{\mathcal{T}or}^A_q(M, o_X) = 0$ for $q \gg 0$. Thus, the results in section 4 apply to this particular A.

Apply pr_{2*} to the sequence (5-5). This gives an exact sequence of graded right $\mathcal{B}$-modules. Passing to the category of A-modules this gives a resolution of $\mathcal{O}_X$ by induced modules, namely

$$0 \to pr_{2*}\mathcal{E}_{n+1} \otimes_X A(-n-1) \to \cdots \to pr_{2*}\mathcal{E}_0 \otimes_X A \to \mathcal{O}_X \to 0.$$

It is a minimal resolution because each $pr_{2*}\mathcal{E}_i$ is concentrated in degree zero. By Lemma 2.6, $\underline{\mathcal{T}or}^A_q(\mathcal{O}_X, o_X) \cong pr_{2*}\mathcal{E}_q(-q)$ so the result is a consequence of Theorem 4.7 and Proposition 4.8. $\qquad\square$

As remarked after Definition 5.6, usual projective space bundles are quantum $\mathbb{P}^n$-bundles, so Theorem 5.8 computes the Grothendieck group of a usual projective space bundle.

6. Intersection theory on quantum ruled surfaces

In this section, let X be a smooth projective curve over an algebraically closed field k. We apply our computation of the Grothendieck groups to a quantum ruled surface (defined below) and then use this to develop an intersection theory for quantum ruled surfaces.

First we recall Van den Bergh's definition of a quantum ruled surface [15] (see also [7], [10], [13]). Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-bimodule. By [15, Section 3], $\mathcal{E}$ has both a right and a left adjoint, $\mathcal{E}^*$ and $^*\mathcal{E}$, each of which is a locally free $\mathcal{O}_X$-bimodule. An invertible $\mathcal{O}_X$-subbimodule $\mathcal{Q} \subset \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E}$ is nondegenerate if the composition

$$\mathcal{E}^* \otimes_{\mathcal{O}_X} \mathcal{Q} \to \mathcal{E}^* \otimes_{\mathcal{O}_X} \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E} \to \mathcal{E}$$

induced by the canonical map $\mathcal{E}^* \otimes_{\mathcal{O}_X} \mathcal{E} \to \mathcal{O}_\Delta$, and the composition

$$\mathcal{Q} \otimes_{\mathcal{O}_X} {}^*\mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E} \otimes_{\mathcal{O}_X} {}^*\mathcal{E} \to \mathcal{E}$$

induced by the canonical map $\mathcal{E} \otimes_{\mathcal{O}_X} {}^*\mathcal{E} \to \mathcal{O}_\Delta$ are isomorphisms.

Definition 6.1. [15] A quantum ruled surface over X is a quasi-scheme $\mathbb{P}(\mathcal{E})$ where

- $\mathcal{E}$ is a locally free $\mathcal{O}_X$-bimodule of rank 2,
- $\mathcal{Q} \subset \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E}$ is a nondegenerate invertible $\mathcal{O}_X$-subbimodule,
- $\mathcal{B} = T(\mathcal{E})/(\mathcal{Q})$ where $T(\mathcal{E})$ is the tensor algebra of $\mathcal{E}$ over X, and
- $\mathsf{Mod}\mathbb{P}(\mathcal{E}) = \mathsf{Proj}_{\mathsf{nc}}\mathcal{B}$.

We define $\mathcal{O}_{\mathbb{P}(\mathcal{E})} := \pi(pr_{2*}\mathcal{B}) = \pi(\mathcal{O}_X \otimes_X \mathcal{B})$.

By [15, 4.2.1], $\mathbb{P}(\mathcal{E})$ is independent of the choice of a nondegenerate $\mathcal{Q}$. In fact, $\mathcal{Q}$ is not even needed to define $\mathbb{P}(\mathcal{E})$ (loc. cit.).

Theorem 6.2. *If $\mathbb{P}(\mathcal{E})$ is a quantum ruled surface over X, then*

$$K_0(\mathbb{P}(\mathcal{E})) \cong \frac{K_0(X)[T]}{(F)} \tag{6-1}$$

where (F) is the ideal generated by $F = [\mathcal{O}_X] - [pr_{2*}\mathcal{E}]T + [pr_{2*}\mathcal{Q}]T^2$.

