DIFFERENTIAL OPERATORS ON THE AFFINE AND PROJECTIVE LINES IN CHARACTERISTIC $p > 0$

by

S.P. SMITH

Let k be a field, and denote by $\mathbb{A}^1$ (or $\mathbb{A}^1_k$) and $\mathbb{P}^1$ (or $\mathbb{P}^1_k$) the affine and projective lines over k. When k is of characteristic 0 the rings of differential operators on $\mathbb{A}^1$ and $\mathbb{P}^1$ (which we denote $D(\mathbb{A}^1_0)$ and $D(\mathbb{P}^1_0)$) have been extensively studied, and are considered to be well understood. In contrast, if char $k = p > 0$, the rings of differential operators on $\mathbb{A}^1$ and $\mathbb{P}^1$ (which we denote $D(\mathbb{A}^1_p)$ and $D(\mathbb{P}^1_p)$) have not been studied at all. The purpose of this note is to begin an investigation into $D(\mathbb{A}^1_p)$ and $D(\mathbb{P}^1_p)$.

Before we outline some of our results, we give a brief account of the wider context in which $D(\mathbb{A}^1_0)$ and $D(\mathbb{P}^1_0)$ appear (and which accounts for their significance). First, if one is to study differential operators on any affine or projective variety then $D(\mathbb{A}^1)$ and $D(\mathbb{P}^1)$ are the first cases to examine. However, another important motivation is the connection of $D(\mathbb{A}^1_0)$ and $D(\mathbb{P}^1_0)$ with the representation theory of finite dimensional Lie algebras in characteristic zero. The recent history of $D(\mathbb{A}^1_0)$ (known as the Weyl algebra) begins with Dixmier's papers [3] and [4]. He showed that if $\underset{\sim}{g}$ is a finite dimensional nilpotent Lie algebra over $\mathbb{C}$, then the primitive factor rings of $U(\underset{\sim}{g})$, the enveloping algebra of $\underset{\sim}{g}$, are of the form $D(\mathbb{A}^n_{\mathbb{C}}) \cong D(\mathbb{A}^1_{\mathbb{C}}) \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} D(\mathbb{A}^1_{\mathbb{C}})$. Hence, the irreducible representations of $\underset{\sim}{g}$ are precisely the simple modules over $D(\mathbb{A}^n_{\mathbb{C}})$ for various n. For example, if $\underset{\sim}{g}$ is the 3-dimensional Heisenberg Lie algebra then the infinite dimensional irreducible representations of $\underset{\sim}{g}$ are precisely the simple modules over $D(\mathbb{A}^1_{\mathbb{C}})$.

The ring $D(\mathbb{P}^1_{\mathbb{C}})$ arises in a similar way. Let G be a connected complex semisimple Lie group with Borel subgroup B; then G/B is a complex projective algebraic variety ($\mathbb{P}^1_{\mathbb{C}}$ arises as SL(2)/B), and the ring of global regular differential operators on G/B, D(G/B), is isomorphic to a primitive factor ring of $U(\underset{\sim}{g})$ where $\underset{\sim}{g}$ is the Lie algebra of G. See [1] where this idea is exploited to verify the

Kazhdan-Lusztig conjectures on Verma modules.

The corresponding connections between representations of characteristic p Lie algebras and modules over $D(\mathbb{A}_p^1)$ and $D(\mathbb{P}_p^1)$ are not studied here. Rather, we concern ourselves with the ring theoretic properties of $D(\mathbb{A}_p^1)$ and $D(\mathbb{P}_p^1)$ and examine to what extent their structure parallels or diverges from $D(\mathbb{A}_0^1)$ and $D(\mathbb{P}_0^1)$. It is largely a matter of taking a result in characteristic zero and asking whether the same result holds in characteristic p, and if not, in what sense is it false.

In Table 1, below, the properties of $D(\mathbb{A}^1)$ in characteristics zero and p are set out side by side. Let us mention just a few of them. $D(\mathbb{A}_0^1)$ is finitely generated and Noetherian - both these are false for $D(\mathbb{A}_p^1)$. Much of the "bad" behaviour of $D(\mathbb{A}_p^1)$ can be attributed to the lack of some sort of finiteness condition (in particular, the question of whether every endomorphism of a simple $D(\mathbb{A}_p^1)$-module is algebraic over k, is difficult because one has no finiteness condition which might allow a result concerning generic flatness of the associated graded algebra to be established). For a similar reason Gelfand-Kirillov dimension, which is an effective tool for $D(\mathbb{A}_0^1)$, does not seem to be useful for $D(\mathbb{A}_p^1)$. But, all is not lost. For example, if k[t] denotes the co-ordinate ring of $\mathbb{A}^1$, and if $0 \neq f \in k[t]$ then $k[t,f^{-1}]$ is a $D(\mathbb{A}^1)$-module. In characteristic zero, $k[t,f^{-1}]$ is an Artinian module, and the usual proof involves Gelfand-Kirillov dimension. Nevertheless, in characteristic p, $k[t,f^{-1}]$ is also an Artinian $D(\mathbb{A}_p^1)$-module, and the proof makes use of one structural feature of $D(\mathbb{A}_p^1)$ that has no analogue in $D(\mathbb{A}_0^1)$. Namely that $D(\mathbb{A}_p^1) = \bigcup_{n=0}^{\infty} \mathrm{End}_{k[t^{p^n}]} k[t]$, is a union of matrix algebras over commutative rings (whereas $D(\mathbb{A}_0^1)$ is a domain). One question which appears in [3] and remains unanswered to date, is whether $D(\mathbb{A}_0^1)$ has a proper subring isomorphic to $D(\mathbb{A}_0^1)$. It is quite easy to construct a proper subring of $D(\mathbb{A}_p^1)$ which is isomorphic to $D(\mathbb{A}_p^1)$.

Although $D(\mathbb{P}_{\mathbb{C}}^1)$ is a primitive factor ring of $U(sl(2,\mathbb{C}))$, the natural map from $Hyp(sl(2,k))$, the hyperalgebra of $sl(2,k)$, to $D(\mathbb{P}_k)$ is not surjective if char $k = p>0$.

$D(\mathbb{P}_0^1)$ has a unique two sided ideal (apart from 0 and $D(\mathbb{P}_0^1)$) and this ideal is of codimension 1; the analogous statement for $D(\mathbb{P}_p^1)$ is also true. Whereas $K_0(D(\mathbb{P}_0^1)) = \mathbb{Z} \oplus \mathbb{Z}$, $K_0(D(\mathbb{P}_p^1)) = \mathbb{Z} \oplus \mathbb{Z}[1/p]$; the lattice of order ideals in $K_0(D(\mathbb{P}_p^1))$ is isomorphic to the lattice of two sided ideals in $D(\mathbb{P}_p^1)$.

TABLE 1

Properties of $D(\mathbb{A}^1_k)$

Characteristic zero	Characteristic $p > 0$
finitely generated	not finitely generated
Noetherian	not Noetherian
simple ring	simple ring
domain	not a domain
gl.dim. = 1	gl.dim. = 1
K.dim. = 1	K.dim. does not exist
GK.dim. = 2	GK. dim. = 1.
centre = k	centre = k
$K_0 = \mathbb{Z}$	$K_0 = \mathbb{Z}[1/p]$
Every derivation is inner	There exists a non-inner derivation
If I is a left ideal with $I \cap k[t] \neq 0$ and $I \cap k[d/dt] \neq 0$, then $I = D(\mathbb{A}^1)$	If char k = 2 then $Dt + Dx_1 \neq D(\mathbb{A}^1)$
If $0 \neq f \in k[t]$ then $k[t,f^{-1}]$ is Artinian	If $0 \neq f \in k[t]$ then $k[t,f^{-1}]$ is of finite length
k[t] is a simple module	k[t] is a simple module
D/Dt is a simple module	D/Dt is a simple module
Open question whether $D(\mathbb{A}^1)$ has a proper subalgebra isomorphic to $D(\mathbb{A}^1)$	$D(\mathbb{A}^1)$ contains a proper subalgebra isomorphic to $D(\mathbb{A}^1)$ viz $k[t^p, x_p, x_{2p}, x_{3p}, \ldots]$
If M is a simple module $\mathrm{End}_D M$ is algebraic over k	Not known

My initial interest in these ideas was aroused during conversations and correspondence with Ken Goodearl. I am indebted to him for his generous comments and assistance, especially relating to matters concerning K-theory. My thanks also go to C.R. Hajarnavis for many useful conversations during the preparation of these notes.

