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Central extensions of three dimensional Artin–Schelter regular algebras

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Contents

1 Introduction

The class of 3-dimensional Artin–Schelter regular algebras was introduced by M. Artin and W. Schelter [2] in 1987. Subsequent papers by M. Artin, J. Tate and M. van den Bergh [3], [4] showed that these algebras had some extremely interesting properties, not least of which was the delicate interplay between the representation theory of the algebra, and the geometry of

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an associated cubic divisor in $\mathbb{P}^2$. This theme was further developed in [17], [8], [18] for a class of algebras discovered by E.K. Sklyanin. These Sklyanin algebras are examples of 4-dimensional Artin–Schelter regular algebras. It is our intention here to examine another large class of 4-dimensional Artin–Schelter regular algebras, namely algebras D with the property that D has a central regular element z of degree 1 such that $A := D/\langle z\rangle$ is a 3-dimensional Artin–Schelter regular algebra. We call D a *central extension* of A.

All the algebras under discussion are graded algebras, generated over a field k by their degree 1 component. The 3-dimensional Artin–Schelter regular algebras are either generated by two elements, or by three elements. We restrict our attention to those having three generators. Let A be such an algebra. We briefly recall some properties of A. Firstly A is a quadratic algebra, meaning that A has a presentation $A = T(V)/\langle R\rangle$ where $R \subset V \otimes V$. Secondly A has excellent homological properties: for example, A is a Koszul algebra, meaning that $\mathrm{Ext}^*_A(k,k) \cong A^!$ where $k = A/A^+$ is the trivial left A-module and $A^! := T(V^*)/\langle R^\perp\rangle$ is the dual quadratic algebra. The Hilbert series of A is $H_A(t) = (1-t)^{-3}$. There are two important classes of graded A-modules, the point modules and line modules (see [4] for definitions and basic properties). The point modules are parametrized by either a degree 3 divisor in $\mathbb{P}^2$, or by $\mathbb{P}^2$ itself, and the line modules are naturally in bijection with the lines in $\mathbb{P}^2$.

Section 2 contains some results about when a central extension of a Koszul algebra is again a Koszul algebra. In fact, we work in the following more general context. Let D be a quadratic algebra with a normal element $z \in D_1$, and suppose that $D/\langle z\rangle$ is a Koszul algebra. Theorem 2.6 gives precise conditions for D to be a Koszul algebra. This result should be useful in situations other than that considered here. In particular it may be used to recover Sridharan's classification of filtered algebras whose associated graded ring is a polynomial ring [19].

Section 3 classifies the central extensions of generic 3-dimensional Artin–Schelter regular algebras.

Section 4 determines the point modules for these central extensions, and Sect. 5 determines the line modules.

Typical examples for our results are the central extensions of generic Type A algebras (or three dimensional Sklyanin algebras, see [2] or [13][14] for precise definitions). In this case the defining relations for D are given by Theorem 3.2.6:

$$\begin{aligned}
&cx_1^2 + ax_2x_3 + bx_3x_2 + l_{11}x_1z + l_{12}x_2z + l_{13}x_3z + \alpha_1 z^2 = 0\\
&cx_2^2 + ax_3x_1 + bx_1x_3 + l_{21}x_1z + l_{22}x_2z + l_{23}x_3z + \alpha_2 z^2 = 0\\
&cx_3^2 + ax_1x_2 + bx_2x_1 + l_{31}x_1z + l_{32}x_2z + l_{33}x_3z + \alpha_3 z^2 = 0 \qquad (1.1)\\
&zx_i - x_iz = 0 \quad i = 1,2,3\ .
\end{aligned}$$

Here $(l_{ij})_{ij} \in M_3(k)$ is a symmetric matrix and $a, b, c, \alpha_1, \alpha_2, \alpha_3 \in k$. If we allow for isomorphism then we obtain a 7-dimensional family of algebras.

Let D be an algebra having equations as in (1.1). Assume furthermore that all the scalars are generic. In that case, the points of $\mathbb{P}(D_1^*)$ that correspond to point modules (see Sects. 4 and 5 for definitions) form a smooth elliptic curve in $\mathscr{V}(z)$ (say E) together with eight special points, not lying in $\mathscr{V}(z)$ (Proposition 4.3.9). The lines in $\mathbb{P}(D_1^*)$ that correspond to line modules come in two families (Example 5.2.5): (1) the lines in $\mathscr{V}(z)$, and (2) for each point p of E a pair of lines going through p, continuously varying with p. In some cases the lines in the line pair coincide and they may also lie in $\mathscr{V}(z)$, in which case they also belong to the first family. This gives rise to some special case analysis, which is preformed in Sect. 5.

It is perhaps interesting to compare this example with that of the 4-dimensional Sklyanin algebra [8]. There the point modules are parametrized by an elliptic curve of degree *four* (three in our example) together with *four* special points. Passing through a general point of $\mathbb{P}(D_1^*)$ there are two lines corresponding to line modules, whereas in our example, usually, there are none.

If the scalars are generic, or if we look at central extensions of other Artin–Schelter-regular algebras, different phenomena may occur. We can still classify the line modules (Theorems 5.1.6 and 5.1.11). However, the description of the point modules, while possible in principle (Theorem 4.2.2, Corollary 4.2.3) falls apart in many special cases, and we do not undertake an exhaustive analysis. Nevertheless we study sufficiently many cases that the general phenomena become apparent (e.g. Propositions 4.3.7 and 4.3.9).

Unless stated otherwise, k is an arbitrary field in Sect. 2 and Sect. 3.1. In Sect. 3.2 and subsequent sections we assume k to be algebraically closed of characteristic zero.

2 Koszul properties

As explained in the introduction this section concerns a quadratic algebra D having a normal element $z \in D_1$ such that $A := D/\langle z \rangle$ is a Koszul algebra. Theorem 2.6 gives precise conditions for D to be a Koszul algebra also. Finally in Corollary 2.7 we specialize to the situation where A is a 3-dimensional Artin–Schelter-regular algebra, and explain how good homological properties of A ensure that D also has good homological properties.

We will always work with algebras over a fixed base field k.

First we consider the Koszul property. If A is a quadratic algebra with defining relations $R_A \subset A_1 \otimes A_1$, then the *dual algebra* is $A^! := T(A_1^*)/\langle R_A^\perp \rangle$. Fix a basis $\{x_m\}$ for A_1 and let $\{\xi_m\}$ be the dual basis. Define $e_A := \Sigma_m x_m \otimes \xi_m \in A \otimes_k A^!$. The *Koszul complex* for A is the complex of free left A-modules

$$\cdots \to A \otimes (A_n^!)^* \to \cdots \to A \otimes (A_1^!)^* \to A \otimes (A_0^!)^* \to {}_A k \to 0$$

where the differential is right multiplication by e_A. It will be denoted by $\mathbf{K}_\bullet(A)$. If $\mathbf{K}_\bullet(A)$ is exact then A is called a *Koszul algebra*. See for example [12] which proves a number of fundamental properties of Koszul algebras.

A homogeneous element z of a graded algebra D is said to be *n-regular* if both right and left multiplication by z is injective on D_i for all $i \leqq n$.

The proof of the next result is similar to the proof of [17, Theorem 5.4].

Theorem 2.1 *Let D be a finitely generated quadratic k-algebra with a normal 1-regular element $z \in D_1$, and define $A = D/\langle z \rangle$. Suppose that*

1. *A is a Koszul algebra,*
2. *$H_{D^!}(t) = (1+t)H_{A^!}(t)$, and*
3. *$A \otimes_D \mathbf{K}_\bullet(D)$ is exact in degree $\geqq 3$, namely*

$$\cdots \to A \otimes_D \mathbf{K}_n(D) \to \cdots\cdots \to A \otimes_D \mathbf{K}_2(D)$$

is exact.

Then D is also a Koszul algebra, and z is a regular element of D.

Proof. The Koszul complex for D in degree n is a direct sum of bounded complexes of finite dimensional vector spaces, namely

$$0 \to D_0 \otimes (D_n^!)^* \to \cdots \to D_n \otimes (D_0^!)^* \to {}_D k \cdot \delta_{n0} \to 0 .$$

Write P_{ij} for the homology group of this complex at $D_i \otimes (D_j^!)^*$, and $p_{ij} = \dim(P_{ij})$. Write $d_i = \dim(D_i)$ and $\delta_j = \dim(D_j^!)^*$. Then

$$d_0\delta_n - d_1\delta_{n-1} + \cdots + (-1)^n d_n\delta_0 = p_{0n} - p_{1,n-1} + \cdots + (-1)^n p_{n0} + (-1)^n\delta_{n0} .$$

Define integers $c_m := \dim(A_0 + \cdots + A_m)$. Thus $\sum_{m\geqq 0} c_m t^m = (1-t)^{-1}H_A(t)$. Because A is Koszul $H_A(t) \cdot H_{A^!}(-t) = 1$. Therefore, hypothesis 2 implies

$$\left(\sum_{m=0}^{\infty} c_m t^m\right) \cdot H_{D^!}(-t) = (1-t)^{-1}H_A(t)H_{D^!}(-t) = H_A(t)H_{A^!}(-t) = 1 .$$

In particular, $c_0\delta_n - c_1\delta_{n-1} + \cdots + (-1)^n c_n\delta_0 = (-1)^n\delta_{n0}$. Hence

$$\begin{aligned}(d_0 - c_0)\delta_n - (d_1 - c_1)\delta_{n-1} + \cdots + (-1)^n(d_n - c_n)\delta_0 \\ = p_{0n} - p_{1,n-1} + \cdots + (-1)^n p_{n0} .\end{aligned}$$

Although we will not use this fact, it is easy to show that z is regular on $D_0 + D_1 + \cdots + D_{n-1}$ if and only if $d_0 - c_0 = d_1 - c_1 = \cdots = d_n - c_n = 0$. Write $W_m = \{x \in D_m | zx = 0\}$ and $w_m = \dim(W_m)$. The regularity of z is equivalent to $w_m = 0$ for all m.

Claim 1 If $j \geqq 2$ then $p_{i-1,j} = w_i = 0 \Rightarrow p_{ij} = 0$.

Proof. Let d denote the differential on the Koszul complex for D. Let $u \in D_i \otimes (D_j^!)^*$ be such that $d(u) = 0$. We must show that u is in the image of d. Since $A \otimes_D (D^!)^*$ is exact at $A \otimes (D_j^!)^*$ by Hypothesis 3, there exists $v \in D_{i-1} \otimes (D_j^!)^*$ and $v' \in D_{i-1} \otimes (D_{j+1}^!)^*$ such that $u = zv + d(v')$. Clearly $z \cdot d(v) = 0$ so $d(v) \in W_i \otimes (D_{j-1}^!)^*$. But this is zero by hypothesis, so

$d(v) = 0$ i.e. v gives a class in $P_{i-1,j} = 0$. Thus $v \in \mathrm{im}(d)$, and it follows that u is also in the image of d.

Claim 2 For all i and for all $j \geqq 2$ we have $p_{i0} = p_{i1} = p_{0j} = p_{1j} = 0$.

Proof. The first two are zero simply because the right hand end of the Koszul complex for D, namely $D \otimes (D_2^!)^* \to D \otimes (D_1^!)^* \to D \otimes (D_0^!)^* \to {}_D k \to 0$, is exact. To prove the other two homology groups are zero we use the previous claim. Since z is 1-regular, $w_0 = w_1 = 0$. Since $p_{-1,j} = 0$ the previous claim gives $p_{0j} = 0$. Now $p_{0j} = w_1 = 0$ allows us to use the previous claim again, to conclude that $p_{1j} = 0$.

The theorem will now be proved by induction. We will say that $H(n)$ is true if

$$d_0 - c_0 = d_1 - c_1 = \cdots = d_{n-1} - c_{n-1} = 0$$

and

$$w_0 = w_1 = \cdots = w_{n-2} = 0$$

and

$$p_{0j} = p_{1j} = \cdots = p_{n-2,j} = 0 \quad \text{for all } j \geqq 2 \, .$$

We have already seen that $H(n)$ is true for $n \leqq 2$. So suppose that $n \geqq 2$ and that $H(n)$ is true. We will prove that $H(n+1)$ is true. By the first paragraph of the proof, and the induction hypothesis

$$(-1)^n(d_n - c_n)\delta_0 = p_{0n} - p_{1,n-1} + \cdots + (-1)^n p_{n0} = (-1)^n p_{n-2,2}$$

whence $d_n - c_n = p_{n-2,2}$. The map 'multiplication by z' yields an exact sequence $0 \to W_{n-1} \to D_{n-1} \to D_n \to A_n \to 0$. Hence $w_{n-1} = d_{n-1} - d_n + (c_n - c_{n-1})$. But $d_{n-1} - c_{n-1} = 0$ by the induction hypothesis, so $w_{n-1} = -(d_n - c_n)$ and $p_{n-2,2} = -w_{n-1}$. Both $p_{n-2,2}$ and w_{n-1} are non-negative since they are dimensions of vector spaces, so $p_{n-2,2} = w_{n-1} = 0$. It follows that $d_n - c_n = 0$ also. Finally, for all $j \geqq 2$ we have $p_{n-2,j} = w_{n-1} = 0$ so by Claim 1 it follows that $p_{n-1,j} = 0$. Hence $H(n+1)$ is true.

In particular, all the homology groups P_{ij} are zero and z is regular. □

Remark 2.2 Let D be as in Theorem 2.1, but suppose now that $z \in D_2$. If Hypotheses 1 and 3 are the same, and Hypothesis 2 is replaced by the hypothesis that $H_{D^!}(t) = (1 - t^2)H_{A^!}(t)$ then an almost identical proof will show that D is a Koszul algebra, and z is regular.

By combining the previous Theorem with the results in [11] and [7] we have the following consequences. The definition of an Auslander-regular algebra and the Cohen–Macaulay property can be found in [7].

Corollary 2.3 *If the hypotheses of the previous Theorem apply, then*

1. *D is noetherian if and only if A is noetherian;*
2. *D is a domain if and only if A is a domain;*
3. *D is Auslander-regular if and only if A is Auslander-regular;*
4. *D satisfies the Cohen–Macaulay property if and only if A does.*

We now seek conditions on D which ensure that the hypotheses of Theorem 2.1 hold.

Let D be a finitely generated quadratic algebra over the field k, and suppose that $z \in D_1$ is a 1-regular normal element. Since z is normal there is a linear map $\phi \in \mathrm{End}(D_1)$ with the property that $zd = \phi(d)z$ for all $d \in D_1$. Since z is 1-regular there is only one such map and it is bijective. We will also assume that ϕ extends to an algebra automorphism of D. Define $\phi^! \in \mathrm{End}(D_1^*)$ by the requirement that $\langle \phi^!(\beta), \phi(d)\rangle = -\langle \beta, d\rangle$ for all $\beta \in D_1^*$ and all $d \in D$. It is clear that $\phi^!$ extends to an algebra automorphism of $D^!$ because ϕ is assumed to preserve the relations in D. Since $\phi(z) = z$ it follows that $\phi(z^\perp) = z^\perp$.

Define $A := D/\langle z\rangle$. Since $\phi(zD) \subset zD$, it follows that ϕ induces an automorphism of A; we will still write ϕ for this automorphism. Similarly there is an induced automorphism $\phi^!$ of $A^!$. The natural map $A_1^* \to D_1^*$ induces an algebra homomorphism $\Phi : A^! \to D^!$ such that $\Phi(A^!) = k[z^\perp]$, the subalgebra of $D^!$ generated by $z^\perp$. It is easy, but important to observe that Φ is injective in degrees 1 and 2. Finally, we note that the automorphisms $\phi_A^!$ of $A^!$ and $\phi_D^!$ of $D^!$ are compatible with Φ in the sense that $\Phi \circ \phi_A^! = \phi_D^! \circ \Phi$.

Proposition 2.4 *Let D be a finitely generated quadratic algebra, and suppose that $z \in D_1$ is a 1-regular normal element. Fix $\omega \in D_1^*$ such that $\omega(z) = 1$. Then*

1. *ω is 1-regular;*
2. *the defining relations for $D^!$ consist of*
   (i) *the relations for $A^!$;*
   (ii) *$\{\omega\beta - \phi^!(\beta)\omega + \psi_\beta | \beta \in z^\perp$, and certain $\psi_\beta \in (z^\perp)^2\}$;*
   (iii) *$\omega^2 - \omega\nu - \varphi$ where $\nu \in z^\perp$ satisfies $\phi^!(\nu - \omega) = \omega$ and φ is some element of $(z^\perp)^2$;*
3. *$D_k^! = \omega(z^\perp)^{k-1} + (z^\perp)^k = (z^\perp)^{k-1}\omega + (z^\perp)^k$ for all $k \geqq 1$.*

Proof. We have

$$D_1 \otimes D_1 = k(z \otimes z) \oplus (z \otimes \omega^\perp) \oplus (\omega^\perp \otimes z) \oplus (\omega^\perp \otimes \omega^\perp)$$

$$D_1^* \otimes D_1^* = k(\omega \otimes \omega) \oplus (\omega \otimes z^\perp) \oplus (z^\perp \otimes \omega) \oplus (z^\perp \otimes z^\perp)\,.$$

The relations in D are of two types. Those in $R_D \cap ((z \otimes \omega^\perp) \oplus (kz \otimes z) \oplus (\omega^\perp \otimes z))$ are of the form $z \otimes d - \phi(d) \otimes z$ where $d \in \omega^\perp$. Modulo these the other relations are of the form $f_\lambda + z \otimes v_\lambda + \alpha_\lambda z \otimes z$ for various $f_\lambda \in \omega^\perp \otimes \omega^\perp$, $v_\lambda \in \omega^\perp$, $\alpha_\lambda \in k$ where $\lambda \in \Lambda$ some index set. Furthermore, we may assume that $\{f_\lambda | \lambda \in \Lambda\}$ is linearly independent (because the 1-regularity of z ensures that there is no relation of the form $z \otimes v + \alpha z \otimes z$ with $v + \alpha z \neq 0$).

