ELSEVIER

Journal of Pure and Applied Algebra 140 (1999) 1–21

JOURNAL OF PURE AND APPLIED ALGEBRA

Injective resolutions of some regular rings

K. Ajitabh[a], S.P. Smith[b,*,1], J.J. Zhang[b,2]

[a] *Department of Mathematics, Florida International University, Miami, FL 33199, USA*

[b] *Department of Mathematics, University of Washington, Box 354350, Seattle, WA 98195, USA*

Communicated by C.A. Weibel; received 3 March 1997

Abstract

Let A be a noetherian Auslander regular ring and δ the canonical dimension function on A-modules, which is defined as $\delta(M) = d - j(M)$ where d is the global dimension of A and $j(M)$ is the grade of M. An A-module is *s-pure* if $\delta(N) = s$ for all its non-zero noetherian submodules N, and is *essentially s-pure* if it contains an essential submodule which is s-pure. Consider a minimal injective resolution of A as an A-module

$$0 \to A \to I^0 \to I^1 \to \cdots \to I^d \to 0.$$

We say A has a *pure* (resp. *essentially pure*) injective resolution if I^i is $(d-i)$-pure (resp. essentially $(d-i)$-pure). We show that several classes of Auslander regular rings with global dimension at most 4 have pure or essentially pure injective resolutions.

MSC: 16E10; 18G10

0. Introduction

The initial motivation for this work came from [1] where the purity of resolution was used in a crucial way to answer a question of M. Artin on the residue complex for quantum planes. Recently, Yekutieli [17] has incorporated the purity property in his proposed definition for the residue complex of a non-commutative graded ring.

* Corresponding author.

E-mail addresses: ajitabhk@servms.fiu.edu (K. Ajitabh), smith@math.washington.edu (S.P. Smith)

1 Supported by NSF grant 9400524.

2 Supported by an NSF Postdoctoral Fellowship.

PII: S0022-4049(99)00049-3

It is well known that if $0 \to R \to I^\bullet$ is a minimal injective resolution of a commutative noetherian ring R, then

$$I^i \cong \bigoplus_{\mathfrak{p}\in \mathrm{Spec}\ R} E(R/\mathfrak{p})^{\mu_i(\mathfrak{p})},$$

where $E(-)$ denotes an injective hull, $k(\mathfrak{p})$ denotes the residue field at $\mathfrak{p}$, and the multiplicity $\mu_i(\mathfrak{p}) = \dim_{k(\mathfrak{p})} \mathrm{Ext}^i_{R_\mathfrak{p}}(k(\mathfrak{p}), R_\mathfrak{p})$. If $\mathrm{injdim}\, R < \infty$, then every $E(R/\mathfrak{p})$ occurs somewhere in the resolution. If R is local and Gorenstein, then $\mu_i(\mathfrak{p})$ is 1 if $\mathrm{height}\,\mathfrak{p} = i$, and is zero otherwise; so each $R/\mathfrak{p}$ appears exactly once in the injective resolution, and I^i is the direct sum of the injective hulls of $R/\mathfrak{p}$ where $\mathfrak{p}$ runs through the set of prime ideals of height i. Thus there is a certain homogeneity to I^i which we refer to as purity: every non-zero finitely generated submodule of I^i has Krull dimension equal to $(\mathrm{K\,dim}\, R - i)$. There is a similar notion of purity for noncommutative rings (which we define below), and one can ask if a minimal injective resolution of an Auslander–Gorenstein ring is pure. In [2, 4.2] we showed that an Auslander–Gorenstein, grade-symmetric ring satisfying a polynomial identity has a pure injective resolution. On the other hand, M. Artin and J. T. Stafford gave examples of Auslander regular rings which do not have essentially pure injective resolution [2, 5.2 and 5.3]. In this paper we show that minimal injective resolutions of some noncommutative rings of low global dimension (≤ 4) are pure or essentially pure.

Definition 0.1. Let A be a ring. The *grade* of an A-module M is

$$j(M) := \min\{i \mid \mathrm{Ext}^i_A(M, A) \neq 0\}$$

or ∞ if no such i exists.

Definition 0.2. We say that a ring A

- is *quasi-Frobenius* (QF) if it is left and right artinian and left and right self-injective;
- satisfies the *Auslander condition* if for every noetherian A-module M and for all $i \ge 0$, $j(N) \ge i$ for all submodules $N \subset \mathrm{Ext}^i(M, A)$;
- is *Auslander–Gorenstein* (AG) if A is left and right noetherian, satisfies the Auslander condition, and has finite left and right injective dimension;
- is *Auslander regular* if it is Auslander–Gorenstein, and has finite global dimension.

Let ∂ be a dimension function on A-modules, in the sense of [11, 6.8.4]. Recall that ∂ is called *exact* if $\partial(M) = \sup\{\partial(N), \partial(M/N)\}$ whenever N is a submodule of M. Krull dimension (Kdim) in the sense of Rentschler–Gabriel is always, and Gelfand–Kirillov dimension (GK dim) is often, exact.

By [18, Lemma A], the injective dimension of the module A_A is equal to the injective dimension of the module ${}_AA$ if both are finite. If A has injective dimension $d < \infty$, we define $\delta(M) = d - j(M)$ for all A-modules M. Note that δ is *not* a dimension function in general. It is a simple observation that for any ring A, and any A-module M, $j(M) \ge \inf\{j(N), j(M/N)\}$ whenever N is a submodule of M. Therefore, for any ring

A of finite injective dimension, the inequality $\delta(M) \le \sup\{\delta(N), \delta(M/N)\}$ always holds, whenever N is a submodule of M. It follows that δ is automatically exact whenever it is a dimension function. If A is AG, then δ is known to be an exact dimension function, by [9, 4.5], [4, 1.8]; we call it the *canonical dimension function.*

Definition 0.3. Let A be a noetherian ring with finite injective dimension. We say that a module M is

- *s-pure* if $\delta(N) = s$ for all non-zero noetherian submodules $N \subset M$;
- *essentially s-pure* if it contains an essential submodule which is s-pure;
- *s-critical* if it is s-pure and $\delta(M/N) < s$ for all non-zero submodules $N \subset M$.

Let

$$0 \to A \to I^0 \to \cdots \to I^d \to 0 \tag{0.1}$$

be a minimal injective resolution of A as a right A-module. We say this resolution is

- *pure* if each I^i is $(d-i)$-pure.
- *essentially pure* if each I^i is essentially $(d-i)$-pure.

Definition 0.4. We say that A is *Cohen–Macaulay with respect to a dimension function* ∂ (or, ∂-CM, in short) if

$$j(M) + \partial(M) = \partial(A) < \infty$$

for every non-zero noetherian A-module M. When we say A is *Cohen–Macaulay* (CM) without reference to any dimension function, we mean A is Cohen–Macaulay with respect to GK dim (assuming tacitly that A is an algebra over a field).

Note the following simple facts. Trivially, an AG ring is CM with respect to the canonical dimension function δ. If a ring A is CM with respect to a dimension function ∂, then ∂ is automatically exact. If A is a ring with finite injective dimension, then A is CM with respect to some dimension function ∂ if and only if δ is a dimension function; indeed, then $\partial(M) = \delta(M) + (\partial(A) - d)$. Our main results are the following.

Proposition A. 1. *If A is a domain of GK-dimension* 2, *generated by two elements subject to a quadratic relation*, *then A has a pure injective resolution.*

2. *The enveloping algebra of a Lie algebra of dimension* ≤ 3 *has an essentially pure injective resolution.*

For graded algebras, we prove the following results, the first of which extends [1, 3.2].

Theorem B. *Every Artin–Schelter graded regular ring of dimension* 3, *generated by three elements of degree* 1, *has a pure graded injective resolution.*

Theorem C. 1. *The homogenization of the universal enveloping algebra of a three-dimensional Lie algebra has an essentially pure graded injective resolution.*

2. *The four-dimensional Sklyanin algebra has an essentially pure graded injective resolution.*

3. *The four-dimensional Sklyanin algebra has a pure graded injective resolution if and only if it satisfies a polynomial identity.*

The organization of the paper is as follows. Sections 1 and 2 contain preparatory results. In Section 1 we examine the effect of localization on injective dimension and injective resolutions. Generally, injective dimension does not behave well under arbitrary localizations, but the situation is good when localizing at normal elements (Proposition 1.3). The rings we examine later (in Sections 3 and 4) tend to be Auslander–Gorenstein and/or Cohen–Macaulay, and Section 2 contains some general results about localizations of such rings. For example, the Auslander–Gorenstein condition is preserved under arbitrary localizations, and the Cohen–Macaulay condition is preserved under certain kinds of localization. In Sections 3 and 4, we study the purity of resolutions for ungraded and graded algebras, respectively, and prove our main results. The final result in the paper deals with the purity of four-dimensional Sklyanin algebras. The main step is a computation of $\mathrm{Ext}^1_A(L,N)$ where L is a line module and N a point module. This computation is of independent interest. Indeed, it is important to compute all $\mathrm{Ext}^i_A(L,N)$ when L and N are linear modules over a quantum polynomial ring A.