§1. DIFFERENTIAL OPERATORS

Let k be any commutative ring, and A any commutative k-algebra. Then $End_k A$ may be made into an $A \otimes_k A$-module by defining $((a \otimes b)\theta)(c) = a\theta(bc)$ for $\theta \in End_k A$ and $a,b,c \in A$. We write $[a,\theta]$ for $(a \otimes 1 - 1 \otimes a)\theta$, so $[a,\theta](b) = a\theta(b) - \theta(ab)$.

DEFINITION 1.1 The space of k-*linear differential operators of order* $\leq n$ *on* A, $Diff_k^n A$, is defined inductively by $Diff_k^{-1} A = 0$, and for $n \geq 0$, $Diff_k^n A = \{\theta \in End_k A \mid [a,\theta] \in Diff_k^{n-1} A \text{ for all } a \in A\}$. The *ring of* k-*linear differential operators on* A is $D(A) = \bigcup_{n=0}^{\infty} Diff_k^n A$. If X is an affine algebraic variety over the field k with ring of regular functions A, we write $D(X) = D(A)$.

REMARK 1.2 (1) $Diff_k^n A$ is an $A \otimes A$-submodule of $End_k A$

(2) If $\theta \in End_k A$, then $\theta \in Diff_k^n A$, if and only if,

$$[a_0[a_1 \ldots [a_n,\theta]\ldots]] = 0 \text{ for all } a_0,a_1,\ldots,a_n \in A.$$

(3) We refer the reader to [10] for a more comprehensive introduction to rings of differential operators on commutative rings.

(4) It is an easy exercise to verify that if k is a field of characteristic zero, and k[t] is the ring of regular functions on $\mathbf{A}_k^1$, then $D(\mathbf{A}_k^1) = k[t,d/dt]$ where d/dt is the usual differentiation operator acting on the polynomial ring k[t]. As elements of $End_k k[t]$ one has $(d/dt)t - t(d/dt) = 1$.

DEFINITION 1.3 Denote by $\mu: A \otimes_k A \to A$ the multiplication map $\mu(a \otimes b) = ab$. This is a k-algebra map (also an A-module map for either the right or left A-module structure on $A \otimes_k A$). Put $I = \ker \mu$.

THEOREM 1.4 (Heynemann-Sweedler [9], Grothendieck [8]). *Let* $\theta \in End_k A$. *Then* $\theta \in Diff_k^n A$, *if and only if,* $I^{n+1}.\theta = 0$.

§2. PROPERTIES OF $D(\mathbf{A}_p^1)$

Write $D = D(\mathbf{A}_p^1)$, and consider D as the ring of k-linear differential operators on k[t], the polynomial ring in t, over the field k of characteristic $p > 0$.

The following result was arrived at during conversation and correspondence with

Ken Goodearl, and I am grateful for his allowing me to include it here.

PROPOSITION 2.1 $D = \bigcup_{n=0}^{\infty} \mathrm{End}_{k[t^{p^n}]}k[t]$ *and* $\mathrm{Diff}_k^{p^n-1}\, k[t] = \mathrm{End}_{k[t^{p^n}]}k[t]$.

Proof Let $\theta \in \mathrm{End}_k k[t]$. Notice that $I = \ker(\mu: k[t] \otimes_k k[t] \to k[t])$ is generated as an ideal by $1 \otimes t - t \otimes 1$. Hence I^{p^n} is generated by $(1 \otimes t - t \otimes 1)^{p^n} = 1 \otimes t^{p^n} - t^{p^n} \otimes 1$. So $\theta \in \mathrm{Diff}_k^{p^n-1}\, k[t]$, if and only if, $I^{p^n}.\theta = 0$. That is, if and only if, $0 = (1 \otimes t^{p^n} - t^{p^n} \otimes 1).\ \theta = \theta t^{p^n} - t^{p^n}\theta$. So θ is a differential operator of order $\leq p^n-1$, if and only if $\theta \in \mathrm{End}_{k[t^{p^n}]}k[t]$. This proves the result. □

We shall write $D_n = \mathrm{Diff}_k^{p^n-1}\, k[t]$. So we have just shown that $D_n \cong M_{p^n}(k[t^{p^n}])$, the $p^n \times p^n$ matrix ring over $k[t^{p^n}]$.

COROLLARY 2.2 (1) D *is not a finitely generated* k*-algebra;*

(2) D *does not contain any primitive idempotents; in fact if* $0 \neq e \in D$ *is idempotent then there exists a set of* p *mutually orthogonal idempotents* $e_1,\ldots,e_p$ *such that* $e = e_1 + \ldots + e_p$;

(3) D *contains an infinite direct sum of non-zero left ideals;*

(4) D *is not Noetherian;*

(5) D *does not have Krull dimension (in the sense of Gabriel and Rentschler).*

Proof (3), (4), (5) are immediate consequences of (2), and (1) is obvious, since any finite set of elements of D lies in some D_n, and so can at best generate D_n which is a proper subalgebra of D.

To prove (2), let $0 \neq e \in D$ be an idempotent. Suppose $e \in D_n = \mathrm{End}_{k[t^{p^n}]}k[t]$. Write $k[t] = U \oplus V$, a direct sum of $k[t^{p^n}]$-submodules, where $e|_U = \mathrm{Id}|_U$ and $e(V) = 0$. As $e \neq 0$, U is non-zero, and as a $k[t^{p^{n+1}}]$-module, $U = U_1 \oplus \ldots \oplus U_p$ is a direct sum of p non-zero $k[t^{p^{n+1}}]$-modules. Now $e = e_1 + \ldots + e_p$ where e_j is the projection of k[t] onto U_j with kernel $V \oplus U_1 \oplus \ldots \oplus \hat{U}_j \oplus \ldots \oplus U_p$ (omit U_j from the sum). One checks that each e_j is a $k[t^{p^{n+1}}]$-module map, hence an element of D_{n+1}, and that the e_j are mutually orthogonal idempotents. □

A concrete illustration of (2) above, is the following: if $e_n: k[t] \to k[t]$ is the

$k[t^{p^n}]$-linear map defined by $e_n(t^i) = \delta_{i,p^n-1}t^i$ for $0 \le i \le p^n$, then $\{e_1,e_2,\ldots\}$ is an infinite set of mutually orthogonal idempotents.

PROPOSITION 2.3 $K_0(D) \cong \mathbb{Z}[1/p]$

Proof $D_n \cong M_{p^n}(k[t^{p^n}])$ and one has that $K_0(D_n) = K_0(k[t^{p^n}])$ (as K_0 is defined in terms of the category of modules over D_n) and it is known that $K_0(k[t^{p^n}]) = \mathbb{Z}$. The inclusions $D_1 \to D_2 \to D_3 \to \ldots$ induce maps on the K_0 groups $\mathbb{Z} \xrightarrow{p} \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \ldots$. The maps are multiplication by p. As K_0 commutes with direct limits [7] we get $K_0(D) = \mathbb{Z}[1/p]$. □

An order unit is $1 = [R]$, and the order relation is the usual order relation on $\mathbb{Z}[1/p]$.

PROPOSITION 2.4 *Not every derivation of* D *is inner.*

Proof Define $\Delta: D \to D$ by $\Delta(d) = [t + t^p + t^{p^2} + \ldots, d]$. This actually makes sense: for $n \gg 0$, $d \in D_{n+1}$ and so d commutes with $t^{p^{n+1}}$, and hence with t^{p^m} for all $m > n$; therefore $\Delta(d) = [t + t^p + \ldots t^{p^n}, d]$ for $d \in D_{n+1}$.