Proof. By [15, Theorem 1.2], $\mathcal{B}$ is noetherian. By [15, Theorem 6.1.2], each $\mathcal{B}_n$ is a locally free coherent $\mathcal{O}_X$-bimodules of rank $n+1$ for $n \geq 0$, and there is an exact sequence

$$0 \to \mathcal{Q} \otimes_{\mathcal{O}_X} \mathcal{B}(-2) \to \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{B}(-1) \to \mathcal{O}_\Delta \otimes_{\mathcal{O}_X} \mathcal{B} \to \mathcal{O}_\Delta \to 0$$

of graded $\mathcal{O}_X$-$\mathcal{B}$ bimodules. By symmetry of the definition of $\mathcal{B}$, there is an exact sequence

$$0 \to \mathcal{B} \otimes_{\mathcal{O}_X} \mathcal{Q}(-2) \to \mathcal{B} \otimes_{\mathcal{O}_X} \mathcal{E}(-1) \to \mathcal{B} \otimes_{\mathcal{O}_X} \mathcal{O}_\Delta \to \mathcal{O}_\Delta \to 0$$

of graded $\mathcal{B}$-$\mathcal{O}_X$ bimodules. It follows that $\mathbb{P}(\mathcal{E})$ is a quantum $\mathbb{P}^1$-bundle over X in the sense of Definition 5.6, so Theorem 5.8 applies and gives the desired result. $\square$

The structure map

$$f : \mathbb{P}(\mathcal{E}) \to X$$

for a quantum ruled surface over X is given by the adjoint pair of functors

$$f_* : \mathsf{Mod}\mathbb{P}(\mathcal{E}) \to \mathsf{Mod}X$$
$$f_*\mathcal{M} = (\omega\mathcal{M})_0,$$
$$f^* : \mathsf{Mod}X \to \mathsf{Mod}\mathbb{P}(\mathcal{E})$$
$$f^*\mathcal{F} = \pi(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{B}).$$

We have $f^*\mathcal{O}_X = \mathcal{O}_{\mathbb{P}(\mathcal{E})}$ and $f_*\mathcal{O}_{\mathbb{P}(\mathcal{E})} = \mathcal{O}_X$. The next lemma, which is a simple consequence of results of Nyman and Van den Bergh, shows among other things that $R^1 f_*\mathcal{O}_{\mathbb{P}(\mathcal{E})} = 0$.

Notice that f^* is exact because $- \otimes_{\mathcal{O}_X} \mathcal{B} : \mathsf{Mod}X \to \mathsf{GrMod}\mathcal{B}$ and $\pi : \mathsf{GrMod}\mathcal{B} \to \mathsf{Mod}\mathbb{P}(\mathcal{E})$ are exact. Moreover, $f^*\mathcal{F} \in \mathsf{mod}\mathbb{P}(\mathcal{E})$ for all $\mathcal{F} \in \mathsf{mod}X$ by [8, Lemma 2.17], and $R^q f_* \mathcal{M} \in \mathsf{mod}X$ for all $\mathcal{M} \in \mathsf{mod}\mathbb{P}(\mathcal{E})$ and $q \geq 0$ by [9, Corollary 3.3].

The isomorphism in Theorem 6.2 is induced by the map $K_0(X)[T, T^{-1}] \to K_0(\mathbb{P}(\mathcal{E}))$, $[\mathcal{F}]T^i \mapsto [f^*\mathcal{F}(-i)]$. Since $\mathcal{Q}$ is invertible, $[pr_{2*}\mathcal{Q}]$ is a unit in $K_0(X)[T, T^{-1}]$, and we obtain an isomorphism $K_0(\mathbb{P}(\mathcal{E})) \cong K_0(X) \oplus K_0(X)T$.

