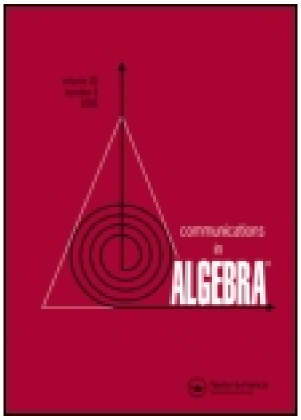

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Auslander-gorenstein rings

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AUSLANDER-GORENSTEIN RINGS

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ABSTRACT. We study basic properties of Auslander-Gorenstein rings related to duality, localization and purity of minimal injective resolutions.

Key words and phrases. injective dimension, injective resolution, Auslander-Gorenstein condition, Cohen-Macaulay condition, pure (or essentially pure) module, duality.

Contents

0. Introduction and definitions

An Auslander-Gorenstein ring can be viewed as a noncommutative analogue of a commutative local Gorenstein ring and as a generalization of a quasi-Frobenius ring. Familiar examples include Weyl algebras, universal enveloping algebras of finite dimensional Lie algebras, three-dimensional Artin-Schelter regular algebras and the Sklyanin algebras. Several recent results in noncommutative ring theory suggest that the Auslander-Gorenstein property is a fundamental homological property that relates to other properties such as being domain, localizable, etc. This paper studies several topics about Auslander-Gorenstein rings.

Definition 0.1. Let A be a ring. The **grade** of an A-module M is

$$j(M) := \min\{\, i \mid \mathrm{Ext}^i_A(M, A) \neq 0\}$$

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or ∞ if no such i exists. We say that A

- satisfies the **Auslander condition** if for every noetherian A-module M and for all $i \geq 0$, we have $j(N) \geq i$ for all submodules $N \subset \mathrm{Ext}^i(M, A)$;
- is **Auslander-Gorenstein** (AG) if A is two-sided noetherian, satisfies the Auslander condition, and has finite left and right injective dimension;
- is **Auslander regular** if it is Auslander-Gorenstein, and has finite global dimension;
- is **grade-symmetric** if $j(M_A) = j({}_AM)$ for every (A, A)-bimodule M finitely generated on both sides.

The injective dimension of a module M is written injdim M. By [Za, Lemma A], injdim A_A is equal to injdim ${}_AA$ if both are finite. If A is commutative, noetherian, and has finite injective dimension, then A is AG. A ring is **quasi-Frobenius** (QF) if it is left and right artinian, and left and right self-injective. It is easy to see that every QF ring is AG.

The plan of the paper is as follows. We start, in section 1, with duality aspects between left and right modules. Let $\mathcal{M} - A$ denote the category of finitely generated right A-modules. If A is QF, then $\mathrm{Hom}(-, A)$ gives a duality between $\mathcal{M} - A$ and $\mathcal{M} - A^{op}$ [Fa, 24.4]. We prove that if A is an AG ring of injective dimension d, then $\mathcal{M} - A$ and $\mathcal{M} - A^{op}$ are in $(d+1)$-step duality [Theorem 1.2]. Using this we recover a result of Roos [Bj2] that the left and right Krull dimensions of an AG ring are bounded above by its injective dimension [Corollary 1.3].

In sections 2-4, we study different aspects of injective resolutions of the ring. Section 2 contains the preliminaries, where we also prove that if A is a noetherian ring of finite injective dimension, then every indecomposable injective module does appear in a minimal injective resolution of A [Theorem 2.3]. This indicates that, in some sense, a *minimal injective resolution of A* contains a lot of information about A-modules. Section 3 gives detailed information about the last term of a minimal injective resolution. Our objective in section 4 is to study the injective resolutions with respect to *purity*, which we now explain.

Let ∂ denote a *dimension function* on A-modules, in the sense of [MR, 6.8.4]. We say ∂ is **exact** if for all A-modules M, $\partial(M) = \max\{\partial(N), \partial(M/N)\}$ whenever N is a submodule of M. The standard example is Krull dimension (Kdim) due to Rentschler-Gabriel. For an algebra over a field, we also have the notion of Gelfand-Kirillov dimension (GKdim). Krull dimension is always, and GK-dimension is often, exact.

Definition 0.2. Given a dimension function ∂, exact or not, we say that a module M is

- **s-pure with respect to ∂** if $\partial(N) = s$ for all non-zero noetherian submodules $N \subset M$;
- **essentially s-pure with respect to ∂** if it contains an essential submodule which is s-pure with respect to ∂;
- **s-critical with respect to ∂** if it is s-pure and $\partial(M/N) < s$ for all non-zero submodules $N \subset M$.

The word 'pure' is a substitute for the word 'homogeneous' used in [MR], and we prefer the former because this has been frequently used in recent literature. If ∂ is exact, then a non-zero submodule of an s-critical module is s-critical, and critical modules are uniform.

Definition 0.3. Suppose that injdim $A_A = d < \infty$, and let

$$0 \longrightarrow A_A \longrightarrow I^0 \longrightarrow \cdots \longrightarrow I^d \longrightarrow 0. \tag{0-1}$$

be a minimal injective resolution. We say this resolution is

- **pure** with respect to ∂ if each I^i is $(\partial(A) - i)$-pure with respect to ∂.

• **essentially pure** with respect to ∂ if each I^i is essentially $(\partial(A)-i)$-pure with respect to ∂.

Let A be a noetherian ring of finite injective dimension d. We define $\delta(M) = d - j(M)$, for a (left or right) noetherian A-module M; we say δ is exact if $\delta(M) = \sup\{\delta(N), \delta(M/N)\}$, or equivalently, $j(M) = \inf\{j(M), j(M/N)\}$ for all submodules N of a noetherian module M. Note that, in general, δ is *not* a dimension function. It is a simple fact that for any ring A and A-module M, $j(M) \geq \inf\{j(N), j(M/N)\}$ where N is a submodule of M; therefore, for a ring A of finite injective dimension, the inequality $\delta(M) \leq \sup\{\delta(N), \delta(M/N)\}$ always holds for A-modules M and submodules $N \subset M$. It follows that whenever δ is a dimension function, it is exact. If A is AG then, by [Le, 4.5] and [Bj, 1.8], δ is a dimension function, and therefore exact; we call it the **canonical dimension function.** One consequence of this is that if M is an A/P-module where $P \in \operatorname{Spec} A$, then $j(M) > j(A/P)$ if and only if M is a torsion A/P-module, and $j(M) = j(A/P)$ otherwise. Levasseur [Le, 4.5] also shows that δ is **finitely partitive,** meaning that if M is noetherian then any chain of submodules $M = M_0 \supset M_1 \supset \cdots$, for which $\delta(M_i/M_{i+1}) = \delta(M)$ for all i, is necessarily finite.

Whether δ is a dimension function or not, we still define that a module M is

• **s-pure** if $\delta(N) = s$ for all non-zero noetherian submodules $N \subset M$;

• **essentially s-pure** if it contains an essential submodule which is s-pure;

• **s-critical** if it is s-pure and $\delta(M/N) < s$ for all non-zero submodules $N \subset M$.

Whenever we use the terms s-pure or s-critical without reference to any particular dimension function, we mean pure or critical with respect to δ, irrespective of whether δ is a dimension function or not. For example, it is easy to see that A is always d-pure both as a left and a right A-module.

Let A be a commutative local noetherian Gorenstein ring. It is well-known that if $0 \to A \to I^\bullet$ is a minimal injective resolution of the A-module A, then

$$I^i \cong \bigoplus_{\mathfrak{p} \in \operatorname{Spec} A, \text{height } \mathfrak{p} = i} E(A/\mathfrak{p}),$$

where $E(-)$ denotes an injective hull. As a consequence, every non-zero finitely generated submodule of I^i has Krull dimension equal to $\operatorname{Kdim} A - i$. Thus, in our language, A has a pure minimal injective resolution with respect to Krull dimension. In the noncommutative case, Artin and Stafford have given examples of AG rings which do not have pure or even essentially pure minimal injective resolutions (Examples 5.2 and 5.3). So it is natural to ask under what hypotheses AG rings have pure or essentially pure injective resolutions. We prove that if A is an AG, grade-symmetric ring satisfying a polynomial identity, then A has a pure resolution [Theorem 4.2]. In [ASZ], we have also proved that many AG rings with small injective dimension have pure or essentially pure injective resolution. Conversely, under a reasonable hypothesis, essential purity can occur only for AG rings [Theorem 4.4]; and if A has an essentially pure minimal injective resolution with respect to an exact dimension function ∂, then in fact ∂ is equal to δ up to some additive constant [Proposition 4.3]. It is in this sense that δ behaves like a canonical dimension function.

In section 6, we describe certain conditions for the existence of quotient rings of AG rings. We prove that a grade-symmetric AG ring has a QF quotient ring [Remark to Theorem 6.1], and as a corollary, a grade-symmetric Auslander regular algebra is semiprime [Corollary 6.3]. This last result is a noncommuatative analog of the fact if A is a noetherian commutative ring of finite global dimension, then A has no nilpotent elements. We give an easy example to show that an

Auslander regular ring need not be grade-symmetric, and, for such a ring, the quotient ring (if it exists) need not be QF or semiprime [Example 5.4.2].

Another property often appearing together with the AG property is the Cohen-Macaulay property.

Definition 0.4. We say that A is **Cohen-Macaulay with respect to a dimension function** ∂ (or, ∂-CM, in short) if

$$j(M) + \partial(M) = \partial(A) < \infty$$

for every non-zero noetherian A-module M. When we say A is **Cohen-Macaulay** (CM) without reference to any particular dimension function, we mean A is Cohen-Macaulay with respect to GKdim (assuming tacitly that A is an algebra over a field). For an algebra A over a field k, we say that A is **Cohen-Macaulay at zero** (CM_0) provided that, for a noetherian A-module M, $j(M) = \operatorname{injdim} A$ if and only if M is finite k-dimensional.

It is a tautology that an AG ring is CM with respect to the canonical dimension function δ. If a ring is CM with respect to a dimension function ∂, then ∂ has to be exact. If A is a noetherian ring with finite injective dimension d, then A is CM with respect to some dimension function ∂ if and only if δ is a dimension function: indeed, then $\partial(M) = \delta(M) + (\partial(A) - d)$.

In section 7, we examine the relation between the groups $\operatorname{Ext}^i_A(M, N)$ and $\operatorname{Ext}^i_A(N, M)$ under some hypotheses such as CM_0 and commutative Gorenstein [Propositions 7.1 and 7.7].

Conventions and Notations. Throughout the paper A will be a left and right noetherian ring. Unless otherwise specified, we work with right modules. We will usually omit the subscript A from $\operatorname{Ext}^i_A(M, N)$. We will often write E^pM for $\operatorname{Ext}^p(M, A)$, and $E^{pq}M$ for $\operatorname{Ext}^p(\operatorname{Ext}^q(M, A), A)$.