Suppose Δ is inner, say $\Delta = ad(y)$ for some $y \in D$. Let $y \in D_n$. As $\Delta(t) = 0$, y commutes with k[t], hence $y \in k[t]$. For all n we have $\Delta - ady|_{D_{n+1}} = 0$ but we have just seen that $\Delta|_{D_{n+1}} = ad(t + t + \ldots + t^{p^n})$. Hence $ad(t+t^p+\ldots+t^{p^n}-y)|_{D_{n+1}}=0$, and so $t + t^p + \ldots t^{p^n} - y$ belongs to the centre of D_{n+1} $(= k[t^{p^{n+1}}])$ for all n; this is impossible. □

PROPOSITION 2.5 Centre (D) = k.

Proof Centre $(D_n) = k[t^{p^n}]$ and $\bigcap_{n=0}^{\infty} k[t^{p^n}] = k$. The proposition is an immediate consequence. □

Another description of D is also useful. For each $i \in \mathbb{N}$, let x_i be the k-linear map on k[t] given by $x_i(t^m) = \binom{m}{i}t^{m-i}$ where the binomial coefficient $\binom{m}{i}$ is evaluated (mod p). One should think of x_i as acting like $(1/i!)\partial^i/\partial t^i$; even though $1/i!$ does not make sense in k if $i \ge p$, this analogy can be made rigorous, as in Theorem 2.7 below. The analogy is useful in noticing relationships such as $x_i x_j = \binom{i+j}{i}x_{i+j}$.

THEOREM 2.6 $D_n = k[t,x_1,x_2,\ldots,x_{p^n-1}]$ *and* $D = k[t,x_1,x_2,\ldots]$.

Proof To see that x_m is a differential operator of order $\leq m$, notice that $x_0 = 1 \in D_0$, and $[x_m,t] = x_{m-1}$ then use the inductive Definition 1.1. Thus $k[t,x_1,\ldots,x_{p^n-1}] \subset D_n$.

Viewing $D_n \cong M_{p^n}(k[t^{p^n}])$, there is a basis for D_n as a $k[t^{p^n}]$-module given by the maps $\theta_{ij}: k[t] \to k[t]$ for $0 \leq i, j < p^n$ where θ_{ij} is the $k[t^{p^n}]$-module map defined by $\theta_{ij}(t^m) = \delta_{jm}\, t^{m+i-j}$ for $0 \leq m < p^n$. The θ_{ij} are just the matrix units (for the basis $1,t,\ldots,t^{p^n-1}$ of $k[t]$ as a $k[t^{p^n}]$-module).

One computes that $\theta_{ij} = t^i x_{p^n-1}\, t^{p^n-1-j}$ (the point being that $\binom{\ell}{p^n-1}$ is zero for all $\ell \in \mathbb{N}$ unless $\ell = p^n-1$). Thus $\theta_{ij} \in k[t,x_1,\ldots,x_{p^n-1}]$. This completes the proof. □

Recall that $D(\mathbb{Q}[t]) = \mathbb{Q}[t,\partial/\partial t]$. One can easily check that the $\mathbb{Z}$-module spanned by all elements of the form $t^j(1/i!)\partial^i/\partial t^i$ is in fact a $\mathbb{Z}$-subalgebra; write $S = \mathbb{Z}[t,\partial/\partial t,(1/2!)\partial^2/\partial t^2,\ldots]$. Of course $S = D(\mathbb{Z}[t])$, the ring of $\mathbb{Z}$-linear differential operators on $\mathbb{Z}[t]$. The following is straightforward.

THEOREM 2.7 $D(k[t]) \cong k \otimes_{\mathbb{Z}} S$ *where the isomorphism is given by* $x_i \to 1 \otimes (1/i!)\partial^i/\partial_t{}^i$ *and* $t \to t$.

The proof that D is a simple ring is inevitably a little more complicated than the proof in the characteristic zero case - if one recalls the characteristic zero proof, one part of it is the observation that if I is a non-zero ideal and $0 \neq a \in I$ then for some n, $\mathrm{ad}^n(\partial/\partial t)(a) \in k[\partial/\partial t]\backslash 0$, so there exists $0 \neq b \in I$ with $b \in k[\partial/\partial_t]$ and then for some m $\mathrm{ad}^m(t)(b) \in k\backslash\{0\}$, so I contains a scalar. However, if char $k = 2$, $\mathrm{ad}(\partial/\partial t)(t^2) = 0$.

Hence we require the following technical result.

LEMMA 2.8 $[x_\ell, t^m] = \sum_{j=1}^{\ell} (-1)^{j+1}\binom{m}{j} x_{\ell-j}\, t^{m-j}$ *for all* m,ℓ.

Proof Evaluate both sides at t^n, and the lemma reduces to checking the identity

$$\binom{m+n}{\ell} - \binom{n}{\ell} = \sum_{j=1}^{\ell} (-1)^{j+1}\binom{m}{j}\binom{m+n-j}{\ell-j} \text{ for all } m,n,\ell .$$

This is standard. □

PROPOSITION 2.9 D *is a simple ring.*

Proof Let $0 \neq I$ be a two-sided ideal of D. For some n, $I \cap D_n \neq 0$. A non-zero two sided ideal of a matrix ring over a ring R contains a non-zero ideal of R. Hence, for some n, $I \cap k[t^{p^n}] \neq 0$.

Choose $0 \neq f \in I \cap k[t]$, of lowest degree in t. Write $f = \alpha + g$ with $g \in tk[t]$, $\alpha \in k$. If $g = 0$ then $I \cap k \neq 0$, hence $I = D$, and the proof is complete. Suppose then, that $g \neq 0$, and let t^r be the lowest degree term appearing in g. Pick n, with $p^n \leq r < p^{n+1}$. Consider $[x_{p^n}, f] = [x_{p^n}, g] \in I$.

If $m \geq p^n$, then by Lemma 2.8, $[x_{p^n}, t^m] = \sum_{j=1}^{p^n} (-1)^{j+1}\binom{m}{j} x_{p^n-j} t^{m-j} = (-1)^{p^n+1}$ $\binom{m}{p^n} t^{m-p^n}$ since $\binom{m}{j} = 0 \pmod p$ for $j < p^n \leq m$. Also notice that as $p^n \leq r < p^{n+1}$ $\binom{r}{p^n} \neq 0 \pmod p$. So in particular $[x_{p^n}, t^r] \neq 0$; thus $[x_{p^n}, g]$ is of lower degree than f and is non-zero. This contradicts the choice of f. Thus $g = 0$, and the proof is complete. □

PROPOSITION 2.10 D *contains a proper subalgebra isomorphic to* D, *namely* $k[t^p, x_p, x_{2p}, x_{3p}, \ldots]$

Proof Notice that for all i,j $x_{ip}(t^{jp}) = \binom{jp}{ip} t^{(j-i)p}$ and that $\binom{jp}{ip} = \binom{j}{i} \pmod p$. Hence the natural action of x_{ip} on $k[t]$ maps $k[t^p]$ into $k[t^p]$, and so each x_{ip} is a differential operator on $k[t^p]$. After Theorem 2.6 $D(k[t^p]) = k[t^p, y_1, y_2, \ldots]$ where $y_i(t^{jp}) = \binom{j}{i}(t^p)^{j-i}$. As each x_{ip} acts as does y_i, we conclude that $D(k[t^p]) \cong k[t^p, x_p, x_{2p}, \ldots]$; of course $D(k[t]) \cong D(k[t^p])$ so we have shown that $D \cong k[t^p, x_p, x_{2p}, \ldots]$.