Let $\beta \in D_1^*$. If $\omega\beta = 0$ then $\omega \otimes \beta$ vanishes on all the relations for D and in particular on all the relations of the first type. Hence $\beta(d) = 0$ for all $d \in D_1$, so $\beta = 0$. Similarly, if $\beta\omega = 0$ then $\beta = 0$, so ω is 1-regular.

Now we show that $\omega^2 \in \omega(z^\perp) + (z^\perp)^2$. First define $\nu \in D_1^*$ by requiring that $\nu(z) = 0$ and $\nu(d) = -\omega(\phi(d))$ for all $d \in \omega^\perp$. Now let $\varphi \in z^\perp \otimes z^\perp$ be such that $\varphi(f_\lambda) = \alpha_\lambda - \nu(v_\lambda)$ for all $\lambda \in \Lambda$. It is straightforward to check that $(\omega \otimes \omega - \omega \otimes \nu - \varphi)(R_D) = 0$ and hence $\omega^2 = \omega\nu + \varphi \in \omega(z^\perp) + (z^\perp)^2$. A similar argument shows that $\omega^2 \in (z^\perp)\omega + (z^\perp)^2$.

Now we show that $\omega(z^\perp) \subset (z^\perp)\omega + (z^\perp)^2$. Let $\beta \in z^\perp$. Choose $\psi_\beta \in z^\perp \otimes z^\perp$ such that $\psi_\beta(f_\lambda) = -\beta(v_\lambda)$ for all $\lambda \in \Lambda$. Then $(\omega \otimes \beta - \phi^!(\beta) \otimes \omega + \psi_\beta)(R_D) = 0$ so in $D^!$ we have $\omega\beta = \phi^!(\beta)\omega - \psi_\beta$. Thus $\omega(z^\perp) \subset (z^\perp)\omega + (z^\perp)^2$ and a similar argument also proves that $(z^\perp)\omega \subset \omega(z^\perp) + (z^\perp)^2$.

An easy induction argument proves that $D^!_k = \omega(z^\perp)^{k-1} + (z^\perp)^k = (z^\perp)^{k-1}\omega + (z^\perp)^k$ for all $k \geqq 1$.

It remains to show that $\phi^!(\nu - \omega) = \omega$. By definition $\langle \phi^!(\nu), \phi(d)\rangle = -\langle \nu, d\rangle = -\langle \omega, \phi(d)\rangle$ for all $d \in \omega^\perp$, so $\langle \phi^!(\nu) - \omega, \phi(\omega^\perp)\rangle = 0$. Also $\langle \phi^!(\nu) - \omega, z\rangle = \langle \phi^!(\nu), \phi(z)\rangle - \langle \omega, z\rangle = \langle \nu, z\rangle - \langle \omega, z\rangle = -1$. But $\langle \phi^!(\omega), \phi(\omega^\perp)\rangle = 0$ and $\langle \phi^!(\omega), z\rangle = -1$ also. Hence $\phi^!(\nu) - \omega = \phi^!(\omega)$ as required. □

Proposition 2.5 *Let D be a finitely generated quadratic algebra, and suppose that $z \in D_1$ is a 1-regular normal element. Fix $\omega \in D_1^*$ such that $\omega(z) = 1$. Define $A = D/Dz$. Suppose that the induced map $A^!_3 \to D^!_3$ is injective. Then*

1. *the map $A^! \to D^!$ is injective, and $A^! \cong k[z^\perp]$;*
2. *$D^! = \omega A^! \oplus A^! = A^!\omega \oplus A^!$ and $D^!$ is a free right (and left) $A^!$-module with basis $\{1, \omega\}$;*
3. *the hypotheses of Theorem 2.1 and its Corollary hold, so that D is a Koszul algebra if A is.*

Proof. Since the image of $\Phi : A^! \to D^!$ is $k[z^\perp]$ and $D^! = k[z^\perp][\omega]$, it follows that there is a surjective algebra homomorphism $A^! \amalg_k k[X] \to D^!$ from the coproduct of $A^!$ and the polynomial ring $k[X]$, which sends X to ω. By Proposition 2.4, the kernel of this map is the ideal generated by $\{X\beta - \phi^!(\beta)X - \psi_\beta | \beta \in A_1^*\} \cup \{X^2 - X\nu - \varphi\}$ where ψ_β, ν and φ are defined in Proposition 2.4.

We first consider the quotient of $A^! \amalg k[X]$ by the ideal generated by all but the last of these elements. Define $\delta : A^!_1 \to A^!_2$ by

$$\delta(\beta) = \omega\beta - \phi^!(\beta)\omega .$$

Although the computation on the right hand side takes place in $D^!$, it is equal to (the image of) $-\psi_\beta$ which is in $(z^\perp)^2$ which is identified with $A^!_2$ through Φ. Hence δ is well-defined. We will show that δ extends to a $\phi^!$-derivation of $A^!$. To do this it suffices to prove that if $\beta_i, \gamma_i \in A^!_1$ satisfy $\Sigma_i \beta_i\gamma_i = 0$ in $A^!$, then $\Sigma_i(\delta(\beta_i)\gamma_i + \phi^!(\beta_i)\delta(\gamma_i))$ is zero in $A^!_3$. Since the map $A^!_3 \to D^!_3$ is injective, it is enough to show that this expression is zero in $D^!$. However, this expression equals

$$\sum_i (\omega\beta_i\gamma_i - \phi^!(\beta_i)\omega\gamma_i + \phi^!(\beta_i)\omega\gamma_i - \phi^!(\beta_i)\phi^!(\gamma_i)\omega) = -\sum_i \phi^!(\beta_i)\phi^!(\gamma_i)\omega$$

which is zero because $\phi^!$ is an algebra homomorphism.

Hence $A^!\amalg k[X]/\langle X\beta-\phi^!(\beta)X-\psi_\beta|\beta\in A_1^*\rangle\cong A^![X;\phi^!,\delta]$ the Ore extension with respect to $(\phi^!,\delta)$. This Ore extension is a free $A^!$-module (on both the right and the left) with basis $1,X,X^2,\ldots$.

Our next goal is to prove that $X^2-Xv-\varphi$ is a normal element in $A^![X;\phi^!,\delta]$. Some preliminary calculations are required.

Claim 1 $\phi^!(\varphi)=\varphi+\delta(v)$.

Proof. Applying $\phi^!$ to the relation $\omega(\omega-v)-\varphi=0$ in $D^!$ and using Proposition 2.4 we have $0=-\phi^!(\omega)\omega-\phi^!(\varphi)=(\omega-\phi^!(v))\omega-\phi^!(\varphi)=\omega v+\varphi-\phi^!(v)\omega-\phi^!(\varphi)=\delta(v)+\varphi-\phi^!(\varphi)$ as required.

Claim 2 For all $\beta\in z^\perp,(\delta\phi^!+\phi^!\delta)(\beta)=\phi^!(v\beta-\phi^!(\beta)v)$.

Proof. The right hand side of this expression equals $(\omega+\phi^!(\omega))\phi^!(\beta)-\phi^{!2}(\beta)(\omega+\phi^!(\omega))=\omega\phi^!(\beta)-\phi^{!2}(\beta)\omega+\phi^!(\omega\beta-\phi^!(\beta)\omega)$ which is precisely the left hand expression in the claim.

Claim 3 If $\beta\in A_1^!$ then $(X^2-Xv-\varphi)\beta-\phi^{!2}(\beta)(X^2-Xv-\varphi)\in A_3^!$.

Proof. Using $1,X,X^2,\ldots$ as a basis for $A^![X;\phi^!,\delta]$ as a left $A^!$ module we must show that the components in $A_1^!X^2$ and $A_2^!X$ are both zero. Since $X\alpha=\delta(\alpha)+\phi^!(\alpha)X$ for all $\alpha\in A_1^!$, the coefficient of X^2 is zero. The coefficient of X is $(\delta\phi^!+\phi^!\delta)(\beta)-\phi^!(v\beta)+\phi^{!2}(\beta)\phi^!(v)=0$ by Claim 2.

Claim 4 $(X^2-Xv-\varphi)X-(X+\phi^{!2}(v)-\phi^!(v))(X^2-Xv-\varphi)\in A_3^!$.

Proof. As in Claim 3 we want to show that the coefficients of X,X^2,X^3 are all zero. It is easy to see that the coefficient of X^3 is zero. The coefficient of X^2 is $-\phi^!(v)-(\phi^{!2}(v)-\phi^!(v))+\phi^{!2}(v)=0$. The coefficient of X is $-\delta(v)-\varphi+(\delta\phi^!+\phi^!\delta)(v)+(\phi^{!2}(v)-\phi^!(v))\phi^!(v)+\phi^!(\varphi)$. This is zero by Claim 1, and Claim 2 (with $\beta=v$).

Claim 5 $\phi^!$ extends to an automorphism of $A^![X;\phi^!,\delta]$ with $\phi^!(X)=-X+\phi^!(v)$.

Proof. It suffices to prove that the relation $X\beta-\delta(\beta)-\phi^!(\beta)X=0$ for $\beta\in A_1^!$ is preserved by $\phi^!$. However applying $\phi^!$ to this gives $(-X+\phi^!(v))\phi^!(\beta)-\phi^!\delta(\beta)-\phi^{!2}(\beta)(-X+\phi^!(v))$. By Claim 2 this is zero.

Claim 6 $X^2-Xv-\varphi$ is a normal element of $A^![X;\phi^!,\delta]$.

Proof. We have just shown that $(X^2-Xv-\varphi)\alpha-\phi^{!2}(\alpha)(X^2-Xv-\varphi)\in A_3^!$ for all $\alpha\in A_1^!\oplus kX$. However, this element clearly maps to zero in $D^!=A^![X;\phi^!,\delta]/\langle\omega^2-\omega v-\varphi\rangle$. Since the map $A_3^!\to D_3^!$ is assumed to be injective it follows that $(X^2-Xv-\varphi)\alpha=\phi^{!2}(\alpha)(X^2-Xv-\varphi)$ as required.

Hence $A^![X;\phi^!,\delta]/\langle X^2-Xv-\phi\rangle$ is a free right and left $A^!$-module with basis $\{1,X\}$. This proves both the first and the second claim.

It is now immediate that $H_{D^!}(t)=(1+t)H_{A^!}(t)$. Furthermore, $A\otimes_D\mathbf{K}_\bullet(D)=A\otimes D^!=(A\otimes\omega A^!)\oplus(A\otimes A^!)$ and the induced differential is given by right multiplication by $e_D:=z\otimes\omega+\Sigma x_i\otimes\xi_i$ where x_i is a basis for $\omega^\perp$ and ξ_i

is the dual basis for $z^{\perp}$. However, since z annihilates A, e_D acts just as right multiplication by $e_A := \Sigma x_i \otimes \xi_i$ where x_i and ξ_i are dual bases for A_1 and A_1^* respectively. Thus $A \otimes_D \mathbf{K}_\bullet(D)$ is just a direct sum of two copies of the Koszul complex for A, one of which is shifted in degree. Hence the second and third hypotheses of Theorem 2.1 hold. □

Theorem 2.6 *Let D be a finitely generated quadratic algebra. Suppose that*

- *there is a 1-regular normal element $z \in D_1$, and $\phi \in \mathrm{Aut}(D)$ such that $zd = \phi(d)z$ for all $d \in D$;*
- *$A := D/\langle z \rangle$ is a Koszul algebra.*

Then the following are equivalent:

1. *D is a Koszul algebra and z is regular;*
2. *$H_{D^!}(t) = (1+t)H_{A^!}(t)$;*
3. *the natural map $A^! \to D^!$ is injective;*
4. *the natural map $A_3^! \to D_3^!$ is injective;*
5. *the image of $(D_1 \otimes R_D) \cap (R_D \otimes D_1)$ under the natural map $D_1^{\otimes 3} \to A_1^{\otimes 3}$ is $(A_1 \otimes R_A) \cap (R_A \otimes A_1)$.*

Proof. Proposition 2.5 proved that (4) implies (1), (2) and (3). Obviously (3) implies (4), so these conditions are equivalent.

Notice that (4) and (5) are equivalent. First observe that

$$D_3^! = (D_1^!)^{\otimes 3}/R_D^{\perp} \otimes D_1^* + D_1^* \otimes R_D^{\perp} = (D_1^{\otimes 3})^*/(R_D \otimes D_1 \cap D_1 \otimes R_D)^{\perp}$$
$$\cong (R_D \otimes D_1 \cap D_1 \otimes R_D)^* .$$

Similarly

$$A_3^! \cong (R_A \otimes A_1 \cap A_1 \otimes R_A)^* .$$

Since the map $A_3^! \to D_3^!$ is induced by the map $(A_1^{\otimes 3})^* \to (D_1^{\otimes 3})^*$ which is dual to the natural map $D_1^{\otimes 3} \to A_1^{\otimes 3}$, the equivalence of (4) and (5) follows.

Suppose that (2) holds. Then $\dim(D_3^!) = \dim(A_3^!) + \dim(A_2^!)$. However, by Proposition 2.4, there is a surjective linear map $A_3^! \oplus A_2^! \to D_3^!$. Hence this map must be an isomorphism, and its restriction $A_3^! \to D_3^!$ is injective. Hence (4) holds. We have shown that (2), (3), (4) and (5) are equivalent.

Finally, suppose that (1) holds. Since z is regular $H_A(t) = (1-t)H_D(t)$. Thus (2) follows from the functional equation for Koszul algebras. □

Corollary 2.7 *Let D be a finitely generated quadratic algebra. Suppose that*

- *there is a 1-regular normal element $z \in D_1$, and $\phi \in \mathrm{Aut}(D)$ such that $zd = \phi(d)z$ for all $d \in D$;*
- *$A := D/\langle z \rangle$ is an Artin–Schelter regular algebra [2] with Hilbert series $H_A(t) = (1-t)^{-3}$;*
- *the image of $(D_1 \otimes R_D) \cap (R_D \otimes D_1)$ under the natural map $D_1^{\otimes 3} \to A_1^{\otimes 3}$ is $(A_1 \otimes R_A) \cap (R_A \otimes A_1)$.*

Then D is a noetherian domain with Hilbert series $H_D(t) = (1-t)^{-4}$, and D is an Auslander-regular, Koszul algebra with the Cohen–Macaulay property.

Remark 2.8 Consider a quadratic algebra D with a normal element $0 \neq z \in D_1$ and $A := D/\langle z \rangle$. In the previous Corollary the hypotheses that z is 1-regular, and the existence of ϕ, are easily checked if D is given in terms of generators and relations. However, if one knows in advance that z is regular then (as T. Levasseur kindly pointed out to us) the proof in [15] adapts to the non-commutative case, and D is Koszul if and only if A is Koszul. In particular, if D is Koszul with Hilbert series $(1-t)^{-n}$ then A is Koszul with Hilbert series $(1-t)^{-(n-1)}$. If D is given by generators and relations, then it may be very difficult to decide if $z \in D_1$ is regular; however, sometimes one might be able to use the Diamond Lemma to check this easily.

Remark 2.9 From Theorem 2.6 we may recover Sridharan's classification [19] of filtered algebras whose associated graded ring is a polynomial algebra.

3 Central extensions of three dimensional Artin–Schelter regular algebras

3.1 Generalities

Definition 3.1.1 *A central extension of a graded algebra A is a graded algebra D with a central element $z \in D_1$ such that z is (left and right) regular and $A \cong D/\langle z \rangle$.*

If A is a 3-dimensional Artin–Schelter regular algebra then a regular central extension of A is a four dimensional Artin–Schelter regular algebra D which is a central extension of A.

Notice that there is nothing to be gained by letting z be normal since a normal regular element of degree one may always be turned into a central one via a twist (see [4, Section 8]).

Let A be a three dimensional Artin–Schelter regular k-algebra with Hilbert series $(1-t)^{-3}$, as classified in [3][2]. Our aim is to classify regular central extensions of A. To be more precise, for a given A we will classify pairs (D, θ) where D is a four dimensional Artin–Schelter regular algebra and θ is a surjective graded k-algebra map $D \to A$, whose kernel is generated by a central element in degree one. Two such pairs (D, θ) and (D', θ') will be called *equivalent* if there is an isomorphism of graded k-algebras $\psi : D \to D'$ such that $\theta = \theta'\psi$. Note that equivalence is stronger than just isomorphism of D and D'.

Remark 3.1.2 It follows from Remark 2.8 that if D is a Koszul algebra with Hilbert series $H_D(t) = (1-t)^{-4}$ and $z \in D_1$ is a regular central regular element, then $D/\langle z \rangle$ is a Koszul algebra with Hilbert series $(1-t)^{-3}$. Thus D is a central extension of an Artin–Schelter regular algebra. Hence after this paper, attention should probably be focused on those 4-dimensional Artin–Schelter regular Koszul algebras having no normal elements in degree one (except 0).