We will often use Ischebeck's spectral sequence: if A is noetherian with $\operatorname{injdim} A = d$, and M a noetherian right A-module, there is a convergent spectral sequence

$$E_2^{pq} = \operatorname{Ext}^p_A(\operatorname{Ext}^q_A(M, A), A) \Longrightarrow \mathbf{H}^{p-q}(M) := \begin{cases} 0 & \text{if } p \neq q, \\ M & \text{if } p = q. \end{cases} \tag{0-2}$$

Thus, on the E_∞-page only the diagonal terms are non-zero. To simplify notation later, we have used a non-standard indexing of E_2^{pq}; with our indexing, the boundary maps on the E_2-page are $E_2^{pq} \to E_2^{p+2,q+1}$. This spectral sequence is functorial in M. Consequently, there is a canonical filtration of M by submodules,

$$M = F^0M \supset F^1M \supset \cdots F^dM \supset F^{d+1}M = 0$$

where $F^iM/F^{i+1}M \cong E_\infty^{ii}$. If A is AG, then by [Bj1, 1.3], F^iM is the largest submodule $X \subset M$ such that $j(X) \geq i$. For each i there is an exact sequence of A-modules:

$$0 \longrightarrow F^iM/F^{i+1}M \longrightarrow \operatorname{Ext}^i(\operatorname{Ext}^i(M, A), A) \longrightarrow Q_{i+2}(M) \longrightarrow 0 \tag{0-3}$$

where $\delta(Q_{i+2}(M)) \leq d - (i+2)$ or, equivalently, $j(Q_{i+2}(M)) \geq i + 2$.

1. Duality between left and right modules

The basic idea of this section appeared implicitly in [Le] and [Bj2]. Let $\mathcal{M} - A$ denote the category of finitely generated right A-modules. If A is AG of injective dimension d, then δ is exact, and we define $\mathcal{M}_i$ to be the full subcategory consisting of modules with $\delta(M) \leq i$. There are inclusions

$$0 = \mathcal{M}_{-1} \subset \mathcal{M}_0 \subset \cdots \subset \mathcal{M}_{d-1} \subset \mathcal{M}_d = \mathcal{M} := \mathcal{M} - A.$$

Similarly $\mathcal{M}^\circ$ and $\mathcal{M}^\circ{}_i$ denote the analogous categories of left A-modules.

An abelian category $\mathcal{C}$ is **artinian** (respectively **noetherian**) if every object in $\mathcal{C}$ is artinian (respectively noetherian) [Po, p. 370]. Since A is noetherian, $\mathcal{M}$ and its subcategories $\mathcal{M}_i$ are noetherian.

A full subcategory $\mathcal{D}$ of an abelian category $\mathcal{C}$ is **dense** if, for every short exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

in $\mathcal{C}$, M is in $\mathcal{D}$ if and only if both L and N are in $\mathcal{D}$ (see [Po, p. 165]). The exactness of δ implies that $\mathcal{M}_i$ is a dense subcategory of $\mathcal{M}$. Hence we may form the quotient category $\mathcal{M}^i := \mathcal{M}_i / \mathcal{M}_{i-1}$, and similarly the quotient category $\mathcal{M}^{\circ i} := \mathcal{M}^\circ{}_i / \mathcal{M}^\circ{}_{i-1}$ for $i = 0, 1, \ldots, d$. The basic properties of quotient categories may be found in [Po, Ch. 4].

Lemma 1.1. *Each $\mathcal{M}^i$ defined above is a noetherian category.*

Proof. Let $\overline{\mathcal{M}}_{i-1}$ denote the category of those right A-modules all of whose finitely generated submodules belong to $\mathcal{M}_{i-1}$ and $\overline{\mathcal{M}}$ the category of all right A-modules. Since $\mathcal{M}_{i-1}$ is dense in $\mathcal{M}$, so is $\overline{\mathcal{M}}_{i-1}$ dense in $\overline{\mathcal{M}}$, whence the quotient category $\overline{\mathcal{M}} / \overline{\mathcal{M}}_{i-1}$ may also be formed. If M is any A-module then the sum of all its finitely generated submodules having δ-dimension $\leq i-1$ is in $\overline{\mathcal{M}}_{i-1}$. In particular, if $M \in \overline{\mathcal{M}}$, there is a largest submodule of M belonging to $\overline{\mathcal{M}}_{i-1}$. Hence the quotient functor $\overline{\mathcal{M}} \to \overline{\mathcal{M}} / \overline{\mathcal{M}}_{i-1}$ has a right adjoint [Po, 4.5.2]. Therefore, if M is a noetherian object it is also noetherian as an object in the quotient category $\overline{\mathcal{M}} / \overline{\mathcal{M}}_{i-1}$ [Po, 5.8.3]. In particular, each $M \in \mathcal{M}_i \subset \overline{\mathcal{M}}$ is noetherian as an object in $\mathcal{M}^i \subset \overline{\mathcal{M}} / \overline{\mathcal{M}}_{i-1}$. □

We say that two categories $\mathcal{C}$ and $\mathcal{D}$ are in **duality** if there are contravariant functors $F : \mathcal{C} \longrightarrow \mathcal{D}$ and $G : \mathcal{D} \longrightarrow \mathcal{C}$ such that $FG \cong Id_{\mathcal{D}}$ and $GF \cong Id_{\mathcal{C}}$. The functors F and G are called **dualities**. It is easy to see that $\mathcal{C}$ is noetherian if and only if the dual category $\mathcal{D}$ is artinian. Two categories $\mathcal{C}$ and $\mathcal{D}$ are in n-**step duality** if there are dense subcategories

$$0 \subset \mathcal{C}_1 \subset \cdots \subset \mathcal{C}_n = \mathcal{C} \qquad \text{and} \qquad 0 \subset \mathcal{D}_1 \subset \cdots \subset \mathcal{D}_n = \mathcal{D}$$

such that the quotient categories $\mathcal{C}^i := \mathcal{C}_i / \mathcal{C}_{i-1}$ and $\mathcal{D}^i := \mathcal{D}_i / \mathcal{D}_{i-1}$ are in duality for all $i = 1, \cdots, n$. The following result shows that AG rings have a duality analogous to QF rings.

Theorem 1.2. *Let A be an AG ring of injective dimension d. Then $\mathcal{M}^i$ and $\mathcal{M}^{\circ i}$ are in duality for all $i = 0, 1 \cdots, d$. As a consequence, $\mathcal{M} - A$ and $\mathcal{M} - A^{op}$ are in $(d+1)$-step duality.*

Proof. By the Auslander condition, $\mathrm{Ext}^{d-i}(-, A)$ is a contravariant functor from $\mathcal{M}_i$ to $\mathcal{M}^\circ{}_i$ so induces a contravariant functor from $\mathcal{M}_i$ to the quotient category $\mathcal{M}^{\circ i}$. If $\delta(M) \leq i$, the Auslander condition implies that $\mathrm{Ext}^l(M, A)$ is in $\mathcal{M}^\circ{}_{i-1}$ whenever $l > d - i$, whence $\mathrm{Ext}^l(-, A)$ is zero as a functor from $\mathcal{M}_i$ to $\mathcal{M}^{\circ i}$ whenever $l \neq d - i$. By the long exact sequence for $\mathrm{Ext}^*(-, A)$, $\mathrm{Ext}^{d-i}(-, A)$ is an exact functor from $\mathcal{M}_i$ to $\mathcal{M}^{\circ i}$ sending objects of $\mathcal{M}_{i-1}$ to 0, so it induces a functor from $\mathcal{M}^i$ to $\mathcal{M}^{\circ i}$ [Po, 4.3.11]. It follows from (0-3) that for each $M \in \mathcal{M}_i$, there is a short exact sequence, natural in M, of the form

$$0 \to M / F^{d-i+1} M \to \mathrm{Ext}^{d-i}(\mathrm{Ext}^{d-i}(M, A), A) \to Q_{d-i+2}(M) \to 0, \tag{1-1}$$

where $F^{d-i+1} M$ is the largest submodule of M with δ-dimension $\leq i-1$ and $Q_{d-i+2}(M)$ is a module with δ-dimension $\leq i-2$. Hence (1-1) yields a natural transformation from the identity functor to the functor $\mathrm{Ext}^{d-i}(\mathrm{Ext}^{d-i}(-, A), A)$ for all modules in $\mathcal{M}_i$ and this natural transformation becomes a natural isomorphism at the level of the quotient categories $\mathcal{M}^i$ and $\mathcal{M}^{\circ i}$. Hence the induced functors of $\text{Ext}^{d-i}(-, A_A)$ and $\text{Ext}^{d-i}(-, {}_AA)$ give a duality between $\mathcal{M}^i$ and $\mathcal{M}^{\circ i}$. □

Since $\mathcal{M}^i$ and $\mathcal{M}^{\circ i}$ are noetherian by Lemma 1.1, they are also artinian by duality. Hence, by induction, $\text{Kdim}\, M \leq i$ for all $M \in \mathcal{M}_i$ and we reprove a result of Roos [Bj2].

Corollary 1.3. *Let A be an AG ring of injective dimension d. If M is a finitely generated A-module, then $\text{Kdim}\, M \leq \delta(M) = d - j(M)$. Hence the left and right Krull dimensions of A are bounded above by its injective dimension. In particular, if $j(M) = d$ then M is artinian.* □

Over a finitely generated commutative k-algebra, $\text{Kdim}\, M = \text{GKdim}\, M$, and in most reasonable noncommutative rings $\text{Kdim}\, M \leq \text{GKdim}\, M$. In the presence of the CM property Corollary 1.3 gives such an inequality. If A is AG, then

$$\text{injdim}\, A = \max\{j(M) \mid 0 \neq M \in \mathcal{M}\}.$$

Hence the CM property ensures that $\text{GKdim}\, A - \text{injdim}\, A$ is a non-negative integer. The following corollary is an immediate consequence of Corollary 1.3 and the equalities

$$\delta(M) = \text{injdim}\, A - j(M) = \text{GKdim}\, M - (\text{GKdim}\, A - \text{injdim}\, A).$$

Corollary 1.4. *Let A be an AG and CM ring.*

1. *For every non-zero noetherian A-module M,*

$$\text{Kdim}\, M \leq \text{GKdim}\, M - (\text{GKdim}\, A - \text{injdim}\, A),$$

where $\text{GKdim}\, A - \text{injdim}\, A \geq 0$.

2. *GK-dimension is left and right finitely partitive.* □

The inequalities in Corollary 1.3 and Corollary 1.4.1 may be strict: if A is the enveloping algebra of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$, then $\text{Kdim}\, A = 2$ whereas $\text{injdim}\, A = \text{GKdim}\, A = 3$.