Lemma 6.3. *Let $\mathcal{L}$ be a locally free coherent $\mathcal{O}_X$-module. Then*

$$f_*((f^*\mathcal{L})(n)) \cong \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{B}_n \text{ for all } n \in \mathbb{Z},$$

$$R^1 f_*((f^*\mathcal{L})(n)) \cong \begin{cases} 0 & \text{if } n \geq -1, \\ \mathcal{L} \otimes_{\mathcal{O}_X} (\mathcal{B}_{-2-n})^* & \text{if } n \leq -2, \end{cases}$$

$$R^i f_* = 0 \text{ for } i \geq 2.$$

Furthermore, $f^* f_* = \mathrm{id}$, $R^1 f_* \circ f^* = 0$, and $Rf_* \circ f^* = \mathrm{id}_{\mathsf{D}^b(X)}$, where $\mathsf{D}^b(X)$ denotes the bounded derived category of coherent $\mathcal{O}_X$-modules.

Proof. By [9, Theorem 2.5],

$$\tau(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{B}) = 0,$$

$$R^1\tau(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{B}) = 0,$$

$$(R^2\tau(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{B}))_n \cong \begin{cases} 0 & \text{if } n \geq -1, \\ \mathcal{L} \otimes_{\mathcal{O}_X} (\mathcal{B}_{-2-n})^* & \text{if } n \leq -2, \end{cases}$$

$$R^i\tau = 0 \text{ for } i \geq 3.$$

If $\mathcal{M} \in \mathsf{GrMod}\mathcal{B}$, there is an exact sequence

$$0 \to \tau\mathcal{M} \to \mathcal{M} \to \omega\pi\mathcal{M} \to R^1\tau(\mathcal{M}) \to 0,$$

and isomorphisms $R^i\omega(\pi\mathcal{M}) \cong R^{i+1}\tau\mathcal{M}$ for $i \geq 1$ in $\mathsf{GrMod}\mathcal{B}$ [14, section 3.8]. Since $R^i f_*(\pi\mathcal{M}(n)) \cong R^i\omega(\pi\mathcal{M})_n$ for $i \geq 0$, the result follows.

Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. There is an exact sequence $0 \to \mathcal{L}' \to \mathcal{L} \to \mathcal{F} \to 0$ with $\mathcal{L}'$ and $\mathcal{L}$ locally free $\mathcal{O}_X$-modules. Because f^* is exact, $f_* f^*$ is left exact; because $R^1 f_* \circ f_*$ vanishes on locally free sheaves, there is an exact sequence $0 \to f_* f^*\mathcal{L}' \to f_* f^*\mathcal{L} \to f_* f^*\mathcal{F} \to 0$. Because $f_* f^*$ is the identity on locally free sheaves it follows that $f_* f^*\mathcal{F} \cong \mathcal{F}$. Hence $f_* f^*$ is isomorphic to the identity functor. It follows that $Rf_* \circ f^* = \mathrm{id}_{\mathsf{D}^b(X)}$.

Because $R^2 f_* = 0$, $R^1 f_*$ is right exact. Because $R^1 f_* \circ f^*$ vanishes on locally free sheaves it vanishes on all $\mathcal{O}_X$-modules. $\qquad\square$

Because $R^2 f_* = 0$, there is a group homomorphism $Rf_* : K_0(\mathbb{P}(\mathcal{E})) \to K_0(X)$.

Intersection theory. Let Y be a noetherian quasi-scheme over a field k such that

- $\dim_k \mathrm{Ext}^i_Y(\mathcal{M}, \mathcal{N}) < \infty$ for all $i \geq 0$, and
- $\mathrm{Ext}^i_Y(\mathcal{M}, \mathcal{N}) = 0$ for all $i \gg 0$,

for all $\mathcal{M}, \mathcal{N} \in \mathsf{mod}Y$. A quantum ruled surface satisfies the first of these conditions by [9, Corollary 3.6], and satisfies the second by Proposition 6.4 below. We define the Euler form on the Grothendieck group of Y by

$$(-, -) : K_0(Y) \times K_0(Y) \to \mathbb{Z}.$$

$$([\mathcal{M}], [\mathcal{N}]) := \sum_{i=0}^{\infty} (-1)^i \dim_k \operatorname{Ext}_Y^i(\mathcal{M}, \mathcal{N}).$$

We define the intersection multiplicity of $\mathcal{M}$ and $\mathcal{N}$ by

$$\mathcal{M} \cdot \mathcal{N} := (-1)^{\dim \mathcal{N}}(\mathcal{M}, \mathcal{N})$$

for some suitably defined dimension function $\dim \mathcal{N}$. In particular, if $\mathcal{M}$ and $\mathcal{N}$ are "curves" on a "quantum surface" Y, then we define $\mathcal{M} \cdot \mathcal{N} = -(\mathcal{M}, \mathcal{N})$.