That $k[t^p, x_p, x_{2p}, \ldots]$ is a proper subalgebra of D is obvious from the fact that $D = k[t] \oplus k[t]x_1 \oplus k[t]x_2 \oplus \ldots$ (this follows from Theorem 2.7) and $k[t^p, x_p, x_{2p}, \ldots]$ $= k[t^p] \oplus k[t^p]x_p \oplus \ldots$. □

The next example illustrates that one useful technique for studying the Weyl algebra in characteristic zero, is not available in characteristic p. If k is a field with char k = 0, then $D(\mathbb{A}_k^1) \cong k[x,y]$ with $xy - yx = 1$; $D(\mathbb{A}_k^1)$ can be localised at the non-zero elements of k[x] and k[y] respectively. The diagonal embedding of $D(\mathbb{A}_k^1)$ into the direct sum of the localisations, $D(\mathbb{A}_k^1) \to k(x)[y] \oplus k(y)[x]$, is a faithfully flat embedding; the "faithfulness" comes from the fact that if I is a left ideal of $D(\mathbb{A}_k^1)$ with $I \cap k[x] \neq 0$ and $I \cap k[y] \neq 0$ then, in fact, $I = D(\mathbb{A}_k^1)$.

<u>EXAMPLE</u> There is a left ideal I of D, $I \neq D$ such that $I \cap k[t] \neq 0$ and $I \cap k[x_1,x_2,\ldots] \neq 0$.

We construct our example for char k = 2, but a similar example exists for any characteristic.

So, assume p = 2, put $I = Dt^2 + Dx_1$. Recall, from Theorem 2.7 that D is a free left k[t]-module with basis $1,x_1,x_2,\ldots$, so $D = \bigoplus_{n=0}^{\infty} k[t]x_n$. Now $x_nx_1 = \binom{n+1}{1}x_{n+1} = \begin{cases} 0 & n \text{ odd} \\ x_{n+1} & n \text{ even} \end{cases}$, and thus $Dx_1 = \bigoplus_{n \text{ odd}} k[t]x_n$. As p = 2, $[t^2,x_1] = 0$, thus $(k[t] + k[t]x_1)t^2 \subseteq k[t]t^2 + k[t]x_1$. If $n \geq 2$, then $x_nt^2 = t^2x_n + x_{n-2}$, so $k[t]x_nt^2 = k[t](t^2x_n + x_{n-2})$. Consequently,

$$I \subseteq \sum_{n \text{ odd}} k[t]x_n + k[t]t^2 + \sum_{\substack{n \text{ even} \\ n \geq 2}} k[t](t^2x_n + x_{n-2}) =$$

$$k[t]t^2 + k[t]x_1 + k[t](t^2x_2+1) + k[t]x_3 + k[t](t^2x_4+x_2) + \ldots$$

and it is easy to see that $1 \notin I$.

<u>PROPOSITION 2.11</u> k[t] *is a simple* D-*module.*

<u>Proof</u> Let $0 \neq N$ be a submodule of k[t]. We will show $N \cap k \neq 0$ from which the result follows. Suppose $N \cap k = 0$, and choose $f \in N$ of least degree. Let t^r be the highest degree term appearing in f. Choose n such that $p^n \leq r < p^{n+1}$. Then Then $x_{p^n}(t^r) = \binom{r}{p^n}t^{r-p^n}$, and $\binom{r}{p^n} \neq 0 \pmod p$. Hence $x_{p^n}(f) \neq 0$ and is of lower degree than f. This contradicts the choice of f. □

Recall that if k is of characteristic zero then the natural action of $k[t,\partial/\partial t]$ on $k[t]$ extends to an action of $k[t,\partial/\partial t]$ on $k[t,f^{-1}]$ for any $0 \neq f \in k[t]$, and that $k[t,f^{-1}]$ is of finite length as a $k[t,\partial/\partial t]$-module. The usual proof of this [2] uses Gelfand-Kirillov dimension. Although the same tool is no longer available in characteristic $p > 0$, the same result is true (Theorem 2.13). In order to prove this a few preliminary observations are required.

As $D_n \cong M_{p^n}(k[t^{p^n}])$, any non-zero D_n-module has dimension (over k) at least p^n. After Theorem 2.6 (and its proof) we have $D_n = k[t] \oplus x_1 k[t] \oplus \ldots \oplus x_{p^n-1} k[t]$ If $0 \neq f \in k[t]$ with $\deg(f) = F$ then $D_n/D_n f \cong S \oplus x_1 S \oplus \ldots \oplus x_{p^n-1} S$ where, $S = k[t]/(f)$, as a right $k[t]$-module. As $\dim S = F$, $\dim (D_n/D_n f) = p^n F$, and hence by our first observation $\text{length}_{D_n}(D_n f) \leq F$.

<u>LEMMA 2.12</u> *Let* M *be a left* D*-module, with a chain of finite dimensional subspaces* $M_0 \subset M_1 \subset M_2 \subset \ldots$ *such that*

(a) *each* M_n *is a* D_n*-module,*

(b) *for large* n, $\text{length}_{D_n}(M_n) \leq F$ (*fixed* F *for all* $n \gg 0$),

(c) $M = \bigcup_{n=0}^{\infty} M_n$.

Then, as a D*-module,* $\text{length}_D(M) \leq F$.

<u>Proof</u> Suppose $F = 1$. We must show that M is a simple D-module. Choose $0 \neq m \in M$ and choose any $m' \in M$. For all sufficiently large n, m and m' belong to M_n, which is a simple D_n-module by (b). Thus $m' \in D_n m \subset Dm$. Thus M is a simple D-module.

We now prove the result by induction on F. Suppose $F \geq 2$, and that the lemma is true for all numbers less than F. If M is simple as a D-module the proof is finished. If not, choose $0 \neq N$ a proper D-submodule of M. Put $N_n = N \cap M_n$; notice that $N = \bigcup_{n=0}^{\infty} N_n$, and each N_n is a D_n-module. We show that for all large n, $\text{length}_{D_n}(N_n) \leq F-1$. To see this, pick $m \in M$, $m \notin N$. There exists n_0 such that $m \in M_n$ for all $n \geq n_0$, but $m \notin N_n$. Hence, if $n \geq n_0$, $N_n \subsetneq M_n$. Thus $\text{length}_{D_n}(N_n) \leq F-1$ for all large n. By the induction hypotheses $\text{length}_D(N) \leq F-1$.

We have shown that any proper submodule of M has length at most F-1. Hence,

$\text{length}_D(M) \le F$. □

THEOREM 2.13 *Let* $0 \neq f \in k[t]$. *Then the* D-*module* $k[t,f^{-1}]$ *is of finite length (in fact, of length* $\le \deg(f) + 1$).

Proof As $k[t]$ is a simple D-submodule of $k[t,f^{-1}]$, it is enough to show that $M = k[t,f^{-1}]/k[t]$ is of length $\le \deg(f)$.

For each n, let M_n be the D_n-submodule of M generated by the image of f^{-p^n}. If $gf^{-m} \in M$ with $g \in k[t]$, there exists an n, with $m < p^n$; then $gf^{-m} = gf^{p^n-m}f^{-p^n} \in M_n$. Hence $M = \bigcup_{n=0}^{\infty} M_n$.

Put $F = \deg(f)$. We will show that $\text{length}_{D_n}(M_n) \le F$, and the theorem will follow from Lemma 2.12. Recall that a non-zero D_n-module has dimension at least p^n, so it will suffice to show that $\dim_k M_n \le Fp^n$.

Recall that $D_n = k[t] \oplus k[t]x_1 \oplus \ldots \oplus k[t]x_{p^n-1}$, so if one has $x_j(f^{-p^n}) = 0$ for $1 \le j < p^n$, then $M_n = D_n.f^{-p^n} = k[t].f^{-p^n}$, and as $f^{p^n}.f^{-p^n} = 0$ (remember $M = k[t,f^{-1}]/k[t]$), it would follow that $\dim_k(M_n) = \dim_k(k[t]/\langle f^{p^n}\rangle) = Fp^n$.

So the theorem is complete if $x_j(f^{-p^n}) = 0$ for $1 \le j < p^n$. However, $f^{p^n} \in k[t^{p^n}]$, and as $x_j \in D_n$, x_j commutes with multiplication by f^{p^n}. Thus $x_j(f^{-p^n}) = f^{-p^n}x_j(1) = 0$, for $1 \le j < p^n$. □

The following is well known and is useful in deciding whether x_ix_j is zero or not.