According to [2] we can, for fixed generators x_1,x_2,x_3 of A, choose a basis of the 3-dimensional vector space of quadratic relations say f_1,f_2,f_3 such that there is a 3×3 matrix M with entries in A and a matrix $Q\in GL_3(k)$ such that the relations f_1,f_2,f_3 may be written as $f=Mx$ and $x^tM=(Qf)^t$ where $f=(f_1,f_2,f_3)^t$, and $x=(x_1,x_2,x_3)^t$. These are equations in matrices over the free algebra $k\langle x_1,x_2,x_3\rangle$. Associated to this presentation of A there is an element $w\in A_1^{\otimes 3}$ such that $w=x^tMx=x^tf=f^t(Q^tx)$. Clearly $w\in R_A\otimes A_1\cap A_1\otimes R_A$. Furthermore it is proved in [2] that $\dim(R_A\otimes A_1\cap A_1\otimes R_A)=1$ whence w is uniquely determined (up to a scalar multiple) by A. We also introduce the element $x^*=(x_1^*,x_2^*,x_3^*)^t\in A_1^3$ defined by $x^*=Q^tx$. Thus $x^tf=f^tx^*$.

Now suppose that D is a central extension of A. Choose representatives for x_1,x_2,x_3 in D_1. We will also denote these by x_1,x_2,x_3 and will consider their span in D_1 as being a copy of A_1. The defining equations for D will therefore be of the form

$$\begin{aligned} g_j := f_j + zl_j + \alpha_j z^2 &= 0 \qquad j=1,\ldots,3 \\ zx_i - x_iz &= 0 \qquad i=1,\ldots,3 \end{aligned} \tag{3.1}$$

where $f_1,f_2,f_3\in A_1\otimes A_1$ are defining equations for A, and $l_1,l_2,l_3\in A_1$ and $\alpha_1,\alpha_2,\alpha_3\in k$. Two such sets of equations will describe equivalent central extensions if they may be transformed into each other via substitutions of the form $x_i\to x_i+u_iz$, $z\to vz$ for scalars $(u_i)_i, v$.

Theorem 3.1.3 *The equations* (3.1) *define a four dimensional regular algebra if and only if there exist* $(\gamma_j)_j\in k$ *that form a solution to the following system of linear equations in* $A_1^{\otimes 2}, A_1$ *and* k.

$$\begin{aligned} \sum_j \gamma_j f_j &= \sum_j (x_jl_j - l_jx_j^*) \\ \sum_j \gamma_j l_j &= \sum_j \alpha_j(x_j - x_j^*) \\ \sum_j \gamma_j\alpha_j &= 0\,. \end{aligned} \tag{3.2}$$

If such $(\gamma_j)_j$ *exist, then they are uniquely determined by* $(l_j)_j$.

Proof. By Theorem 2.6 equations (3.1) define a regular algebra with z a non-zero divisor if and only if there exists $w'\in D_1\otimes R_D\cap R_D\otimes D_1$, mapping to w. Let w' be such an element. Then it may be written as

$$w' = \sum_j a_j(f_j + l_jz + \alpha_jz^2) - \sum_i b_i(zx_i - x_iz) \tag{3.3}$$

for some $a_j,\ b_i \in D_1$, and as

$$w' = \sum_j (f_j + l_j z + \alpha_j z^2)c_j - \sum_i (zx_i - x_i z)d_i \tag{3.4}$$

for some $c_j,\ d_i \in D_1$.

Write $a_j = a_j^{(1)} + a_j^{(2)}z$, $c_j = c_j^{(1)} + c_j^{(2)}z$ with $a_j^{(1)}, c_j^{(1)} \in \sum kx_i$, and $a^{(2)}$, $c_j^{(2)} \in k$. By reduction mod(z) we find that $a_j^{(1)} = x_j$, $b_j^{(1)} = x_j^*$ (up to a scalar multiple which does not matter). Comparing (3.3) and (3.4) we now find the following identity in $D_1^{\otimes 3}$.

$$\begin{aligned}\sum_j x_j(l_j z + \alpha_i z^2) + \sum_j a_j^{(2)} z(f_j + l_j z + \alpha_j z^2) - \sum_i b_i(zx_i - x_i z)\\ = \sum_j (l_j + \alpha_i z^2)x_j^* + \sum_j c_j^{(2)}(f_j + l_j z + \alpha_j z^2)z - \sum_i (zx_i - x_i z)d_i\end{aligned} \tag{3.5}$$

Conversely w' will exist if there exist $a_j^{(2)}$, $c_j^{(2)}$, b_i, d_i such that (3.5) holds. Now let $T = k\langle x_1,x_2,x_3\rangle[z]$. The existence of $a_j^{(2)}$, $c_j^{(2)}, b_i$, d_i such that (3.5) holds is equivalent with the existence of $a_j^{(2)}$, $c_j^{(2)} \in k$ such that

$$\begin{aligned}\sum_j x_j(l_j z + \alpha_i z^2) + \sum_j a_j^{(2)} z(f_j + l_j z + \alpha_j z^2) = \sum_j (l_j z + \alpha_i z^2)x_j^*\\ + \sum_j c_j^{(2)}(f_j + l_j z + \alpha_j z^2)z\end{aligned} \tag{3.6}$$

in T_3.

Define $\gamma_j = c_j^{(2)} - a_j^{(2)}$. Now (3.2) is obtained by comparing terms in (3.6) with the same z-degree.

Since the $(f_j)_j$ are linearly independent in $A_1^{\otimes 2}$, the first of these three equations shows that the $(\gamma_j)_j$ are uniquely determined by the l_j. $\square$

Remark 3.1.4 Equations (3.2) may be rewritten as (with $\gamma = (\gamma_1, \gamma_2, \gamma_3)^t$)

$$g^t x^* = (x - z\gamma)^t g$$

in $k\langle x_1,x_2,x_3\rangle[z]$, which is perhaps more elegant.

Lemma 3.1.5 *If we make a substitution $x \to x + uz$ and $z \to vz$ where $u = (u_1, u_2, u_3)^t$, then*

$$\begin{aligned} l_j &\to v l_j + \sum_i u_i \frac{\partial f_j}{\partial x_i}\\ \alpha_j &\to v^2 \alpha_j + v l_j(u) + f_j(u)\\ \gamma &\to v\gamma + (u^* - u)\end{aligned} \tag{3.7}$$

where $u^* = Q^t u$.

Proof. If we use the notation $f_j(x)$ and $l_j(x)$ to emphasize the dependency of f_j and l_j on x, then there are linear forms $\frac{\partial f_j}{\partial x_i}$ in x such that

$$f_j(x+ut) = f_j(x) + \sum_i u_i \frac{\partial f_j}{\partial x_i} z + f_j(u) z^2 .$$

Similarly $l_j(x+uz) = l_j(x) + l_j(u)z$. It follows at once from this that l_j and α_j are transformed as in (3.7). To see that γ_j is transformed into $v\gamma_j + u_j^* - u_j$ requires more work. First, as remarked before, γ_j is completely determined by l_j. Hence it suffices to check that $(v\gamma_j + u_j^* - u_j)_j$ is a solution to (3.2) for the transformed l and α. To do so involves some tedious calculations using the fact that $x^t f = f^t x^*$, and that $\Sigma_i u_i \frac{\partial f_j}{\partial x_i} = f_j(u,x) + f_j(x,u)$ (we are viewing f_j as a bilinear form in the obvious way). □

We are now in a position to draw a first conclusion from what we have done so far. We can describe those sets of equations (3.1) having no linear terms ($l_j = 0$ for all j) which give rise to a regular algebra.

Theorem 3.1.6 *If there are no linear terms in equation* (3.1) (*i.e.* $l_j = 0$ *for all* j), *then these equations define a regular algebra if and only if*

$$(I - Q)\alpha = 0 . \tag{3.8}$$

Furthermore, any equivalent algebra which also has no linear terms, will have the same α (*up to a scalar multiple*).

Proof. Since the f_j are linearly independent, the only possible solution to the first of the three equations in (3.2) is given by $\gamma_j = 0$ for all j. But $\gamma_j = 0$ will be a solution to (3.2) if and only if α satisfies

$$\sum_j \alpha_j (x_j - x_j^*) = 0 .$$

However, this is equivalent to (3.8).

It follows from (3.7) that a substitution $x \to x + uz, z \to uz$ preserving the property $l_j = 0$, satisfies

$$\sum_i u_i \frac{\partial f_j}{\partial x_i} = 0 \quad \forall j \tag{3.9}$$

Since $\sum_i u_i \frac{\partial f_j}{\partial x_i} = f_j(u,x) + f_j(x,u)$ it follows that $f_j(u) = 0$. Therefore, according to (3.7) α will only be multiplied by the scalar v^2. □

3.2 Computations for the generic three dimensional types

In this section we compute the regular central extensions of the generic types in [2]. We need the following technical notion.

Definition 3.2.1 *A solution to* (3.2) *will be called simple if* $\gamma = 0$.

The existence of simple solutions is of practical importance since (if they exist) they depend only on Q and not on the f_j, so they are easier to determine. Furthermore as we will see below, in some instances all solutions to (3.2) are equivalent to simple ones.

Lemma 3.2.2 *Define the following subspaces of* $k\langle x_1, x_2, x_3\rangle_2$:

$$U = kf_1 + kf_2 + kf_3$$

$$V = \sum_{i,j} k(x_j x_i - x_i x_j^*)$$

$W =$ *subspace spanned by the entries of the vector* $(Q - I)f$.

Then

1. $W \subset U \cap V$;
2. *every solution to* (3.2) *is equivalent to a simple solution if*

$$W = U \cap V \,. \tag{3.10}$$

Proof. 1. Let $u = (u_i)_i$ be arbitrary. Under the substitution $x \to x + uz$ (z central) the identity $x^t f = f^t x^*$ becomes $(x^t + u^t z) f(x + uz) = f(x + uz)^t (x^* + u^* z)$. Comparing the coefficients of z it follows that

$$(u^t - u^{*t})f = \sum_i u_i \sum_j \frac{\partial f_j}{\partial x_i} x_j^* - x_j \frac{\partial f_j}{\partial x_i}$$

But $u^t - u^{*t} = u(I - Q)$. Since u is arbitrary, it follows that

$$((I - Q)f)_i = \sum_j \frac{\partial f_j}{\partial x_i} x_j^* - x_j \frac{\partial f_j}{\partial x_i}$$

which proves that $W \subset V$. Hence $W \subset U \cap V$.

2. Suppose that $U \cap V = W$. Let $(\gamma_j)_j$ be a solution to (3.2). It is clear from (3.7) that (γ_j) is equivalent to a simple one if and only if there exists u such that $\gamma = u^* - u = (Q^t - I)u$. Since the f_j are linearly independent this is equivalent to the condition that $\sum \gamma_j f_j = u^t(Q - I)f$ for some u. However, this is equivalent to the requirement that $\sum_j \gamma_j f_j \in W$. But $\sum_j \gamma_j f_j \in U \cap V$, so by hypothesis it is also in W. $\square$

From now on we will assume that k is an algebraically closed field of characteristic 0 and that Q is diagonal, say $Q = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$.

Lemma 3.2.3

$$\dim V = 3 + |\{i | \lambda_i \neq 1\}| + |\{(i, j) | i \neq j, \lambda_i \lambda_j \neq 1\}|$$

$$\dim W = |\{i | \lambda_i \neq 1\}|$$

It follows that $\dim U + \dim V - \dim W \leqq 9$, so (3.10) is never excluded for numerical reasons.

Corollary 3.2.4 *Assume that for all* $i, \lambda_i \neq 1$ *and for all* $i \neq j$, $\lambda_i\lambda_j \neq 1$. *Then every solution to* (3.2) *is equivalent to a simple solution with*

$$(\alpha_j)_j = 0, \qquad (l_j)_j = 0, \qquad (\gamma_j)_j = 0\,.$$

Proof. By Lemma 3.2.3 it follows that $\dim V = 9$ and $\dim W = 3$. Thus $W = U \cap V = U$. Hence every solution is equivalent to a simple one by Lemma 3.2.2. Taking such a simple solution $(\gamma_j) = 0$, it follows from (3.2) that every $l_j = 0$ since the $x_ix_j - x_jx_i^*$ are linearly independent. Similarly all $\alpha_j = 0$ because the $x_j - x_j^*$ are linearly independent. □

As an application of our results we will now analyze the generic types in [2, Table 3.11]. Recall that these all have the property that Q is diagonal.

Lemma 3.2.5 *If A is a generic Artin–Schelter regular algebra with* $H_A(t) = (1-t)^{-3}$, *then every solution to* (3.2) *is equivalent to a simple solution.*

Proof. For all the generic types of [2] it is somewhat tedious but uneventful to verify that $U \cap V = W$. Hence Lemma 3.2.2 applies. □

Theorem 3.2.6 *Let A be a generic Artin–Schelter regular algebra with* $H_A(t) = (1-t)^{-3}$. *The number of central extensions of A is given by Table* 3.1 *below. An entry* -1 *means that there exist no nontrivial D* (*i.e. every central extension of A is equivalent to a polynomial extension*), *and* 0 *means that there exists exactly one such D up to equivalence.*

Proof. It follows from the previous lemma that to classify all the central extensions up to equivalence, we may (*and we will*) restrict ourselves to classifying the simple solutions to (3.2).

In what follows ζ_m denotes a primitive m^{th} root of 1.

Type A $Q = I$ and the defining equations for A are

$$\begin{aligned} f_1 &= cx_1^2 + ax_2x_3 + bx_3x_2 \\ f_2 &= cx_2^2 + ax_3x_1 + bx_1x_3 \\ f_3 &= cx_3^2 + ax_1x_2 + bx_2x_1\,. \end{aligned}$$

Table 3.1. Central extensions of A

Type (A)	Number of moduli	Number of moduli if $l = 0$
A	5	2
B	2	1
E	-1	-1
H	0	0
S_1	-1	-1
S_1'	0	0
S_2	-1	-1

To find all simple extensions D, we must find all l and α such that (3.2) is satisfied for $\gamma = 0$. First observe that $x^* = x$ since Q is the identity matrix. Hence if we write $l = (l_1, l_2, l_3)^t = H_3 x$ for some 3×3-matrix H_3 then the first equation in (3.2) says that H_3 must be symmetric. The second equation in (3.2) says that α may be chosen freely. Given such α and l (and hence A), by making a substitution of the form $x \to x + uz$ we can transform A to an equivalent algebra in which $\alpha = 0$. In general this will be possible in a finite number of ways. Since we may still perform the substitution $z \to vz$ we see that generically our equivalence classes are parametrized by projectivized symmetric 3×3-matrices, i.e. a 5 dimensional family. If we now consider those with $l = 0$ then α may be arbitrary, and allowing for substitutions of the form $z \to vz$, it follows that these form a 2-dimensional family.

Type B $Q = \operatorname{diag}(1, 1, -1)$ and the defining equations for A are

$$\begin{aligned} f_1 &= x_1x_2 + x_2x_1 + x_2^2 - x_3^2 \\ f_2 &= x_1^2 + x_2x_1 + x_1x_2 - ax_3^2 \\ f_3 &= x_3x_1 - x_1x_3 + ax_3x_2 - ax_2x_3 \,. \end{aligned}$$

Since we restrict ourselves to simple solutions we find from (3.2) that $\alpha_3 = 0$, $l_3 = 0$, and there is a symmetric 2×2-matrix H_2 such that $(l_1, l_2)^t = H_2(x_1, x_2)^t$. We may still perform the substitutions

$$x_1 \to x_1 + u_1 z \quad x_2 \to x_2 + u_2 z \quad x_3 \to x_3 \tag{3.11}$$

without destroying the simpleness of our solution. We can use (3.11) to normalize to $\alpha = 0$ in a finite number of ways i.e. now the equivalence classes are generically parametrized by projectivized symmetric 2×2-matrices i.e. a three dimensional family.

Type E $Q = \operatorname{diag}(\zeta_9, \zeta_9^4, \zeta_9^7)$ and the defining equations are

$$\begin{aligned} f_1 &= x_3x_1 + \zeta_9^8 x_1x_3 + \zeta_9^4 x_2^2 \\ f_2 &= x_1x_2 + \zeta_9^5 x_2x_1 + \zeta_9^7 x_3^2 \\ f_3 &= \zeta_9 x_1^2 + x_2x_3 + \zeta_9^2 x_3x_2 \,. \end{aligned}$$

Corollary 3.2.4 applies, so D is equivalent to $A[z]$.

Type H $Q = \operatorname{diag}(1, -1, \zeta_A)$ and the defining equations are

$$\begin{aligned} f_1 &= x_1^2 - x_2^2 \\ f_2 &= x_1x_2 - x_2x_1 + \zeta_4 x_3^2 \\ f_3 &= x_2x_3 - \zeta_4 x_3x_2 \,. \end{aligned}$$

It follows from (3.2) that $\alpha_2 = \alpha_3 = 0$, $l_2 = l_3 = 0$ and l_1 is a scalar multiple of x_1. Making a suitable substitution $x_1 \to x_1 + u_1 z$, $x_2 \to x_2$, $x_3 \to x_3$, $z = vz$ we can make $\alpha_3 = 0$ also, and make this scalar equal to 0 or 1. Hence there is (up to equivalence) a unique algebra D which is not of the form $A[z]$; it is given by $\alpha = 0$, $l_1 = x_1$, $l_2 = l_3 = 0$. Finally, the extension with $l_1 = l_2 = l_3 = 0$, $\alpha_2 = \alpha_3 = 0$ and $\alpha_1 = 1$ is not equivalent to $A[z]$, and this algebra gives the unique D (up to equivalence) having no linear terms.