1. Preliminaries

We begin by examining the behavior of injective dimension under localization. If S is a multiplicatively closed Ore set of regular elements in a right noetherian ring A, then $\operatorname{injdim} AS^{-1} \le \operatorname{injdim} A$ because every finitely generated right AS^{-1}-module is of the form MS^{-1}, and

$$\mathrm{Ext}^i_{AS^{-1}}(MS^{-1}, AS^{-1}) \cong AS^{-1} \otimes_A \mathrm{Ext}^i_A(M,A), \tag{1.1}$$

where we have used the fact that AS^{-1} is a flat A-module. Despite this, the injective dimension of a module does *not* always behave well under localization because a localization of an injective need not be injective. In [5,6] Goodearl and Jordan showed that one must localize at normal elements for injective dimension to behave well.

Let A be noetherian, and S a multiplicatively closed Ore set of regular normal elements. If E is an injective A-module, then ES^{-1} is an injective AS^{-1}-module [5, Theorem 1.3] and, if $L \subset M$ is an essential extension of A-modules, then $LS^{-1} \subset MS^{-1}$ is an essential extension of AS^{-1}-modules [5]. It follows that if $0 \to M \to E^\bullet$ is a minimal injective resolution of an A-module M, then $0 \to MS^{-1} \to E^\bullet S^{-1}$ is a minimal injective resolution of the AS^{-1}-module MS^{-1}. Therefore

$$\operatorname{injdim} MS^{-1} = \min\{n \mid E^n S^{-1} = 0\} \le \operatorname{injdim} M. \tag{1.2}$$

Proposition 1.3 describes more precisely the relation between these two injective dimensions. First we recall the following version of the well-known Rees' Lemma.

Lemma 1.1. *Let M and N be A-modules. Let g be a non-unit regular normal element of A, acting faithfully on M, and annihilating N. Then*

$$\mathrm{Ext}^i_{A/(g)}(N, M/Mg) \cong \mathrm{Ext}^{i+1}_A(N, M).$$

Proof. This follows from the collapsing of the spectral sequence

$$\mathrm{Ext}^p_{A/(g)}(N, \mathrm{Ext}^q_A(A/(g), M)) \Rightarrow \mathrm{Ext}^n_A(N, M). \qquad \square$$

The isomorphism in Lemma 1.1 is as abelian groups, but there is a stronger isomorphism when M is the ring itself. Given an automorphism σ of A, and a right A-module M, define the twisted module M^σ by $m * a = m\sigma(a)$ for all $m \in M$ and $a \in A$. Then $M \mapsto M^\sigma$ is an automorphism of the category of right A-modules. If M is a bimodule, so is M^σ, and the corresponding functor is an automorphism of the bimodule category.

Lemma 1.2. *Let g be a regular normal element of A and σ the automorphism determined by $ag = g\sigma(a)$ for all $a \in A$. If N is a right $A/(g)$-module, then there are natural left $A/(g)$-module isomorphisms*

$$\mathrm{Ext}^i_{A/(g)}(N, A/(g)) \cong \mathrm{Ext}^{i+1}_A(N, A^{\sigma^{-1}}) \cong \mathrm{Ext}^{i+1}_A(N^\sigma, A)$$

for all $i \geq 0$.

Proposition 1.3. *Let S be a multiplicatively closed Ore set of regular normal elements in a noetherian ring A. If M is a right A-module on which each $g \in S$ acts faithfully, then*

$$\operatorname{injdim} M = \max\{\operatorname{injdim}_{AS^{-1}} MS^{-1}, \operatorname{injdim}_{A/(g)}(M/Mg) + 1 \mid g \in S\}. \tag{1.3}$$

Proof. If $g \in S$, then $\operatorname{injdim}_{A/(g)} M/Mg \leq \operatorname{injdim} M - 1$ by Lemma 1.1. Combining this with (1.2), we have

$$\operatorname{injdim} M \geq \max\{\operatorname{injdim} MS^{-1}, \operatorname{injdim}(M/Mg) + 1 \mid g \in S\}.$$

Let $0 \to M \to E^0 \to \cdots$ be a minimal injective resolution of M, and define $\Omega^i := \ker(E^i \to E^{i+1})$. Let $d \leq \operatorname{injdim} M$. If $E^d S^{-1} = 0$, then E^d contains a non-zero submodule N such that $Ng = 0$ for some $g \in S$. Replacing N by $N \cap \Omega^d \neq 0$, there is a non-split extension $0 \to \Omega^{d-1} \to E \to N \to 0$, whence

$$0 \neq \mathrm{Ext}^1_A(N, \Omega^{d-1}) \cong \mathrm{Ext}^2_A(N, \Omega^{d-2}) \cong \cdots \cong \mathrm{Ext}^d_A(N, M) \cong \mathrm{Ext}^{d-1}_{A/(g)}(N, M/Mg),$$

whence $1 + \operatorname{injdim}_{A/(g)} M/Mg \geq d$. On the other hand, if $E^d S^{-1} \neq 0$ for $d \leq \operatorname{injdim} M$, then $\operatorname{injdim} MS^{-1} \geq d$ because $E^\bullet S^{-1}$ is a minimal resolution for MS^{-1}. Thus the reverse inequality also holds, hence the result. $\square$

Proposition 1.3 is used in Sections 3 and 4 in the following way: the rings there have sufficiently many normal elements, so to a large extent questions about A can be reduced to questions about $A/(g)$ which has lower injective dimension.

Write Ω_s for the image of the boundary map $I^{s-1} \to I^s$ in (0.1). Thus $\Omega_0 = A$, and there are exact sequences

$$0 \longrightarrow \Omega_{s-1} \longrightarrow I^{s-1} \longrightarrow \Omega_s \longrightarrow 0$$

for all $s = 1, \dots, d$, with each I^s an essential extension of Ω_s. Recall the following basic facts.

Proposition 1.4 (Ajitabh et al. [2, 2.4 and 2.5]). *Let A be a right noetherian ring with* $\operatorname{injdim} A_A = d$, *and let N be a noetherian right A-module.*

1. *If N embeds in Ω_i, then* $\operatorname{Ext}^i(N,A) \neq 0$, *whence $j(N) \leq i$.*
2. *If every non-zero submodule of N is $(d-i)$-critical, then N embeds in Ω_i.*
3. *I^0 is essentially d-pure (and d-pure if δ is exact).*
4. *If A has a QF quotient ring Q, then $I^0 \cong Q$ is d-pure and every torsion module M (i.e., a module such that $M \otimes_A Q = 0$) has $j(M) \geq 1$. As a consequence, Ω_1 is $(d-1)$-pure, and I^1 is essentially $(d-1)$-pure. Furthermore, if A is semiprime and δ is exact, then I^1 is $(d-1)$-pure.*
5. *If A is AG, then I^d is a direct sum of injective hulls of* 0*-critical modules, hence is essentially* 0*-pure.*

2. Localization of Auslander–Gorenstein and Cohen–Macaulay rings

In this section, we study localizations of Auslander–Gorenstein and Cohen–Macaulay rings. The final result shows that the Cohen–Macaulay condition implies the Auslander–Gorenstein condition if $\operatorname{injdim} A \leq 3$.

Note that if S is a multiplicatively closed Ore set of regular elements in a right noetherian ring A, and M is a right noetherian A-module, then it follows from (1.1) that

$$j(M_A) \leq j(MS^{-1}_{AS^{-1}}). \tag{2.1}$$

Levasseur [9] notes that the Auslander–Gorenstein property is preserved when factoring out by a normal regular element; it is also preserved under arbitrary localizations.

Proposition 2.1. *Let A be an AG ring. If S is a multiplicatively closed Ore set of regular elements in A, then AS^{-1} is AG; and if g is a regular normal element of A, then $A/(g)$ is AG.*

Proof. By (1.1), for any AS^{-1}-submodule L of $\operatorname{Ext}^i(MS^{-1}, AS^{-1})$, there is a submodule K of $\operatorname{Ext}^i(M,A)$ such that $L = S^{-1}K$. For all $j < i$, $\operatorname{Ext}^j(L, AS^{-1}) = \operatorname{Ext}^j(K,A) \otimes AS^{-1} = 0$ by the Auslander condition for A, so AS^{-1} is AG.

Let M be a right $A/(g)$-module. For every left $A/(g)$-module $N \subset \operatorname{Ext}^i_{A/(g)}(M, A/(g)) \cong \operatorname{Ext}^{i+1}_A(M^\sigma, A)$, we have

$$\operatorname{Ext}^{j+1}_A({}^\sigma N, A) \cong \operatorname{Ext}^{j+1}(N, {}^{\sigma^{-1}}A) \cong \operatorname{Ext}^{j+1}(N, A) = 0$$

for all $j < i$, by the Auslander condition on A. By Rees' Lemma, $\operatorname{Ext}^j_{A/(g)}(N, A/(g)) = 0$ for $j < i$, so $A/(g)$ is AG. $\square$

Lemma 2.2. *Let A be AG and S a multiplicatively closed subset of regular normal elements.*

1. *If every simple A-module M with $j(M) = \operatorname{injdim} A$ is S-torsion (equivalently, $Mg = 0$ for some $g \in S$), then $\operatorname{injdim} AS^{-1} < \operatorname{injdim} A$.*
2. *If M is a noetherian S-torsion-free critical A-module, then $j(MS^{-1}) = j(M)$.*

Proof. 2. Write $s = j(M)$; by Proposition 1.4.2, there is an injective map $M \to \Omega_s$, hence an injective map $MS^{-1} \to \Omega_s S^{-1}$. By Proposition 1.4.1 applied to MS^{-1}, $j(MS^{-1}) \le s = j(M)$, so we obtain equality by (2.1).