LEMMA 2.14 *If* $a,b \in \mathbb{N}$ *and the* p-*adic expansions are* $a = a_0 + a_1p + a_2p^2 + \ldots$, $b = b_0 + b_1p + b_2p^2 + \ldots$ *then* $\binom{a}{b} \equiv \prod_{j=1} \binom{a_j}{b_j} (\text{mod } p)$.

LEMMA 2.15 *For* $m \ge n$, D_m *is free as a* D_n-*module (on either the right or the left) of rank* p^{m-n}. *A basis for* D_m *as a* D_n *-module is given by* $1, x_{p^n}, x_{2p^n}, \ldots, x_{(p^m-1)p^n}$.

Proof Recall the description of D_n and D_m given in Theorem 2.6. If $0 \le j \le p^n-1$, and $0 \le i \le p^m-1$ then $x_jx_{ip^n} = \binom{j+ip^n}{j}x_{j+ip^n}$. However, writing j and ip^m in their p-adic form, Lemma 2.14 ensures that $x_jx_{ip^n} \neq 0$. The Lemma follows. □

The following consequence of Lemma 2.12 is useful.

LEMMA 2.16 *If* N *is a* D_n*-module of finite length, then* $D \otimes_{D_n} N$ *is of finite length as a* D*-module.*

Proof If N were a faithful D_n-module then D_n would be artinian (which it is not). So $I = \text{ann}_{D_n}(N) \neq 0$. But a non-zero ideal of $D_n = M_{p^n}(k[t^{p^n}])$ intersects $k[t^{p^n}]$ in a non-zero ideal. Thus N is a finitely generated module over the finite dimensional algebra $M_{p^n}(k[t^{p^n}]/I \cap k[t^{p^n}]) = D_n/I$. Thus $\dim_k N < \infty$.

Let $m \geq n$. As D_m is a free D_n-module of rank p^{m-n}, $D_m \otimes_{D_n} N$ is of dimension $\leq p^{m-n} \dim_k N$. As a non-zero D_m-module has dimension $\geq p^m$, $\text{length}_{D_m}(D_m \otimes_{D_n} N) \leq p^{-n} \dim_k N$. The lemma follows from Lemma 2.12 by observing that $D \otimes_{D_n} N = \bigcup_{m \geq n} D_m \otimes_{D_n} N$. □

We next show that gl.dim.D = 1. As the comments and example following Proposition 2.10 indicate, the proof that $\text{gl.dim.}(D(A_k^1)) = 1$ when k is of characteristic zero cannot be used. The following preparatory lemma is required (and allows us in the proof of Theorem 2.18 to make frequent use of the fact that for a finitely generated D_n-module the concepts of torsion submodule coincide whether we consider torsion with respect to the regular elements of D_n, or with respect to the non-zero elements of k[t] when the D_n-module is viewed as a k[t]-module).

LEMMA 2.17 *Let* M *be a finitely generated* D_n*-module. Let* M_1 *be the torsion submodule of* M *with respect to the regular elements of* D_n*; let* M_2 *be the torsion submodule of* M *with respect to* $k[t^{p^n}]$*; let* M_3 *be the torsion submodule of* M *with respect to* k[t]. *Then* $M_1 = M_2 = M_3$.

Proof As $k[t] \subset D_n$ and D_n is a free k[t]-module, $k[t]\backslash\{0\}$ consists of regular elements in D_n. Hence $M_3 \subset M_1$. Similarly $M_2 \subset M_3 \subset M_1$.

Write Q_n for the ring of fractions of D_n. That is, $Q_n = M_{p^n}(k(t^{p^n})) = k(t^{p^n}) \otimes_{k[t^{p^n}]} D_n$, where $k(t^{p^n})$ denotes the field of rational functions in t^{p^n}. Now $Q_n \otimes_{D_n} M_1 = 0$. Hence $k(t^{p^n}) \otimes_{k[t^{p^n}]} M_1 = 0$, and it follows that $M_1 \subset M_2$. □

THEOREM 2.18 gl.dim. D = 1.

Proof As D is not semi-simple artinian, gl.dim. $D \geq 1$. So it is enough to show that every left ideal of D is projective. Let I be a left ideal.

Put $I_n = I \cap D_n$, and define I'_n to be the left ideal of D_n containing I_n such that I'_n/I_n is the torsion submodule of the D_n-module D_n/I_n. Put $T_n = DI'_n \cap I$.

We claim that $T_n \subset T_{n+1}$. To see this it is enough to check that $I'_n \subset I'_{n+1}$. But $I'_n + I_{n+1}/I_{n+1} \cong I'_n/I'_n \cap I_{n+1}$ which is a homomorphic image of I'_n/I_n. As I'_n/I_n is k[t]-torsion so is $I'_n + I_{n+1}/I_{n+1}$. Thus $I'_n \subset I'_{n+1}$.

We claim that T_n is a finitely generated left ideal. Notice that $T_n/DI_n \subseteq DI'_n/DI_n \cong D \otimes_{D_n} (I'_n/I_n)$. By Lemma 2.16 this latter D-module is of finite length since I'_n/I_n is of finite length as a D_n-module. The truth of the claim follows from the fact that DI_n is finitely generated, and that T_n/DI_n is of finite length.

Consider T_{n+1}/T_n. As both these left ideals are finitely generated there exists $m \in \mathbb{N}$ with $T_{n+1}/T_n = D(T_{n+1} \cap D_m)/D(T_n \cap D_m)$. Now $T_{n+1} \cap D_m/T_n \cap D_m \cong T_n + (T_{n+1} \cap D_m)/T_n$ which is a submodule of $I/T_n = I/I \cap DI'_n \cong I + DI'_n/DI'_n$ which is a submodule of $D/DI'_n \cong D \otimes_{D_n} (D_n/I'_n)$. However, as a k[t]-module D_n/I'_n is torsion-free, hence so is D/DI'_n. Thus $T_{n+1} \cap D_m/T_n \cap D_m$ is torsion-free as a D_m-module. But D_m is a hereditary Noetherian prime ring, so by [5, Theorem 2.1] a torsion-free D_m-module is projective. Hence there is a left ideal J of D_m with $T_{n+1} \cap D_m = T_n \cap D_m \oplus J$. Thus (as D is free as a D_m-module) $D(T_{n+1} \cap D_m) = D(T_n \cap D_m) \oplus DJ$. In particular, there is a finitely generated left ideal S_n with $T_{n+1} = T_n \oplus S_n$.

Now $I = \bigcup_{n=0}^{\infty} DI_n = \bigcup_{n=0}^{\infty} T_n = T_0 + T_1 + \ldots = S_0 \oplus S_1 \oplus S_2 \oplus \ldots$. But each S_n is finitely generated hence projective (because $S_n \cong D \otimes_{D_m} (D_m \cap S_n)$, and $D_m \cap S_n$ is a projective D_m-module). Thus I is projective. □

Goodearl has pointed out the following way of viewing D. Let B denote the subring $k[x_1, x_2, \ldots]$ of D; B is isomorphic to the factor ring of a commutative polynomial ring $k[X_1, X_2, \ldots]$ modulo the ideal generated by $X_i X_j - \binom{i+j}{i} X_{i+j}$. The inner derivation $ad(t) = [t,-]$ of D maps B into itself, so $ad(t)$ acts as a derivation on B, and D may be viewed as $B[t]$, the extension of B by the derivation $ad(t)$. Now it is easy to see gl.dim. $B = \infty$ because there exist non-split exact sequences:

$$0 \to X_1B \to B \to x_{p-1}B \to 0$$

$$0 \to x_{p-1}B \to B \to x_1B \to 0.$$

Hence this gives an example of a commutative ring of infinite global dimension such that an extension by a derivation has finite global dimension (the first such example appears in [6]).

As D is a ring of differential operators it has a filtration given by the order of the operators. As x_n is of order n, the filtration is given by $F_nD = k[t] \oplus k[t]x_1 \oplus \dots \oplus k[t]x_n$, and the associated graded algebra grD is isomorphic to B[s] where s is a commuting indeterminate. Hence although gl.dim $D = 1$, gl.dim (grD) $= \infty$.

Notice that the exact sequences over D corresponding to those for B given above are split. This is because Dx_{p-1} is projective (being generated by the idempotent $t^{p-1}x_{p-1}$).