Type S_1 $Q = \mathrm{diag}(\alpha, \beta, (\alpha\beta)^{-1})$ with α and β generic, and the defining equations are

$$f_1 = x_2x_3 + a\beta x_3x_2$$

$$f_2 = \alpha x_3x_1 + ax_1x_3$$

$$f_3 = x_1x_2 + ax_2x_1 \ .$$

Corollary 3.2.4 applies, so D is equivalent to $A[z]$.

Type S_1' $Q = \mathrm{diag}(\alpha, \alpha^{-1}, 1)$ and the defining equations for A are

$$f_1 = x_2x_3 + a\alpha^{-1}x_3x_2$$

$$f_2 = \alpha x_3x_1 + ax_1x_3$$

$$f_3 = x_3^2 + x_1x_2 + ax_2x_1 \ .$$

This is similar to Type H. There is a unique algebra not of the form $A[z]$ and it is equivalent to an extension with no linear term.

Type S_2 $Q = \mathrm{diag}(\alpha, -\alpha, \alpha^{-2})$ with α generic and the defining equations are

$$f_1 = x_3x_1 + \alpha^{-1}x_1x_3$$

$$f_2 = x_3x_2 - \alpha^{-1}x_2x_3$$

$$f_3 = x_1^2 - x_2^2$$

Corollary 3.2.4 applies so D is equivalent to $A[z]$. □

4 The point variety

4.1 General results

Let $T = k\langle x_0, \ldots, x_n\rangle$ be the free algebra. If g is a homogeneous element of degree d in T then we will denote by g also, the image of g in the commutative polynomial ring $k[x_0, \ldots, x_n]$, i.e. g becomes a section of $\mathcal{O}(d)$ on $\mathbb{P} = \mathbb{P}(T_1^*)$.

On the other hand if $g = \Sigma\, a_{ij}x_ix_j$ then $g^{(1,2)}$ will denote the multilinearization of g in the sense of [3]; that is, $g^{(1,2)} = \Sigma\, a_{ij}x_i^{(1)}x_j^{(2)}$ where $(x_i^{(1)})_i, (x_i^{(2)})_i$ represent homogeneous coordinates on $\mathbb{P}$ corresponding to the basis $(x_i)_i$ of T_1. Then $g^{(1,2)}$ defines a section of $\mathcal{O}(1,1)$ on $\mathbb{P} \times \mathbb{P}$.

Suppose that $D = T/(g_1, \ldots, g_m)$ is a quadratic algebra. Then we will write Γ_D for the scheme defined by $(g_i^{(1,2)})_{i=1,\ldots,m}$. Thus $\Gamma_D \subset \mathbb{P} \times \mathbb{P}$. Furthermore we define $\mathscr{P}_D = \mathrm{pr}_1(\Gamma_D) \subset \mathbb{P}$, the projection onto the first coordinate.

Recall that a point module over D is a graded left D-module $M_0 \oplus M_1 \oplus \cdots$ generated in degree zero such that $\dim M_i = 1$ for all $i \geqq 0$. This definition may, in an obvious way, be extended to families so as to define a functor from Sch/k to Sets (see [3]). Our definition of $\mathcal{P}_D$ may then be justified by the following specialization of a result in [3].

Theorem 4.1.1 *Suppose that* $\mathcal{P}_D = \mathrm{pr}_1(\Gamma_D) = \mathrm{pr}_2(\Gamma_D)$ *and that* Γ_D *is the graph of an automorphism* $\sigma_D : \mathcal{P}_D \to \mathcal{P}_D$. *Then*

1. $\mathcal{P}_D$ *represents the functor of point modules, and the truncation functor* $M \mapsto M_{\geqq 1}(1)$ *is represented by* σ^{-1}.
2. *Every point module of* D *is of the form* $D/Dy_1 + \cdots + Dy_n$ *where* $(y_i)_i \in D_1$. *The corresponding point in* $\mathcal{P}_D$ *is given by the common zero of* $(y_i)_i$.

Proof. 1. The first statement is given in [3, Corollary 3.13] and the remarks thereafter.

2. Suppose to the contrary that $M := D/Dy_1 + Dy_2 + \cdots + Dy_n$ is not a point module. It certainly maps onto the point module corresponding to $\mathcal{V}(y_1, \ldots, y_n)$, and hence (after shifting) has a cyclic subquotient, N say, with $\dim N_0 = \dim N_1 = 1$ and $\dim N_2 = 2$. Choose elements e_0, e_1, e_{21}, e_{22} such that $N_0 = ke_0$, $N_1 = ke_1$, $N_2 = ke_{21} + ke_{22}$. Then there are points $p, q_1, q_2 \in D_1^*$ such that $x \cdot e_0 = x(p)e_1$ and $x \cdot e_1 = x(q_1)e_{21} + x(q_2)e_{22}$ for all $x \in D_1$. Since N is cyclic q_1 and q_2 are linearly independent. Since N/ke_{22} and N/ke_{21} are truncated point modules of length 3 the points (p, q_1) and (p, q_2) (viewed in $\mathbb{P} \times \mathbb{P}$) are both in Γ_D. But $q_1 \neq q_2$ and this contradicts the fact that Γ_D is the graph of an automorphism of $\mathcal{P}_D$. □

If the hypotheses of Theorem 4.1.1 apply (and they always will in the examples we consider in this paper) then we will call the pair $(\mathcal{P}_D, \sigma_D)$ the *point variety* of D. If $p \in \mathcal{P}_D$ then the corresponding point module will be denoted $M(p)$. Thus (4.1.1.1) says that $M(p)_{\geqq 1}(1) \cong M(p^{\sigma^{-1}})$.

Lemma 4.1.2 *Suppose that* $(\Gamma_D)_{\mathrm{red}}$ *defines an isomorphism between* $\mathrm{pr}_1(\Gamma_D)_{\mathrm{red}}$ *and* $\mathrm{pr}_2(\Gamma_D)_{\mathrm{red}}$. *Then* Γ_D *defines an isomorphism between* $\mathrm{pr}_1(\Gamma_D)$ *and* $\mathrm{pr}_2(\Gamma_D)$.

Proof. This is a consequence of [3, Proposition 3.6]. If $p \in \mathrm{pr}_1(\Gamma_D)$ then the preimage of p is (scheme-theoretically) a linear space, and by hypothesis is 0-dimensional. Hence the preimage consists of a single reduced point. Hence by the argument in [3] pr_1 is an isomorphism in a neighbourhood of p. The same argument applies to pr_2, whence both projections are isomorphisms from Γ_D. □

We will need the following result in the next subsection:

Theorem 4.1.3 *Suppose that* D *is an Artin–Schelter regular algebra with Hilbert series* $(1-t)^{-4}$. *Suppose furthermore that* $(\Gamma_D)_{\mathrm{red}}$ *defines an isomorphism between* $\mathrm{pr}_1(\Gamma_D)_{\mathrm{red}}$ *and* $\mathrm{pr}_2(\Gamma_D)_{\mathrm{red}}$. *Then* $\mathrm{pr}_1(\Gamma_D) = \mathrm{pr}_2(\Gamma_D)$ *and* Γ_D *defines an automorphism of* $\mathcal{P}_D = \mathrm{pr}_1(\Gamma_D)$

Proof. Let $x = (x_1, \ldots, x_4)^t$ be the generators of D. It follows from the Gorenstein property, and the hypothesis that $H_D(t) = (1-t)^{-4}$ and that ${}_Dk$

has a minimal resolution of the form

$$0 \longrightarrow D \xrightarrow{x^t} D^4 \xrightarrow{P(x)} D^6 \xrightarrow{Q(x)} D^4 \xrightarrow{x} D \longrightarrow k \longrightarrow 0$$

where we have written $P(x), Q(x)$ to emphasize the dependence on x. Hence R_D is spanned by the entries of the vectors x^tP and Qx.

The fact that $(\Gamma_D)_{\mathrm{red}}$ defines an isomorphism means that if $0 \neq \zeta = (\zeta_1, \ldots, \zeta_4)^t \in L^4$ is a vector with entries in some field extension L/k then

$$\operatorname{rk} P(\zeta) \geqq 3 \quad \text{and} \quad \operatorname{rk} Q(\zeta) \geqq 3\,. \tag{4.1}$$

We now show that for any local k-algebra R and any R-point (p', p'') of Γ_D there exist R-points of the form (p, p') and (p'', p''') in Γ_D.

Let $p' := x^{(1)} = (x_1^{(1)}, \ldots, x_4^{(1)}) \in R^4$, $p'' := x^{(2)} = (x_1^{(2)}, \ldots, x_4^{(2)}) \in R^4$. Then there is a complex

$$R^4 \xrightarrow{P(x^{(1)})} R^6 \xrightarrow{Q(x^{(2)})} R^4 \tag{4.2}$$

However, using (4.1), we see that (4.2) is exact after restricting to the residue class field of R. Therefore (4.2) itself is split exact. Hence we may find $x^{(0)} = (x_1^{(0)}, \ldots, x_4^{(0)})$ and $x^{(3)} = (x_1^{(3)}, \ldots, x_4^{(3)}) \in R^4$ such that

$$0 \longrightarrow R \xrightarrow{x^{(0)t}} R^4 \xrightarrow{P(x^{(1)})} R^6$$

and

$$R^6 \xrightarrow{Q(x^{(2)})} R^4 \xrightarrow{x^{(3)}} R \longrightarrow 0$$

are exact and the $x_i^{(0)}, x_i^{(3)}$ are independent after tensoring with the residue class field of R. Putting $p := x^{(0)t}$, $p''' := x^{(3)t}$ yields the desired result.

This shows that $\mathrm{pr}_1(\Gamma_D) = \mathrm{pr}_2(\Gamma_D)$. The last assertion of the Theorem follows from Lemma 4.1.2. □

4.2 Point varieties of central extensions of three dimensional Artin–Schelter regular algebras

In this section D will be a central extension of a three dimensional Artin–Schelter regular algebra $A = D/(z)$ with Hilbert series $(1-t)^{-3}$. We will use the notation of Section 3.

In particular $A = k\langle x_1, x_2, x_3\rangle/(f_1, f_2, f_3)$ where $f = Mx$. However in this section we will *not* assume that M is normalized in such a way that there exists a Q such that $x^tM = (Qf)^t$. The equations of D are

$$\begin{aligned} g_j := f_j + zl_j + \alpha_j z^2 &= 0 \quad j = 1,2,3 \\ zx_i - x_i z &= 0 \quad i = 1,2,3 \end{aligned}$$

where $(l_j)_j, (\alpha_j)_j$ satisfy the conditions of Theorem 3.1.3.

Recall that by [3] the equation of $\mathscr{P}_A$ is given by $\det M = 0$. If $\det M$ is identically zero then one says that A is *linear*. Otherwise one says that A is *elliptic* because then $\det M$ defines a divisor of degree of 3 in $\mathbb{P}^2$, i.e. a scheme of arithmetic genus one.

Clearly $\mathscr{P}_A \subset \mathscr{P}_D$. Our aim will be to describe how much $\mathscr{P}_D$ differs from $\mathscr{P}_A$. To this end we have to introduce some auxilliary forms on $\mathbb{P}(D_1^*)$.

Lemma 4.2.1 *Let $g = (g_1, g_2, g_3)^t, l = (l_1, l_2, l_3)^t, \alpha = (\alpha_1, \alpha_2, \alpha_3)^t$ and let M_1, M_2, M_3 be the columns of M, i.e. $M = [M_1\, M_2\, M_3]$. Define*

$$h_1 = x_1 \det M + z \det [l\, M_2\, M_3] + z^2 \det [\alpha\, M_2\, M_3]$$

$$h_2 = x_2 \det M + z \det [M_1\, l\, M_3] + z^2 \det [M_1\, \alpha\, M_3]$$

$$h_3 = x_3 \det M + z \det [M_1\, M_2\, l] + z^2 \det [M_1\, M_2\, \alpha]$$

Then h_i is in the ideal generated by (g_1, g_2, g_3).

Proof. By definition $h_1 = \det[x_1 M_1 + zl + \alpha z^2, M_2, M_3]$ (expand down the first column). Since $f = Mx = x_1 M_1 + x_2 M_2 + x_3 M_3$, it follows that $h_1 = \det[f + zl + \alpha z^2, M_2, M_3]$ also (we have just added to the first column a linear combination of the other columns). But this is just $h_1 = \det[g, M_2, M_3]$ and by expanding down the first column, one sees that h_1 is in the ideal (g_1, g_2, g_3). Explicitly

$$h_i = \hat{M}_{1i} g_1 + \hat{M}_{2i} g_2 + \hat{M}_{3i} g_3 \tag{4.3}$$

where $\hat{M}_{ij}$ is the minor obtained by deleting the i'th row and the j'th column from M. $\square$

Theorem 4.2.2 1. *The hypotheses in Theorem* 4.1.1 *hold for D. In particular, the point variety $(\mathscr{P}_D, \sigma_D)$ of D exists.*

2. *$\mathscr{P}_A = \mathscr{P}_D \cap \mathscr{V}(z)$. On $\mathscr{P}_D \cap \mathscr{V}(z)$, σ_D restricts to σ_A and on $\mathscr{P}_D \cap \mathscr{V}(z)^c, \sigma_D$ is the identity.*

3. *$(\mathscr{P}_D)_{\text{red}} = (\mathscr{P}_A)_{\text{red}} \cup \mathscr{V}(g_1, g_2, g_3)_{\text{red}}$.*

4. *The equations for $\mathscr{P}_D$ are as follows:*

(a) *On $\mathscr{P}_D \cap \mathscr{V}(z)^c : g_1 = g_2 = g_3 = 0$.*

(b) *On $\mathscr{P}_D \cap \mathscr{V}(x_i)^c : zg_1 = zg_2 = zg_3 = h_i = 0$.*

5. *Suppose that A is linear. Then there is a vector μ (unique up to scalar multiples) of independent linear forms such that $\mu^t M = 0$. Define $q := \mu^t(l + \alpha z)$. Then the equations of $\mathscr{P}_D$ are*

$$zq = zg_1 = zg_2 = zg_3 = 0\,.$$

Furthermore q, g_1, g_2, g_3 are related by the identity:

$$zq = \mu^t g\,. \tag{4.4}$$

Proof. The defining equations of Γ_D are

$$M^{(1)}x^{(2)} + l^{(1)}z^{(2)} + \alpha z^{(1)}z^{(2)} = 0 \tag{4.5}$$

$$z^{(1)}x^{(2)} - x^{(1)}z^{(2)} = 0 \tag{4.6}$$

with the obvious notation. Because of (4.6) one can replace the $l^{(1)}z^{(2)}$ term in (4.5) by $z^{(1)}l^{(2)}$. It follows from these equations that $\mathscr{P}_D \cap \mathscr{V}(z^{(1)})$ is given by the equation $\det M^{(1)} = 0$ which is the defining equation of $\mathscr{P}_A$. On $\mathscr{P}_D \cap \mathscr{V}(z)^c$ we can take $z^{(1)} = 1$, and it then follows from these equations that this part of $\mathscr{P}_D$ is defined by $g^{(1)} = 0$. Thus (3) holds. Consequently the hypotheses for Theorem 4.1.3 hold. From this theorem we then deduce (1). The fact that σ_D exists implies that $\mathscr{P}_A = \mathscr{P}_D \cap \mathscr{V}(z)$ and that σ_D restricts to σ_A. It follows directly from (4.5) and (4.6) that σ_D is the identity on $\mathscr{P}_D \backslash \mathscr{V}(z)$. Hence (2) is true. It remains to prove (4) and (5).

Fix a point $p \in (\mathscr{P}_D)_{\mathrm{red}}$ with coordinates (x_0, z_0). Let (p, p') be the closed point of Γ_D projecting to p. Write (x', z') for the coordinates of p'. We want to determine the equations in $(x^{(1)}, z^{(1)})$ (locally around p) such that there is a non-trivial solution to equations (4.5) and (4.6).

Suppose that $z_0 \neq 0$. Then $z' \neq 0$ for otherwise (4.6) forces $x' = 0$ whence $p' = (0,0)$ which is absurd. Hence we may assume that $z^{(1)} = z^{(2)} = 1$ locally on Γ_D around (p, p'). Then by (4.6) $x^{(2)} = x^{(1)}$. Substituting this in (4.5) we get $g^{(1)} = 0$, i.e. $g_1 = g_2 = g_3 = 0$. This proves (4a).