Remark: If M is a finitely generated right (respectively left) A-module with $j(M) = j$, define $M^\vee$ to be the left (respectively right) A-module $\text{Ext}^j(M, A)$. By the proof of Theorem 1.2, $j(M) = j(M^\vee)$ or equivalently $\delta(M) = \delta(M^\vee)$. If $\delta(M) = 0$, then by Corollary 1.3, M and $M^\vee$ are artinian. Since $\mathcal{M}_0$ and $\mathcal{M}^\circ{}_0$ are in duality, M is simple if and only if $M^\vee$ is simple. We also have $(M^\vee)^\vee \cong M$ for all M with $\delta(M) = 0$.

2 Injective resolutions

Write Ω_s for the image of the boundary map $I^{s-1} \to I^s$ in (0-1). Thus $\Omega_0 = A$, and there are exact sequences

$$0 \longrightarrow \Omega_{s-1} \longrightarrow I^{s-1} \longrightarrow \Omega_s \longrightarrow 0 \tag{2-1}$$

for all $s = 1, \cdots, d$, with each I^s an essential extension of Ω_s. By the definition of minimal injective resolution, the injective dimension of A is d if and only if $\Omega_d = I^d$.

Lemma 2.1. *For any A-module N, and any $i \geq 0$,*

1. $\text{Ext}^{i+1}(N, A) \cong \text{Ext}^1(N, \Omega_i)$, *and*
2. *if* $\text{Hom}(N, I^{i-1}) = 0$, *then* $\text{Ext}^i(N, A) \cong \text{Hom}(N, \Omega_i)$.

Proof. The long exact sequence for $\mathrm{Ext}^*(N,-)$ applied to (2-1) gives isomorphisms

$$\mathrm{Ext}^i(N,\Omega_s) \cong \mathrm{Ext}^{i+1}(N,\Omega_{s-1})$$

for $1 \le s \le d$, and $i \ge 1$. Furthermore, if $\mathrm{Hom}(N, I^{s-1}) = 0$, then

$$\mathrm{Hom}(N,\Omega_s) \cong \mathrm{Ext}^1(N,\Omega_{s-1}).$$

The result follows by induction. □

There is an obvious generalization of this, applying to $\mathrm{Ext}^i(N,M)$, which involves $\Omega_i(M)$, the cosyzygies of M.

Lemma 2.2. *Let M be a noetherian module. Then there exist $f_1,\cdots,f_n \in \mathrm{Hom}(M,\Omega_i)$ such that for every $N \subset \bigcap_j ker(f_j)$, the natural map $\mathrm{Ext}^i(M,A) \to \mathrm{Ext}^i(N,A)$ is zero, or equivalently, the natural map $\mathrm{Ext}^i(M/N,A) \to \mathrm{Ext}^i(M,A)$ is surjective.*

Proof. We prove that the natural map from $\mathrm{Ext}^i(M/N,A)$ to $\mathrm{Ext}^i(M,A)$ is surjective. From the injective resolution of A, we see that

$$\mathrm{Ext}^i(M,A) \cong \mathrm{Hom}(M,\Omega_i)/im(\mathrm{Hom}(M,I^{i-1})).$$

Suppose that $\mathrm{Ext}^i(M,A)$ is generated as a left A-module by maps $f_1,\cdots,f_n \in \mathrm{Hom}(M,\Omega_i)$. Because $N \subset \ker f_j$, there are corresponding maps $g_1,\cdots,g_n \in \mathrm{Hom}(M/N,\Omega_i)$, so the natural map $\mathrm{Ext}^i(M/N,A) \to \mathrm{Ext}^i(M,A)$ sending g_j to f_j is surjective. □

The next result appears as Corollary 4.6 of [Miy], but the proof below is shorter. The commutative case of the result is well-known [Bass].

Theorem 2.3. *Let A be a noetherian ring with* injdim $A = d$.

1. *Then for each non-zero noetherian right A-module M there is a non-zero submodule N of M which embeds in some Ω_i.*
2. *Every indecomposable injective module appears in the minimal injective resolution of A.*

Proof. Part 2 is an obvious consequence of part 1 and we prove part 1 next.

Replacing by a submodule, we may assume M is uniform. Suppose to the contrary that no non-zero submodule N of M embeds in any Ω_i. Under this hypothesis we claim that there is a chain of non-zero submodules

$$M = M_{-1} \supset M_0 \supset M_1 \supset \cdots \supset M_d$$

such that $F^{d-t}M_t = 0$ for all $t = -1, 0, \cdots, d$. If $t = -1$, $F^{d+1}M_{-1} = 0$ by definition. Now suppose $F^{d-t}M_t = 0$ for some t. As in Lemma 2.2, $\mathrm{Ext}^i(M_t,A)$ is generated as a left A-module by $f_1^i,\cdots,f_{n_i}^i \in \mathrm{Hom}(M_t,\Omega_i)$. Let $M_{t+1} = \bigcap_{i,j} ker(f_j^i)$. Since M_t does not embed in any Ω_i, $ker(f_j^i) \ne 0$ for all i,j, whence $M_{t+1} \ne 0$ because M is uniform. By Lemma 2.2, the natural map $\mathrm{Ext}^i(M_t,A) \to \mathrm{Ext}^i(M_{t+1},A)$ is zero for all i, so the natural map $\mathrm{Ext}^p(\mathrm{Ext}^q(M_{t+1},A),A) \to \mathrm{Ext}^p(\mathrm{Ext}^q(M_t,A),A)$ is zero for all p and q. Therefore, by the spectral sequence (0-2), the natural maps from $F^iM_{t+1}/F^{i+1}M_{t+1}$ to $F^iM_t/F^{i+1}M_t$ are zero for all i. In particular, the natural map from $F^{d-(t+1)}M_{t+1}/F^{d-t}M_{t+1}$ to $F^{d-(t+1)}M_t/F^{d-t}M_t$ is zero. The filtration is functorial, so the inclusion $M_{t+1} = \mathbb{H}(M_{t+1}) \to M_t = \mathbb{H}(M_t)$ embeds F^iM_{t+1} in F^iM_t. Hence $F^{d-t}M_{t+1} \subset F^{d-t}M_t = 0$, so the morphism from $F^{d-(t+1)}M_{t+1}$ to $F^{d-(t+1)}M_t$ is zero. Therefore $F^{d-(t+1)}M_{t+1} = 0$ and we have proved our claim by induction. Letting $t = d$ we obtain $M_d = F^0M_d = 0$, a contradiction. Therefore part 1 follows. □

Proposition 2.4. *Let A be a right noetherian ring with* injdim $A_A = d$, *and let N be a noetherian right A-module.*

1. If N embeds in Ω_i, then $\mathrm{Ext}^i(N, A) \neq 0$, whence $j(N) \leq i$.

2. If every non-zero submodule of N is $(d-i)$-critical, then N embeds in Ω_i.

3. I^0 is essentially d-pure (and d-pure if δ is exact).

4. If A has a QF quotient ring Q, then $I^0 \cong Q$ is d-pure and every torsion module M (i.e., a module such that $M \otimes_A Q = 0$) has $j(M) \geq 1$. As a consequence, Ω_1 is $(d-1)$-pure, and I^1 is essentially $(d-1)$-pure. Furthermore, if A is semiprime and δ is exact, then I^1 is $(d-1)$-pure.

Proof. 1. The statement is obvious for $\Omega_0 = A$. If $i \geq 1$ and N is a submodule of Ω_i, then there is a nonsplit exact sequence $0 \longrightarrow \Omega_{i-1} \longrightarrow E \longrightarrow N \longrightarrow 0$, so $\mathrm{Ext}^1(N, \Omega_{i-1}) \neq 0$. By Lemma 2.1.1, $\mathrm{Ext}^i(N, A) \cong \mathrm{Ext}^1(N, \Omega_{i-1}) \neq 0$.

2. If M is $(d-i)$-critical, then $j(M/L) \geq i$ for all submodules $L \subset M$, so $\mathrm{Hom}(M, \Omega_{i-1}) = 0$, by part 1. However, if $\mathrm{Hom}(N, I^{i-1}) \neq 0$, then for some non-zero submodule $M \subset N$, $\mathrm{Hom}(M, \Omega_{i-1}) \neq 0$ which contradicts the previous sentence; hence $\mathrm{Hom}(N, I^{i-1}) = 0$. By Lemma 2.1.2, $\mathrm{Hom}(N, \Omega_i) \cong \mathrm{Ext}^i(N, A)$, which is non-zero because $j(N) = i$, so there is a non-zero map $f : N \longrightarrow \Omega_i$. But N is critical and Ω_i contains no submodule X with $\delta(X) < d - i$, so f is injective.

3. Because I^0 is an essential extension of the d-pure module A, I^0 is essentially d-pure. If δ is exact, then essentially d-pure is the same as d-pure.

4. Since Q_Q is injective and Q_A is flat, Q_A is injective. Since A is essential in Q_A, Q is an injective hull of A; that is $I^0 \cong Q$. For every finitely generated submodule $M \subset Q_A$, there is a regular element c such that $cM \subset A$. Hence $j(M) = 0$ and $\delta(M) = d$, whence Q_A is d-pure.

If M were a torsion module with $j(M) = 0$, then some non-zero quotient of M would be both torsion and a submodule of A. But this cannot happen, so torsion modules have grade ≥ 1.

Since $\Omega_1 = Q/A$ is torsion, every submodule $M \subset \Omega_1$ is too, so has grade ≥ 1. Combining this with part 1 gives $j(M) = 1$, whence Ω_1 is $(d-1)$-pure. Now I^1, being an essential extension of Ω_1, is essentially $(d-1)$-pure. If in addition A is semiprime, then an essential extension of a torsion module is torsion, so I^1 is torsion. Thus every finite submodule M of I^1 has $j(M) \geq 1$ or $\delta(M) \leq d - 1$. By exactness of δ, $\delta(M) = d - 1$, whence I^1 is $(d-1)$-pure. □

If injdim $A_A = 0$ (i.e., A is QF [Fa, 24.5]), then it is obvious that A_A has a pure injective resolution. If A is a semiprime noetherian ring of injective dimension 1, then its injective resolution is $0 \to A_A \to Q \to Q/A \to 0$, where $Q = \mathrm{Fract}\, A$, so A has a pure injective resolution by Proposition 2.4.4; one may also check that A is AG in this case.

Proposition 2.5. *Let A be AG with* injdim $A = d$. *Then*

1. I^d is a direct sum of injective hulls of 0-critical modules, hence essentially 0-pure;

2. $\mathrm{Ext}^d(M, A) \cong \mathrm{Ext}^d(F^d M, A)$ where $F^d M$ is the largest submodule of grade $\geq d$.

For next two parts we further assume that A has finite global dimension.

3. pdim $M = d$ *if and only if $F^d M \neq 0$;*

4. if L is a right A-module and $I^d(L)$ is the d-th term of the minimal injective resolution of L, then $I^d(L)$ is either 0 or essentially 0-pure.