The homological dimension of $\mathcal{M} \in \mathsf{mod}Y$ is

$$\operatorname{hd}(\mathcal{M}) = \sup\{i \mid \operatorname{Ext}_Y^i(\mathcal{M}, \mathcal{N}) \neq 0 \text{ for some } \mathcal{N} \in \mathsf{mod}Y\},$$

and the homological dimension of Y is

$$\operatorname{hd}(Y) = \sup\{\operatorname{hd}(\mathcal{M}) \mid \mathcal{M} \in \mathsf{mod}Y\}.$$

Proposition 6.4. *The quantum ruled surface $\mathbb{P}(\mathcal{E})$ over X has finite homological dimension.*

Proof. First we show that $\operatorname{hd}(f^*\mathcal{F}) \leq 2$ if $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module. Let $\mathcal{M} \in \mathsf{mod}\mathbb{P}(\mathcal{E})$.

Since f_* is right adjoint to the exact functor f^*, it preserves injectives, so there is a Grothendieck spectral sequence

$$\operatorname{Ext}_X^p(\mathcal{F}, R^q f_*\mathcal{M}) \Rightarrow \operatorname{Ext}_{\mathbb{P}(\mathcal{E})}^{p+q}(f^*\mathcal{F}, \mathcal{M})$$

for any $\mathcal{F} \in \mathsf{mod}X$ and $\mathcal{M} \in \mathsf{mod}\mathbb{P}(\mathcal{E})$. Since $R^q f_* = 0$ for $q \geq 2$ by Lemma 6.3, we get an exact sequence

$$0 \to \operatorname{Ext}_X^1(\mathcal{F}, f_*\mathcal{M}) \to \operatorname{Ext}_{\mathbb{P}(\mathcal{E})}^1(f^*\mathcal{F}, \mathcal{M}) \to \operatorname{Hom}_X(\mathcal{F}, R^1 f_*\mathcal{M}) \to$$
$$\to \operatorname{Ext}_X^2(\mathcal{F}, f_*\mathcal{M}) \to \operatorname{Ext}_{\mathbb{P}(\mathcal{E})}^2(f^*\mathcal{F}, \mathcal{M}) \to \operatorname{Ext}_X^1(\mathcal{F}, R^1 f_*\mathcal{M}) \to$$
$$\to \operatorname{Ext}_X^3(\mathcal{F}, f_*\mathcal{M}) \to \operatorname{Ext}_{\mathbb{P}(\mathcal{E})}^3(f^*\mathcal{F}, \mathcal{M}) \to \operatorname{Ext}_X^2(\mathcal{F}, R^1 f_*\mathcal{M}) \to \cdots.$$

Since X is a smooth curve, $\operatorname{Ext}_X^p = 0$ for $p \geq 2$, so $\operatorname{Ext}_{\mathbb{P}(\mathcal{E})}^p(f^*\mathcal{F}, -) = 0$ for $p \geq 3$ and $\operatorname{hd}(f^*\mathcal{F}) \leq 2$.

By Theorem 6.2, $K_0(\mathbb{P}(\mathcal{E}))$ is generated by $[(f^*\mathcal{F})(n)]$ for $\mathcal{F} \in \mathsf{mod}X$ and $n \in \mathbb{Z}$. Let $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ be an exact sequence in $\mathsf{mod}\mathbb{P}(\mathcal{E})$. If two of $\mathcal{M}, \mathcal{M}', \mathcal{M}''$ have finite homological dimension, so does the other, so it is enough to show that $(f^*\mathcal{F})(n)$ has finite homological dimension. But we have just seen that $\operatorname{hd}((f^*\mathcal{F})(n)) \leq 2$ so the result follows. $\square$