Suppose that $(x_0)_1 \neq 0$. Then we may assume that $x_1^{(1)} = 1$ locally around p. From (4.6) we deduce that $z^{(2)} = z^{(1)}x_1^{(2)}$ on Γ_D around (p, p'). By resubstituting in (4.6) this gives $z^{(1)}(x_k^{(2)} - x_1^{(2)}x_k^{(1)}) = 0$ for all k. If $p \neq p'$ we may proceed as follows. As $p \neq p'$ some $x_k^{(2)} - x_1^{(2)}x_k^{(1)}$ will be invertible in a neighbourhood of (p, p'). Hence locally around (p, p') we must have $z^{(1)} = 0$ and therefore Γ_D has equations $z^{(1)} = z^{(2)} = M^{(1)}x^{(2)} = 0$ around (p, p'). As $x^{(2)}$ cannot be the zero vector, this implies that $\mathscr{P}_D$ has defining equations

$$z^{(1)} = 0, \qquad \det M^{(1)} = 0 \tag{4.7}$$

around p. These equations clearly imply the equations of (4b). Conversely, as $p \neq p'$, p cannot be a common zero of the g_i. So locally some g_i is invertible giving $z = 0$ whence (4.7). This proves 4b when $p \neq p'$.

Suppose that $p = p'$. If $z_0 \neq 0$ then as before $g_1 = g_2 = g_3 = 0$ whence also $h_1 = 0$ by the foregoing lemma. Hence the difficult case is when $z_0 = 0$.

So assume $z_0 = z' = 0$ and $x_1^{(1)} = x_1^{(2)} = 1$ locally around (p, p'). As before we have $z^{(1)} = z^{(2)}$ and $z^{(1)}(x^{(2)} - x^{(1)}) = 0$ around (p, p'). Define $\varepsilon_k = x_k^{(2)} - x_k^{(1)}$; notice in particular that $\varepsilon_1 = 0$ in a neighbourhood of (p, p'). The defining equations for Γ_D can be rewritten as

$$M^{(1)}x^{(1)} + M^{(1)}\varepsilon + l^{(1)}z^{(1)} + \alpha z^{(1)2} = 0 \tag{4.8}$$

$$z^{(1)}\varepsilon = 0 \tag{4.9}$$

where $\varepsilon = (0, \varepsilon_2, \varepsilon_3)^t$. At p, the columns $M_2^{(1)}$ and $M_3^{(3)}$ are linearly independent for otherwise $M^{(1)}x^{(2)} = 0$ would have another solution with $x_1^{(2)} = 0$

(contradicting the fact that $p' = p$). So some 2×2 minor of $[M_2^{(1)}, M_3^{(1)}]$ is invertible in a neighbourhood of p. Hence we can pick two equations of (4.8) and solve for $\varepsilon_2, \varepsilon_3$; substituting these in the other equation of (4.8) yields the equations in (4b).

For example suppose that $\hat{M}_{31} = m_{12}m_{23} - m_{22}m_{13}$ is invertible around p (we will drop the superscripts (1) temporarily). Then the first two equations of (4.8) can be solved locally and give us formulas for ε_2 and ε_3 viz.

$$\hat{M}_{32}x_1 - \hat{M}_{31}x_2 + \hat{L}_{32}z + \hat{K}_{32}z^2 = \hat{M}_{31}\varepsilon_2 \tag{4.10}$$

$$\hat{M}_{33}x_1 - \hat{M}_{31}x_3 + \hat{L}_{33}z + \hat{K}_{33}z^2 = \hat{M}_{31}\varepsilon_3 \tag{4.11}$$

where $L = [l, M_2, M_3]$, $K = [\alpha, M_2, M_3]$ and $\hat{L}_{ij}$ and $\hat{K}_{ij}$ denote the minors. Substituting these values in the third equation of (4.8) gives

$$(m_{31}\hat{M}_{31} + m_{32}\hat{M}_{32} + m_{33}\hat{M}_{33})x_1 + (l_3\hat{M}_{31} + m_{32}\hat{L}_{32} + m_{33}\hat{L}_{33})z$$
$$+ (\alpha_3\hat{M}_{31} + m_{32}\hat{K}_{32} + m_{33}\hat{K}_{33})z^2 = 0 .$$

But this expression is simply $\det(M)x_1 + \det(L)z + \det(K)z^2 = h_1$ so $h_1 = 0$ around p. If (4.8) is multiplied by $z^{(1)}$ and (4.9) is substituted, then one obtains $zg^{(1)} = 0$. This shows that the equations in (4b) do indeed vanish in a neighbourhood of p.

Now assume that A is linear. Then $\mathscr{P}_A = \mathbb{P}(A_1^*) \cong \mathbb{P}^2$. We will write σ for the linear automorphism of A_1 which induces σ_A. We transfer this automorphism to A_1 by defining $x_i^\sigma(p) = x_i(p^\sigma)$ where x_1, x_2, x_3 is the basis for A_1. Because Γ_A is the graph of σ, it follows that $\mu^t M = 0$ where

$$\mu^t = (x_1^{\sigma^{-1}}, x_2^{\sigma^{-1}}, x_3^{\sigma^{-1}}) .$$

The uniqueness of μ follows from the fact that $\operatorname{rank} M = 2$ at all points of $\mathbb{P}^2$. Since $\det M = 0$ we also have $(\hat{M}_{1i}, \hat{M}_{2i}, \hat{M}_{3i})M = 0$. Since $\operatorname{rank} M = 2$ at all points it follows that $(\hat{M}_{1i}, \hat{M}_{2i}, \hat{M}_{3i}) = y\mu^t$ for some linear form y. Pick a point p such that $x_i(p) = 0$. Suppose that $\{i,j,k\} = \{1,2,3\}$. It follows that columns j and k of M are dependent at $p^{\sigma^{-1}}$ and hence that $\hat{M}_{1i}, \hat{M}_{2i}$ and $\hat{M}_{3i}$ all vanish at $p^{\sigma^{-1}}$. Thus these minors vanish along the line where x_i^σ vanishes. Therefore

$$(\hat{M}_{1i}, \hat{M}_{2i}, \hat{M}_{3i})^t = x_i^\sigma \, \mu \tag{4.12}$$

up to a scalar multiple. From the definition of q we have

$$zq = \mu^t(lz + \alpha z^2) = \mu^t(g - f) = \mu^t(g - Mx) = \mu^t g$$

as claimed. By Lemma 4.2.1 we have

$$h_i = \hat{M}_{1i}g_1 + \hat{M}_{2i}g_2 + \hat{M}_{3i}g_3 = x_i^\sigma \mu^t g = x_i^\sigma zq .$$

Using (4.4), (4.12) and (4.3) one deduces (5) from (4). □

This theorem can be understood through its ensuing corollary:

Corollary 4.2.3 *Let*

$$Y = \begin{cases} \mathscr{V}(g_1, g_2, g_3) & \text{if } A \text{ is elliptic} \\ \mathscr{V}(q, g_1, g_2, g_3) & \text{if } A \text{ is linear}. \end{cases}$$

Then there is an exact sequence

$$0 \longrightarrow \mathscr{O}_Y(-1) \xrightarrow{\phi} \mathscr{O}_{\mathscr{P}_D} \xrightarrow{\theta} \mathscr{O}_{\mathscr{P}_A} \longrightarrow 0 \tag{4.13}$$

where θ is the restriction map and ϕ is induced by multiplication by z : $\mathscr{O}_{\mathbb{P}(D_1^*)}(-1) \longrightarrow \mathscr{O}_{\mathbb{P}(D_1^*)}$.

Proof. If A is linear this follows directly from Theorem 4.2.2.5. If A is not linear then we may check the existence and exactness of (4.13) locally. Of course we use the open cover in Theorem 4.2.2.4. Notice that in the elliptic case, the fact that $\det M \neq 0$, is used crucially on $\mathscr{V}(x_i)^c$. The result now follows from (4.2.2.4) and Lemma 4.2.1. □

Hence in a certain sense $\mathscr{P}_D$ is the union of $\mathscr{P}_A$ and the (scheme-theoretic) base locus of three or four quadrics.

Remark 4.2.4 1. If A is elliptic then Y represents the functor of non-trivial one-dimensional representations of D, so it has a simple interpretation. The appearance of q in the linear case seems harder to understand. In the case of homogenizations of enveloping algebras of three dimensional Lie algebras $\mathscr{V}(q)$ was seen to represent codimension one Lie algebras [9] [10]. However it is not clear to us whether, in general, $\mathscr{V}(q)$ or $\mathscr{V}(q, g_1, g_2, g_3)$ represents a similar, easy to understand functor.

2. Unlike the 3-dimensional case, the point variety may have embedded components. For example, if $\mathscr{V}(g_1, g_2, g_3)$ contains a zero dimensional component lying in $\mathscr{V}(z)$.

3. One of the conclusions of Theorem 4.2.2 (through Theorem 4.1.1), namely that point modules for D are of the form $D/Du + Dv + Dw$, u, v, $w \in D_1$ was proved in greater generality in [8]. In our case (i.e. for central extensions of three dimensional Artin–Schelter regular algebras) it can also be easily proved directly.

4. Theorem 4.2.2 seems to load to two different descriptions of $\mathscr{P}_D \cap \mathscr{V}(x_i)^c \cap \mathscr{V}(x_j)^c$. It is a pleasant exercise, left to the reader, to check that these descriptions are the same.

The defining relations for a 3-dimensional Artin–Schelter regular algebra can be described in geometric terms [3]: a tensor $f \in A_1 \otimes A_1$ belongs to R_A if and only if f vanishes on $\Gamma_A \subset \mathbb{P}^2 \times \mathbb{P}^2$. The next result shows that a similar result is true for D (at least when A is elliptic).

Proposition 4.2.5 *Suppose that D is a central extension of a three dimensional elliptic regular algebra $A = D/(z)$ with Hilbert series $(1-t)^{-3}$. Then a tensor $g \in D_1 \otimes D_1$ belongs to R_D if and only if g vanishes on Γ_D.*

Proof. If $g \in R_D$ then $g(\Gamma_D) = 0$ by definition. Conversely, suppose that $g \in D_1 \otimes D_1$ and $g(\Gamma_D) = 0$. Write

$$g = f + l \otimes z + \alpha z \otimes z, \quad f \in A_1 \otimes A_1, \ l \in A_1 .$$

In particular g vanishes on $\Gamma_A = \Gamma_D \cap \mathrm{pr}_1^{-1}(\mathscr{V}(z)) \cap \mathrm{pr}_2^{-1}(\mathscr{V}(z))$ whence $f \in R_A$. Hence after adding a suitable linear combination of the relations g_1, g_2, g_3 to g we may assume that $f = 0$. Thus $(l^{(1)} + \alpha z^{(1)})z^{(2)}$ vanishes on Γ_D.

The proof of (4.2.2.4) showed that $z^{(1)} = z^{(2)}$ on Γ_D. Thus $(l^{(1)} + \alpha z^{(1)})z^{(1)}$ vanishes on Γ_D and hence on $\mathscr{P}_D$. We may drop the superscripts and say that $(l + \alpha z)z$ vanishes on $\mathscr{P}_D$. By the short exact sequence (4.13) this implies that $l + \alpha z$ vanishes on the subscheme $Y = \mathscr{V}(g_1, g_2, g_3)$. However, by Lemma 4.2.6 below this scheme is not contained in a plane, so $l + \alpha z = 0$. Therefore the original g belongs to R_D. □

Lemma 4.2.6 *The scheme-theoretic intersection of three quadrics, g_1, g_2, g_3 in $\mathbb{P}^3$, is not contained in a plane.*

Proof. Suppose that the lemma is false. We may, without loss of generality, assume that the three quadrics are independent.

By assumption there is a linear form l such that $\mathscr{V}(g_1, g_2, g_3) \subset \mathscr{V}(l)$. If $\mathscr{V}(g_1, g_2, g_3)$ is zero-dimensional then $\mathscr{V}(g_1, g_2, g_3) = \mathscr{V}(g_1, g_2, g_3) \cap \mathscr{V}(l)$ cannot consist of eight points (with multiplicity counted) since it is the intersection of three conics in a plane. Hence $\mathscr{V}(g_1, g_2, g_3)$ is either a non-degenerate conic in $\mathscr{V}(l)$ or contains at least a $\mathbb{P}^1$.

Suppose that the linear system spanned by g_1, g_2, g_3 contains a pair of planes, say $\mathscr{V}(g_1 = uv)$. Then $\mathscr{V}(l) \supset \mathscr{V}(uv, g_2, g_3) \supset \mathscr{V}(u, g_2, g_3) \cup \mathscr{V}(v, g_2, g_3)$. Suppose that $\mathscr{V}(v) \neq \mathscr{V}(l)$. Then on the plane $\mathscr{V}(v)$ the scheme theoretic intersection of the conics $\mathscr{V}(v, g_2)$ and $\mathscr{V}(v, g_3)$ is contained in the line $\mathscr{V}(v, l)$. Since the scheme theoretic intersection of two coplanar conics can not be contained in a line it follows that $\mathscr{V}(v) = \mathscr{V}(l)$. Similarly $\mathscr{V}(u) = \mathscr{V}(l)$. Thus the only possible pair of planes in this linear system is the double plane $\mathscr{V}(l^2)$.

Suppose that $\mathscr{V}(g_1, g_2, g_3)$ is a non-degenerate conic. Then there exists a degree 2 form g, and linear forms x_i, such that (up to a scalar multiple) $g_i = g + lx_i$. But then the linear system contains $l(x_1 - x_2)$ and $l(x_1 - x_3)$ which are independent since the g_i are independent. This contradicts the last paragraph.

Hence $\mathscr{V}(g_1, g_2, g_3)$ contains a line L. The set of all quadrics in $\mathbb{P}^3$ containing L forms a $\mathbb{P}^6$. The subset of quadrics which are the union of two planes, one of which contains L, forms a 4-dimensional family in $\mathbb{P}^6$. By assumption g_1, g_2, g_3 spans a $\mathbb{P}^2$ in $\mathbb{P}^6$. Hence the linear system contains a pair of planes, which by the above must be $\mathscr{V}(l^2)$. We may assume that $g_1 = l^2$. Since no linear combination of g_2 and g_3 contains a plane, $\mathscr{V}(l, g_2, g_3)$ is the intersection of two independent coplanar conics that have a common line L. Hence $\mathscr{V}(l, g_2, g_3)$ is L together with a point (possibly embedded). Specifically $\mathscr{V}(l, g_2, g_3)$ is either $\mathscr{V}(l, xy, xz)$ or $\mathscr{V}(l, xy, x^2)$ where l, x, y, z are independent linear forms. In both cases $\mathscr{V}(l^2, g_2, g_3)$ will be strictly bigger than $\mathscr{V}(l, g_2, g_3)$. This yields a contradiction. □

Remark 4.2.7 1. We think that Proposition 4.2.5 is also true when A is linear. This would amount to showing that the scheme $\mathscr{V}(q, g_1, g_2, g_3)$ is not contained in a plane, but we don't see how to do this. Presumably (4.4) will be important.

2. Lemma 4.2.6 is probably a special case of a more general result, but we have not found an appropriate reference.

3. In Remark 4.3.6 below we will show that there may be tensors $g \in D_1 \otimes D_1$ which vanish on $(\Gamma_D)_{\text{red}}$ but are not contained in R_D. This illustrates the importance of considering the full scheme Γ_D rather than just $(\Gamma_D)_{\text{red}}$.

4.3 Explicit computations in the generic cases

This section computes (the reduced part of) the point variety $\mathscr{P}_D$. For simplicity we sometimes restrict attention to the case where there are no linear terms in the relations of D (i.e. $l_1 = l_2 = l_3 = 0$ in the notation of Section 3), because this allows us to use a well known classification of pencils and nets of conics in the plane [20]. By Theorem 4.2.2.3 $(\mathscr{P}_D)_{\text{red}} = (\mathscr{P}_A)_{\text{red}} \cup \mathscr{V}(g_1, g_2, g_3)_{\text{red}}$ so to compute $(\mathscr{P}_D)_{\text{red}}$ we must compute the (reduced) base locus of these three quadrics. Usually we will denote $(\mathscr{P}_D)_{\text{red}}$ by $\mathscr{P}_D$ and if σ_D is defined then we will denote $(\sigma_D)_{\text{red}}$ by σ_D also.

First, a general remark.

Lemma 4.3.1 *Let $D = k\langle x_1, \ldots, x_n\rangle/(g_1, \ldots, g_m)$ be a quadratic algebra. Write $\mathscr{N}_D$ for the linear system of quadrics in $\mathbb{P}(D_1^*)$ spanned by the $\mathscr{V}(g_i)$. There is a bijection between the following five sets:*

1. *base points of $\mathscr{N}_D$;*
2. *point modules which have a non-trivial* 1*-dimensional quotient module;*
3. *two-sided ideals J of D such that D/J is isomorphic to a polynomial ring in* 1 *indeterminate;*
4. *points of $(\Gamma_D \cap \Delta)_{\text{red}}$ where $\Delta \subset \mathbb{P}(D_1^*) \times \mathbb{P}(D_1^*)$ is the diagonal;*
5. *the fixed points of σ_D (if σ_D is defined).*

Proof. Write I for the ideal of D generated by the image of the skew symmetric tensors in D. Thus D/I is the largest commutative quotient of D. It is clear that each base point gives a point module for D/I which is necessarily of the form D/J for some two-sided ideal J, and that such a point module has a non-trivial 1-dimensional quotient. Conversely if M is a point module with a non-trivial 1-dimensional quotient, then as in [8, Proposition 5.9], $M \cong D/J$ for some two sided ideal J and $D/J \cong k[X]$. Since D/J is commutative $I \subset J$, whence M corresponds to a point in the base locus of $\mathscr{N}_D$. Hence there is a bijection as claimed.