Proof. 1. We need to show that I^d contains no s-critical submodules for $s > 0$. If $s > 0$ and M is s-critical then, by (0-3), there is an exact sequence

$$0 \longrightarrow F^d M \longrightarrow \mathrm{Ext}^d(\mathrm{Ext}^d(M, A), A) \longrightarrow Q_{d+2}(M) \longrightarrow 0.$$

But $F^d M = 0$ because M is s-critical with $s > 0$, and $Q_{d+2}(M) = 0$ because $j(X) \leq d$ for all $X \neq 0$, so $\mathrm{Ext}^d(\mathrm{Ext}^d(M, A), A) = 0$. But $j(\mathrm{Ext}^d(M, A)) \geq d$ by the Auslander condition, whence $\mathrm{Ext}^d(M, A) = 0$. So M is not a submodule of $\Omega_d = I^d$, by Proposition 2.4.1.

2. Applying $\operatorname{Ext}^*(-,A)$ to the exact sequence $0 \longrightarrow F^dM \longrightarrow M \longrightarrow M/F^dM \longrightarrow 0$ gives an exact sequence

$$\operatorname{Ext}^d(M/F^dM, A) \longrightarrow \operatorname{Ext}^d(M,A) \longrightarrow \operatorname{Ext}^d(F^dM, A) \longrightarrow 0,$$

so it suffices to show that $\operatorname{Ext}^d(M/F^dM, A) = 0$. Since $L = M/F^dM$ is noetherian and δ is exact, no submodule of L has δ-dimension 0. Further, since δ is finitely partitive, there is a finite chain of submodules $\{L_i\}$ such that each L_i/L_{i+1} is s_i-critical for $s_i > 0$. By part 1 and Proposition 2.4.1, $\operatorname{Ext}^d(L_i/L_{i+1}, A) = 0$ whence $\operatorname{Ext}^d(L,A) = 0$ by induction, using the long exact sequence.

3. ($\Leftarrow$) This is clear because $\operatorname{Ext}^d(M,A) \cong \operatorname{Ext}^d(F^dM, A) \neq 0$.

($\Rightarrow$) If pdim $M = d$, then $\operatorname{Ext}^d(M,N) \neq 0$ for some N. Writing N as a quotient of a free module, and using the fact that $\operatorname{Ext}^{d+1}(M,-) = 0$, it follows from the long exact sequence for $\operatorname{Ext}^*(M,-)$ that $\operatorname{Ext}^d(M,A) \neq 0$. Hence, by part 2, $F^dM \neq 0$.

4. Suppose to the contrary that $I^d(L)$ contains an s-critical module M for some $s > 0$. By part 2, pdim $M < d$. By an analogous version of Proposition 2.4.1, valid for an injective resolution of L (and which can be proved in the same way), we have $\operatorname{Ext}^d(M,L) \neq 0$, contradicting pdim $M < d$. □

Part 1 of Theorem 2.5 appears as Theorem 6 of [IS].

3. The last term in the injective resolution

In this section we will study AG and CM_0 algebras over a field k. A finitely generated CM k-algebra of finite injective dimension, which equals GK-dimension, is CM_0 and a connected graded AG ring is graded CM_0.

Let A be an AG and CM_0 k-algebra and let I^d be the last term of the minimal injective resolution of A. By Proposition 2.5.1, I^d is a direct sum of the injective hulls of finite dimensional simple modules. Let S be a finite dimensional simple module and $E(S)$ its injective hull. Then the multiplicity of $E(S)$ in I^d is equal to $\dim \operatorname{Hom}(S, I^d)/\dim \operatorname{Hom}(S,S)$. It is easy to see that $\operatorname{Hom}(S, I^d) \cong \operatorname{Ext}^d(S,A) = S^\vee$ and that $\operatorname{Hom}(S,S) = k$ if k is algebraically closed.

Theorem 3.1. *Let A be an AG and CM_0 k-algebra with* injdim $A = d$. *Then*

$$I^d = \bigoplus_S E(S)^{m_S}$$

where S runs over all finite dimensional simple modules, and $m_S = \dim S^\vee/\dim \operatorname{End}(S_A)$. As a consequence, the injective module $E(S)$ appears in I^d finitely many times. □

For a finite dimensional module M, define $M^* := \operatorname{Hom}_k(M,k)$. If M is a left A-module, M^* is a right A-module (and conversely), and M is simple if and only if M^* is simple. Since $(M^*)^* \cong M$, the functor $M \mapsto M^*$ is a duality between $\mathcal{M}_0$ and $\mathcal{M}^\circ{}_0$. Composing $^\vee$ and * yields auto-equivalences of $\mathcal{M}_0$ and $\mathcal{M}^\circ{}_0$ (Recall that $M^\vee = \operatorname{Ext}^j(M,A)$ where $j = j(M)$). In general this auto-equivalence is not equivalent to the identity functor. Write $M' := (M^\vee)^*$. The following example shows that M' need not be isomorphic to M.

Example 3.2. Let L be the 2-dimensional solvable Lie algebra over $\mathbb{C}$ generated by x and y and subject to the relation $[x,y] = x$. Let A be the enveloping algebra $U(L)$. The finite dimensional simple A-modules are $\{S_r := A/(x, y-r) \mid r \in \mathbb{C}\}$. Each S_r is an A-A-bimodule and $S_r^* \cong S_r$. The projective resolution of S_r as a right A-module is

$$0 \longrightarrow A \longrightarrow A \oplus A \longrightarrow A \longrightarrow S_r \longrightarrow 0 \tag{3-1}$$

where the boundary map from $A \oplus A$ to A is defined by $f_1(a,b) = xa + (y-r)b$ and the boundary map from A to $A \oplus A$ is defined by $f_2(a) = ((y-r-1)a, -xa)$. Applying $\mathrm{Hom}(-, A)$ to (3-1), we obtain the complex

$$0 \longleftarrow A \longleftarrow A \oplus A \longleftarrow A \longleftarrow 0$$

where the boundary maps are $f_1^*(a) = (ax, a(y-r))$ and $f_2^*(a,b) = a(y-r-1) - bx$. Hence $\mathrm{Ext}^i(S_r, A) = 0$ if $i \neq 2$, and $S_r^\vee := \mathrm{Ext}^2(S_r, A) \cong S_{r+1}$. Therefore $S_r' \ncong S_r$. □

It is trivial that $A := M_2(k) \oplus k$ is AG, CM, and gldim $A = 0$. There are two finite dimensional simple modules with dimensions 1 and 2 respectively. There is an obvious auto-equivalence of $\mathcal{M}$ (which is also equal to $\mathcal{M}_0$) which exchanges the 1-dimensional and the 2-dimensional simple modules. Hence an auto-equivalence of $\mathcal{M}_0$ may change the dimension of simple modules. Next we will show that the auto-equivalence $'$ preserves the dimension of finite dimensional modules in some rings (including enveloping algebras).

Theorem 3.3. *Let A be a filtered ring such that the associated graded algebra grA is a connected graded AG and CM k-algebra of injective dimension d. If M is a finite dimensional right A-module, then* $\dim M^\vee = \dim M$.

Proof. By [SfZ, 4.4], A is a noetherian, AG and CM k-algebra of injective dimension at most d. Since grA is connected and noetherian, it is finitely generated, and hence A is finitely generated. If M is a finite dimensional right A-module, then $j(M) = d$ by the CM property, so injdim $A = d$. Since M is finitely generated it has a good filtration so, by [Bj1, 3.1], $gr(\mathrm{Ext}_A^d(M,A))$ is a subquotient of $\mathrm{Ext}_{grA}^d(grM, grA)$. Since grA is connected graded and AG, $k^\vee \cong k$. By induction we obtain that $\dim M^\vee = \dim M$ for every finite dimensional graded grA-module M. In particular,

$$\dim(grM)^\vee = \dim grM = \dim M.$$

Hence

$$\dim M^\vee = \dim \mathrm{Ext}_A^d(M,A) = \dim gr\,\mathrm{Ext}_A^d(M,A) \leq \dim(grM)^\vee = \dim M.$$

But $M^{\vee\vee} \cong M$, so $\dim M = \dim M^{\vee\vee} \leq \dim M^\vee$. Therefore $\dim M = \dim M^\vee$. □

Applying Theorems 3.1 and 3.3, we have the following immediate corollary.

Corollary 3.4. *Let L be a d-dimensional Lie algebra over an algebraically closed field. Then the last term in the minimal injective resolution of its enveloping algebra is*

$$I^d \cong \bigoplus_S E(S)^{\dim S},$$

where S runs over all finite dimensional simple modules, and $E(S)$ is the injective hull of S. □

Remark: For the universal enveloping algebra $U(L)$ of a finite dimensional solvable or semisimple Lie algebra L, Theorem 3.3 and Corollary 3.4 can be proved without using the filtration on $U(L)$ (still under the hypothesis that k is algebraically closed). First, suppose L is a solvable Lie algebra. Every simple right (or left) $U(L)$-module S is 1-dimensional. Since $S^\vee$ is also simple, it is 1-dimensional too. Hence $\dim S = \dim S^\vee$. Therefore $\dim M = \dim M^\vee$ for all finite dimensional modules M. Second, suppose L is semisimple. Non-isomorphic finite dimensional simple modules S and T are annihilated by different central elements. Since $S^\vee \cong \operatorname{Ext}^d(S, A)$ and $S \cong \operatorname{Ext}^d(S^\vee, A)$, S and $S^\vee$ are annihilated by the same central elements. Therefore $S \cong S' := (S^\vee)^*$. Hence $\dim S = \dim S^\vee$ for finite dimensional simples and hence for all finite dimensional modules. In this case we have a stronger statement: if L is semisimple, then $S \cong S'$ for every finite dimensional simple module S.

If A is Auslander regular and CM_0, then analog of Theorem 3.1 holds for injective resolutions of all A- modules.

Theorem 3.5. *Let A be Auslander regular with* gldim $A = d$ *and CM_0 and L a finitely generated right A-module. Then the d-th term in a minimal injective resolution of L is*

$$I^d(L) = \bigoplus_S E(S)^{l_S}$$

where S runs over all finite dimensional simple modules, $E(S)$ is an injective hull of S, and $l_S = \dim \operatorname{Ext}^d(S, L)/\dim \operatorname{Hom}(S, S)$. As a consequence, the injective module $E(S)$ appears in $I^d(L)$ finitely many times.

Proof. By Proposition 2.5.4, $I^d(L)$ is essentially 0-pure and hence $I^d(L) = \bigoplus_S E(S)^{l_S}$ by CM_0. The multiplicity can easily be seen to be $l_S = \dim \operatorname{Ext}^d(S, L)/\dim \operatorname{Hom}(S, S)$. Since $\operatorname{Ext}^i(S, A)$ is finite dimensional by CM_0 (in fact, the only nonzero Ext is $\operatorname{Ext}^d(S, A)$), $\operatorname{Ext}^i(S, M)$ is finite dimensional for all i and M, by induction on pdim M. Therefore l_S is finite. □

4. Purity

It is well-known that a commutative noetherian ring of finite injective dimension is AG, and has a pure injective resolution. The PI (polynomial identity) rings will be the next natural cases to analyze. But we do not know whether an AG, PI ring has a pure injective resolution. Nevertheless, we prove that an AG, PI, grade-symmetric ring has a pure injective resolution, part of which is a generalization of [SfZ, 6.4].