1. If $d = \operatorname{injdim} A$, then $\operatorname{injdim} AS^{-1} \le d$ by (1.2). Since AS^{-1} is AG, it suffices to show that $j(H) < d$ for all non-zero AS^{-1}-modules H. Suppose to the contrary that $j(H) = d$. By [2, 1.3], H is artinian, so we may assume that H is simple. If M is a finitely generated critical A-submodule of H, then $H = MS^{-1}$ because H is simple. By Proposition 1.3, $j(M) = j(H) = d$, so M is artinian by [2, 1.3]. Therefore the hypotheses ensure that $Mg = 0$ for some $g \in S$, whence $0 = MS^{-1} = H$, which is a contradiction. $\square$

Next, we show that the CM condition is preserved under certain kinds of normal localizations. A finite dimensional subspace $W \subset A$ is a *subframe* if $1 \in W$. We say $x \in A$ is a *local* normal element if for every subframe $W \subset A$, there is a subframe $W' \supset W$ such that $xW' = W'x$ [8, p. 209]. Trivially, every central element is local. By [8, Theorem 2], a multiplicatively closed subset S of A, consisting of regular local normal elements, is Ore and $\operatorname{GK\,dim}(AS^{-1}) = \operatorname{GK\,dim} A$. The next lemma says that this statement is also true for modules.

Lemma 2.3. *Let S be a multiplicatively closed subset of A consisting of regular local normal elements. If M is an A-submodule of an AS^{-1}-module N such that $N = MS^{-1}$, then $\operatorname{GK\,dim}(N_{AS^{-1}}) = \operatorname{GK\,dim} N_A = \operatorname{GK\,dim} M_A$.*

Proof. Since we are working with GK dim, we may assume that both M and N are finitely generated. Let $V \subset M$ be a generating set of M as an A-module (so it is also a generating set of N as an AS^{-1}-module). Every subframe of AS^{-1} is generated by $W \cup \{t^{-1} | t \in T\}$ where W is a subframe of A and where T is a finite subset of S. Let $x = \prod_{t \in T} t$ (so $x \in S$). Let W' be a subframe of A containing W and x such that $xW' = W'x$. Every subframe of AS^{-1} is generated by $W' \cup \{x^{-1}\}$. Hence

$$\operatorname{GK\,dim}(N) = \sup_{W,x} \overline{\lim_{n\to\infty}} \log_n \{\dim(V(W \cup \{x^{-1}\})^n)\}$$

and

$$\operatorname{GK}\dim M = \sup_W \overline{\lim_{n\to\infty}} \log_n \{\dim(VW^n)\},$$

where $x \in S$ and W is a subframe of A such that $xW = Wx$. Note that the dimension of a vector space over k is always denoted by dim. It is clear that $\operatorname{GK}\dim(N_{AS^{-1}}) \geq \operatorname{GK}\dim M_A$. By direct computation, we have

$$V(W \cup \{x^{-1}\})^n \cdot x^n \subset VW^{2n}.$$

Hence $\operatorname{GK}\dim(N_{AS^{-1}}) \leq \operatorname{GK}\dim M_A$. $\square$

Theorem 2.4. *Let A be an AG ring, and S a multiplicatively closed set of local normal elements in A. If A is CM, so is AS^{-1}.*

Proof. Since A is CM, GK-dimension differs from the canonical dimension by a constant, so is exact and finitely partitive on A-modules, and is therefore exact and finitely partitive on AS^{-1}-modules by Lemma 2.3. By the noetherian hypothesis, to prove that AS^{-1} is CM, i.e., $\operatorname{GK}\dim N + j(N) = \operatorname{GK}\dim AS^{-1}$ for all noetherian AS^{-1}-modules N, it suffices to show this equality for some non-zero AS^{-1}-submodule $N' \subset N$. Let M be a critical A-submodule of N. Then $j(M) = j(MS^{-1})$ by Lemma 2.2, and $\operatorname{GK}\dim M = \operatorname{GK}\dim MS^{-1}$ by Lemma 2.3. Let $N' = MS^{-1}$; we have

$$\operatorname{GK}\dim N' + j(N') = \operatorname{GK}\dim M + j(M) = \operatorname{GK}\dim A = \operatorname{GK}\dim AS^{-1}.$$

Therefore AS^{-1} is CM. $\square$

Theorem 2.5 (Ajitabh et al. [2, 6.2]). *A noetherian CM ring with finite injective dimension has a QF quotient ring.*

We finish this section by showing that the CM property implies the AG property in low dimensions. We will use Ischebeck's spectral sequence: if A is noetherian with $\operatorname{injdim} A = d$, and M a noetherian right A-module, there is a convergent spectral sequence

$$E_2^{pq} = \operatorname{Ext}_A^p(\operatorname{Ext}_A^q(M,A),A) \Rightarrow \mathbb{H}^{p-q}(M) := \begin{cases} 0 & \text{if } p \neq q, \\ M & \text{if } p = q. \end{cases} \tag{2.2}$$

Thus, on the E_∞-page only the diagonal terms can be non-zero. To simplify notation later, we have used a non-standard indexing of E_2^{pq}; with our indexing, the boundary maps on the E_2-page are $E_2^{pq} \to E_2^{p+2,q+1}$.

Proposition 2.6. *Let A be a noetherian ring which is CM with respect to some dimension function. Then A is AG if*

1. $\operatorname{injdim} A \leq 2$, *or*
2. $\operatorname{injdim} A = 3$ *and A has a QF quotient ring.*
3. $\operatorname{injdim} A = 3$ *and A is CM (with respect to GK dim).*

Proof. First, recall that if A is CM with respect to a dimension function ∂, then ∂ and δ are both exact and differ by a constant.

1. For $\operatorname{injdim} A < 2$, the proof is easy, so we assume that $\operatorname{injdim} A = 2$. By the spectral sequence (2.2), the E_2 table for M is

$$\begin{array}{ccc} E^{00} & E^{10} & E^{20} \\ E^{01} & E^{11} & E^{21} \\ E^{02} & E^{12} & E^{22} \end{array}$$

with $E^{12} = E^{02} = E^{10} = E^{20} = 0$ and $E^{01} \hookrightarrow E^{22}$ and $E^{00} \twoheadrightarrow E^{21}$. Hence $\delta(\operatorname{Ext}^2(M,A)) \le 0$ for all M. Since δ is a dimension function, $\delta(N) \le 0$ for all $N \subset \operatorname{Ext}^2(M,A)$. Now E^{01} is a submodule of E^{22} and $\delta(E^{22}) \le 0$, hence $\delta(E^{01}) \le 0$. If $E^{01} \ne 0$, then $\operatorname{Ext}^2(E^{01},A) \ne 0$. But this contradicts the fact $E^{20}(\operatorname{Ext}^1(M,A)) = 0$. Hence $E^{01} = 0$ and $\delta(\operatorname{Ext}^1(M,A)) \le 1$. Since δ is a dimension function, $\delta(N) \le 1$ for all $N \subset \operatorname{Ext}^1(M,A)$. Therefore A is AG.

2. By the spectral sequence (2.2), the E_2 table for M now looks like

$$\begin{array}{cccc} E^{00} & E^{10} & E^{20} & E^{30} \\ E^{01} & E^{11} & E^{21} & E^{31} \\ E^{02} & E^{12} & E^{22} & E^{32} \\ E^{03} & E^{13} & E^{23} & E^{33} \end{array}$$

with $E^{13} = E^{03} = E^{30} = E^{20} = 0$, $E^{02} \twoheadrightarrow E^{23}$, and $E^{12} \hookrightarrow E^{33}$. Since $Q = \operatorname{Fract} A$ is self-injective, $\operatorname{Ext}^i(M,A) \otimes_A Q = \operatorname{Ext}^i(M \otimes Q, Q) = 0$ if $i > 0$. Hence, by Proposition 1.4.4, $E^{0i} := \operatorname{Hom}(\operatorname{Ext}^i(M,A),A) = 0$ if $i > 0$. Therefore $E^{23} = 0$ and it follows that $\delta(E^3(M)) \le 0$, whence $\delta(E^{12}) \le 0$. If $E^{12} \ne 0$, then $E^{312} := \operatorname{Ext}^3(E^{12},A) \ne 0$; but $E^{312} = E^{31}(\operatorname{Ext}^2(M,A)) \cong E^{10}(\operatorname{Ext}^2(M,A)) = \operatorname{Ext}^1(E^{02},A) = 0$, a contradiction. Hence $E^{12} = 0$. Combining all these, we have proved that $\delta(\operatorname{Ext}^i(M,A)) \le 3 - i$ for all i. Since δ is a dimension function, we also have $\delta(N) \le 3 - i$ for every submodule $N \subset \operatorname{Ext}^i(M,A)$, so A is AG.