To verify the third statement, it suffices to observe that a non-trivial 1-dimensional module is a quotient of a point module. This is easy, and is proved in [16, Proposition 2.2] (also see [8, Proposition 5.9]).

The fourth and the fifth statements are clear. □

Table 4.1. Pencils of conics

Type	s	d	b
α	3	0	4
β	2	0	3
γ	2	1	2
δ	1	0	2
ε	1	1	1
ϵ	∞	0	$1+\infty$
ζ	∞	2	1
η	∞	1	∞

Table 4.2. Nets of conics

Type	d	b	locus of the singular conics
A	0	0	smooth cubic
B	1	0	cubic with a node dl
B^*	0	1	cubic with a node
C	1	1	cubic with a cusp dl
D	2	0	conic+line intersecting in 2 dl
D^*	0	2	conic+line intersecting twice
E	3	0	triangle with vertices dl
E^*	0	3	triangle
F	2	1	line+double line containing 2 dl away from intersection
F^*	1	2	conic+tangent line meeting in dl
G	2	1	line+double line containing 2 dl including intersection
G^*	1	2	line+double line containing dl away from intersection
H	1	1	triple line containing one dl
I	∞_c	1	plane containing conic of dl
I^*	1	∞	plane containing one dl

We will use the classification of pencils (i.e. 1-dimensional linear systems) of conics in $\mathbb{P}^2$ (classical) and nets (i.e. 2-dimensional linear systems) of conics in $\mathbb{P}^2$ (obtained by C.T.C. Wall [20]). Tables 4.1 and 4.2 below summarize the results; more details can be found in [20].

Pencils of Conics Over k there are 8 types of pencils of conics in $\mathbb{P}^2$, see e.g. [20, Table 0]. We will use the following notation for them (differing from that of Wall):

s is the number of singular conics in the pencil,
d is the number of double lines in the pencil,
b is the number of basepoints of the pencil,
∞ denotes a line.

We remark that the base points in type β are not collinear, and no 3 of the basepoints in type α are collinear.

Nets of Conics Over k there are 15 types [20, Table 1 and Table 2] of nets of conics. This classification is determined by the type of the discriminant curve which is a divisor of degree 3 (in all except two cases). As before d is the number of double lines in the net of conics, b is the number of basepoints of the net and the last column of Table 4.2 describes the discriminant locus and which points of it are the double lines (our abbreviation for 'double line' is 'dl'). We denote a conic by ∞_c.

The baselocus is collinear for all nets except those of type E^*.

It seems to be a delicate problem to determine which nets of conics can occur as $\mathcal{N}_A$ for a three dimensional Artin–Schelter regular algebra A. However, for generic Artin–Schelter regular algebras one can use the defining equations (given in Section 3.2) to determine the type of $\mathcal{N}_A$. First, however, we remark that if A is a generic Artin–Schelter regular algebra of type B then the linear system $\mathcal{N}_A$ is a pencil of conics (rather than a net) of type α. One obtains:

For non-generic algebras other types of $\mathcal{N}_A$ can occur.

Example 4.3.2 There are 3-dimensional Artin–Schelter regular algebras A such that $\mathcal{N}_A$ has net-type A, B, D, E and I^*.

Take for A the enveloping algebra of a $(3,3)$-quadratic Lie algebra, in the terminology of [5]. By [5, Proposition 1.7] they are 3-dimensional Artin–Schelter regular and by [5, p. 163] the net-types A, B, D and E occur. To be specific, let $A = \mathbb{C}[x, y, z]$. If the defining relations are $xy + yx = yz + zy = zx + xz = 0$ then $\mathrm{Type}(\mathcal{N}_A) = E$. If the defining relations are $x^2 - z^2 = xy + yx = xz + zx - 2y^2 + bx^2 = 0$ then $\mathrm{Type}(\mathcal{N}_A) = B$ when $b^2 = 1$, $\mathrm{Type}(\mathcal{N}_A) = D$ when $b = 0$, and $\mathrm{Type}(\mathcal{N}_A) = A$ when $b^2 \neq 0, 1$.

Now let $A = \mathbb{C}[x, y, z]$ with relations $xy - \lambda yx = yz - \lambda zy = zx - xz - y^2 = 0$ with $0 \neq \lambda \in \mathbb{C}$. It follows from the Diamond Lemma that A has Hilbert series $(1-t)^{-3}$, and that y is a normal 1-regular element with $A/\langle y \rangle$ a polynomial ring. Since there is a $\phi \in \mathrm{Aut}(A)$ such that $yd = \phi(d)y$ for all $d \in A_1$, we may apply Theorem 2.6, to see that A is Koszul, and hence Artin–Schelter regular. It is clear that $\mathcal{N}_A$ is of type I^*.

Now we will determine $\mathcal{P}_D$ for central extensions of generic Artin–Schelter regular algebras with no linear terms. The first case we discuss is that of a polynomial extension.

Proposition 4.3.3 *Let A be an Artin–Schelter regular algebra with $H_A(t) = (1-t)^{-3}$, and let $D = A[z]$ be the polynomial extension of A. Then P_D is the union of P_A and the cone over the base locus of $\mathcal{N}_A$ (which lies in $\mathcal{V}(z)$) with 'center' $(0,0,0,1)$. In particular:*

1. *if A is generic of type A or E then $\mathcal{N}_A$ has no base points, so $P_D = P_A \cup \{(0,0,0,1)\}$;*

2. *if A is generic of type B then $\mathcal{N}_A$ has four base points, so P_D is the union of P_A and four lines through $(0,0,0,1)$.*

3. *if A is generic of type H or S_1' then P_D is the union of P_A and two lines through $(0,0,0,1)$;*

4. *if A is generic of type S_1 or S_2 then P_D is the union of P_A and three lines through* $(0,0,0,1)$.

Proof. Since the extra relations for D are all of the form $az - za$ for $a \in A_1$ and these all map to zero in the symmetric algebra $S(D_1)$, the equations of the quadrics in $\mathcal{N}_D$ are the same as the equations for the conics in $\mathcal{N}_A$. Hence each quadric in $\mathcal{N}_D$ is the cone (with center $(0,0,0,1)$) over the corresponding conic in $\mathcal{N}_A$. Hence the base locus of $\mathcal{N}_D$ is the cone over the base locus of $\mathcal{N}_A$. Now apply the previous result. Each of the four cases is obtained by using Tables 4.1 and 4.3 to compute the baselocus of $\mathcal{N}_A$. □

Remark 4.3.4 By Theorem 3.2.6, if A is a generic 3-dimensional algebra of type E, S_1 or S_2 then every central extension of A is trivial i.e. it is a polynomial ring $D = A[z]$. Hence the previous result describes the point modules for D when A is generic of type E, S_1, S_2.

We now attend to the case when D is a non-trivial extension of A having no linear terms in z in the defining equations (i.e. $l = 0$ in the notation of the previous section). However, first we require the following result in order to compare $\mathcal{N}_D$ and $\mathcal{N}_A$.

Proposition 4.3.5 *Let x_0, x_1, x_2, z be homogeneous coordinates on $\mathbb{P}^3$. Fix linearly independent quadratic forms $q_1, q_2, q_3 \in \mathbb{C}[x_0, x_1, x_2]$ and scalars $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$ which are not all zero. Define conics $C_i = \mathcal{V}(q_i)$ in $\mathbb{P}^2$ and quadrics $Q_i = \mathcal{V}(q_i + \alpha_i z^2)$ in $\mathbb{P}^3$. Let $\mathcal{N}_A$ (respectively $\mathcal{N}_D$) denote the net of conics (respectively, quadrics) generated by C_1, C_2, C_3 (respectively, Q_1, Q_2, Q_3). Suppose that $\mathcal{N}_A$ is not of type I or I^*. Then*

1. *the locus of singular quadrics in $\mathcal{N}_D$ is a degree 4 divisor of the form $L + \Delta$ where:*
   (i) *L is the pencil of quadrics in $\mathcal{N}_D$ which contain* $(0,0,0,1)$;
   (ii) *Δ is isomorphic to the locus of singular conics in $\mathcal{N}_A$;*
2. *all the quadrics in L are singular: they are the cones with center $(0,0,0,1)$ over the conics in a pencil $L' \subset \mathcal{N}_A$;*
3. *if L' has b_1 basepoints and $\mathcal{N}_A$ has b_2 basepoints, then $\mathcal{N}_D$ has $2b_1 - b_2$ basepoints;*
4. *the number of basepoints of $\mathcal{N}_D$ which do nôt lie in $\mathcal{V}(z)$ is $2(b_1 - b_2)$.*

Table 4.3. Net types for generic Artin–Schelter regular algebras

Type(A)	Type($\mathcal{N}_A$)
A	A
B	α
E	A
H	F^*
S_1	E^*
S_1'	D^*
S_2	E^*

For each pair $(\mathcal{N}_A, L')$, Table 4.4 below gives the number of points in the baselocus of $\mathcal{N}_D$. ∞_c denotes a plane conic and ∞_s denotes a line pair. A blank entry means that the pair $(\mathcal{N}_A, L')$ cannot occur.

Proof. Let M_i be the symmetric 3×3 matrix corresponding to the conic C_i. Then the symmetric 4×4 matrix corresponding to the quadric $Q = \alpha Q_1 + \beta Q_2 + \gamma Q_3$ in $\mathcal{N}_D$ is

$$\begin{pmatrix} \alpha M_1 + \beta M_2 + \gamma M_3 & 0 \\ 0 & \alpha\alpha_1 + \beta\alpha_2 + \gamma\alpha_3 \end{pmatrix}.$$

The singular quadrics $Q \in \mathcal{N}_D$ are those for which the determinant of this matrix vanishes. This determinant is

$$(\alpha\alpha_1 + \beta\alpha_2 + \gamma\alpha_3) \cdot \det(\alpha M_1 + \beta M_2 + \gamma M_3)$$

and since A is not of type I on I^*, $\det(\alpha M_1 + \beta M_2 + \gamma M_3)$ does not vanish identically. The first two parts of the proposition follow at once.

We may choose the basis for $\mathcal{N}_A$ such that Q_1 and Q_2 pass through $(0,0,0,1)$ and hence determine L. Then L' is the pencil spanned by C_1 and C_2. Now $Q_1 \cap Q_2$ consists of b_1 lines through $(0,0,0,1)$ and the basepoints of L'. A simple calculation shows that the third quadric Q_3 intersects each of these lines in two distinct points, neither of which lies in $\mathcal{V}(z)$, unless the line passes through a basepoint of $\mathcal{N}_A$; in that case Q_3 meets the line at a single point with multiplicity 2, and that point lies in $\mathcal{V}(z)$. This gives the number of basepoints for $\mathcal{N}_D$ and also counts those which don't lie in $\mathcal{V}(z)$.

To illustrate how Table 4.4 is obtained we discuss the row labelled F^*. From [20, Table 1], we may choose coordinates such that the net $\mathcal{N}_A$ is all $\lambda x_0^2 + 2\mu x_0 x_1 + \nu(x_1^2 + x_2^2)$ for $(\lambda, \mu, \nu) \in \mathbb{P}^2$. The locus of singular conics is the union of a conic $C := \mathcal{V}(\lambda\nu - \mu^2)$ and the tangent line $\ell := \mathcal{V}(\nu)$ to C at the point x_0^2. The only double line in the net is x_0^2. There are two basepoints, p_1 and p_2 of $\mathcal{N}_A$. The tangent line ℓ is spanned by x_0^2 and $2x_0x_1$.

Table 4.4. Baselocus of $\mathcal{N}_D$ for the pair $(\mathcal{N}_A, L')$

	α	β	γ	δ	ε	ε	ζ	η
A	8	6	–	4	–	–	–	–
B	8	6	4	4	2	–	–	–
B^*	7	5	–	3	–	–	–	–
C	7	5	3	3	1	–	–	–
D	8	6	4	–	2	–	2	–
D^*	6	4	–	2	–	$2 + \infty_c$	–	–
E	8	–	4	–	–	–	2	–
E^*	5	3	–	–	–	$1 + \infty_c$	–	–
F	–	5	3	3	–	$2 + \infty_s$	1	–
F^*	6	4	2	–	–	–	–	∞_c
G	–	5	3	–	1	–	1	∞_s
G^*	–	4	2	2	–	$1 + \infty_s$	–	∞_c
H	–	–	–	3	1	–	–	∞_s
I	–	–	–	–	–	–	1	∞_s
I^*	–	–	–	–	–	$2 + \infty$	–	∞

Now L determines, and is determined by a pencil in $\mathcal{N}_A$, which we label L'. There are 4 possibilities for L': (i) L' is in general position relative to $\Delta = C \cup \ell$; (ii) $L' \neq \ell$ is tangent to C; (iii) $L' \neq \ell$ passes through x_0^2; (iv) $L' = \ell$. In case (i) $|L' \cap \Delta| = 3$ and L' has no double lines, so Type$(L') = \alpha$ and $b_1 = 4$. In (ii) $|L' \cap \Delta| = 2$ and L' has no double lines, so Type$(L') = \beta$ and $b_1 = 3$. In (iii) $|L' \cap \Delta| = 2$ and L' contains 1 double line, so Type$(L') = \gamma$ and $b_2 = 3$. In (iv) every conic in L' is singular and L' contains 1 double line, so Type$(L') = \eta$.

Suppose that Type$(L') = \alpha$. Let p_1, p_2, p_3, p_4 be the basepoints of L'. Then the basepoints of $\mathcal{N}_D$ consist of p_1, p_2 lying in $\mathcal{V}(z)$, two points on the line through p_3 and $(0,0,0,1)$, and two points on the line through p_4 and $(0,0,0,1)$. Of these the 4 not lying in $\mathcal{V}(z)$ are coplanar. This gives the entry in (F^*, α). The cases (F^*, β) and (F^*, γ) are similar. Now suppose that Type$(L') = \eta$. Then $\mathcal{N}_D$ is spanned by $q_1 = x_0^2$, $q_2 = 2x_0x_2$ and $q_3 + z^2$ where $q_3 \in C$. It is clear that the baselocus of $\mathcal{N}_D$ lies in the plane $x_0 = 0$ and is the intersection of this plane with the quadric $q_3 + z^2 = 0$. This gives the (F^*, η) entry. □

Remark 4.3.6 If D is not of type $(A,\alpha), (A,\beta), (B,\alpha), (B,\beta), (B^*,\alpha), (C,\alpha), (D,\alpha), (D,\beta)$ or (E,α), then there are tensors in $D_1 \otimes D_1$ which vanish on $(\Gamma_D)_{\text{red}}$, but which do not belong to R_D. In particular, in all except the above nine cases, there is a linear form u such that $0 \neq uz$ vanishes on $\mathcal{V}(g_1, g_2, g_3)_{\text{red}}$, and hence on P_D. Therefore $u \otimes z$ vanishes on $(\Gamma_D)_{\text{red}}$ but $u \otimes z \notin R_D$. Contrast this with Proposition 4.2.5 above, which says that the defining relations of D are precisely the tensors in $D_1 \otimes D_1$ which vanish on (the non-reduced scheme) Γ_D. In Proposition 4.3.9 we prove that such situations can occur when D is a central extension of a type A Artin–Schelter regular algebra, and in particular when A is elliptic.

The point variety for the central extensions of generic Artin–Schelter regular algebras can be determined with the help of Table 4.4.

Proposition 4.3.7 1. *If A is generic of type A and D is a generic central extension of A, then P_D is the union of P_A and eight additional points.*

2. *If A is generic of type B, and D is a generic central extension of A, then P_D is the union of P_A and an elliptic space curve of degree* 4.

3. *If A is generic of type H, and D is the unique non-trivial central extension of A, then P_D is the union of P_A and a plane conic.*

4. *If A is generic of type S_1' and D is the unique non-trivial central extension of A, then P_D is the union of P_A, a plane conic and two additional points.*

Proof. By Theorem 4.2.2.3, $P_D = P_A \cup \{\text{basepoints of } \mathcal{N}_D\}$, so we have to compute this base locus.

A generic type A algebra is a Sklyanin algebra (see below). In Proposition 4.3.9 we discuss the "constant" central extensions of a Sklyanin algebra. We find that in type (A,α) there are eight additional points. Since eight is the maximum by Bezout's theorem, a generic central extension will also have eight additional points.

Suppose now that A is generic of type B. It follows from Theorem 3.2.6 that $D = k[x_1, x_2, x_3, z]$ has defining relations of the form $f_1 + z(bx_1 + cx_2) + \alpha_1 z^2, f_2 + z(cx_1 + dx_2) + \alpha_2 z^2, f_3$ where $\alpha, b, c, d, \alpha_1, \alpha_2 \in k$ and f_1, f_2, f_3 are the defining relations for A given in Theorem 3.2.6. Since f_3 is a symmetric tensor, it follows that $\mathcal{N}_D$ is the pencil of quadrics spanned by $2x_1x_2 + x_2^2 - x_3^2 + z(bx_1 + cx_2) + \alpha_1 z^2$ and $2x_1x_2 + x_1^2 - ax_3^2 + z(cx_1 + dx_2) + \alpha_2 z^2$. Since the A and D are generic, the scalars $a, b, c, d, \alpha_1, \alpha_2$ are in general position. It follows that the intersection of these two quadrics is a smooth elliptic curve (cf. [17, Proposition 2.5]).