Lemma 4.1. *Suppose A_A and ${}_AA$ have essentially pure injective resolutions with respect to δ. Then A has a pure injective resolution if and only if every essential extension of a module of grade s has grade $\geq s$.*

Proof. Suppose that A has a pure injective resolution, and let M be a module of grade s. Then M is an essential extension of a direct sum of uniform modules, each of which has grade $\geq s$. By Theorem 2.3, each of these uniform modules embeds in I^t for an appropriate $t \geq s$, hence M, and therefore any essential extension of it, embeds in a finite direct sum of copies of various I^t with $t \geq s$. By hypothesis every non-zero finitely generated submodule of I^t has grade t; therefore every non-zero finitely generated submodule of this direct sum of various I^t also has grade $\geq s$; in particular, every finitely generated essential extension of M has grade $\geq s$. The converse is trivial. □

Theorem 4.2. *Suppose that A is AG, and satisfies a polynomial identity.*

1. *If A is grade-symmetric, i.e., $j({}_AM) = j(M_A)$ for every noetherian A-bimodule M, then*
   - *grade is constant on the cliques of A, and*
   - *A has a pure injective resolution.*
2. *If A is CM with respect to GK-dimension or Krull dimension, then A is grade-symmetric.*

Proof. 1. If there is a link $Q \rightsquigarrow P$, then there is an A/Q-A/P bimodule, B say, finitely generated and torsion-free on both sides; since δ is a dimension function and ${}_{A/Q}B$ is torsion-free, $j(A/Q) = j({}_AB)$; similarly, $j(B_A) = j(A/P)$; by hypothesis, $j({}_AB) = j(B_A)$, so $j(A/Q) = j(A/P)$. Hence, by induction, grade is constant on cliques.

Next we show that A has an essentially pure injective resolution. It suffices to show that if M is a critical submodule of Ω_i, then $j(M) = i$. Since M is critical it is uniform, so embeds in A/P where $P = \operatorname{Ann} M$ is a prime ideal of A. Therefore, by [Br, 2.3], since M embeds in I^i, $\operatorname{Ext}^i(A/P_A, A)$ is not torsion as a right A/P-module. By [SfZ, 3.5], $\operatorname{Ext}^i(A/P_A, A)$ is a noetherian right A/P-module. Hence, by [SfZ, 3.10], $j(\operatorname{Ext}^i(A/P_A, A)) = j(A/P)$ as a right A-module. By the Auslander condition $j(\operatorname{Ext}^i(A/P_A, A)) \geq i$ as a left A-module, and hence as a right module by the hypothesis on bimodules. Thus $j(A/P) \geq i$ as a right A-module, whence $j(M) \geq i$; the reverse inequality is given by Proposition 2.4.1, so $j(M) = i$.

Finally, we show that an essential extension of a module of grade s has grade $\geq s$. As in the proof of Lemma 4.1, we need only prove this for a uniform module. Suppose that M is an essential extension of a uniform module U. Then M has a unique associated prime, $P := \operatorname{Ann} U$. Since A is a PI ring, it satisfies the strong second layer condition so, by [GW, 11.4], there is a chain of submodules $0 = M_0 \subset \cdots \subset M_n = M$ such that the annihilators of the various M_i/M_{i-1} are primes ideals, say P_i, belonging to the clique containing P; moreover, each M_i/M_{i-1} is a torsion-free A/P_i-module, so $j(M_i/M_{i-1}) = j(A/P_i) = j(A/P)$. Thus $j(M) = j(A/P) = j(U)$.

The last two paragraphs allow us to apply Lemma 4.1 to conclude that A has a pure injective resolution.

2. Since GK-dimension [MR, 8.3.14] and Krull dimension [MR, 6.4.13] are symmetric for noetherian A-bimodules (the latter because A is a PI ring), A is grade-symmetric by the CM property. □

Remark: If A is a noetherian PI ring of injective dimension d and A_A has an essentially pure injective resolution with respect to a dimension function ∂, then $I^i = \oplus_P E_P^{N_P}$ where P runs over all primes with $\partial(R/P) = d - i$, where E_P is the injective hull of a non-zero uniform right ideal of R/P and N_P is a positive integer. The proof of this is the same as the second part of the proof of [SfZ, 3.15]. This remark applies to the rings in Theorem 4.2.

Example 5.4.2 in the next section shows that there is an AG, PI, but not grade-symmetric, ring of global dimension 1 which has a pure injective resolution.

Next we show that if A has an essentially pure injective resolution with respect to ∂ and δ is exact, then ∂ is essentially the canonical dimension δ and A is AG.

Proposition 4.3. *Let A be a noetherian ring with finite injective dimension d and suppose that A_A has an essentially pure injective resolution with respect to an exact dimension function ∂ and that δ is exact. Then*

$$\partial(M) = \delta(M) + \partial(A) - d$$

for all noetherian right A-modules M. In particular, δ is a dimension function.

Proof. Replacing ∂ by the function $M \mapsto \partial(M) + d - \partial(A)$ we can, and do, assume that $\partial(A) = d$.

By exactness and the noetherian property, it suffices to show $\partial(N) = \delta(N)$ for some nonzero submodule $N \subset M$. By Theorem 2.3.1, there is a non-zero submodule $M' \subset M$ which embeds in some Ω_i and by purity, there is a nonzero submodule $N \subset M' \subset \Omega_i$ which is $(d-i)$-pure with respect to ∂. By purity of the injective resolution, $\operatorname{Hom}(N, I^s) = 0$ for all $s < i$, whence

$j(N) \geq i$. By Proposition 2.4.1, $j(N) \leq i$. Hence $j(N) = i$ and $\delta(N) = d - i = \partial(N)$. Therefore the results follows. $\square$

Theorem 4.4. *Let A be a noetherian ring with finite injective dimension d. Suppose that A_A and ${}_AA$ have essentially pure injective resolutions with respect to exact dimension function ∂ and that δ is exact. Then A is AG.*

Remark: In Theorem 4.4 the condition that δ is exact is necessary as Example 5.4.1 shows.

Proof. By Proposition 4.3, we may assume $\partial = \delta$. Recall that the exactness of δ can be expressed by $j(M) = \min\{j(N), j(M/N)\}$ whenever N is a submodule of M, and we simply say that j is exact when this condition is satisfied. Because j is exact, to prove that A is AG we need only show that $j(E^sM) \geq s$ for all s and all M. This is true if $s = d+1$, so we fix s, and suppose inductively that $j(E^iM) \geq i$ for all $i > s$ and all M.

If $j(M) > s$, then $j(E^sM) = \infty > s$, so we will argue by downward induction on $j(M)$. We treat the case $j(M) = s$.

The E_2-page of the double-Ext spectral sequence for M lives in rows $s, s+1, \ldots, d$ and, by the induction hypothesis for $i > s$, looks like

$$\begin{array}{ccccccccc}
E^{0\,s}M & E^{1\,s}M & \cdots & E^{s-1\,s}M & E^{s\,s}M & \cdots & \cdots & \cdots & E^{d\,s}M \\
0 & 0 & \cdots & 0 & 0 & E^{s+1\,s+1}M & \cdots & \cdots & E^{d\,s+1}M \\
0 & 0 & \cdots & 0 & 0 & 0 & E^{s+2\,s+2}M & \cdots & E^{d\,s+2}M \\
\cdots & \cdots & \text{etc} & \cdots & \cdots & \cdots & \cdots & \text{etc} & \cdots
\end{array}$$

In row s, $E^{0\,s}M = E^{1\,s}M = \cdots = E^{s-2\,s}M = 0$, since all these terms survive to the E_∞ page which is zero off the main diagonal. Hence we must show that $E^{s-1\,s}M = 0$.

Because it is an off diagonal term, $E_\infty^{s-1\,s}M = 0$. Hence, because of the zeroes in rows $0, \ldots, s-1$, there is a finite filtration

$$E_2^{s-1\,s}M \supset E_3^{s-1\,s}M \supset \cdots \tag{4-1}$$

with embeddings of the factors

$$E_r^{s-1\,s}M/E_{r+1}^{s-1\,s}M \to E_r^{s+r-1\,s+r-1}M \tag{4-2}$$

for $r \geq 2$; however, the right hand term in (4-2) is a subquotient of $E_2^{s+r-1\,s+r-1}M$, so has grade $\geq s+r-1 \geq s+1$ by the induction hypothesis; by exactness of j, so does the left hand term in (4-2), and from the filtration (4-1), it follows that $j(E_2^{s-1\,s}M) \geq s+1$ also.

Write $N = E^sM$ and $t = j(E^{s-1}N) \geq s+1$. We have already shown that $j(N) \geq s-1$, so the E^2-page of the double-Ext spectral sequence for N lives in rows $s-1, s, \ldots, d$ and looks like

$$\begin{array}{ccccccccccc}
0 & \cdots & 0 & 0 & 0 & \cdots & \cdots & 0 & E^{t\,s-1}N & \cdots & E^{d\,s-1}N \\
E^{0\,s}N & \cdots & E^{s-2\,s}N & E^{s-1\,s}N & E^{s\,s}N & \cdots & \cdots & \cdots & \cdots & \cdots & E^{d\,s}N \\
0 & \cdots & 0 & 0 & 0 & E^{s+1\,s+1}N & \cdots & \cdots & \cdots & \cdots & E^{d\,s+1}N \\
\cdots & \text{etc} & \cdots & \cdots & \cdots & \cdots & \text{etc} & \cdots & \cdots & \cdots & \cdots
\end{array}$$

If $q \geq t$, then $E_\infty^{q\,s-1}N = 0$ because it is not on the diagonal, so there is a finite filtration

$$E_2^{q\,s-1}N \supset E_3^{q\,s-1}N \supset \cdots \tag{4-3}$$

with embeddings of the factors

$$E_r^{q\,s-1}N/E_{r+1}^{q\,s-1}N \to E_r^{q+r\,s+r-2}N$$

for $r \geq 2$; by the induction hypothesis the successive factors in the filtration (4-3) have grade $\geq q+r \geq t+2$, whence $j(E_2^{q\,s-1}N) \geq t+2$. Thus $j(E_2^{q\,s-1}N) \geq t+2$ for all q; therefore, by the induction hypothesis for $i > s$, $j(E^p(E^{q\,s-1}N)) \geq t+2$ for all p and q. Hence by the convergence of the double-Ext spectral sequence for $E^{s-1}N$, $j(E^{s-1}N) \geq t+2$. But this contradicts the fact that $j(E^{s-1}N) = t$ unless $t = \infty$, so we conclude that $E^{s-1}N = 0$ as required; that is, $E^{s-1\,s}M = 0$, whence $j(E^sM) \geq s$ whenever $j(M) = s$.