We now briefly turn our attention to the ring of fractions of D. As D is a free k[t]-module, $k[t]\backslash\{0\}$ consists of regular elements of D. Hence Fract D contains k(t). As $D_n \cong M_{p^n}(k[t^{p^n}])$, Fract $D_n \cong M_{p^n}(k(t^{p^n}))$. Thus we have

THEOREM 2.19 *The ring of fractions* Q, *of* D, *is equal to* $k(t)[x_1,x_2,\dots]$ *and* $Q = \bigcup_{n=0}^{\infty} Q_n$ where $Q_n = \mathrm{End}_{k(t^{p^n})}k(t) = \mathrm{Fract}\ D_n$.

In particular Q is a union of simple artinian rings, so is von Neumann regular. As Q is flat as a D-module, gl.dim. Q $\le$ gl.dim. D. But Q is not semi-simple artinian, so gl. dim. $Q = 1$.

PROPOSITION 2.20 Q *is not self-injective.*

Proof It is sufficient to find a left ideal J of Q, and a Q-module map $\phi: J \to Q$ which is not the restriction of a Q-module map $\psi: Q \to Q$.

Put $J = Qx_1 + Qx_2 + \dots$; consider the formal sum $y = \sum_{j=0}^{\infty} x_{p^j-1}$, and define $\phi: J \to Q$ by $\phi(r) = ry$. This does make sense: notice that $x_i x_{p^j-1} = 0$ if i is fixed and j is sufficiently large, thus ry is actually a finite sum for $r \in J$. So ϕ is a bona-fide Q-module homomorphism.

Suppose that $\psi: Q \to Q$ is a left Q-module map. Then ψ is just right multiplication by $z = \psi(1)$. So if $\phi = \psi|_J$ then, in particular, $x_i(y-z) = 0$ for all $i \geq 1$. Suppose $z = a_0 + x_1 a_1 + \dots + x_n a_n$ with each $a_j \in k[t]$, and $a_n \neq 0$. Suppose $p^m-1 > n$. Then $x_{p^m} \cdot y = \sum_{j=0}^{m} x_{p^m} \cdot x_{p^j-1}$, and $x_{p^m} \cdot x_{p^m-1} \neq 0$, but $x_{p^m}\text{-}z$ cannot contain a term involving x_{2p^m-1} since $n < p^m-1$. Hence $x_{p^m} \cdot y \neq x_{p^m} z$, and thus $\phi \neq \psi|_J$. □

§3. PROPERTIES OF $D(\mathbf{P}_p^1)$

We begin by defining $D(\mathbb{P}_p^1)$. Let $\mathcal{D}$ be the sheaf of differential operators on $\mathbf{P}^1$, and define $D(\mathbf{P}^1) = \Gamma(\mathbb{P}^1, \mathcal{D})$. As $\mathcal{D}$ is the unique quasi-coherent sheaf of $\mathcal{O}_{\mathbf{P}^1}$-modules such that for every open affine $U \subset \mathbf{P}^1$, $\Gamma(U, \mathcal{D})$ is the ring of differential operators on $\mathcal{O}(U)$ (the ring of regular functions on U) to compute the global sections of $\mathcal{D}$ we may proceed as follows. Let U_+, U_- be two copies of $\mathbf{A}^1$ covering $\mathbf{P}^1$ such that $\mathcal{O}(U_+) = k[t]$, $\mathcal{O}(U_-) = k[t^{-1}]$ and let D^+, D^- denote the rings of differential operators on U^+ and U^- respectively. If D^+ and D^- are considered as subalgebras of $D(k(t))$, we have $D(\mathbf{P}^1) = D^+ \cap D^-$. As $D^+ = \{\theta \in D(k(t)) \mid \theta(k[t]) \subset k[t]\}$ and $D^- = \{\theta \in D(k(t)) \mid \theta(k[t^{-1}]) \subset k[t^{-1}]\}$ we have $D(\mathbf{P}^1) = \{\theta \in D(k(t)) \mid \theta(k[t]) \subset k[t]$ and $\theta(k[t^{-1}]) \subset k[t^{-1}]\}$. Thus we obtain (for k a field of characteristic $p > 0$)

<u>LEMMA 3.1</u> *Fix* n, *put* $q = p^n$ *and let* $\theta \in D_n^+$ (using the notation of §2). *Then* $\theta \in D(\mathbf{P}_k^1)$, *if and only if*

(1) $\theta(1) \in k$

(2) $\theta(t^j) \in \text{lin.span} \langle 1, t, t^2, \dots, t^q \rangle$ *for all* j, $0 < j < q$.

<u>Proof</u> Suppose θ satisfies the conditions. First observe that θ extends to a $k[t^q]$-linear differential operator on $k(t)$ (since $\theta \in D_n^+$). Pick $i > 0$; we show that $\theta(t^{-i}) \in k[t^{-1}]$.

Pick m such that $mq < i \leq (m+1)q$. Then $0 \leq (m+1)q-i < q$, so by (2), $\theta(t^{(m+1)q-i}) \in$ lin.span $<1,t,t^2,\ldots,t^q>$. But $\theta(t^{(m+1)q-i}) = t^{(m+1)q}\theta(t^{-i})$, hence $\theta(t^{-i}) \in$ lin.span $t^{-(m+1)q}<1,t,\ldots,t^q> \subset k[t^{-1}]$. This and (1) ensure that $\theta(k[t^{-1}]) \subset k[t^{-1}]$, and so $\theta \in D(\mathbf{P}_k^1)$. The conditions are therefore sufficient.

On the other hand, if $\theta \in D(\mathbf{P}_k^1)$, then certainly $\theta(k[t] \cap k[t^{-1}]) \subset k[t] \cap k[t^{-1}]$, so (1) is necessary. Also if $0 < j < q$, then $\theta(t^{-j}) \in k[t^{-1}]$, and hence $\theta(t^{q-j}) = t^q\theta(t^{-j}) \in t^q k[t^{-1}] \cap k[t] =$ lin.span. $<1,t,\ldots,t^q>$. So (2) is necessary. □

Put $D(\mathbf{P}^1)_n = D(\mathbf{P}^1) \cap D_n^+$; that is, $D(\mathbf{P}^1)_n$ is the differential operators in $D(\mathbf{P}^1)$ of order $\leq n$. Notice that after the lemma, $\dim_k D(\mathbf{P}^1)_n = 1 + (p^n-1)(p^n+1) = p^{2n}$, so $D(\mathbf{P}^1)$ is a union of finite dimensional subalgebras.

<u>LEMMA 3.2</u> *The nilpotent radical of* $D(\mathbf{P}^1)_n$ *is the span of those* θ *which satisfy*

(1) $\theta(1) = 0$

(2) $\theta(t^j) \in$ lin.span $<1,t^{p^n}>$ *for all* $0 < j < p^n$.

<u>Proof</u>

Put $q = p^n$. First the span of such θ is an ideal of $D(\mathbf{P}^1)_n$. If $\psi \in D(\mathbf{P}^1)_n$, then $\psi\theta(1) = \theta\psi(1) = 0$; and for $0 < j < q$, one has $\psi\theta(t^j) \subset$ lin.span $<\psi(1),\psi(t^q)> =$ lin.span $<\psi(1), t^q\psi(1)> \subset$ lin.span $<1,t^q>$ by Lemma 3.1(1); also $\theta\psi(t^j) \subset$ lin.span $<\theta(1),\theta(t),\ldots,\theta(t^q)> \subset$ lin.span $<1,t^q>$ as $\theta(t^q) = t^q\theta(1) = 0$. We have shown that if θ satisfies (1) and (2), so do $\theta\psi$ and $\psi\theta$. Hence the span of such θ is an ideal.

The square of this ideal is zero: if θ and ψ satisfy (1) and (2) then $\psi\theta(1) = 0$ and for $0 < j < q$, $\psi\theta(t^j) \subset$ lin.span $<\psi(1),\psi(t^q)> = 0$.