Now suppose that A is generic of type H or S_1'. By Theorem 3.2.6, A has a unique non-trivial central extension and it has no linear terms in z (i.e. $l = 0$ in the notation of the previous section). Hence $\mathcal{N}_D$ is obtained from $\mathcal{N}_A$ as in Proposition 4.3.5. One verifies that the pair $(\mathcal{N}_A, L')$ is of type (F^*, η) if A has type H and is of type (D^*, ε) if A is of type S_1'. One may now read off the description of the base locus of $\mathcal{N}_D$ from Table 4.4. □

Recall that a three-dimensional *Sklyanin algebra* [13] [14] is an Artin–Schelter regular algebra, with Hilbert series $(1-t)^{-3}$ such that P_A is a smooth cubic and σ_A is translation by a point which is not 3-torsion. We will take as defining equations of a Sklyanin algebra, the equations given for generic Type A algebras in Theorem 3.2.6.

It is easy to see that a Sklyanin algebra is of type A and furthermore a type A algebra is a Sklyanin algebra if and only if $(3abc)^3 \neq (a^3 + b^3 + c^3)^3$ (with notation as in Theorem 3.2.6).

Lemma 4.3.8 *If A is a 3-dimensional Sklyanin algebra, then*

1. *$\mathcal{N}_A$ is a basepoint free net of conics;*
2. *$\mathcal{N}_A$ is either of type A or type E.*

Proof. It follows from Lemma 4.3.1 that $\mathcal{N}_A$ is basepoint free, since σ_A does not have a fixed point. Hence $\mathcal{N}_A$ cannot be a pencil of conics. Since A has 3 defining equations it follows that $\mathcal{N}_A$ is a net of conics in $\mathbb{P}(A_1^*) \cong \mathbb{P}^2$.

A basis for $\mathcal{N}_A$ consists of the conics

$$ax_0^2 + 2gx_1x_2 = 0, \qquad ax_1^2 + 2gx_0x_2 = 0, \qquad ax_2^2 + 2gx_0x_1 = 0$$

where $g = \frac{1}{2}(b + c)$. The discriminant divisor has equation

$$(a^3 + 2g^3)x_0x_1x_2 - ag^2(x_0^3 + x_1^3 + x_2^3).$$

It is an easy exercise to see that there are precisely 4 singular curves in the pencil of cubics $\lambda x_0x_1x_2 + \mu(x_0^3 + x_1^3 + x_2^3)$ and that these are given by $\mu = 0$ and $(\lambda/\mu)^3 = 27$. Each of these singular curves is a triangle.

Hence, when the discriminant curve is smooth $\mathrm{Type}(\mathcal{N}_A) = A$ and when it is not smooth $\mathrm{Type}(\mathcal{N}_A) = E$ since the net is basepoint free. □

Proposition 4.3.9 *Let A be a 3-dimensional Sklyanin algebra and D a non-trivial central extension of A having no linear terms in z. Then P_D is the*

Table 4.5. Types of $\mathcal{N}_D$ for Sklyanin algebras

Type	geometry (C + L)	points
(A, α)	elliptic curve + transversal line	8
(A, β)	elliptic curve + tangent in non-flex	6
(A, δ)	elliptic curve + tangent in flex	4_{vp}
(E, α)	triangle + line avoiding vertices	8
(E, γ)	triangle + line through 1 vertex	4_{vp}
(E, ζ)	triangle + baseline	2_{vp}

union of P_A and either 2, 4, 6 *or* 8 *additional points according to the type of* $(\mathcal{N}_A, L')$ *determining the net of quadrics* $\mathcal{N}_D$*:*
Each of these cases does occur.

Proof. By Lemma 4.3.8, $\mathcal{N}_A$ is either of type A or E. Table 4.4 then gives the possibilities for the type of L'. We give examples to illustrate that each possibility does occur. Let D be the algebra determined by the equations:

$$ax_1^2 + bx_2x_3 + cx_3x_2 + dz^2 = 0$$

$$ax_2^2 + bx_3x_1 + cx_1x_3 + ez^2 = 0$$

$$ax_3^2 + bx_1x_2 + cx_2x_1 + fz^2 = 0$$

and z central. Set $g := \frac{1}{2}(b + c)$ and $N := a^3 + 2g^2$.

Type	a	b	c	d	e	f
(A, β)	a	b	c	$Nac - 3a^3g^2$	$Nac - 3b^3g^2$	$Nab - 3c^3g^2$
(A, δ)	a	b	c	$-3ag^2$	$3ag^2$	0
(E, γ)	a	b	$-b$	1	1	0
(E, ζ)	a	b	$-b$	1	0	0

Types (A, α) and (E, α) arise when d, e, f are generic. $\square$

5 The line modules

This section classifies the line modules for central extensions of 3-dimensional Artin–Schelter regular algebras. Section 5.1 handles the general case, and Section 5.2 specializes to the extensions in which $l = 0$. Throughout this section D is a central extension of an Artin–Schelter regular algebra $A = D/(z)$ with Hilbert series $(1 - t)^{-3}$.

The key result is Theorem 5.1.6. The statement of the theorem involves certain quadrics Q_p, one for each $p \in \mathcal{P}_A$. Each Q_p passes through p and belongs to linear system $\mathcal{N}_D$ introduced in Section 4. The Theorem states that the line modules for D correspond to the lines in $\mathcal{V}(z)$ and the lines lying on Q_p which pass through p. The line modules of the first type are precisely those line modules which are A-modules, so this is the trivial case. Hence a finer analysis of the line modules reduces to the analysis of the Q_p e.g. its rank etc.

5.1 General results

Notation. We will just write σ for the maps $\sigma_A, \sigma_D, (\sigma_A)_{\mathrm{red}}$ and $(\sigma_D)_{\mathrm{red}}$.

Recall that a graded D-module M is a *line module* if M is cyclic and has Hilbert series $H_M(t) = (1-t)^{-2}$. Obviously the best thing would be to determine the structure of the scheme (a Hilbert scheme) representing the functor of line modules, as we did for point modules. Unfortunately, our methods are not sufficient to treat this problem. Instead we classify the line modules directly. It will be clear that they occur in certain families, which would correspond to the irreducible components of the Hilbert scheme. However, we make no attempt to formalize this point.

Since D has all the good properties stated in Corollary 2.7, the results in [8, Sect. 2] apply. These are summarized in Proposition 5.1.1.

Proposition 5.1.1 *Let D be an Auslander regular noetherian domain with Hilbert series $(1-t)^{-4}$ and suppose that D also has the Cohen–Macaulay property.*

1. *Every line module M for D is a critical module of $GK\dim(M) = 2$, multiplicity $e(M) = 1$ and is a Cohen–Macaulay module.*
2. *Every line module is of the form $D/Du + Dv$ for some $u, v \in D_1$. Consequently, the line modules are in bijection with certain lines $\mathcal{V}(u,v)$ in $\mathbb{P}^3 = \mathbb{P}(D_1^*)$.*
3. *If M is a line module for D, then there is a unique (up to scalar multiples) element $a \otimes u - b \otimes v \in R_D$ such that $M \cong D/Du + Dv$. Conversely, if R_D contains such an element then $D/Du + Dv$ is a line module. Thus there is a bijection between line modules for D and the projectivized space of rank 2 tensors in R_D.*

Proof. Most of this is in [8, Proposition 2.8], and all that remains to be shown is that the element $a \otimes u - b \otimes v$ is unique (up to scalar multiples). Suppose that $M \cong D/Du_i + Dv_i$ $(i = 1,2)$ and that $a_i \otimes u_i - b_i \otimes v_i \in R_D$. Since $ku_1 + kv_1 = ku_2 + kv_2$ it follows that $a_2 \otimes u_2 - b_2 \otimes v_2 = a_3 \otimes u_1 - b_3 \otimes v_1$ for some $a_3, b_3 \in D_1$. Hence in D we have $a_1u_1 = b_1v_1$ and $a_3u_1 = b_3v_1$. If $a_1 = \mu a_3$ for some $\mu \in k$ then it follows that $b_1 = \mu b_3$ also since D is a domain–the 'uniqueness' now follows. Hence the result is true if $ka_1 = ka_3$ or if $kb_1 = kb_3$.

Suppose that $ka_1 \neq ka_3$ and $kb_1 \neq kb_3$. By [8, Sect. 2], $\mathrm{Ext}^2_D(M,D)(2)$ is a right line module, and is isomorphic to $D/a_1D + b_1D$ and to $D/a_3D + b_3D$. Hence $ka_1 + kb_1 = ka_3 + kb_3$ and therefore $ka_1 + ka_3 = kb_1 + kb_3$. If we write W for this vector space, then $Wu_1 = Wv_1$. Using the fact that D is a domain, we may define a linear map $\psi : W \to W$ by $wv_1 = \psi(w)u_1$. However, if

w is an eigenvector of ψ then it follows that w is a zero divisor in D. This contradicts the fact that D is a domain. □

There are some obvious line modules: every line module for $A = D/\langle z\rangle$ is a line module for D. By [4] these line modules are in bijection with the lines in $\mathbb{P}^3 = \mathbb{P}(D_1^*)$ which actually lie in $\mathscr{V}(z)$. Indeed, since a line module M is critical, either $z \cdot M = 0$ or z acts as a non-zero divisor on M. If $z \cdot M = 0$ then M is a line module for A, so it is the others which we must classify. These line modules are closely related to point modules over A for the following simple reason. If the line module $M(\ell) = D/Du + Dv$ corresponds to a line $\ell = \mathscr{V}(u,v)$ which does not lie in the plane $\mathscr{V}(z)$ then M/zM is a point module for A; that is $\ell \cap \mathscr{V}(z) \in \mathscr{P}_A$.

Just as line modules for D correspond to rank 2 tensors in R_D, so too do point modules for A correspond to rank two tensors in R_A. That is, if $u, v \in A_1$ are linearly independent, then $\{p\} = \mathscr{V}(u,v,z)$ belongs to $\mathscr{P}_A$ if and only if there exist $a, b \in A_1$ such that $a\otimes v - b\otimes u \in R_A$. This rank 2 tensor is uniquely determined (up to a scalar multiple) by p; this may be proved as in (5.1.1.3) or as a consequence of (5.1.2.2). Define $p^\vee$ as $\mathscr{V}(a,b,z)$. Clearly $p^\vee$ is uniquely determined by p. It should also be remarked that $p^\vee$ may be characterized by the fact that $\mathrm{Ext}^2_A(M(p),A)(2)$ is the right point module corresponding to $p^\vee$.

Proposition 5.1.2 *Let $a\otimes v - b\otimes u \in R_A$ with $\mathscr{V}(u,v,z) = p$. Then*

1. *There is an exact sequence*

$$0 \to A/Aa(-1) \to A/Au \to A/Au + Av \to 0\,.$$

2. *If $\mathscr{V}(u,z)$ is contained in $\mathscr{P}_A$ then $\mathscr{V}(a,z) = \mathscr{V}(u,z)^{\sigma^{-1}}$. Otherwise let $S = \mathscr{V}(u,z) \cap \mathscr{P}_A$. This is a scheme of length 3 which is the direct union of the spectra of serial k-algebras. Then $\mathscr{V}(a,z)$ is the unique line containing $(S-p)^{\sigma^{-1}}$. Furthermore, in this case $p^\vee$ is the point in $\mathscr{V}(a,z)\cap\mathscr{P}_A$, not contained in $(S-p)^{\sigma^{-1}}$.*

3. *If p is fixed then a and u determine each other up to a scalar multiple.*

4. *If A is elliptic then $p^\vee = (p^{\eta^{-1}})^{\sigma^{-1}}$ where η is as defined in* [4, (6.26)]. *If A is linear then $p^\vee = p^{\sigma^{-1}}$.*

5. *The map $p \to p^\vee$ is an automorphism of $\mathscr{P}_A$ as a k-scheme. This automorphism commutes with σ.*

Proof. 1. The existence of the complex is clear. The exactness follows from the fact that line modules are critical.

2. This is [4, Prop. 6.24] translated to left modules. (Actually [4, Prop. 6.24] was stated for elliptic A, but the slight generalization given here is proved in the same way.)

3. Clear from (2).

4. If A is elliptic this follows from (2) and [4, Prop. 6.24]. If A is linear then according to (2), $p^\vee$ lies in $\mathscr{V}(u,z)^{\sigma^{-1}} \cap \mathscr{V}(v,z)^{\sigma^{-1}} = p^{\sigma^{-1}}$.

5. This follows from (4) and [4, (6.26), Lemma 5.10(ii)]. □

Definition 5.1.3 *Let $p \in \mathscr{P}_A \subset \mathscr{V}(z)$. Choose $a, b, u, v \in A_1$ such that $p = \mathscr{V}(u, v, z)$ and $a \otimes v - b \otimes u \in R_A$. Let $l \in A_1$, $\lambda \in k$ be such that $a \otimes v - b \otimes u + l \otimes z + \lambda z \otimes z \in R_D$.*

Define the scheme $Q_p := \mathscr{V}(av - bu + lz + \lambda z^2) \subset \mathbb{P}(D^*)$. It follows from (5.1.1.3) *that Q_p is uniquely determined by p.*

Lemma 5.1.4 *If $p = p^\vee$ then $\mathscr{V}(z)$ is tangent to the scheme Q_p at p.*

Proof. Since $p = p^\vee$, the defining equation of Q_p is

$$(\alpha u + \beta v)u + (\gamma u + \delta v)v + lz + \lambda z^2 = 0$$

for some $l \in A_1$ and some $\alpha, \beta, \gamma, \delta, \lambda \in k$. If $(\alpha u + \beta v)u + (\gamma u + \delta v)v = 0$ then the result is obvious. If $(\alpha u + \beta v)u + (\gamma u + \delta v)v \neq 0$ then $\mathscr{V}(z) \cap Q_p = \mathscr{V}((\alpha u + \beta v)u + (\gamma u + \delta v)v)$ is singular (as a scheme) at $\mathscr{V}(u, v, z) = p$. Since $\mathscr{V}(z) \cap Q_p$ is singular it follows that $\mathscr{V}(z)$ is tangent to Q_p at p. $\square$

Lemma 5.1.5 *Let $p \in \mathscr{P}_A$ such that $p \neq p^\vee$. Then there is a unique (up to a scalar multiple) $u \in A_1$ such that $p \in \mathscr{V}(u)$ and such that there is a rank two tensor in R_A*

$$(\alpha u + \beta v) \otimes v + w \otimes u \tag{5.1}$$

with $p = \mathscr{V}(u, v, z), (u, v, w) \in A_1$, linearly independent and $\alpha, \beta \in k$. Subject to $p = \mathscr{V}(u, v, z)$, v may be chosen freely.

Let $l \in A_1$, $\lambda \in k$ be such that

$$(\alpha u + \beta v) \otimes v + w \otimes u + l \otimes z + \lambda z \otimes z \in R_D$$

and write $l = l_1 u + l_2 v + l_3 w$. Then l_3 is uniquely determined by p and u.

Proof. Let $a \otimes v - b \otimes u \in R_A$ with $p = \mathscr{V}(u, v, w)$. If $a(p) = 0$ then $a \in ku + kv$. If $b(p) = 0$ then we may relabel a, b, u, v so that $a \in ku + kv$. If $a(p) \neq 0$ and $b(p) \neq 0$ then for some $0 \neq \mu \in k(a + \mu b)(p) = 0$ and $(a + \mu b) \otimes v - b \otimes (u + \mu v) \in R_A$ and $a + \mu b \in k(u + \mu v) + kv$. Hence R_A contains a tensor $(\alpha u + \beta v) \otimes v + w \otimes u$ with $p = \mathscr{V}(u, v, w)$. Since $\mathscr{V}(\alpha u + \beta v, w, z) = p^\vee \neq p$ it follows that u, v, w are linearly independent. Clearly $\alpha u + \beta v$ is unique, up to a scalar, since it must be zero on p and $p^\vee$. By Proposition 5.1.2.3. u is therefore unique up to a scalar.

It is clear that we may substitute $v \to \delta_1 v + \delta_2 u$, $\delta_1 \neq 0$, without changing the form of (5.1). Hence v is arbitrary. Furthermore under such a substitution w is transformed to a sum of w and a linear combination of u and v. Hence the coefficient in l of this new w will not change. $\square$

Theorem 5.1.6 *Let $p \in \mathscr{P}_A \subset \mathscr{V}(z)$. A line through p corresponds to a line module if and only if it is contained in either $\mathscr{V}(z)$ or in Q_p.*

Proof. Let L be a line through p, not lying in $\mathscr{V}(z)$ and corresponding to a line module. Choose $u, v \in A_1$ such that $\mathscr{V}(u, v, z) = p$. Then there exist scalars μ, η such that $L = \mathscr{V}(u + \mu z, v + \eta z)$. Since L corresponds to a line module there exist $a, b \in A_1$, $\alpha, \beta \in k$ such that

$$(a + \alpha z) \otimes (v + \eta z) - (b + \beta z) \otimes (u + \mu z) \in R_D$$

which implies that $Q_p = \mathscr{V}((a + \alpha z)(v + \eta z) - (b + \beta z)(u + \mu z))$. Hence $L \subset Q_p$.

Conversely, suppose that $L = \mathscr{V}(u+\mu z, v+\eta z)$ lies in Q_p and not in $\mathscr{V}(z)$, and that $p = \mathscr{V}(u, v, z)$. We will show that L corresponds to a line module.