3. Follows from part 2 and Theorem 2.5. □

3. Purity for ungraded algebras

In this section we examine purity questions for some noetherian domains having injective dimension ≤ 3. First, we recall the following result.

Theorem 3.1 (Ajitabh et al. [2, 4.2]). *An Auslander–Gorenstein, Cohen–Macaulay ring satisfying a polynomial identity, has a pure injective resolution.*

By Proposition 1.4, every AG ring of injective dimension ≤ 1 has an essentially pure injective resolution.

We now consider what happens for injective dimension 2. As [2, 5.5] shows, there are noetherian domains of injective dimension 2 which do not have a pure injective resolution. But we do not know if every noetherian domain with $\operatorname{gldim} A = \operatorname{GK}\dim A = 2$ has a pure injective resolution.

Proposition 3.2. *Let A be an AG ring with* $\operatorname{injdim} A = 2$. *If A has a QF quotient ring, then A has an essentially pure injective resolution.*

Proof. By Proposition 1.4, I^0 is essentially 2-pure, I^1 is essentially 1-pure, and I^2 is essentially 0-pure. $\square$

Next, we show that this 'essential purity' result can be improved to a purity result when there is a plentiful supply of normal elements.

Let A be a noetherian domain. Two normal elements g_1, g_2 are *equivalent* if there is a unit $u \in A$ such that $g_1 = ug_2$. A normal element $g \in A$ is *completely prime* if $A/(g)$ is a domain.

Notational Remark. Whenever $S_g = A - gA$ is a right and left Ore set of regular elements we write

$$A_{(g)} := AS_g^{-1} = S_g^{-1}A,$$

we will drop the subscript g from S_g if there is no confusion. If N is a set of normal regular elements, then there is an Ore set consisting of 1 and the products of the elements of N; by abuse of notation we denote the localization of A with respect to this set by AN^{-1}. Notice that A is a subring of $A_{(g)}$ and AN^{-1}.

Lemma 3.3. *Let A be a noetherian domain and write $Q = \operatorname{Fract} A$.*

1. *For every completely prime normal element g, $S_g = A - gA$ is an Ore set and $A_{(g)}$ is a local algebra with* $\operatorname{gldim} A_{(g)} = 1$.

2. *Let $N \subset A$ be a set of inequivalent completely prime normal elements, and write $B = AN^{-1}$. Let ∂ be an exact dimension function on A-modules. Then there is an exact sequence*

$$0 \to A \to Q \to Q/B \bigoplus \bigoplus_{g \in N} Q/A_{(g)} \to E \to 0, \tag{3.1}$$

where E is a module with $\partial(M) \le \partial(A) - 2$ for all finitely generated submodules $M \subset E$.

Remark. The graded version of Lemma 3.3 also holds. In fact, [1, Section 2] proved the following: let A be a connected graded noetherian domain and let $g \in A$ be a completely prime homogeneous element of positive degree; then the set S of homogeneous elements in $A - gA$ is an Ore set, and $A_{(g)} := S^{-1}A$ is a $\mathbb{Z}$-graded local ring with graded global dimension 1. There is also a graded version of part 2.

Proof. 1. The proof in [1, Section 2] works in the ungraded case.

2. To prove that (3.1) is exact, it suffices to show that $B \cap_{g \in N} A_{(g)} \subset A$. For every $y \in B \cap_{g \in N} A_{(g)}$, we can write $y = a g_1^{-1} \cdots g_n^{-1}$ where $a \in A$ and $g_i \in N$; we may assume that $a \notin g_1 A$ by assuming n is minimal. Since $y \in A_{(g_1)}$, we can write y as $s^{-1}b$ where $b \in A$ and $s \in A - g_1 A$. Hence $sa = bg_n \cdots g_1 \in g_1 A$, which contradicts the fact that A/g_1A is a domain, unless $n = 0$ and $y = a \in A$. Hence (3.1) is exact.

Now we prove the last statement in part 2. Denote by N^q ($q \geq 0$) the set consisting of products of q elements of N, where we also put $N^0 = \{1\}$. Observe that every element $x \in A$ can be written as $x = yz$ where $y \in N^q$ (for a unique value of q) and $z \in \bigcap_{g \in N} S_g$. This and the fact that elements of N are completely prime imply that $(Q/A_{(g)}) \otimes_A B = 0$ and $(Q/B) \otimes_A A_{(g)} = 0$ for all $g \in N$, and $(Q/A_{(g)}) \otimes_A A_{(g')} = 0$ for two inequivalent elements $g, g' \in N$. Now tensoring the exact sequence (3.1) with B and $A_{(g)}$, we see that $E \otimes_A B = E \otimes_A A_{(g)} = 0$ for all $g \in N$. For every finitely generated submodule M of E, $M \otimes_A B = 0$. Hence there is a $y \in N^q$ such that $My = 0$. By replacing M by its Kdim-critical subquotients, we may assume that $Mg = 0$ for some $g \in N$. But $M \otimes A_{(g)} = 0$ implies that $Ms = 0$ for some $s \in A - gA$. Hence M is a quotient of $A/(gA + sA)$ which has ∂-dimension at most $\partial(A) - 2$. $\square$

As a consequence of Lemma 3.3, we obtain the following.

Proposition 3.4. *Let A be an AG domain of* $\operatorname{injdim} A = 2$, *and suppose that every simple module M with $\delta(M) = 0$ is annihilated by some completely prime normal element of A. Then A has a pure injective resolution.*

Proof. Let $0 \to A \to I^0 \to I^1 \to I^2 \to 0$ be a minimal injective resolution. By Proposition 1.4, I^0 and I^1 are pure and I^2 is essentially pure, so it remains to show that I^2 is pure, or equivalently, $\delta(M) = 0$ for all finitely generated $M \subset I^2$. Let N be the set of all non-equivalent completely prime normal elements. By Lemma 2.2, $B = AN^{-1}$ has injective dimension ≤ 1. By Lemma 3.3, $\operatorname{injdim} A_{(g)} = 1$ whenever $g \in N$. Hence Q/B and $Q/A_{(g)}$ are injective modules over B and $A_{(g)}$ respectively. By [7, 9.16], these are injective A-modules, so

$$0 \to A \to Q \to Q/B \bigoplus \bigoplus_{g \in N} Q/A_{(g)} \to I^2 \to 0$$

is an injective resolution of A. Thus $E \cong I^2$ (see (3.1)), and purity follows from Lemma 3.3. $\square$

Next, we describe some domains to which Proposition 3.4 applies. Let A be a domain, generated by two elements x, y subject to a relation of degree two, say $ax^2 + bxy + cyx + dy^2 + ex + fy + g = 0$, with $(a, b, c, d) \neq (0, 0, 0, 0)$. We assume that k is algebraically closed from now until Corollary 3.6. By changing variables, the relation can be put in one of the following forms:

(1) $xy - qyx$, where $q \neq 0$,
(2) $xy - qyx - 1$, where $q \neq 0$,
(3) $xy - yx - x^2$,
(4) $xy - yx - x^2 - 1$,
(5) $xy - yx - x$.

We denote by R_i the algebra subject to the relation (i). If $q = 1$, then R_1 is the commutative polynomial ring and R_2 is the first Weyl algebra, both of which have pure injective resolutions by Theorem 3.1 and Proposition 1.4. So we further assume that $q \neq 1$ in cases 1 and 2.

Proposition 3.5. *R_i is Auslander regular, CM, and has a pure injective resolution* ($i = 1, \dots, 5$).

Proof. If we filter R_i in the obvious way by defining $\deg x = 1$ and $\deg y = 2$ then the associated graded algebra is isomorphic to either $k_q[x, y]$ or $k[x, y]$. Therefore, by [16, 4.4], R_i is an Auslander regular, Cohen–Macaulay, noetherian domain of GK dim 2.

If R_i satisfies a polynomial identity the result is given by Theorem 3.1. So suppose R_i is not PI: thus, in cases (1) and (2) we assume q is not a root of 1, and in cases (3)–(5) we assume that $\operatorname{char} k = 0$. We will show that R_i has at most two completely prime normal elements, and every simple R_i-module of GK dim 0 is annihilated by one of them; the result will then follow from Proposition 3.4. Because the annihilator of a GK dim-zero simple module is a non-zero prime ideal it suffices to show that every non-zero prime ideal contains a completely prime normal element. We proceed case-by-case.