Finally, to prove that $j(E^sM) \geq s$ for a general M, we argue by induction on Kdim M; so suppose that Kdim $M = p$ and that the result is true for modules having Kdim $< p$. By noetherian property and the exactness of j, we may assume that M is Kdim-critical and pure and embeds in Ω_i for some i [Theorem 2.3.1]. By Proposition 2.4.1, $j(M) \leq i$ and hence $j(M) = i$, since A has essentially pure injective resolution. We may assume $i < s$ because the result has been proved for $j(M) \geq s$. Let $f_1, \ldots, f_n \in \operatorname{Hom}(M, \Omega_s)$ be as in Lemma 2.2, and set $L = \cap_{j=1}^n \ker f_j$; thus there is a surjection $E^s(M/L) \to E^sM$. By hypothesis Ω_s is an essential extension of a pure module of grade s; since M is pure of grade $< s$ it does not embed in Ω_s. Hence $\ker f_j \neq 0$ for all j, whence $L \neq 0$ because M is uniform. But M is Kdim-critical so $\operatorname{Kdim}(M/L) < \operatorname{Kdim} M$, whence $j(E^s(M/L)) \geq s$ by the induction hypothesis. Since E^sM is a quotient of $E^s(M/L)$, it too has grade $\geq s$. □

Corollary 4.5. *If ${}_AA$ and A_A have essentially pure injective resolutions with respect to some (not necessarily exact) dimension function ∂ and if δ is exact, then A is AG.*

Proof. Let ∂ denote the dimension function. By [MR, 6.8.9], there is an exact dimension function ∂^* such that $\partial(M) = \partial^*(M)$ for pure modules M. Then A_A and ${}_AA$ have essentially pure injective resolution with respect to the exact dimension function ∂^*, so the result follows from Theorem 4.4. □

5. Examples

We first give examples constructed by M. Artin and J. T. Stafford, which show that the converse of Theorem 4.4 does not hold in general: not every AG ring has an essentially pure injective resolution.

Lemma 5.1. *(Artin) Let A be an AG ring and $0 \to A_A \to I^0 \to \cdots \to I^d \to 0$ a minimal injective resolution of A. Let M be a non-zero finitely generated right A-module, and assume that M has no submodule N with $j(N) = d$. If $\operatorname{Hom}_A(M, I^d) \neq 0$, then $\operatorname{Hom}_A(M, I^{d-1}) \neq 0$.*

Proof. By hypothesis, $F^d\,M = 0$ and by Proposition 2.5.2, $\operatorname{Ext}^d(M, A) = 0$. Since $\operatorname{Ext}^d(M, A) \cong \operatorname{Hom}(M, I^d)/im(\operatorname{Hom}(M, I^{d-1}))$ and by hypothesis $\operatorname{Hom}(M, I^d) \neq 0$, $\operatorname{Hom}(M, I^{d-1}) \neq 0$. □

Let k be a field of characteristic zero and $\mathfrak{sl}_2$ the special linear Lie algebra over k. Consider $A := U(\mathfrak{sl}_2)^{op} \otimes U(\mathfrak{sl}_2)$, which is isomorphic to the universal enveloping algebra $U(\mathfrak{sl}_2 \oplus \mathfrak{sl}_2)$. Let $B = U(\mathfrak{sl}_2)/(\Omega)$, where Ω denotes the Casimir element. It is standard that B has exactly one non-zero ideal P, the augmentation ideal, and that $B/P \cong k$. Thus, as a right A-module, B is non-split of length two, with factor module finite dimensional and submodule of GK-dimension two. Now, as is standard, A is Auslander-regular and CM with injective dimension 6 . We may apply Lemma 5.1 to this example with $M = B$ to conclude $\operatorname{Hom}_A(B, I^5) \neq 0$. Since non-zero finite dimensional modules have grade 6, I^5 contains no non-zero finite dimensional submodules, whence B in fact embeds into I^5. Since $j(B) = 6 - \operatorname{GKdim}(B) = 4 < 5$, we conclude:

Example 5.2. *(Stafford) The universal enveloping algebra $U(\mathfrak{sl}_2 \oplus \mathfrak{sl}_2)$ does not have an essentially pure injective resolution.* □

We may obtain a comparable example for the homogenized universal enveloping algebra $H(\mathfrak{sl}_2 \oplus \mathfrak{sl}_2)$ of the Lie algebra $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$. Let L be any finite dimensional Lie algebra over k. Then $H(L)$ is the subring of $U(L)[t]$ generated by $\{t\} \cup \{tL\}$. With $\deg t = 1$, $H(L)$ is a connected graded Auslander regular and CM algebra of global dimension $\dim L + 1$. It is easy to see that $H(L)$ contains a special central element t such that $H(L)[t^{-1}] = U(L)[t, t^{-1}]$ and that $H(L)/(t)$ is the commutative polynomial ring $k[L]$. By a graded version of [ASZ, 3.7], $H(L)$ has an essentially pure graded injective resolution if and only if $U(L)$ has an essentially pure (ungraded) injective resolution. By Example 5.2, we get:

Example 5.3. *(Stafford) The homogenized enveloping algebra $H(\mathfrak{sl}_2 \oplus \mathfrak{sl}_2)$ does not have an essentially pure graded minimal injective resolution.* □

As a graded ring $H(\mathfrak{sl}_2 \oplus \mathfrak{sl}_2)$ has injective dimension 7, and I^6 is essentially pure, but I^5 is not.

Second we give examples showing that (a) the hypothesis of δ being exact is necessary in Theorem 4.4, (b) an AG ring need not be grade-symmetric, and (c) the artinian quotient ring of an AG ring A (if it exists) may not be QF, and in particular, need not be isomorphic to I^0.

Example 5.4. *1. There is a noetherian ring A of global dimension 1 such that*

(a) δ is not exact and hence A is not AG,

(b) the injective hulls of of A_A and ${}_AA$ are different, whence there is no bimodule resolution which is a minimal injective resolution of A on both left and right sides, and

(c) A_A and ${}_AA$ have pure injective resolutions with respect to some dimension function.

2. There is an AG ring A of global dimension 1 such that

(a) A is its own left and right ring of fractions, which is not injective on either side, and

(b) A has a minimal prime such that $\delta((A/P)_A) = \delta(A)$ and $\delta({}_A(A/P)) < \delta(A)$, i.e., A is not grade-symmetric.

Fix a field k, an integer $n > 0$, and let $V_n \cong k^n$. Define

$$A = \begin{pmatrix} k & V_n \\ 0 & k \end{pmatrix}.$$

Since A is artinian, it is its own ring of fractions. By [MR, 7.5.1], gldim $A = 1$. The simple right A-modules are $S_1 = (0, k)$ and $S_2 = (k, 0)$. It is clear that S_1 is a direct summand of A, so is projective and $j(S_1) = 0$. It is easy to see that S_2 does not embed in A, so $j(S_2) = 1$.

We may view A as a subring of the matrix ring $B = M_{n+1}(k)$ via the embedding

$$A \cong \begin{pmatrix} kI_n & c_n \\ 0r_n & k \end{pmatrix} \subset \begin{pmatrix} M_n(k) & c_n \\ r_n & k \end{pmatrix} = B,$$

where $c_n \cong k^n$ is the column space $(k, k, \cdots, k)^{\mathsf{T}}$, and $r_n \cong k^n$ is the row space $(k, k \cdots, k)$. With this point of view, S_1 is isomorphic to $((0, 0 \cdots, 0), k)$ and S_2 is isomorphic to $(e_i k, 0) = ((0, 0, \cdots, k, \cdots, 0), 0)$ for all i. The minimal injective resolution of S_1 is

$$0 \longrightarrow S_1 \longrightarrow R = (r_n, k) \longrightarrow S_2^n \longrightarrow 0.$$

If $n > 1$, then $\mathrm{Hom}(R, A) = 0$, so $j(R) = 1 > j(S_1) = 0$. Thus δ is not a dimension function and hence A is not AG. This proves 1(a).

Since A is non-singular on both sides the injective envelope of A on either side has a natural ring structure extending that on A; hence the injective envelopes are bimodules, and it is natural to ask whether this bimodule structure makes the injective envelope an injective envelope on both sides simultaneously. This question is natural given that the ring of fractions of a semi-prime noetherian ring is simultaneously the left and right injective envelope of the ring. In any case, for this A the answer is no. The minimal injective resolution of A_A is

$$0 \to A_A \to B_A \to (B/A)_A \to 0,$$

but ${}_AB$ is not injective if $n > 1$. This proves 1(b).

To prove 1(c) we define $\delta_m(M) = \max\{\delta(N) \mid N \subset M\}$ for every noetherian module M. Then $\delta_m(M) = 1$ for a right A-module if and only if there is a subquotient of M isomorphic to S_1. Since $\delta_m(M)$ is either 0 or 1, $\delta_m(M) = \max\{\delta_m(N), \delta_m(M/N)\}$ for every submodule N. Other axioms of a dimension function (see [MR, 6.8.4]) can be checked trivially and hence δ_m is an exact dimension function. It is easy to see that $(B/A)_A$ is isomorphic to $S_2^{(n^2+n-1)}$. Thus A_A has a pure injective resolution with respect to δ_m. Similarly ${}_AA$ has a pure injective resolution with respect to δ_m; therefore 1(c).

Now assume $n = 1$. Then A_A and ${}_AA$ have the same injective hull and the same minimal injective resolution, namely

$$0 \longrightarrow \begin{pmatrix} k & k \\ 0 & k \end{pmatrix} \longrightarrow \begin{pmatrix} k & k \\ k & k \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 0 \\ k & 0 \end{pmatrix} \longrightarrow 0.$$

It is easy to check that A is AG. Notice that the canonical dimension δ is not equal to either GKdim or Kdim (but equal to δ_m). Finally, if $P = \begin{pmatrix} k & k \\ 0 & 0 \end{pmatrix}$, then $\delta((A/P)_A) = 1$ and $\delta({}_A(A/P)) = 0$, proving 2(b). By 1(c) A has a pure injective resolution. □

Our last example shows that even a nice AG ring can only have an essentially pure but not a pure injective resolution.

Example 5.5. Let Ω be the Casimir element in the enveloping algebra $U(\mathfrak{sl}_2)$ over a field of characteristic zero. We show that $U(\mathfrak{sl}_2)/(\Omega)$ has an essentially pure but not a pure injective resolution.

As is standard, $U(\mathfrak{sl}_2)/(\Omega)$ is an AG and CM domain of injective dimension two, and hence has an essentially pure injective resolution [Propositions 2.4.4 and 2.5.1]. By Lemma 4.1, we only need to show that $U(\mathfrak{sl}_2)/(\Omega)$ has a module of GK-dimension zero having an essential extension of GK-dimension ≥ 1.