Lemma 6.5. *Let $\mathbb{P}(\mathcal{E})$ be a quantum ruled surface over X. If $\mathcal{F} \in \mathsf{mod}X$ and $\mathcal{M} \in \mathsf{mod}\mathbb{P}(\mathcal{E})$, then*

(1) *$(f^*\mathcal{F}, \mathcal{M}) = (\mathcal{F}, f_*\mathcal{M}) - (\mathcal{F}, R^1 f_*\mathcal{M})$ and*
(2) *if $\mathcal{L}$ is a locally free coherent $\mathcal{O}_X$-module and $n \geq -1$, then*

$$(f^*\mathcal{F}, (f^*\mathcal{L})(n)) = (\mathcal{F}, \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{B}_n).$$

Proof. (1) From the long exact sequence in the previous proof, we obtain an exact sequence

$$0 \to \operatorname{Ext}_X^1(\mathcal{F}, f_*\mathcal{M}) \to \operatorname{Ext}_{\mathbb{P}(\mathcal{E})}^1(f^*\mathcal{F}, \mathcal{M}) \to \operatorname{Hom}_X(\mathcal{F}, R^1 f_*\mathcal{M}) \to 0$$

and an isomorphism

$$\operatorname{Ext}_{\mathbb{P}(\mathcal{E})}^2(f^*\mathcal{F}, \mathcal{M}) \cong \operatorname{Ext}_X^1(\mathcal{F}, R^1 f_*\mathcal{M}).$$

Combining this with the adjoint isomorphism

$$\operatorname{Hom}_{\mathbb{P}(\mathcal{E})}(f^*\mathcal{F}, \mathcal{M}) \cong \operatorname{Hom}_X(\mathcal{F}, f_*\mathcal{M}),$$

we get

$$(f^*\mathcal{F}, \mathcal{M}) = (\mathcal{F}, f_*\mathcal{M}) - (\mathcal{F}, R^1 f_*\mathcal{M}).$$

(2) By Lemma 6.3, $f_*((f^*\mathcal{L})(n)) \cong \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{B}_n$ and $R^1 f_*((f^*\mathcal{L})(n)) = 0$ so part (1) gives $(f^*\mathcal{F}, (f^*\mathcal{L})(n)) = (\mathcal{F}, \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{B}_n)$. $\qquad\square$

Our remaining goal is Theorem 6.8 which shows that the intersection theory on $\mathbb{P}(\mathcal{E})$ is like that for a commutative ruled surface.

First we recall the following standard result that describes the Euler form on the smooth projective curve X.

Lemma 6.6. *Let g be the genus of X, and p and q closed points of X. Then*

$$(\mathcal{O}_X, \mathcal{O}_X) = 1 - g.$$
$$(\mathcal{O}_X, \mathcal{O}_p) = 1.$$
$$(\mathcal{O}_p, \mathcal{O}_X) = -1.$$
$$(\mathcal{O}_p, \mathcal{O}_q) = 0.$$

Lemma 6.7. *Let p be a closed point of X and n an integer ≥ -1. Then*

(1) $R^1 f_*((f^*\mathcal{O}_p)(n)) = 0$,
(2) $[f_*((f^*\mathcal{O}_p)(n))] = \sum_{i=0}^{n} [\mathcal{O}_{p_i}]$ *in $K_0(X)$ for some closed points $p_0, \cdots, p_n \in X$, and*
(3) $(\mathcal{O}_{\mathbb{P}(\mathcal{E})}, (f^*\mathcal{O}_p)(n)) = n + 1$.

Proof. Let $n \in \mathbb{Z}$. Since f^* is exact, the exact sequence $0 \to \mathcal{O}_X(-p) \to \mathcal{O}_X \to \mathcal{O}_p \to 0$ in $\mathsf{mod}X$ induces an exact sequence

$$0 \to (f^*(\mathcal{O}_X(-p)))(n) \to (f^*\mathcal{O}_X)(n) \to (f^*\mathcal{O}_p)(n) \to 0$$

in $\mathsf{mod}\mathbb{P}(\mathcal{E})$. Applying f_* to this produces a long exact sequence

$$0 \to f_*((f^*\mathcal{O}_X(-p))(n)) \to f_*((f^*\mathcal{O}_X)(n)) \to f_*((f^*\mathcal{O}_p)(n))$$
$$\to R^1 f_*((f^*\mathcal{O}_X(-p))(n)) \to R^1 f_*((f^*\mathcal{O}_X)(n)) \to R^1 f_*((f^*\mathcal{O}_p)(n)) \to \cdots$$

in $\mathsf{mod}X$.