Case 1: $R_1 = k\langle x, y\rangle/(xy - qyx)$. It is easy to see that x and y are the only completely prime normal elements of R_1. We will show that every non-zero prime ideal contains either x or y. Suppose I is a non-zero prime ideal of R_1 such that I does not contain x and y. Consider the conjugation by x (i.e., the map $a \mapsto xax^{-1}$), which is determined by $x \mapsto x$ and $y \mapsto qy$. Let $f(x, y)$ be an element in I with minimal degree in y. Then $xf(x, qy) = f(x, y)x \in I$. Since x is normal and $x \notin I$, $f(x, qy) \in I$. Hence $f(x, qy) - q^d f(x, y)$ (where $d = \deg_y(f(x, y))$) has lower degree in y. By the choice of $f(x, y)$, $f(x, qy) = q^d f(x, y)$ which implies that $f(x, y)$ is of the form $g(x)y^d$. If $d \neq 0$, then $g(x) \in I$ because $y \notin I$ and I is prime. Repeating the same argument for $g(x)$, this time using conjugation by y, we obtain that $g(qx) = q^d g(x)$ if $g(x)$ has a minimal degree in x among all such elements in I. Hence $g(x) = cx^d$. But x is normal and I is prime, thus $x \in I$, which is a contradiction. As a consequence, note that every non-zero ideal of R_i contains $x^i y^j$ for some i, j.

Case 2: The only completely prime normal element in R_2 is $g := xy - yx$. We will prove that every non-zero prime ideal contains g. The subalgebras $B_1 = k\langle x, g\rangle$ and $B_2 = k\langle y, g\rangle$ are skew polynomial rings of the type examined in Case 1. The algebra R_2 is $\mathbb{Z}$-graded if $\deg(x) = 1$ and $\deg(y) = -1$. Let I be a non-zero prime ideal of R_2 which does not contain g. Since $ga_i = q^i a_i g$, for all a_i of degree i, I is graded. Hence $I \cap B_1$ is not zero. By Case 1, every non-zero ideal of B_1 contains $x^i g^j$ for some i, j. Hence I contains x^i because $g \notin I$ and I is prime. By the induction on i and the relation $xy - qyx = 1$, we obtain $1 \in I$, so $I = R_2$. This is a contradiction.

Cases 3,4,5: We consider the algebra $R = k\langle x, y\rangle/(yx - xy - r(x))$ for some polynomial $r(x)$. Since k is algebraically closed, $r(x) = \prod_i (x - a_i)^{t_i}$. It is easy to check that $x - a_i$ are completely prime normal elements. We claim that every non-zero prime ideal I of R contains some $x - a_i$. If not, then conjugating by $x - a_i$, we can show (as we did in Case 1) that I contains a polynomial $f(x)$. Hence I contains $yf(x) - f(x)y = f'(x)r(x)$. Since $r(x)$ is normal in R and I contains no $x - a_i$, I contains $f'(x)$. By induction on $\deg f(x)$, we obtain $I = R$, a contradiction. □

Corollary 3.6. *The enveloping algebra of a two-dimensional Lie algebra over an algebraically closed field has a pure injective resolution.*

Proof. Such a Lie algebra is either abelian or solvable, so its enveloping algebra is isomorphic to either R_1 (with $q=1$) or R_5. $\square$

Next we study algebras with injective dimension 3. First we need a lemma.

Lemma 3.7. *Let A be a right noetherian ring with* $\operatorname{injdim} A_A = d$, *and let M be a finitely generated uniform module. Then M embeds in Ω_i if and only if for every $0 \neq N \subset M$, the natural map $\operatorname{Ext}^i(M,A) \to \operatorname{Ext}^i(N,A)$ is non-zero. In particular, M does not embed in I^i if and only if* $\lim_{N \subset M} \operatorname{Ext}^i(N,A) = 0$.

Proof. ($\Rightarrow$) This is trivially true if $i=0$, so suppose that $i>0$. By [2, 2.1.1], $\operatorname{Ext}^i(M,A) = \operatorname{Ext}^1(M,\Omega_{i-1})$. If M embeds in Ω_i, then there is a nonsplit exact sequence

$$0 \longrightarrow \Omega_{i-1} \longrightarrow E \longrightarrow M \longrightarrow 0. \tag{3.2}$$

Let $0 \neq f \in \operatorname{Ext}^1(M,\Omega_{i-1})$ correspond to this extension. If $0 \neq N \subset M$, and $E' \subset E$ is the preimage of N under (3.2), then the image of f under that natural map $\operatorname{Ext}^i(M,A) \to \operatorname{Ext}^i(N,A)$ corresponds to the extension $0 \longrightarrow \Omega_{i-1} \longrightarrow E' \longrightarrow N \longrightarrow 0$. This is still non-split since Ω_{i-1} is essential in E, and hence in E'. Therefore $\operatorname{Ext}^i(M,A) \to \operatorname{Ext}^i(N,A)$ is non-zero.

($\Leftarrow$) Conversely if M does not embed in Ω_i, then every map from M to Ω_i has a non-zero kernel. By [2, 2.2], there exist $f_1,\ldots,f_n \in \operatorname{Hom}(M,\Omega_i)$ and submodule $N := \cap \ker(f_j) \subset M$ such that the natural map $\operatorname{Ext}^i(M,A) \to \operatorname{Ext}^i(N,A)$ is zero. But each $\ker(f_j)$ is non-zero and M is uniform, so $N = \cap \ker(f_j)$ is non-zero. $\square$

Proposition 3.8. *Let A be an AG ring, and let N be a set of regular normal elements in A. Then A has an essentially pure injective resolution if and only if AN^{-1} and $A/(g)$, for all $g \in N$, do.*

Proof. ($\Leftarrow$) We suppose that I^i is not essentially $(d-i)$-pure, and seek a contradiction. By Proposition 1.4.1, I^i contains a critical submodule M with $\delta(M) > d-i$ or $j(M) < i$. If $MN^{-1} \neq 0$, then MN^{-1} is a submodule of I^iN^{-1}, and $j(MN^{-1}) = j(M) < i$ by Lemma 2.2.2, so AN^{-1} does not have an essentially pure injective resolution, a contradiction. If $MN^{-1} = 0$, then $Lg = 0$ for some $g \in N$ and some $0 \neq L \subset M$; since g is regular, $\delta(L) < d$, so $j(L_A) > 0$; by Lemma 3.7 and Proposition 1.3,

$$\lim_{N \subset L} \operatorname{Ext}^{i-1}_{A/(g)}(N, A/(g)) \cong \lim_{N \subset L} \operatorname{Ext}^i_A(N,A) \neq 0.$$

So, by Lemma 3.7 again, $L_{A/(g)}$ is contained in the I^{i-1}-term of a minimal injective resolution of $A/(g)$. But $j(L_{A/(g)}) = j(L_A) - 1 < i-1$, so $A/(g)$ does not have an essentially pure injective resolution, a contradiction.

($\Rightarrow$) The proof of the converse also splits into two cases: either AN^{-1} does not have an essentially pure injective resolution, or some $A/(g)$ does not have an essentially pure injective resolution. In each case, the argument in the previous paragraph works in the reverse direction. □

Proposition 3.9. *Let A be an AG domain with* $\operatorname{injdim} A = 3$. *Suppose every simple module of grade* 3 *is annihilated by a non-zero completely prime normal element. Then A has an essentially pure injective resolution.*

Proof. Let N be the set of all inequivalent completely prime normal elements of A. By Lemma 2.2.1, and Proposition 1.3, AN^{-1} and $A/(g)$ $(g \in N)$ have injective dimension <3. Since AN^{-1} and $A/(g)$ are AG domains they have essentially pure injective resolutions by Proposition 3.2, so the result follows from Proposition 3.8. □

Proposition 3.9 applies to enveloping algebras of three-dimensional Lie algebras over an algebraically closed field k. Such a Lie algebra L is isomorphic to one of the following:

(1) $L_1 = L_{ab} = kx + ky + kz$ with $[L_{ab}, L_{ab}] = 0$.
(2) $L_2 = \mathfrak{sl}_2 = ke + kf + kh$ with $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$.
(3) $L_3 = kx + ky + kz$ with $[x, y] = z$, $[x, z] = [y, z] = 0$.
(4) $L_4 = kx + ky + kz$ with $[x, y] = y$, $[x, z] = [y, z] = 0$.
(5) $L_5 = kx + ky + kz$ with $[x, y] = 0$, $[x, z] = bx$, $[y, z] = y$, where $b \neq 0$.
(6) $L_6 = kx + ky + kz$ with $[x, y] = 0$, $[x, z] = x + y$, $[y, z] = y$.

Lemma 3.10. *Let k be an algebraically closed field with* $\operatorname{char} k = 0$, *and A a k-algebra.*

1. *If g is a central element of A, then for every finite-dimensional simple A-module M, there is an $a \in k$ such that $M(g - a) = 0$.*

2. *If $y \in A$ is normal and $yz - zy = y$ for some $z \in A$, then y annihilates every finite-dimensional simple A-module.*

Proof. Let M be a finite-dimensional simple A-module.

1. Consider g as a k-linear map of M. By the Cayley–Hamilton theorem there is a polynomial f such that $Mf(g) = 0$. But k is algebraically closed, so $M(g - a) = 0$ for some $a \in k$.