The factor by this ideal is semi-simple artinian: the factor may be identified with those θ such that $\theta(1) \in k$ and $\theta(t^j) \in$ lin.span $<t,t^2,\ldots,t^{q-1}>$ for $1 \leq j < q$; but this algebra is isomorphic to $(\mathrm{End}_k k) \oplus (\mathrm{End}_k k^{q-1})$. □

Put N_n = nilpotent radical of $D(\mathbf{P}^1)_n$; notice that $\dim N_n = 2(p^n-1)$.

<u>LEMMA 3.3</u> $N_n \cap N_{n+1} = 0$.

<u>Proof</u> Pick $0 \neq \theta \in N_n$. Then $\theta(t^j) \neq 0$ for some $0 < j < p^n$. Hence, if $\theta \in N_{n+1}$,

then $\theta(t^j) \in$ lin. span $<1,t^{p^{n+1}}> \cap$ lin.span $<1,t^{p^n}> = k$. But $0 < j + p^n < p^{n+1}$ and $\theta(t^{j+p^n}) = t^{p^n}\,\theta(t^j) \in kt^{p^n}$. But by applying Lemma 3.2(2) for n+1, one must have $\theta(t^{j+p^n}) \in$ lin.span $<1,t^{p^{n+1}}>$. Thus $\theta(t^{j+p^n}) = 0$, whence $\theta(t^j) = 0$. This contradiction gives the result. □

PROPOSITION 3.4 $D(\mathbf{P}^1)$ *contains no non-zero nilpotent ideal.*

Proof Suppose $N \neq 0$, is a nilpotent ideal. Then $N \cap D(\mathbf{P}^1)_n \neq 0$ for some n. Thus $N \cap D(\mathbf{P}^1)_n$ is a nilpotent ideal of D_n. Similarly $N \cap D(\mathbb{P}^1)_{n+1}$ is a nilpotent ideal of $D(\mathbf{P}^1)_{n+1}$. Hence $0 \neq N \cap D(\mathbf{P}^1)_n \subset N_n \cap N_{n+1}$. This contradicts Lemma 3.3. □

PROPOSITION 3.5 $D(\mathbb{P}^1)$ *is not von Neumann regular.*

Proof Consider $x_1 \in D^+$ (the notation is that of §2). One sees that $x_1 = \partial/\partial t \in D(\mathbf{P}')$. Suppose there exists $a \in D(\mathbb{P}')$ with $x_1 a x_1 = x_1$. Then in particular, as $x_1(t) = 1$, one has $x_1 a(1) = 1$. But if $a \in D(\mathbf{P}')$ then $a(1) = 1$. However, $x_1(k) = 0$, so there exists no $a \in D(\mathbb{P}')$ with $x_1 a(1) = 1$. Hence the result. □

PROPOSITION 3.6 $D(\mathbf{P}^1)$ *is its own ring of fractions.*

Proof This is true of any algebra which is a union of finite dimensional algebras over a field (since an artinian ring is its own ring of fractions). □

PROPOSITION 3.7 (1) $D(\mathbf{P}^1)_n$ *is the sum of the two-sided ideals* $J_n = \{\theta \in D(\mathbf{P}^1)_n \mid \theta(t^j) \in k$ for all $0 \le j < p^n\}$ *and* $Q_n = \{\theta \in D(\mathbf{P}^1)_n \mid \theta(1) = 0\}$. (2) $\dim_k (D(P^1)_n/Q_n) = 1$ (3) $J_n \cap Q_n = N_n$. (4) For $n \ge 1$, J_n/N_n *and* Q_n/N_n *are minimal ideals of* $D(\mathbf{P}^1)_n/N_n$. (5) *Let* $\alpha \in D(\mathbb{P}^1)_n$. *The two sided ideal of* $D(\mathbb{P}^1)_n$ *generated by* α *equals* $D(\mathbb{P}^1)_n$ *if and only if* α *can be written in the form* $\alpha = \beta + \gamma$ *with* $\beta \in J_n \backslash N_n$ *and* $\gamma \in Q_n \backslash N_n$.

Proof After Lemmas 3.1 and 3.2 the proposition is straightforward. □

PROPOSITION 3.8 (Notation as in (3.7)). *Put* $Q = \bigcup_{n=0}^{\infty} Q_n$. *Then* Q *is the unique proper ideal of* $D(\mathbf{P}^1)$, *and* $D(\mathbf{P}^1)/Q \cong k$.

Proof As each $Q_n \subset Q_{n+1}$, and Q_n is an ideal of $D(\mathbf{P}^1)_n$, Q is a two sided ideal of

$D(\mathbf{P}^1)$.

Suppose $\theta \in D(\mathbf{P}^1)_n$ and $\theta \notin Q_n$. Then $D(\mathbf{P}^1)\,\theta\, D(\mathbf{P}^1) = D(\mathbf{P}^1)$. To prove this it is enough to show that $D(\mathbf{P}^1)_{n+1}\,\theta D(\mathbf{P}^1)_{n+1} = D(\mathbf{P}^1)_{n+1}$. As $\theta \notin Q_n$, $\theta(1) \neq 0$. Hence, without loss of generality $\theta(1) = 1$. As θ is $k[t^{p^n}]$-linear, $\theta(t^{p^n}) = t^{p^n}$, and it follows that $\theta \notin J_{n+1}$, and $\theta \notin Q_{n+1}$. Hence by Proposition 3.7(5), the two sided ideal of $D(\mathbf{P}^1)_{n+1}$ generated by θ is $D(\mathbf{P}^1)_{n+1}$ itself.

It follows that any two sided ideal of $D(\mathbf{P}^1)$ not equal to $D(\mathbf{P}^1)$ must be contained in Q.

Suppose now that $\theta \in Q$, $\theta \neq 0$. We show θ generates Q. Suppose $\theta \in D(\mathbf{P}^1)_n$. Hence $\theta(1) = 0$, and as $\theta \neq 0$, $\theta(t^j) \neq 0$ for some j, $0 < j < p^n$. Hence $\theta(t^{j+p^n}) = t^{p^n}\,\theta(t^j) \notin k$. Thus $\theta \notin J_{n+1}$. It follows that $D(\mathbf{P}^1)_{n+1}\theta\, D(\mathbf{P}^1)_{n+1} = Q_{n+1}$. This is true for all $n \gg 0$, so $D(\mathbf{P}^1)\theta D(\mathbf{P}^1) = Q$.

Thus Q is the unique proper ideal of $D(\mathbf{P}^1)$. Finally as $\dim_k(D(\mathbf{P}^1)_n/Q_n) = 1$ for all n, $\dim_k(D(\mathbf{P}^1)/Q) = 1$. □

PROPOSITION 3.9 $D(\mathbf{P}^1)$ *is a primitive ring, and* $k[t]$ *is a faithful module of length* 2*, the submodule being* k.

Proof This is an immediate consequence of Lemma 3.1. □

We now compute $K_0(D(\mathbf{P}^1))$. As K_0 commutes with direct limits, one has $K_0(D(\mathbf{P}^1)) = \varinjlim K_0(D(\mathbf{P}^1)_n)$. We need only consider $n \geq 1$, so henceforth assume $n \geq 1$.

Recall that $D(\mathbf{P}^1)_n/N_n = J_n/N_n \oplus Q_n/N_n$ and $J_n/N_n \cong k$ while $Q_n/N_n \cong M_{p^n-1}(k^{p^n-1})$ (this is implicit in the proof of Lemma 3.2). Hence $K_0(D(\mathbf{P}^1)_n) = \mathbf{Z} \oplus \mathbf{Z}$ with $[D(\mathbf{P}^1)_n] = (1,p^n-1)$. The positive cone in $K_0(D(\mathbf{P}^1)_n)$ is $K_0^+(D(\mathbf{P}^1)_n) = \{(a,b) \in \mathbf{Z} \oplus \mathbf{Z} \mid a \geq 0,\ b \geq 0\}$.

The embedding $D(\mathbf{P}^1)_n \to D(\mathbf{P}^1)_{n+1}$ induces maps $\phi_n : K_0(D(\mathbf{P}^1)_n) \to K_0(D(\mathbf{P}^1)_{n+1})$ given by $\phi_n(1,0) = (1,p-1)$ and $\phi_n(0,1) = (0,p)$.