Now suppose that $n \geq -1$.

Applying Lemma 6.3 to various terms in the previous long exact sequence, we get an exact sequence

$$0 \to \mathcal{O}_X(-p) \otimes_{\mathcal{O}_X} \mathcal{B}_n \to \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{B}_n \to f_*((f^*\mathcal{O}_p)(n)) \to 0,$$

and also see that $R^1 f_*((f^*\mathcal{O}_p)(n)) = 0$. On the other hand, there is an exact sequence

$$0 \to \mathcal{O}_X(-p) \otimes_{\mathcal{O}_X} \mathcal{B}_n \to \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{B}_n \to \mathcal{O}_p \otimes_{\mathcal{O}_X} \mathcal{B}_n \to 0$$

in $\mathsf{mod}X$, so

$$[f_*((f^*\mathcal{O}_p)(n))] = [\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{B}_n] - [\mathcal{O}_X(-p) \otimes_{\mathcal{O}_X} \mathcal{B}_n] = [\mathcal{O}_p \otimes_{\mathcal{O}_X} \mathcal{B}_n]$$

in $K_0(X)$. By [7, Lemma 4.1], the length of $\mathcal{O}_p \otimes_{\mathcal{O}_X} \mathcal{B}_n \in \mathsf{mod}X$ is $n+1$, so $[\mathcal{O}_p \otimes_{\mathcal{O}_X} \mathcal{B}_n] = \sum_{i=0}^{n}[\mathcal{O}_{p_i}]$ in $K_0(X)$ for some closed points $p_0, \cdots, p_n \in X$. In particular,

$$(f^*\mathcal{O}_X, (f^*\mathcal{O}_p)(n)) = (\mathcal{O}_X, f_*((f^*\mathcal{O}_p)(n))) - (\mathcal{O}_X, R^1f_*((f^*\mathcal{O}_p)(n)))$$
$$= \sum_{i=0}^{n}(\mathcal{O}_X, \mathcal{O}_{p_i})$$
$$= n+1$$

by Lemmas 6.5 and 6.6. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The next result says that "fibers do not meet" and "a fiber and a section meet exactly once".

Theorem 6.8. *Let $\mathbb{P}(\mathcal{E})$ be a quantum ruled surface over X, and p and q closed points of X. Define*

$$H := [\mathcal{O}_{\mathbb{P}(\mathcal{E})}] - [\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1)] \in K_0(\mathbb{P}(\mathcal{E})).$$

Then

$$f^*\mathcal{O}_p \cdot f^*\mathcal{O}_q = 0.$$
$$f^*\mathcal{O}_p \cdot H = 1.$$
$$H \cdot f^*\mathcal{O}_p = 1$$
$$H \cdot H = \deg(pr_{2*}\mathcal{E}).$$

Proof. (1) By Lemmas 6.5 (2), 6.6, and 6.7,

$$(f^*\mathcal{O}_p, f^*\mathcal{O}_q) = (\mathcal{O}_p, f_*(f^*\mathcal{O}_q)) - (\mathcal{O}_p, R^1f_*(f^*\mathcal{O}_q)) = (\mathcal{O}_p, \mathcal{O}_q) = 0.$$

(2) By Lemmas 6.5 (3) and 6.6

$$(f^*\mathcal{O}_p, H) = (f^*\mathcal{O}_p, f^*\mathcal{O}_X) - (f^*\mathcal{O}_p, (f^*\mathcal{O}_X)(-1))$$
$$= (\mathcal{O}_p, \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{B}_0) - (\mathcal{O}_p, \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{B}_{-1})$$
$$= (\mathcal{O}_p, \mathcal{O}_X)$$
$$= -1.$$