2. There is a polynomial f (and we can assume that the degree of f is minimal) such that $Mf(y) = 0$. Then $M(f(y)z - zf(y)) = 0$, and $fz - zf = yf'(y)$. Since y is normal, $Mf'(y) \neq 0$, and M is simple, therefore $Mf'(y) = M$. Then $My = Mf'(y)y = Myf'(y) = 0$. □

Theorem 3.11. *Over an algebraically closed field, the enveloping algebra of a three-dimensional Lie algebra has an essentially pure injective resolution.*

Proof. If $\operatorname{char} k > 0$, then $U(L)$ satisfies a polynomial identity, so has pure injective resolution by Theorem 3.1. We now assume that $\operatorname{char} k = 0$.

We check that $U(L_i)$ satisfies the conditions in Proposition 3.9. The universal enveloping algebras are known to be Auslander regular and CM. Hence $\operatorname{GK} \dim M = \delta(M)$

for all finitely generated modules. Now to verify the last condition in Proposition 3.9, it is sufficient (in view of Lemma 3.10) to check that, in $A = U(L_i)$, there is a central element g such that $A/(g-a)$ is a domain for all $a \in k$, or there is a completely prime normal element y and an element z such that $yz - zy = y$. We check case-by-case. Case 1 is trivial, since then $U(L)$ is a commutative polynomial ring. In case 2, $U(\mathfrak{sl}_2)/(\Omega - a)$ is a domain, where Ω is the Casimir element. In case 3, z is central element and $U(L_3)/(z-a)$ is either the first Weyl algebra or the polynomial ring, hence a domain. In case 4, z is central and $U(L_4)/(z-a)$ is isomorphic to the algebra R_5 in the previous section, which is a domain. In cases 5 and 6, y is a completely prime normal element and satisfies the condition $yz - zy = y$. $\square$

4. Purity for graded algebras

We now study the purity of the minimal graded injective resolution for some connected graded algebras of injective dimension ≤ 4. Unless otherwise specified, all modules, rings, and operations are graded and homomorphisms preserve the degree.

There are graded versions of the concepts and results appearing in Sections 1 and 2 for ungraded rings and modules; for the most part these are obvious, and can be obtained by adding the word 'graded' in the appropriate places. There is a notion of minimal *graded* injective resolution of A, and we still denote it by (0.1). By [9, 3.3], for a connected algebra, the graded injective and global dimensions equal the ungraded injective and global dimensions, respectively. When A is graded, and M and N are graded modules with M finitely generated, $\mathrm{Ext}^i_A(M,N)$ has a natural grading; and in this case we denote it by $\underline{\mathrm{Ext}}^i_A(M,N)$.

We recall the basic facts. Let A be a connected graded k-algebra. The linear dual $A^* := \bigoplus_n \mathrm{Hom}_k(A_{-n}, k)$ is an injective hull of the trivial module k. We say that an A-module M is $\mathfrak{m}$-torsion (where $\mathfrak{m} = \bigoplus_{i>0} A_i$), if M is a union of finite-dimensional submodules. We use $M[l]$ to denote the shift of M by degree l; thus, $M[l] = M$ as an A-module, but the grading is defined by $M[l]_i = M_{l+i}$. Every $\mathfrak{m}$-torsion injective module is a (possibly infinite) direct sum of shifts of A^*. As a consequence, an essential extension of an $\mathfrak{m}$-torsion module is $\mathfrak{m}$-torsion. If A is connected and AG with $\operatorname{injdim} A = d$, then by [9, 6.3] A is Artin–Schelter–Gorenstein, and by [20, 0.3(3)], I^d is isomorphic to $A^*[l]$ for some $l \in \mathbb{Z}$. In particular, I^d is 0-pure.

The GK-dimension of a module over a CM ring is an integer. By [9, 3.1], if A is graded AG, then A is ungraded AG. Also, by [9, 5.8], if A is graded AG and CM then A is ungraded CM. Finally, note that if A is CM with $\operatorname{GK}\dim A = \operatorname{injdim} A$, then $\operatorname{GK}\dim M = \delta(M)$ for all finitely generated modules M.

Proposition 4.1. *Let A be a connected graded CM algebra with* $\operatorname{GK}\dim A = \operatorname{injdim} A = 2$. *Then A is AG and has a pure graded injective resolution.*

Proof. By the graded version of Proposition 2.6.1, A is graded AG. Hence, by [9, 3.1 and 5.8], A is AG and CM as an ungraded ring. Let $0 \longrightarrow A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow 0$

be the minimal graded injective resolution of A. By Proposition 1.4, I^0 is 2-pure, and I^1 is essentially 1-pure; so it suffices to show that $\mathrm{GK}\dim I^1 \le 1$. By Theorem 2.5, A has a QF quotient ring Q, and I^0 is the graded quotient ring of A, which embeds in Q, so $\mathrm{GK}\dim I^0/A \le \mathrm{GK}\dim Q/A \le 1$. Thus $\mathrm{GK}\dim I^1 = \max\{\mathrm{GK}\dim I^0/A, \mathrm{GK}\dim I^2\} \le 1$. $\square$

Theorem 4.2. 1. *Let A be a connected graded CM algebra with* $\mathrm{GK}\dim A = \mathrm{injdim}\, A = 3$. *Suppose that A contains a homogeneous regular normal element g of positive degree.*

(a). *A is AG and has an essentially pure graded injective resolution.*

(b). *If $A/(g)$ is a domain, then A has a pure graded injective resolution if and only if $A[g^{-1}]$ does.*

2. *The three-dimensional Artin–Schelter regular algebras which are generated by three elements of degree one over an algebraically closed field have pure graded injective resolutions.*

Proof. 1(a). By Proposition 2.6.3, A is AG. By [9, 5.10], $A/(g)$ is AG and CM of injective dimension 2. By Proposition 4.1, $A/(g)$ has a pure graded injective resolution. By Theorem 2.5, A has a QF quotient ring, say Q. Note that $Q = \mathrm{Fract}\, A[g^{-1}]$ too. By a graded version of Lemma 2.6.1, $\mathrm{injdim}\, A[g^{-1}] \le 2$, and by Proposition 2.5, $A[g^{-1}]$ is AG. By a graded version of Proposition 3.2, $A[g^{-1}]$ has an essentially pure graded injective resolution. Thus, by a graded version of Proposition 3.8, A has an essentially pure graded injective resolution.

1(b). Since $A/(g)$ is a domain and g is regular, A is a domain. Write Q_{gr} for the graded ring of fractions of A. By part 1(a), the minimal graded injective resolution of A is essentially pure; let

$$0 \longrightarrow A \longrightarrow Q_{\mathrm{gr}} \longrightarrow I^1 \longrightarrow I^2 \longrightarrow I^3 \longrightarrow 0$$

be this resolution. The last term I^3 is $A^*[l]$ which is pure of GK dim zero. Since $I^1 \otimes_A Q_{\mathrm{gr}} = 0$, $\mathrm{GK}\dim I^1 \le 2$, so I^1 is pure. Hence A has a pure injective resolution if and only if $\mathrm{GK}\dim I^2 \le 1$.

The graded minimal injective resolution of $A[g^{-1}]$ is

$$0 \longrightarrow A[g^{-1}] \longrightarrow Q_{\mathrm{gr}} \longrightarrow I^1[g^{-1}] \longrightarrow I^2[g^{-1}] \longrightarrow 0 \tag{4.1}$$

and this is essentially pure by the proof of part 1(a). Since g is a homogeneous normal element and A is locally finite, g is *local* normal, whence $A[g^{-1}]$ is AG and CM (Theorem 2.4). Therefore $\mathrm{GK}\dim I^1[g^{-1}] = 2$. Now, $I^2[g^{-1}]$ is the injective hull of a module having GK-dimension one. Thus $A[g^{-1}]$ has a pure injective resolution if and only if $\mathrm{GK}\dim I^2[g^{-1}] = 1$.

Since $I^i[g^{-1}]$ is an injective $A[g^{-1}]$-module, it is an injective A-module. Therefore, $I^1[g^{-1}]$ is the injective hull of $Q_{\mathrm{gr}}/A[g^{-1}]$ both as an $A[g^{-1}]$-module and as an A-module. By Lemma 3.3.1, $\mathrm{gldim}\, A_{(g)}=1$, so $Q_{\mathrm{gr}}/A_{(g)}$ is a graded injective $A_{(g)}$-module, and hence a graded injective A-module. By the graded version of Lemma 3.3, there is an exact sequence

$$0 \longrightarrow A \longrightarrow Q_{\mathrm{gr}} \longrightarrow Q_{\mathrm{gr}}/A[g^{-1}] \bigoplus Q_{\mathrm{gr}}/A_{(g)} \longrightarrow E \longrightarrow 0. \tag{4.2}$$

Thus $I^1 \cong Q_{\mathrm{gr}}/A_{(g)} \oplus I^1[g^{-1}]$. Lemma 3.3 also gives $\operatorname{GK}\dim E \le 1$.