The following construction is well-known. Let $A_1 = k[x, \partial]$ be the first Weyl algebra with relation $\partial x - x\partial = 1$. There is an algebra homomorphism $\varphi : U(\mathfrak{sl}_2) \to B = k[\partial, x\partial, x^2\partial]$ determined by

$$p : e \longmapsto \partial,\ h \longmapsto -2x\partial,\ f \longmapsto -x^2\partial,$$

where e, h, f have the usual meaning for $\mathfrak{sl}_2$. By a direct computation, $\Omega = ef + fe + \frac{1}{2}h^2$ is in the kernel of φ, so there is a surjection $U(\mathfrak{sl}_2)/(\Omega) \to B$; but both are domains of GK-dimension 2, so $U(\mathfrak{sl}_2)/(\Omega) \cong B$. There is a natural action of A_1 on $k[x]$ defined by

$$x \cdot f(x) = xf(x), \quad \text{and} \quad \partial \cdot f(x) = f'(x)$$

for $f(x) \in k[x]$. It is easy to see that k is the only proper B-submodule of $k[x]$, so $k[x]$ is an essential extension of k, whence B does not have a pure injective resolution. □

6. Localization

In this section we prove that, under some extra hypotheses such as grade-symmetric, an AG ring has a QF quotient ring. As a consequence an Auslander regular, grade-symmetric algebra is semiprime. Example 5.4.2 shows that the quotient ring (if it exists) may not be self-injective if the hypotheses fail.

Suppose that A is AG, and let N be its prime radical. Then $N^s = 0$ for some s; because δ is exact

$$\delta(A) = \max\{\delta(N^i/N^{i+1}) \mid 1 \leq i \leq s-1\} = \delta(A/N).$$

We say that N is **right weakly invariant** with respect to δ, if $\delta(M \otimes_A N) < \delta(A/N) = \delta(A)$ whenever M_A is finitely generated and $\delta(M) < \delta(A)$ [MR, 6.8.13]. By [Le, 5.3], an AG and CM ring has a left and right artinian quotient ring; the CM hypothesis ensures that GK-dimension is exact. The next result shows that the CM hypothesis can be weakened; the symmetry of GK-dimension can be replaced by the hypothesis (*) below.

Theorem 6.1. *Let A be a noetherian ring of finite injective dimension, with N as its prime radical, satisfying the following conditions:*

(*)
- *δ is an exact dimension function, i.e., A is CM with respect to some dimension function;*
- *for every bimodule subquotient L of N, $j(L_A) = 0$ if and only if $j({}_AL) = 0$;*
- *if P is a minimal prime, then $j((A/P)_A) = 0$ if and only if $j({}_A(A/P)) = 0$.*

Then

1. *N is weakly invariant;*
2. *if P is a minimal prime of A, then $j(A/P) = 0$;*
3. *A has a left and right artinian quotient ring Q;*
4. *Q is self-injective and hence QF.*

Remark: If A is AG and grade-symmetric, then the hypotheses in Theorem 6.1 hold.

Proof. Let $d = \text{injdim } A$.

1. We will prove that if B is a bimodule subquotient of N and M_A is noetherian with $\delta(M) < d$, then $\delta(M \otimes_A B) < d$. Since δ is exact, we may assume that M is cyclic and critical, and that B is critical as a bimodule; hence the ideals $P = ann(M_A)$, $R = ann(B_A)$ and $L = ann({}_AB)$ are all prime, and B is a fully faithful $(A/L, A/R)$-bimodule. We are done if $\delta((A/R)_A) < d$ because $M \otimes_A B$ is a right A/R-module. So suppose now that $\delta((A/R)_A) = d$. By (*), $\delta({}_A(A/L)) = \delta({}_AB) = \delta(B_A) = d$. If $P \not\subset L$, then PB is non-zero sub-bimodule of B, so

$$\delta(M \otimes B) \leq \delta(A/P \otimes B) = \delta(B/PB) < \delta(B) = d.$$

Now suppose that $P \subset L$; since $\delta({}_A(A/L)) = d$, L is a minimal prime; whence $P = L$. By (*), $\delta((A/P)_A) = \delta({}_A(A/P)) = d$. Since $\delta(M) < \delta(A/P)$, M is a factor module of A'/xA' where $A' = A/P$ and x is a regular element of A'. Since ${}_{A'}B$ is fully faithful, x is a non-zero-divisor on B and hence B/xB is a torsion A/R-module. By [MR, 6.8.4(iii)], $\delta(B/xB) < \delta(A/R) = d$. Therefore

$$\delta(M \otimes B) \leq \delta(A'/xA' \otimes B) = \delta(B/xB) < d,$$

which proves that N is weakly invariant.

2 and 3. By [MR, 6.8.15] and part 1, A has a left and right artinian quotient ring Q, whence $\mathcal{C}(0) = \mathcal{C}(N)$ by [MR, 4.1.4], where $\mathcal{C}(I)$ denotes the set of regular elements in R/I. Let P be a minimal prime ideal of A. If $\delta(A/P) < d$, then $P \cap \mathcal{C}(0) \neq \emptyset$ by [MR, 6.8.14]; but N contains the product of the minimal primes, so $P \cap \mathcal{C}(N) = \emptyset$, so we conclude that $\delta(A/P) = d$ and $j(A/P) = 0$.

4. Since $Q/QN \cong \operatorname{Fract}(A/N)$, every prime ideal of Q is minimal and of the form PQ for some minimal prime $P \subset A$. By part 2, $j(A/P) = 0$ for every such P, so $0 = j(A/P \otimes_A Q) = j(Q/PQ)$. Since every prime factor of Q has grade zero, every non-zero Q-module has grade zero. If $d' := \operatorname{injdim} Q \neq 0$, we pick a noetherian Q-module M such that $\operatorname{Ext}^{d'}(M, Q) \neq 0$. Hence the grade of $\operatorname{Ext}^{d'}_Q(M, Q)$ is zero. But (0-2) implies that $\operatorname{Hom}_Q(\operatorname{Ext}^{d'}_Q(M, Q), Q) = 0$, a contradiction. Therefore $d' = 0$ and Q is self-injective. □

It is easy to produce some corollaries by checking, in certain cases, that $\delta(L_A) = d$ if and only if $\delta({}_AL) = d$ for *every* noetherian bimodule L. One such case occurred in [SfZ] where A is FBN and δ equals the Krull dimension. Another such case occurs when A is AG of injective dimension 1 and CM with respect to Krull dimension: by Lenagan's lemma [GW, 7.10], $\operatorname{Kdim}(L_A) = 0$ if and only if $\operatorname{Kdim}({}_AL) = 0$, whence $\operatorname{Kdim}(L_A) = 1$ if and only if $\operatorname{Kdim}({}_AL) = 1$, so Theorem 6.1 applies. Here are some other special cases.

Corollary 6.2. *Let A be a noetherian ring with finite injective dimension. Then A has a QF quotient ring if*

1. A is CM, or

2. A is commutative, or

3. A is an AG ring with finite GK-dimension and A has a unique minimal prime ideal.

Proof. 1. This is a consequence of Theorem 6.1 and the fact that GKdim is symmetric on noetherian bimodules (see [Le, 5.3]).

2. A commutative noetherian ring is AG if and only if it has finite injective dimension. If A is commutative, (*) in Theorem 6.1 holds trivially.

3. By Theorem 6.1, it is enough to show that $\delta(L_A) = d$ if and only if $\delta({}_AL) = d$ for all noetherian bimodules L. Let N be the unique minimal prime ideal of A. Then N is the prime radical, and $\delta(A/N) = \delta(A)$. Since δ is exact, we may assume that L is a critical fully faithful $(A/P, A/Q)$-bimodule with P, Q prime ideals of A. By [GW, 7.1], $\delta(L_A) = \delta((A/Q)_A)$ and $\delta({}_AL) = \delta({}_A(A/P))$. Therefore, it suffices to show that $P = Q = N$. If $\delta(L_A) = d$, then $\delta((A/Q)_A) = d$ and then Q is a minimal prime of A, namely, $Q = N$. It remains to prove that $P = N$. Suppose not, then P is a prime ideal containing N. Modulo N, we may assume that A is prime and P is a non-zero prime ideal of A and L is a critical and fully faithful $(A/P, A)$-bimodule. If GKdim A is finite this can not happen. □

Another immediate consequence is the following.

Corollary 6.3. *Let A be as in Theorem 6.1 and suppose A has finite global dimension (e.g., A is Auslander regular and grade-symmetric). Then A is semiprime.*

Proof. By Theorem 6.1, A has a QF quotient ring, say Q. Since A has finite global dimension, so has Q and hence Q has global dimension zero, because Q has injective dimension zero. Therefore Q is semisimple artinian. By Goldie's theorem, A is semiprime. □

For other basic properties about localization of AG rings see [ASZ, §2].

7. Dualities on Ext

Another type of duality between left and right modules can be introduced by using an idea similar to the Yoneda product. Let k be a field, and A a CM_0 k-algebra of global dimension d. If S is a finite dimensional right A-module, then $j(M) = d$. By (0-2) we have $\mathrm{Ext}^i(\mathrm{Ext}^d(S, A), A) = 0$ if $i \neq d$, and $\mathrm{Ext}^d(\mathrm{Ext}^d(S, A), A) \cong S$. Consequently, $j(\mathrm{Ext}^d(S, A)) = d$ and hence $\mathrm{Ext}^d(S, A)$ is finite dimensional. We have seen (in section 1) that $\mathcal{M}_0$ and $\mathcal{M}^\circ{}_0$ are in duality via the functors $\mathrm{Ext}^d(-_A, A)$ and $\mathrm{Ext}^d({}_A-, A)$. For a right A-module M, since $pd(M) \leq d$, $\mathrm{Ext}^d(M, -)$ is a right exact covariant functor from $\mathcal{M}$ to mod $- k$. Recall that $M^* = \mathrm{Hom}_k(M, k)$.

Proposition 7.1. *Let A be a CM_0 noetherian k-algebra with* $\mathrm{gldim}\, A = d$. *Let S be a finite dimensional right A-module and N a noetherian right A-module. Then*

1. $\mathrm{Ext}^i(S, N)$ *and* $\mathrm{Ext}^i(N, S)$ *are finite dimensional for all i, and*

2. $\mathrm{Ext}^{d-i}(S, N)^* \cong \mathrm{Ext}^i(N, S')$ *for all i, where* $S' = \mathrm{Ext}^d(S, A)^*$.

Proof. 1. Computing $\mathrm{Ext}^i(N, S)$ by using a finite free resolution of N, we see that $\mathrm{Ext}^i(N, S)$ is finite dimensional for all i. Since A is CM_0, $\mathrm{Ext}^i(S, A) = 0$ for all $i < d$ and $\mathrm{Ext}^d(S, A)$ is finite dimensional. Hence if $N = A$ then $\mathrm{Ext}^i(S, N)$ is finite dimensional for all i. Since every finitely generated projective module is a summand of a finite free module, the statement holds for projective modules N. By induction on the projective dimension of N, the statement follows.