(3) By Lemma 6.7,

$$(H, f^*\mathcal{O}_p) = (f^*\mathcal{O}_X, f^*\mathcal{O}_p) - ((f^*\mathcal{O}_X)(-1), f^*\mathcal{O}_p)$$
$$= (f^*\mathcal{O}_X, f^*\mathcal{O}_p) - (f^*\mathcal{O}_X, (f^*\mathcal{O}_p)(1))$$
$$= 1 - 2$$
$$= -1.$$

(4) Let g be the genus of X. Since $pr_{2*}\mathcal{E}$ is a locally free module of rank 2 on a smooth curve, there is an exact sequence

$$0 \to \mathcal{L}_1 \to pr_{2*}\mathcal{E} \to \mathcal{L}_2 \to 0$$

in which $\mathcal{L}_1$ and $\mathcal{L}_2$ are line bundles [3, Chapter V, Exercise 2.3 (a)]. Thus $[pr_{2*}\mathcal{E}] = [\mathcal{L}_1] + [\mathcal{L}_2]$ and $\deg(pr_{2*}\mathcal{E}) = \deg\mathcal{L}_1 + \deg\mathcal{L}_2$ [3, Chapter II, Exercise 6.12]. By

Riemann-Roch, and Lemmas 6.5 (3) and 6.6, we have

$$
\begin{aligned}
(H, H) =& (f^*\mathcal{O}_X, f^*\mathcal{O}_X) - ((f^*\mathcal{O}_X)(-1), f^*\mathcal{O}_X) \\
& - (f^*\mathcal{O}_X, (f^*\mathcal{O}_X)(-1)) + ((f^*\mathcal{O}_X)(-1), (f^*\mathcal{O}_X)(-1)) \\
=& (\mathcal{O}_X, \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{B}_0) - (\mathcal{O}_X, \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{B}_1) - (\mathcal{O}_X, \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{B}_{-1}) \\
& + (\mathcal{O}_X, \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{B}_0) \\
=& (1 - g) - (1 - g + \deg \mathcal{L}_1) - (1 - g + \deg \mathcal{L}_2) + (1 - g) \\
=& - \deg \mathcal{L}_1 - \deg \mathcal{L}_2 \\
=& - \deg(pr_{2*}\mathcal{E}).
\end{aligned}
$$

$\square$

The previous result is the same as for the commutative case [3, Propositions V.2.8 and V.2.9]. When $\mathcal{E}$ is "normalized", it is reasonable to call H a K-theoretic section; in this case the integer $e := -H \cdot H = -\deg(pr_{2*}\mathcal{E})$ should play an important role in the classification of quantum ruled surfaces analogous to the role that e plays in the commutative theory.

The rank of a coherent $\mathcal{O}_X$-module gives a group homomorphism rank : $K_0(X) \to \mathbb{Z}$. Recall that $K_0(\mathbb{P}(\mathcal{E})) = K_0(X) \oplus K_0(X)T$, where the $K_0(X)$ on the right hand side is really $f^*K_0(X)$. We define rank : $K_0(\mathbb{P}(\mathcal{E})) \to \mathbb{Z}$ by $\text{rank}(a + bT) = \text{rank}(a) + \text{rank}(b)$ where $a, b \in K_0(X)$. Then $\text{rank}(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(n)) = 1$ for all n, so $\text{rank}(H) = 0$, and also $\text{rank}(f^*\mathcal{O}_p(n)) = 0$ for all n and all $p \in X$. We define $F^1K_0(\mathbb{P}(\mathcal{E}))$ to be the kernel of the rank function and think of it as the subgroup of $K_0(\mathbb{P}(\mathcal{E}))$ generated by those $\mathbb{P}(\mathcal{E})$-modules whose "support" is of codimension ≥ 1.