Define $G_n = \mathbf{Z} \oplus \mathbf{Z}$ and let $\psi_n : G_n \to G_{n+1}$ be the group homomorphism $\psi_n(1,0) = (1,0)$, $\psi_n(0,1) = (0,p)$. Define $\delta : \mathbf{Z} \oplus \mathbf{Z} \to \mathbf{Z} \oplus \mathbf{Z}$ by $\delta(1,0) = (1,1)$, $\delta(0,1) = (0,1)$, and extend δ to a group isomorphism. Then $\delta : (K_0(D(\mathbf{P}^1)_n, \phi_n) \to (G_n, \psi_n)$ is a chain isomorphism, so $K_0(D(\mathbf{P}^1)) = \varinjlim (G_n,\psi_n)$. As ψ_n is just the multiplication map $(a,b) \xrightarrow{(1,p)} (a,bp)$ one sees that this direct limit is $\mathbb{Z} \oplus \mathbb{Z}[1/p]$, and that $[D(\mathbf{P}^1)] = (1,p)$.

By chasing the positive cones $K_0^+(D(\mathbf{P}^1)_n)$, one obtains $K_0^+(D(\mathbb{P}^1)) = \{(a,b) \in \mathbb{Z} \oplus \mathbb{Z}[1/p] \mid a \geq 0 \text{ and } b > 0 \text{ or } (a,b) = (0,0)\}$. It is an easy matter now to see that the only order ideal in $K_0(D(\mathbf{P}^1))$ apart from 0 and $K_0(D(\mathbf{P}^1))$ is $\mathbb{Z}[1/p]$.

Hence the lattice of order ideals is isomorphic to the lattice of two sided ideals of $D(\mathbf{P}^1)$. We summarise the above.

THEOREM 3.10 $K_0(D(\mathbf{P}^1)) \cong \mathbb{Z} \oplus \mathbb{Z}[1/p]$, *with* $[D(\mathbf{P}^1)] = (1,p)$. *The lattice of order ideals in* $K_0(D(\mathbf{P}^1))$ *is isomorphic to the lattice of two sided ideals in* $D(\mathbb{P}^1)$; *this lattice is:*

Remark In [7, Corollary 15.21] it is proved that if R is a unit-regular ring there is an isomorphism between the lattice of two sided ideals of R, and the order ideals of $K_0(R)$. Of course after Proposition 3.5, $D(\mathbb{P}^1)$ is not unit-regular.

Recall that if k is a field of characteristic zero, then there is a surjective map $U(sl(2,k)) \to D(\mathbf{P}^1_k)$. This map is given by $e \to t^2\partial/\partial t$, $f \to -\partial/\partial t$, $h \to 2t\partial/\partial t$ where $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the usual basis for $sl(2,k)$. The surjectivity is seen from the fact that $D(\mathbf{P}^1_k) = k[\partial/\partial t, t\partial/\partial t, t^2\partial/\partial t]$, and this equality can be proved by elementary arguments. We show below that, if char $k = p > 0$, then the analogous map does not give a surjection from U_k, the hyperalgebra of $sl(2,k)$, to $D(\mathbf{P}^1_k)$.

So k is once again a field of characteristic $p > 0$. Denote the $\mathbb{Z}$-span of the elements $\frac{f^a}{a!}\binom{h}{b}\frac{e^c}{c!}$ with $a,b,c \in \mathbb{N}$, in $U(sl(2,\mathbb{C}))$ by $U_{\mathbb{Z}}$; this is the Kostant $\mathbb{Z}$-form and is a $\mathbb{Z}$-subalgebra of $U(sl(2,\mathbb{C}))$. The hyperalgebra U_k is defined to be $U_k = k \otimes_{\mathbb{Z}} U_{\mathbb{Z}}$.

$D(\mathbf{P}^1_{\mathbb{Z}})$ is equal to $D(\mathbb{Z}[t]) \cap D(\mathbb{Z}[t^{-1}])$, the intersection being taken inside $D(\mathbb{Z}[t,t^{-1}])$. Hence $D(\mathbf{P}^1_{\mathbb{Z}})$ is precisely those elements of $D(\mathbf{P}^1_{\mathbb{C}})$ which, when acting on $\mathbb{C}[t]$ and $\mathbb{C}[t^{-1}]$, map $\mathbb{Z}[t]$ into $\mathbb{Z}[t]$ and $\mathbb{Z}[t^{-1}]$ into $\mathbb{Z}[t^{-1}]$. The image of $\frac{f^a}{a!}\binom{h}{b}\frac{e^c}{c!}$ in $D(\mathbf{P}^1_{\mathbb{C}})$ is of course $\frac{(-\partial/\partial t)^a}{a!}\binom{2t\,\partial/\partial t}{b}\frac{(t^2\partial/\partial t)^c}{c!}$, and it is easy to check that this differential operator sends $\mathbb{Z}[t]$ to $\mathbb{Z}[t]$ and $\mathbb{Z}[t^{-1}]$ to $\mathbb{Z}[t^{-1}]$. Hence this element belongs to $D(\mathbf{P}^1_{\mathbb{Z}})$. Thus the map $U(sl(2,\mathbb{C})) \to D(\mathbf{P}^1_{\mathbb{C}})$ restricts to give a map $U_{\mathbb{Z}} \to D(\mathbf{P}^1_{\mathbb{Z}})$. This in turn induces a map $\phi: U_k \to D(\mathbf{P}^1_k)$ since $D(\mathbf{P}^1_k) = k \otimes_{\mathbb{Z}} D(\mathbf{P}^1_{\mathbb{Z}})$. This last equality derives from Theorem 2.7.

THEOREM 3.11 *The map $\phi: U_k \to D(\mathbf{P}^1_k)$ is not surjective.*

Proof Give $k[t,t^{-1}]$ the grading where t is of degree 1; define $D(\mathbf{P}^1_k)(j) = \{\theta \in D(\mathbf{P}^1_k) | \theta(kt^i) \subset kt^{i+j}$ for all $i \in \mathbb{Z}\}$. Then $D(\mathbf{P}^1_k) = \bigoplus_{j\in\mathbb{Z}} D(\mathbf{P}^1_k)(j)$ and this gives a grading on $D(\mathbf{P}^1_k)$. Notice that $\phi(e) \in D(\mathbf{P}^1_k)(1)$, $\phi(f) \in D(\mathbf{P}^1_k)(-1)$, $\phi(h) \in D(\mathbf{P}^1_k)(0)$. Likewise, $\phi(\frac{f^a}{a!} \binom{h}{b} \frac{e^c}{c!}) \in D(\mathbf{P}^1_k)(c-a)$.

Consider the element $t^{p-1} \frac{(\partial/\partial t)^p}{p!}$ which belongs to $D(\mathbf{P}^1_k)(1)$. We will show this is not in the image of ϕ. If it were in the image of ϕ, then it would be a linear combination of the image of elements $\frac{f^a}{a!} \binom{h}{b} \frac{e^c}{c!}$ with $c-a = 1$. Notice that $t^{p-1} \frac{(\partial/\partial t)^p}{p!}$ acts on $k[t]$ sending t^p to t^{p-1}. The action of $\frac{(\partial/\partial t)^a}{a!} \binom{2t\partial/\partial t}{b} \frac{(t^2\partial/\partial t)^{a+1}}{(a+1)!}$ sends t^p to $\binom{p+a}{p-1} \binom{2p+2a+2}{b} \binom{p+a+1}{p-1} t^{p-1}$. However, for all $a \in \mathbb{N}$, $\binom{p+a}{p-1}\binom{p+a+1}{p-1} \equiv 0 (\text{mod } p)$. Hence $\phi(\frac{f^a}{a!} \binom{h}{b} \frac{e^{a+1}}{(a+1)!})$ sends t^p to zero. Consequently, no linear combination of these elements can equal $t^{p-1} \frac{(\partial/\partial t)^p}{p!}$ which sends t^p to t^{p-1}. □

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