Suppose first that $p = p^\vee$. If $Q_p = \mathbb{P}^3$ then $u \otimes v - v \otimes u \in R_D$ so $(v + \eta z) \otimes (u + \mu z) - (u + \mu z) \otimes (v + \eta z) \in R_D$ whence L corresponds to a line module. Now suppose that $Q_p \neq \mathbb{P}^3$. Then there exist $l \in A_1$ and $\alpha, \beta, \gamma, \delta, \lambda \in k$ such that

$$f_1' := (\alpha u + \beta v) \otimes u + (\gamma u + \delta v) \otimes v + l \otimes z + \lambda z \otimes z \in R_D .$$

Hence

$$Q_p = \mathscr{V}((\alpha u + \beta v)u + (\gamma u + \delta v)v + lz + \lambda z^2) . \tag{5.2}$$

If p is a smooth point of the scheme Q_p then any line on Q_p passing through p is contained in the tangent plane to Q_p at p. By Lemma 5.1.4 this tangent plane is $\mathscr{V}(z)$, so $L \subset \mathscr{V}(z)$ which contradicts the hypotheses. Hence Q_p is not smooth at p, from which it follows that $l \in ku + kv$. Thus $f_1' \in k\langle u, v, w\rangle$.

Now define f_2', f_3' by

$$f_2' := z \otimes u - u \otimes z ,$$
$$f_3' := z \otimes v - v \otimes z ,$$

and define $A' = k\langle u, v, z\rangle/(f_1', f_2', f_3')$. Then A' is a 3-dimensional Artin–Schelter regular algebra, because $A'/(z)$ is 2-dimensional regular and Theorem 2.6 applies to A'. Furthermore $\mathscr{P}_{A'} \subset \mathbb{P}(A_1'^*)$ is given by

$$z((\alpha u + \beta v)u + (\gamma u + \delta v)v + lz + \lambda z^2) = 0 .$$

On the other hand, L lies in Q_p, given by (5.2). This means that

$$A'/A'(u + \mu z) + A'(v + \eta z)$$

is a point module for A'. Consequently there exist $a, b \in A_1'$, such that

$$a \otimes (v + \eta z) - b \otimes (u + \mu z) \in R_{A'} \subset R_D$$

This shows that L determines a line module.

Now suppose $p \neq p^\vee$. We may assume that u, v, μ, η were chosen so that there is a tensor of the form

$$(\alpha u + \beta v) \otimes v + w \otimes u + (l_1 u + l_2 v + l_3 w) \otimes z + \lambda z \otimes z \tag{5.3}$$

in R_D. Hence $Q_p = \mathscr{V}((\alpha u + \beta v)v + wu + (l_1 u + l_2 v + l_3 w)z + \lambda z^2)$. Since $L \subset Q_p$ this implies that $\mu = l_3$ and

$$\eta(\alpha\mu + \beta\eta) - l_1\mu - l_2\eta + \lambda = 0 .$$

Now by adding a term of the form $h \otimes z - z \otimes h$ to (5.3) we obtain that

$$(\alpha u + \beta v + (l_2 - \beta\eta)z) \otimes (v + \eta z) + (w + (l_1 - \eta\alpha)z) \otimes (u + \mu z) \in R_D$$

which shows that L represents a line module. $\square$

Proposition 5.1.7 *Let $p \in \mathscr{P}_A$, and assume that the equations of D are in standard form (as in the previous sections)*

$$g := f + lz + \alpha z^2 = 0$$
$$zx - xz = 0\,,$$

where $g = (g_1, g_2, g_3)^t$, $l = (l_1, l_2, l_3)^t$, $f = (f_1, f_2, f_3)^t$, $x = (x_1, x_2, x_3)^t$, $f = Mx$, and $x^tM = (Qf)^t$.

Let $\zeta = (\zeta_1, \zeta_2, \zeta_3)^t$ be the coordinates of p^σ. Then $Q_p = \mathscr{V}(\zeta^t Qg)$.

Proof. Since the entries of x^tM span R_A there exists $\gamma = (\gamma_1, \gamma_2, \gamma_3)^t \in k^3$ such that $x^tM\gamma$ is a rank two tensor corresponding to p. This means that p is the common zero of the entries of $M\gamma$. That is $M(p)\gamma = 0$, so γ is the coordinate vector of p^σ, whence $\gamma = \zeta$. Therefore the rank two tensor in R_A corresponding to p is $x^tM\zeta = f^tQ^t\zeta$, and the corresponding tensor in R_D is $g^tQ^t\zeta$. This yields the desired result. $\square$

Remark 5.1.8 1. Notice that $Q_p = \mathbb{P}^3$ if and only if R_D contains a tensor $u \otimes v - v \otimes u$ with $p = \mathscr{V}(u, v, z)$ (and in this case $p = p^\vee$). Otherwise Q_p is a quadric belonging to the linear system $\mathscr{N}_D$. Sometimes Q_p is uniquely determined by the fact that it goes through p and $p^\vee$ and lies in $\mathscr{N}_D$.

2. Suppose that $\mathscr{N}_D$ is a net of quadrics. Then the map $p \to Q_p$ can be interpreted as a map $\mathscr{P}_A \to \mathbb{P}^2$. By (5.1.7) this is a morphism. If A is elliptic then the image of $\mathscr{P}_A$ is degree 3 curve in $\mathscr{N}_D = \mathbb{P}^2$. Since the locus of singular quadrics in $\mathscr{N}_D$ is a degree 4 curve, in general we expect Q_p to be singular for exactly 12 points $p \in \mathscr{P}_A$. The final example of the paper illustrates this clearly.

In order to give a more explicit description of the lines on Q_p which pass through p, we introduce the following definitions.

Definition 5.1.9 *Let $p \in \mathscr{P}_A$. Let T_p be 'the' rank two tensor in R_A corresponding to p. Then p is*

1. *of the first kind if T_p may be written as*

$$v \otimes v + w \otimes u$$

with $u, v, w \in A_1$ linearly independent.

2. *of the second kind if T_p may be written as*

$$u \otimes v + w \otimes u$$

with $u, v, w \in A_1$ linearly independent.

3. *of the third kind if $p = p^\vee$.*

Note that p is always of exactly one kind.

Lemma 5.1.10 *If $p \neq p^\vee$, then Q_p is smooth at p.*

Proof. Let $u, v \in A_1$ be such that $\mathcal{V}(u,v,z) = p$. Since $p \neq p^\vee$, the defining equation for Q_p is of the form $av + bu + lz + \lambda z^2$ with either $a(p) \neq 0$ or $b(p) \neq 0$ for some $a, b, l \in A_1$ and $\lambda \in k$. By changing the choice of u, v we may assume that $a(p) \neq 0$. Then the partial derivative with respect to v does not vanish at p, so Q_p is smooth at p. □

Theorem 5.1.11 *Assume* char$(k) \neq 2$. *Let $p \in \mathcal{P}_A \subset \mathcal{V}(z)$. Then the nature of the lines passing through p which correspond to line modules, and are not in $\mathcal{V}(z)$ is given by the following table. In the table Q_p is treated as a scheme.*

Proof. After Theorem 5.1.6 we need to describe the lines in Q_p which pass through p.

In Case 1, $Q_p = \mathcal{V}(v^2 + uw + lz + \lambda z^2)$ and obviously no lines on Q_p can lie in $\mathcal{V}(z)$.

In Case 2, $Q_p = \mathcal{V}(u(v+w) + z(l + \lambda z))$ and one of the lines on Q_p through p is $\mathcal{V}(u,z)$. When rank $(Q_p) = 3$, the other line on Q_p through p cannot lie in $\mathcal{V}(z)$ since $\mathcal{V}(z) \cap Q_p$ is a pair of lines, one of which does not pass through p. When rank$(Q_p) = 2$ there is only one line on Q_p through p, namely $\mathcal{V}(u,z)$. When rank$(Q_p) = 1$, Q_p is a pair of planes only one of which contains p.

Case 3. The sub-cases $Q_p = \mathbb{P}^3$ and $Q_p = \mathcal{V}(z^2)$ are obvious so suppose that we are in neither of these cases. Therefore $Q_p = \mathcal{V}(au + bv + lz + \alpha z^2)$ with $au + bv \neq 0$, $ka + kb = ku + kv$ and $\mathcal{V}(u,v,z) = \{p\}$. If Q_p is smooth at p, then any line on Q_p through p lies in the tangent space to Q_p at p. By Lemma 5.1.4 this tangent space is $\mathcal{V}(z)$, so there are no new line modules in this case. Suppose that Q_p is not smooth at p. In this case it follows that $l \in ku + kv$. Now Q_p is either a cone or a pair of planes (not necessarily distinct). In the first case there are infinitely many lines on Q_p through p. In the second case both these planes contain p, and at least one plane is not $\mathcal{V}(z)$; that plane gives rise to infinitely many new line modules. □

Table 5.1. Lines through p, not in $\mathcal{V}(z)$, corresponding to line modules

	Nature of p	Nature of Q_p	Lines through p
1.	first kind	rank 3	two distinct lines
		rank 2	a double line
2.	second kind	rank 3	one line
		rank 2	none (double line in $\mathcal{V}(z)$)
		rank 1	a $\mathbb{P}^1$ of lines
3.	third kind	$Q_p = \mathbb{P}^3$	a $\mathbb{P}^2$ of lines
		$Q_p = \mathcal{V}(z^2)$	none
		$\mathbb{P}^3 \neq Q_p$ and smooth at p	none
		$\mathcal{V}(z^2) \neq Q_p$ and non-smooth at p	infinite family

Remark 5.1.12 A useful example to consider is that of 'Homogenized sl(2)' [9]. In this case the point variety for D is $\mathcal{V}(z)$ together with an embedded conic in $\mathcal{V}(z)$ and a point not in $\mathcal{V}(z)$. Since $A = D/(z)$ is commutative, $\mathcal{P}_A = \mathcal{V}(z)$ and every $p \in \mathcal{P}_A$ is of the third kind. Furthermore Q_p is always a union of $\mathcal{V}(z)$ and one other plane. However, p is on both these planes if and only if p lies on the embedded conic. Hence there are no extra line modules through p if p is not on the embedded conic, and there is a $\mathbb{P}^1$ of extra line modules through p if it lies on the embedded conic. In [9] the line modules for D are described as the lines lying on a certain pencil of quadrics; this is different from the description in this paper, and it suggests that for a particular D there may be a more elegant description of the line modules than that given in Theorems 5.1.6 and 5.1.11.

Proposition 5.1.13 *Suppose that $p \in \mathcal{P}_A$. Then, to which kind p belongs, may be found in the rightmost column of Table 5.2 below. The result depends on whether certain conditions are true or not. Y and N mean 'yes' and 'no', and – means that the condition is irrelevant, meaningless, or follows from the other conditions.*

Proof. Let T_p be 'the' rank two tensor in R_A corresponding to p.

1. Suppose to the contrary that p is of the second kind. Then $T_p = u \otimes v + w \otimes u$. Thus $\mathcal{V}(u)$ goes through $p, p^\vee$ and also through p^σ (by Proposition 5.1.2.2). This contradicts the hypothesis that $p, p^\vee, p^\sigma$ are not collinear.

2. In this case A is necessarily elliptic. Suppose to the contrary that $T_p = u\otimes v+w\otimes u$. By (5.1.2.2), $\mathcal{V}(u)\cap\mathcal{P}_A = p+p^\vee+q = p^\vee+(p^\vee)^{\sigma^{-1}}+q^{\sigma^{-1}}$ whence $p+q^\sigma = p^\vee+q$. But this is impossible since $p \nparallel p^\vee$.

3. Suppose to the contrary that $T_p = v\otimes v+w\otimes u$. By (5.1.2.2) v vanishes on $p, p^\vee$ and also on p^σ by the collinearity hypothesis. However, u also vanishes on p and p^σ. Therefore $\mathcal{V}(u) = \mathcal{V}(v)$ contradicting the fact that u and v are independent.

4. Suppose to the contrary that $T_p = v\otimes v+w\otimes u$. If $p \nparallel p^\sigma$ then we may continue as in 3. Assume that $p = p^\sigma$. Then $(p^\vee)^\sigma = p^\vee$ (since σ commutes with the map $p \to p^\vee$) and hence $\mathcal{V}(v)^\sigma = \mathcal{V}(v)$ (since $v(p) = v(p^\vee) = 0$). On the other hand, by (5.1.2.2), $\mathcal{V}(v)^\sigma = \mathcal{V}(u)$. This implies $\mathcal{V}(u) = \mathcal{V}(v)$ which yields a contradiction.

5. This is by definition. □

Table 5.2. The kind to which a point belongs

	$p = p^\vee$	$p = p^\sigma$	line through $p, p^\vee$ contained in $\mathcal{P}_A$	$p, p^\vee, p^\sigma$ collinear	
1.	N	–	–	N	first kind
2.	N	Y	N	–	first kind
3.	N	N	–	Y	second kind
4.	N	–	Y	Y	second kind
5.	Y	–	–	–	third kind

5.2 The constant case ($l = 0$)

We now specialize the results in Section 5.1 to the case when the defining relations of D are of the form $f_j + \alpha_j z^2 = zx_j - x_j z = 0$ ($j = 1,2,3$) i.e. $l = 0$ in the earlier notation.

If we specialize Theorem 5.1.11 to this case, then there is some simplification, as we now show.

Definition 5.2.1 *A point $p \in \mathcal{P}_A \subset \mathcal{V}(z)$ is special if there exist $0 \neq a, b, u, v \in A_1$ such that $\mathcal{V}(u, v, z) = p$ and $a \otimes v - b \otimes u \in R_D$.*

Proposition 5.2.2 *A point $p \in \mathcal{P}_A$ is special if and only if p^σ lies on the line $\mathcal{V}(\alpha^t x, z)$.*

Proof. This is a direct consequence of Proposition 5.1.7. Let $p \in \mathcal{P}_A$ and let $\zeta = (\zeta_1, \zeta_2, \zeta_3)^t$ be the coordinates of p^σ. By definition p is special if and only if z does not occur in the equation for Q_p, but this equation is $\zeta^t Qg$. Therefore p is special if and only if $\zeta^t Q(lz + \alpha z^2) = 0$. But $l = 0$ so by Theorem 3.1.6, $\alpha = Q\alpha$. $\square$

It is clear that if $D = A[z]$ then every point in $\mathcal{P}_A$ is special.

Theorem 5.2.3 *The lines through a point $p \in \mathcal{P}_A$ which do not lie in $\mathcal{V}(z)$ and which correspond to line modules, are given by the following table.*

Proof. This is straightforward after (5.1.11). $\square$

Remark 5.2.4 If $\mathcal{N}_A$ is a net of quadrics, then for every p we have $Q_p = \mathcal{V}(au + bv + \alpha z^2)$ with $au + bv \neq 0$. Hence if p is of the third kind, then Q_p is not smooth at p, and one obtains an infinite family of lines through p which are not in $\mathcal{V}(z)$.

Example 5.2.5 Let A be a 3-dimensional Sklyanin algebra. That is, $\mathcal{P}_A$ is a smooth elliptic curve and σ is a translation of the form $p^\sigma = p + \tau$ with $3\tau \neq 0$. Then $p^\vee = p + 2\tau$.

If $2\tau = 0$ then every point of $\mathcal{P}_A$ is of the third kind, and hence there will be a lot of line modules.

Table 5.3. Lines, not in $\mathcal{V}(z)$, containing p, corresponding to line modules

	Nature of p	Lines through p not in $\mathcal{V}(z)$
1.	first kind, not special	2 lines, not through $(0,0,0,1)$ (1 in characteristic 2)
2.	second kind, not special	none
3.	first kind, special	1 line through $(0,0,0,1)$
4.	second kind, special	a $\mathbb{P}^1$ of lines
5.	third kind	none or an infinite family

If $2\tau \neq 0$ then there will be no point of the third kind. A point p will be of the second kind if and only if $p, p^\vee, p^\sigma$ are collinear i.e. if and only if $p = -\tau + \omega$ with $3\omega = 0$. Consequently (if char. $\neq 3$) there will be nine points of the second kind. All the other points are of the first kind.

Now suppose that $l = 0$, i.e. $g = f + \alpha z^2$ in the defining equations for D. It follows from Theorem 3.1.6 that α, and hence the line containing the special points is arbitrary. Hence generically there will be three distinct special points, not coinciding with any points of the second kind. (The 3 special points and the 9 points of the second kind account for the 12 points where Q_p is singular.) Consequently, in the generic case, there will be two continuous families of lines corresponding to line modules, namely

1. the lines in $\mathscr{V}(z)$, and
2. the lines on Q_p passing through p.

Clearly the lines in the second family are parametrized by some double covering of $\mathscr{P}_A$.

At a special point the two lines in the second family coincide, and at a point of the second kind they lie in $\mathscr{V}(z)$ and hence they already belong to the first family.

Now assume that D is a generic central extension of a generic Sklyanin algebra, with linear terms ($l \neq 0$) in the defining equations. Since $\operatorname{rank}(Q_p) = 3$ for the points of the first kind when $l = 0$ this will still be true for almost all points when $l \neq 0$. Hence by Theorems 5.1.6 and 5.1.11 there will still be two lines through p, not in $\mathscr{V}(z)$, corresponding to line modules for almost all $p \in \mathscr{P}_A$. Hence the above picture, of line modules, i.e. two families, one of which is parametrized by a double covering of $\mathscr{P}_A$, remains valid in the generic, non-constant, case.

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