Consider the cosyzygy modules $\Omega_s = \ker(I^s \to I^{s+1})$. From (4.1) and (4.2) we obtain an exact sequence $0 \to E \to \Omega_2 \to I^2[g^{-1}] \to 0$. Now,

$$\begin{aligned}\operatorname{GK}\dim I^2 &= \max\{\operatorname{GK}\dim \Omega_2, \operatorname{GK}\dim I^3\} = \operatorname{GK}\dim \Omega_2\\ &= \max\{\operatorname{GK}\dim E, \operatorname{GK}\dim I^2[g^{-1}]\}.\end{aligned}$$

Thus $\operatorname{GK}\dim I^2 \le 1$ if and only if $\operatorname{GK}\dim I^2[g^{-1}] \le 1$, from which the result follows.

2. If A is PI, it has a pure graded injective resolution by Theorem 3.1. Suppose A is not PI. If A is an elliptic algebra, the result is proved in [1, 3.2]. If A is not elliptic, then there is a completely prime normal element $g \in A_1$ by [ATV]. By [19, 2.12], replacing A by a suitable twist by an automorphism, g becomes central. Since twisting is a category equivalence [19, 3.1], twisting preserves the minimal injective resolution. By [19, 5.7] twisting preserves GK-dimension, and hence the purity of the injective resolution A. Hence we may assume that g is central. Since A is generated by three elements in degree 1 and is defined by three relations of degree 2, $A[g^{-1}]$ is isomorphic to $B_0[g, g^{-1}]$ where B_0 is a domain generated by two elements and defined by one relation. By Proposition 3.5 and the discussion prior to it, B_0 has a pure injective resolution. By adjoining $[g, g^{-1}]$ we obtain a pure graded injective resolution of $A[g^{-1}] = B_0[g, g^{-1}]$. By part 1(b), A has a pure graded injective resolution. □

By homogenizing $U(\mathfrak{sl}_2)/(\Omega)$ in [2, 5.5], we obtain a connected, graded AG and CM ring A, with $\operatorname{GK}\dim A = \operatorname{injdim} A = 3$, which does not have a pure graded injective resolution. We conjecture that every connected graded algebra of global dimension 3 has a pure graded injective resolution. For four-dimensional regular algebras we do not have purity in general, because the homogenization of $U(\mathfrak{sl}_2)$ is not pure [1, Example after Proposition 2.5]. Next, we prove that some familiar four-dimensional regular algebras have essentially pure injective resolutions.

Lemma 4.3. *Let $A = \oplus_{n\in\mathbb{Z}} A_n$ be a strongly graded algebra.*

1. *Let M be a noetherian graded A-module and M_0 the degree zero part of M. Then $j(M) = j(M_0)$. As a consequence, A is graded AG if and only if A_0 is AG.*

2. *A has a pure (respectively, essentially pure) graded minimal injective resolution if and only if A_0 has a pure (respectively, essentially pure) minimal injective resolution.*

Proof. By the category equivalence $(-)_0$ from graded modules over A to modules over A_0 [12, A.I.3.4], we have $\underline{\operatorname{Ext}}^i_A(M, A)_0 \cong \operatorname{Ext}^i_{A_0}(M_0, A_0)$ for every graded A-module M. A graded A-module L is zero if and only if $L_0 = 0$. Hence $j(M) = j(M_0)$. Again by the equivalence, $j(N) = j(N_0)$ for all $N \subset \underline{\operatorname{Ext}}^i(M, A)$. Hence A is graded AG if and only if A_0 is AG. Part 2 follows from the category equivalence and part 1. □

By [14], the four-dimensional Sklyanin algebra is AG and CM. By standard results on Rees rings, the homogenized universal enveloping algebra $H(L)$ of a finite-dimensional Lie algebra L is AG and CM.

Theorem 4.4. *Let k be an algebraically closed field.*

1. *The four-dimensional Sklyanin algebra has an essentially pure graded injective resolution.*

2. *The homogenized universal enveloping algebra $H(L)$ of a three-dimensional Lie algebra L has an essentially pure graded injective resolutions.*

Proof. 1. Let A denote the four-dimensional Sklyanin algebra. By [15], there are central elements $\Omega_1, \Omega_2 \in A_2$ such that $\{\Omega_1, \Omega_2\}$ is a regular sequence. It follows that $A/(a\Omega_2 + b\Omega_2)$ is a domain for all $a, b \in k$. In particular, $A/(\Omega_1)$ is a graded domain having a regular central element Ω_2. By Theorem 4.2.1, $A/(\Omega_1)$ has a graded essentially pure injective resolution. By a graded version of Proposition 3.8, it remains to show that $B = A[\Omega_1^{-1}]$ has a graded essentially pure injective resolution.

Now B is strongly $\mathbb{Z}$-graded, and AG by Proposition 2.1, so B_0 is AG by Lemma 4.3.1. By Lemma 4.3.2, it suffices to show that B_0 has an essentially pure injective resolution. Note that $g := \Omega_1^{-1}\Omega_2$ is a central element in B_0. For every $a \in k$, $g - a = \Omega_1^{-1}(\Omega_2 - a\Omega_1)$ so $B_0/(g-a)$ embeds in the ring of fractions of $A/(\Omega_2 - a\Omega_1)$; thus $B_0/(g-a)$ is a domain. By Lemma 3.10, for every finite-dimensional simple B_0-module M, $M(g-a) = 0$ for some $a \in k$. Write $N = \{g - a \mid a \in k\}$. By Lemma 2.2.1, $\operatorname{injdim} B_0 = \operatorname{gr.injdim} B \le \operatorname{injdim} A - 1 \le 3$, and $\operatorname{injdim}(B_0 N^{-1}) \le \operatorname{injdim} B_0 - 1 \le 2$. By Proposition 1.3, $\operatorname{injdim}(B_0/(g-a)) \le \operatorname{injdim}(B_0) - 1 \le 2$. Therefore by a graded version of Proposition 3.2, $B_0 N^{-1}$ and $B_0/(g-a)$ have essentially pure injective resolutions. Hence by Proposition 3.8, B_0 has an essentially pure injective resolution.

2. There is a central element t in $H(L)$ such that $H(L)/(t) \cong k[x_1, x_2, x_3]$ and $H(L)[t^{-1}] \cong U(L)[t, t^{-1}]$. By Theorem 3.11, $U(L)$ has an essentially pure injective resolution, and so does $U(L)[t, t^{-1}]$. Since $k[x_1, x_2, x_3]$ also has a pure graded injective resolution, therefore $H(L)$ has an essentially pure graded injective resolution by Proposition 3.8. □

Proposition 4.5. *The minimal graded injective resolution of the four-dimensional Sklyanin algebra $A(E,\tau)$ is pure if and only if the ring is PI.*

Proof. The Sklyanin algebra is AG and CM, so Theorem 3.1 gives purity in the PI case. Therefore, we will show here that purity fails in the non-PI case.

The failure of purity will be proved by constructing an extension of the form $0 \to N[-1] \to X \to M(l) \to 0$ where N is a suitable point module, $M(l)$ is a suitable line module and X contains $N[-1]$ as an essential submodule. By [10] a point module is pure of GK-dimension 1 and a line module is pure of GK-dimension 2, so the injective envelope of N, which appears in the minimal graded injective resolution of A by a graded version of [2, 2.3.2], has finitely generated submodules of GK-dimension ≥ 2, whence purity fails.

Let A be the four-dimensional Sklyanin algebra. Fix a line module $M(l)$ and a point module $M(p)$. The minimal projective resolution of $M(l)$ looks like

$$0 \longrightarrow A[-2] \xrightarrow{(a\,b)} A[-1]A[-1] \xrightarrow{\binom{c}{d}} A \longrightarrow M(l) \longrightarrow 0,$$

where $a,b,c,d \in A_1$ are such that $\mathscr{V}(c,d)=l$ and $\mathscr{V}(a,b)=l'$ where l' is a line in $\mathbb{P}(A_1^*)$ which corresponds to some line module. Thus $\underline{\mathrm{Ext}}_A^1(M(l),M(p))_j$ is the homology of

$$M(p)[2]_j \xleftarrow{\delta_1} M(p)[1]_j M(p)[1]_j \xleftarrow{\delta_0} M(p)_j.$$

where $\delta_1(m',m'') = am' + bm''$ and $\delta_0(m) = (cm, dm)$. Recall that, for $i \geq 0$, $M(p)_{\geq i}$ is again a shift of a point module; for brevity we write p_i for the point satisfying $M(p_i)[-i] \cong M(p)_{\geq i}$. Also remember that if $0 \neq m \in M(p)_0$ and $x \in A_1$, then $xm = 0$ if and only if $p \in \mathscr{V}(x)$. Therefore,

$$\dim(\ker \delta_1)_j = \begin{cases} 0 & \text{if } j < -1, \\ 1 & \text{if } p_{j+1} \notin l' \text{ and } j \geq -1, \\ 2 & \text{if } p_{j+1} \in l' \text{ and } j \geq -1, \end{cases}$$

$$\dim(\mathrm{Im}\,\delta_0)_j = \begin{cases} 0 & \text{if } p_j \in l \text{ or } j < 0, \\ 1 & \text{if } p_j \notin l \text{ and } j \geq 0, \end{cases}$$

so

$$\dim \underline{\mathrm{Ext}}_A^1(M(l),M(p))_j = \begin{cases} 0 & \text{if } p_j \notin l \text{ and } p_{j+1} \notin l' \text{and } j \geq 0, \\ & \text{or } j < -1, \\ 2 & \text{if } p_j \in l \text{ and } p_{j+1} \in l' \text{and } j \geq 0, \\ & \text{or } p_{j+1} \in l' \text{ and } j = -1, \\ 1 & \text{otherwise.} \end{cases}$$

We adopt the usual conventions and notations (see for example [10,15]). Thus $A = A(E,\tau)$ is determined by the elliptic curve E and the point $\tau \in E$. We can write $l = \overline{rs}$, the secant line to E spanned by $r,s \in E$. Hence by [10, Proposition 4.4], $l' = \overline{r + \tau\, s + \tau}$. The point modules are of two types:

- the 'standard' ones, those of the form $M(p)$, $p \in E$, and
- the 'exceptional' ones, of which there are four, one corresponding to each 2-torsion point $\omega \in E$, say $M(e_\omega)$, where $e_\omega \in \mathbb{P}(A_1^*)$ is the singular point of the cone which is the union of the secant lines $\{\overline{rs} \mid r + s = \omega\}$.