The kernel of rank : $K_0(X) \to \mathbb{Z}$ is $F^1K_0(X)$, the subgroup generated by the $\mathcal{O}_X$-modules having support of codimension ≥ 1, and this subgroup is isomorphic to the Picard group Pic X. We have $F^1K_0(\mathbb{P}(\mathcal{E})) = F^1K_0(X) \oplus F^1K_0(X).H \oplus \mathbb{Z}H$. Using the results already obtained, one sees that $F^1K_0(X).H$ is contained in both the left and right radicals of the Euler form restricted to $F^1K_0(\mathbb{P}(\mathcal{E}))$, so the Euler form induces a $\mathbb{Z}$-valued bilinear form on the quotient $F^1K_0(\mathbb{P}(\mathcal{E}))/F^1K_0(X).H$. There are good reasons to think of this quotient as playing the role of the Picard group for $\mathbb{P}(\mathcal{E})$, and if one does this one has Pic $\mathbb{P}(\mathcal{E}) \cong f^*\text{Pic}\,X \oplus \mathbb{Z}.H$ just as in the commutative case (cf. [3, Ch. V, Prop.2.3]); here $f^*\text{Pic}\,X$ denotes the image of $F^1K_0(X)$ under the map $f^* : K_0(X) \to K_0(\mathbb{P}(\mathcal{E}))$.

Modding out all the radical for the Euler form on $F^1K_0(\mathbb{P}(\mathcal{E}))$ produces an analogue of the Neron-Severi group, and this is a free abelian group of rank two with basis $[f^*\mathcal{O}_p]$ and H, and the pairing on this is given by Theorem 6.8. Again, one has a good analogue of the commutative theory.

References

[1] M. Artin, Some problems on three-dimensional graded domains. *Representation theory and algebraic geometry* (Waltham, MA, 1995), 1–19, London Math. Soc. Lecture Note Ser., 238, Cambridge Univ. Press, Cambridge, 1997.

[2] P. Berthelot, Le $K^\bullet$ d'un fibre projectif, Exposé VI, *Théorie des intersections et Théorème de Riemann-Roch,* Séminaire de Géometrie Algébrique Bois-Marie 1966–1967 (SGA 6). Dirigé par P. Berthelot, A. Grothendieck et L. Illusie. Avec la collaboration de D. Ferrand, J. P. Jouanolou, O. Jussila, S. Kleiman, M. Raynaud et J. P. Serre. Lecture Notes in Mathematics, Vol. 225. Springer-Verlag, Berlin-New York, 1971.

[3] R. Hartshorne, *Algebraic Geometry,* Springer-Verlag, 1977.

[4] P. Jörgensen, Intersection theory on non-commutative surfaces, *Trans. Amer. Math. Soc.*, **352** (2000) 5817-5854.

[5] Y. Manin, Lectures on the K-functor in algebraic geometry, *Russ. Math. Surveys,* **24**(5) (1969) 1-89.

[6] I. Mori and S.P. Smith, Bézout's theorem for non-commutative projective spaces, *J. Pure Appl. Algebra*, **157** (2001) 279-299.

[7] A. Nyman, The geometry of points on quantum projectivizations, *J. Algebra* **246** (2001) 761-792.

[8] A. Nyman, Serre duality for non-commutative $\mathbb{P}^1$-bundles, *Trans. Amer. Math. Soc.,* **357** (2005) 1349-1416, RA/0210083.

[9] A. Nyman, Serre finiteness and Serre vanishing for non-commutative $\mathbb{P}^1$-bundles, *J. Algebra,* **278** (2004) 32-42, RA/0210080.

[10] D. Patrick, Quantum ruled surfaces, Ph.D. thesis, M.I.T., 1997.

[11] D. Quillen, Higher Algebraic K-theory I, Algebraic *K*-theory, I: Higher *K*-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), pp. 85–147. Lecture Notes in Math., Vol. 341, Springer, Berlin 1973.

[12] A. Rosenberg, *Non-commutative Algebraic Geometry and representations of quantized algebras*, Math. and its Appl., vol. 330, Kluwer Acad. Publ., Dordrecht, 1995.

[13] M. Van den Bergh, A translation principle for the four-dimensional Sklyanin algebras, *J. Algebra,* **184** (1996) 435-490.

[14] M. Van den Bergh, Blowing up of non-commutative smooth surfaces, *Mem. Amer. Math. Soc.*, **154** (2001).

[15] M. Van den Bergh, Non-commutative $\mathbb{P}^1$-bundles over commutative schemes, RA/0102005.

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