2. First we prove that $\mathrm{Ext}^d(S, N)^* \cong \mathrm{Hom}(N, S')$. Since $\mathrm{Ext}^d(S, -)^*$ is a left exact and contravariant functor from $\mathcal{M}$ to mod $- k$, by Watts' theorem [Ro, 3.36], $\mathrm{Ext}^d(S, -)^* \cong \mathrm{Hom}(-, S')$ where $S' = \mathrm{Ext}^d(S, A)^*$. Part 2 follows because $\{\mathrm{Ext}^{d-i}(S, -)^* \mid i\}$ and $\{\mathrm{Ext}^i(-, S') \mid i\}$ are universal δ-functors, by the CM_0 condition (see [Ha, pp 205-206] for the definition of δ-functor). □

When L is a semisimple Lie algebra, $S' \cong S$ (see Remark after Corollary 3.4), so we have the following.

Corollary 7.2. *Let S be a finite dimensional simple module over a d-dimensional semisimple Lie algebra. Then* $\mathrm{Ext}^{d-i}(S, N)^* \cong \mathrm{Ext}^i(N, S)$ *for all i and all modules N.* □

Next is a dual version of Proposition 2.5.3.

Corollary 7.3. *Let A be an Auslander regular and CM_0 noetherian k- algebra with* $\mathrm{gldim}\, A = d$, *and M a finitely generated right A-module. Then* $\mathrm{injdim}\, M = d$ *if and only if M has a factor module M/N with $\delta(M/N) = 0$.*

Proof. If $\mathrm{injdim}\, M = d$, then $\mathrm{Ext}^d(L, M) \neq 0$ for some L. By Proposition 2.5.3, $\mathrm{Ext}^d(N, M) = 0$ if N contains no finite dimensional module. Hence $\mathrm{Ext}^d(S, M) \neq 0$ for some finite dimensional module S. By Proposition 7.1.2, $\mathrm{Hom}(M, S') \neq 0$ and hence M has a finite dimensional factor module. The converse is similar to prove. □

For a finite dimensional right A-module S and a finitely generated right A-module N, we define

$$\nu_S(N) = (n_0, n_1, \cdots, n_d) \in \mathbf{N}^{d+1} \quad \text{and} \quad c_S(N) = \sum_i (-1)^i n_i \in \mathbf{Z}$$

where $n_i = \dim \mathrm{Ext}^i(S, N)$. For every $v = (n_0, n_1, \cdots, n_d) \in \mathbf{N}^{d+1}$, v^τ denotes the vector $(n_d, \cdots, n_1, n_0)$.

Proposition 7.4. *Let A be a CM_0 noetherian k-algebra with* gldim $A = d$. *Then*

1. $\nu_S(A) = (0, 0, \cdots, \dim S^\vee)$ *and* $c_S(A) = (-1)^d \dim S^\vee$.

2. If N is also finite dimensional, then $\nu_S(N)^\tau = \nu_N(S')$ *and* $c_S(N) = (-1)^d c_N(S')$.

3. Suppose that the Grothendieck group K_0 of A is $\mathbb{Z}$. For every finitely generated right A-module N, $c_S(A)$ divides $c_S(N)$.

4. Suppose that the Grothendieck group K_0 of A is $\mathbb{Z}$, $d > 0$, and that A has an artinian quotient ring. For every finite dimensional module S, $c_S(N) = 0$ if and only if N is torsion.

Proof. Part 1 follows from the definitions. Part 2 follows from the definitions and Proposition 7.1.2.

3. By the long exact sequence for $\mathrm{Ext}^*(S, -)$, we see that $c_S(-)$ is additive on exact sequences. Since K_0 is trivial, every finitely generated module N has a finite free resolution of finite length

$$0 \longrightarrow P_l \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow N \longrightarrow 0 \tag{7-1}$$

where $P_i = A^{\oplus k_i}$. By additivity, $c_S(N) = (\sum_i (-1)^i k_i) c_S(A)$, which is divisible by $c_S(A)$.

4. Let Q be the artinian quotient ring of A. Tensoring (7-1) with Q, we obtain an exact sequence

$$0 \longrightarrow P_l \otimes_A Q \longrightarrow \cdots \longrightarrow P_1 \otimes_A Q \longrightarrow P_0 \otimes_A Q \longrightarrow N \otimes_A Q \longrightarrow 0.$$

Since the length over Q is exact, $N \otimes_A Q = 0$ if and only if $\sum_i (-1)^i k_i = 0$. Hence N is torsion if and only if $c_S(N) = (\sum_i (-1)^i k_i) c_S(A) = 0$. □

As an application of Proposition 7.4, we study the minimal injective resolution of finite dimensional simple modules over $U(\mathfrak{sl}_2)$.

Proposition 7.5. *Let $\mathfrak{sl}_2$ be the special linear Lie algebra over the field $k = \mathbb{C}$. The minimal injective resolution of a finite dimensional simple module S over $U(\mathfrak{sl}_2)$ is of the form*

$$0 \to S \to E(S) \to I \to I \to E(S) \to 0,$$

where

$$I = \bigoplus_N E(N)^{\dim \mathrm{Ext}^1(S,N)}$$

where N runs over all infinite dimensional simple modules.

Proof. Let S and T be two finite dimensional simple modules. Since $\mathfrak{sl}_2$ is simple, $\mathrm{Ext}^1(T, S) = 0$. If $S \not\cong T$, then $\mathrm{Ext}^0(T, S) = 0$. By the Remark after Corollary 3.4, $S' \cong S$ for all finite dimensional simple modules S. By Proposition 7.1.2, $\mathrm{Ext}^2(T, S) \cong \mathrm{Ext}^{3-2}(S, T) = 0$ for all S and T, and $\mathrm{Ext}^3(T, S) = \mathrm{Ext}^0(S, T) = 0$ if $S \not\cong T$. Therefore $\mathrm{Ext}^i(T, S) = 0$ for all i if $S \not\cong T$, and $\mathrm{Ext}^i(S, S) = k$ if $i = 0, 3$ and $\mathrm{Ext}^i(S, S) = 0$ if $i = 1, 2$. Consider the minimal injective resolution of S, say

$$0 \longrightarrow S \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow I^3 \longrightarrow 0. \tag{7-2}$$

Then $I^0 = E(S)$. By [Da], I^0 is locally artinian and hence I^0/S is locally artinian. By [Da], the essential extension I^1 is locally artinian. Similarly, I^i is locally artinian for all i. Since $\mathrm{Ext}^i(T, S) = 0$ for all T for $i = 1, 2$, I^1 and I^2 contain no finite dimensional submodules; since $\mathrm{Ext}^3(T, S) = 0$ if $T \not\cong S$ and $\mathrm{Ext}^3(S, S) = k$, the only finite dimensional simple submodule of I^3 is S, whence $I^3 \cong E(S) \oplus J^3$ where J^3 contains no finite dimensional modules. By Theorem 3.5, $J^3 = 0$ and hence $I^3 = E(S)$. Now let N be an infinite dimensional simple module. Since N is a torsion module and $\mathrm{Hom}(S, N) = \mathrm{Ext}^3(S, N) = 0$, by Proposition 7.4.4,

$$\dim \operatorname{Ext}^1(S, N) = \dim \operatorname{Ext}^2(S, N). \tag{7-3}$$

Then by Corollary 7.2, we also have $\operatorname{Hom}(N, S) = \operatorname{Ext}^3(N, S) = 0$, and $\dim \operatorname{Ext}^1(N, S) = \dim \operatorname{Ext}^2(N, S)$. Since $k = \mathbb{C}$ is uncountable, $\operatorname{Hom}(N, N) \cong k$, so

$$I^1 \cong I^2 \cong \bigoplus_N E(N)^{\nu_S(N)_1}.$$

Hence (7-2) is of the form claimed. □

The previous result is analogous to what happens for connected graded rings of global dimension 3.

Let N be an infinite dimensional simple $U(\mathfrak{sl}_2)$-module. By Corollary 7.3, injdim $N \leq 2$. Suppose the minimal injective resolution of N is

$$0 \longrightarrow N \longrightarrow E(N) \longrightarrow I^1 \longrightarrow I^2 \longrightarrow 0.$$

By [Da], I^i is locally artinian for $i = 1, 2$. By (7-3), $\dim \operatorname{Ext}^1(S, N) = \dim \operatorname{Ext}^2(S, N)$ for every finite dimensional module S. There is a choice of infinite dimensional simple module N such that $\operatorname{Ext}^1(S, N) = 0$ for all finite dimensional S; for such an N, $\operatorname{Ext}^i(S, N) = 0$ for all i; thus $\nu_S(N)$ may be zero for all finite dimensional simple modules S even when N is non-zero.

We can also apply these methods to rings of global dimension 2. For example, we leave the reader to explore what happens when A is AG, CM and gldim $A = 2$ and $K_0(A) \cong \mathbf{Z}$.

If A is commutative we can prove a version of Proposition 7.1 without the CM_0 condition (essentially because a commutative noetherian ring of finite global dimension is Auslander regular). Recall that $\mathcal{M}_i = \{M \in \mathcal{M} \mid \delta(M) \leq i\}$ is a dense subcategory of $\mathcal{M}$ for all i.

Lemma 7.6. *Let A be a commutative noetherian ring of global dimension d. If $M \in \mathcal{M}_i$, then for every noetherian module N, $\operatorname{Ext}^s(M, N)$ and $\operatorname{Ext}^s(N, M)$ are in $\mathcal{M}_i$ for all s.*

Proof. We may assume that M is critical and cyclic, hence isomorphic to A/P for some prime ideal P with $\delta(A/P) \leq i$. Therefore $\operatorname{Ext}^s(M, N)$ and $\operatorname{Ext}^s(N, M)$ are finitely generated A/P-modules, and hence in $\mathcal{M}_i$. □

Proposition 7.7. *Let A be commutative noetherian with* gldim $A = d$, *and let M be a finitely generated A-module such that $j(M) = s = pd(M)$. Then*

1. $\operatorname{Ext}^s(\operatorname{Ext}^s(M, N), A) \cong \operatorname{Hom}(N, M)$ *for all finitely generated A-modules N;*
2. *in the quotient category $\mathcal{M} / \mathcal{M}_{d-s-1}$, there are isomorphisms*

$$\operatorname{Ext}^s(\operatorname{Ext}^{s-i}(M, N), A) \cong \operatorname{Ext}^i(N, M)$$

for all finitely generated N and all $i \geq 0$;

3. *if $s = d$, then the isomorphism in part 2 is as A-modules.*

Proof. The proof is similar to that of Proposition 7.1 and we leave it to the interested reader. □

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