If $p \in E$, then $p_i = p - i\tau$, whereas for the exceptional ones $(e_\omega)_i = e_\omega$. Therefore, for the standard point modules we have

$$\dim \underline{\mathrm{Ext}}^1_A(M(l), M(p))_j = \begin{cases} 0 & \text{if } p - j\tau \notin \{r, s, r+2\tau, s+2\tau\} \\ & \text{and } j \geq 0, \text{ or } j < -1, \\ 2 & \text{if } p - j\tau \in \{r, s\} \cap \{r+2\tau, s+2\tau\} \\ & \text{and } j \geq 0, \text{ or } p \in \{r+\tau, s+\tau\} \\ & \text{and } j = -1, \\ 1 & \text{otherwise} \end{cases}$$

and for the exceptional point modules we have

$$\dim \underline{\mathrm{Ext}}^1_A(M(l), M(e_\omega))_j = \begin{cases} 0 & \text{if } \omega \notin \{r+s, r+s+2\tau\} \text{ and } j \geq 0, \\ & \text{or } j < -1, \\ 2 & \text{if } \omega = r+s = r+s+2\tau \text{ and } j \geq 0, \\ & \text{or } \omega = r+s+2\tau \text{ and } j = -1, \\ 1 & \text{otherwise.} \end{cases}$$

Note the different behavior of the two kinds of point modules when τ has infinite order: $\underline{\mathrm{Ext}}^1_A(M(l), N)$ is finite-dimensional if N is a standard point module, but may be infinite dimensional when N is exceptional.

Recall that A satisfies a polynomial identity if and only if τ is of finite order, so suppose τ is not of finite order. Fix a 2-torsion point ω, and fix $r, s \in E$ such that $r+s+2\tau = \omega$; write $N = M(e_\omega)[-1]$; then $\underline{\mathrm{Ext}}^1_A(M(\overline{rs}), N)_0 = \underline{\mathrm{Ext}}^1_A(M(\overline{rs}), M(e_\omega))_{-1} \cong k^2$; so we may choose a non-split extension

$$0 \to N \to X \to M(\overline{rs}) \to 0$$

in which the maps are of degree zero. Let $\xi \in \underline{\mathrm{Ext}}^1_A(M(\overline{rs}), N)_0$ represent the extension. To show that N is essential as a graded submodule of X, it suffices to show that the sequence

$$0 \to N \to N + Ax \to A\bar{x} \to 0 \tag{4.3}$$

is non-split whenever x is a homogeneous element of X not in N; here $\bar{x}$ denotes the image of x in $M(\overline{rs})$. If $Q = M(\overline{rs})/A\bar{x}$, then there is an exact sequence

$$\underline{\mathrm{Hom}}_A(A\bar{x}, N) \to \underline{\mathrm{Ext}}^1_A(Q, N) \to \underline{\mathrm{Ext}}^1_A(M(\overline{rs}), N) \to \underline{\mathrm{Ext}}^1_A(A\bar{x}, N),$$

so it suffices to show that ξ does not lie in the image of $\underline{\mathrm{Ext}}^1(Q, N)_0$, because then its image in $\underline{\mathrm{Ext}}^1_A(A\bar{x}, N)$ is non-zero.

By [15, 4.4], since $r+s = \omega - 2\tau$, Q has a finite filtration by graded submodules, the successive quotients of which are either finite-dimensional or shifts of point modules; moreover, using the fact that $r+s = \omega - 2\tau$, it follows from [10, Section 5] that these point modules are standard ones. Since N is exceptional, $\underline{\mathrm{Hom}}_A(A\bar{x}, N) = 0$.

There are two linearly independent central elements in A_2, say Ω_1 and Ω_2; every standard point module is annihilated by Ω_1 and Ω_2, but an exceptional point module is not. Hence, there is a homogeneous central element c such that $cQ = 0$, but $cN \neq 0$. It follows that c acts faithfully on N. By a graded version of Rees' lemma, $\underline{\mathrm{Ext}}_A^1(Q,N) \cong \underline{\mathrm{Hom}}_{A/(c)}(Q,N/cN)[d]$, where $d = \deg c$. But Q is a cyclic module generated in degree zero, and the degree d component of N/cN is one-dimensional (because N is generated in degree 1), so $\underline{\mathrm{Hom}}_{A/(c)}(Q,N/cN)_d \cong k$. Thus the image of $\underline{\mathrm{Ext}}_A^1(Q,N)_0$ in $\underline{\mathrm{Ext}}_A^1(M(\overline{rs}),N)_0$ is a one-dimensional subspace of this two-dimensional space. As x varies, so does Q, and the various Q obtained form an inverse system. Hence the $\underline{\mathrm{Ext}}_A^1(Q,N)_0$ form a directed system, so their union is a one-dimensional subspace of $\underline{\mathrm{Ext}}_A^1(M(\overline{rs}),N)_0$. By choosing ξ not in this union, we ensure that (4.3) does not split. This completes the proof that the minimal resolution is not pure. $\square$

References

[1] K. Ajitabh, Residue complex for Sklyanin algebras of dimension three, Adv. Math., to appear.

[2] K. Ajitabh, S.P. Smith, J.J. Zhang, Auslander-Gorenstein rings, Commun. Algebra 26 (7) (1998) 2159–2180.

[4] J.-E. Björk, The Auslander condition on noetherian rings, Séminaire Dubreil-Malliavin 1987–1988, Lecture Notes in Mathematics, vol. 1404, Springer, Berlin, 1989, pp. 137–173.

[5] K.R. Goodearl, D.A. Jordan, Localizations of injective modules, Proc. Edinburgh Math. Soc. 28 (1985) 289–299.

[6] K.R. Goodearl, D.A. Jordan, Localizations of essential extensions, Proc. Edinburgh Math. Soc. 31 (1988) 243–247.

[7] K.R. Goodearl, R.B. Warfield Jr., An introduction to noncommutative noetherian rings, London Math. Soc. Student Texts, vol. 16, Cambridge Univ. Press, Cambridge, 1989.

[8] A. Leroy, J. Matczuk, J. Okninski, On the Gelfand-Kirillov dimension of normal localizations and twisted polynomial rings, in: F. van Oystaeyen, L. Le Bruyn (Eds.), Perspectives in Ring Theory, Kluwer Academic Publishers, Dordrecht, 1988, pp. 205–214.

[9] T. Levasseur, Some properties of noncommutative regular rings, Glasgow J. Math. 34 (1992) 277–300.

[10] T. Levasseur, S.P. Smith, Modules over the four-dimensional Sklyanin algebra, Bull. Soc. Math. France 121 (1993) 35–90.

[11] J.C. McConnell, J.C. Robson, Non-commutative noetherian Rings, Wiley-interscience, Chichester, 1987.

[12] C. Năstăcescu, F. Van Oystaeyen, Graded Ring Theory, North-Holland, Amsterdam, 1982.

[14] S.P. Smith, J.T. Stafford, Regularity of the four-dimensional Sklyanin algebra, Compos. Math. 83 (1992) 259–289.

[15] S.P. Smith, J.M. Staniszkis, Irreducible representations of the four-dimensional Sklyanin algebra at points of infinite order, J. Algebra 160 (1993) 57–86.

[16] J.T. Stafford, J.J. Zhang, Homological properties of (graded) noetherian PI rings, J. Algebra 168 (1994) 988–1026.

[17] A. Yekutieli, The residue complex of a noncommutative graded algebra, J. Algebra 186 (2) (1996) 522–543.

[18] A. Zaks, Injective dimension of semiprimary rings, J. Algebra 3 (1969) 73–89.

[19] J.J. Zhang, Twisted graded algebras and equivalences of graded categories, Proc. London Math. Soc. 72 (3) (1996) 281–311.

[20] J.J. Zhang, Connected graded Gorenstein algebras with enough normal elements, J. Algebra 189 (2) (1997) 390–405.