PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 90, Number 1, January 1984

GELFAND-KIRILLOV DIMENSION OF RINGS OF FORMAL DIFFERENTIAL OPERATORS ON AFFINE VARIETIES

S. P. SMITH

ABSTRACT. Let A be the coordinate ring of a smooth affine algebraic variety defined over a field k. Let D be the module of k-linear derivations on A and form $A[D]$, the ring of differential operators on A, as follows: consider A and D as subspaces of $\operatorname{End}_k A$ (A acting by left multiplication on itself), and define $A[D]$ to be the subalgebra generated by A and D. Let rk D denote the torsion-free rank of D (that is, $\operatorname{rk} D = \dim_F F \otimes_A D$ where F is the quotient field of A). The ring $A[D]$ is a finitely generated k-algebra so its Gelfand-Kirillov dimension $\operatorname{GK}(A[D])$ may be defined. The following is proved.

THEOREM. $\operatorname{GK}(A[D]) = \operatorname{tr deg}_k A + \operatorname{rk} D = 2 \operatorname{tr deg}_k A$.

Actually we work in a more general setting than that just described, and although a more general result is obtained, this is the most natural and important application of the main theorem.

1. Introduction. Let A be a finitely generated commutative algebra over the field k. In the terminology of [**9**] let D be a (k, A)-Lie algebra. We recall the definition.

(i) D is a Lie algebra over k, with the Lie product denoted [,];

(ii) D is an A-module;

(iii) there is an A-module homomorphism, θ: $D \to \operatorname{Der} A$ (the module of k-linear derivations on A); we denote $\theta(d)(a)$ by $d(a)$ for $d \in D$, $a \in A$;

(iv) these structures are related by the requirement that $[d_1, ad_2] = a[d_1, d_2] + d_1(a)d_2$ for $d_1, d_2 \in D$ and $a \in A$.

The most natural examples of (k, A)-Lie algebras are simply submodules D of $\operatorname{Der} A$, which are closed under the Lie bracket on $\operatorname{Der} A$.

Given a (k, A)-Lie algebra D we form the ring of *formal differential operators*, $A\langle D\rangle$, as follows: it is the factor ring, $T_A(D)/J$, of the tensor algebra of the A-module D, by the ideal J generated by the relations $da - ad = d(a)$ for all $a \in A$, $d \in D$ and $d_1 d_2 - d_2 d_1 = [d_1, d_2]$ (for all $d_1, d_2 \in D$). When D is an abelian Lie algebra and a finitely generated free A-module the ring $A\langle D\rangle$ has been studied by a number of authors [**1**], [**3–5**].

We shall always assume D is a finitely generated A-module, in which case $A\langle D\rangle$, being a factor of $T_A(D)$, is a finitely generated k-algebra. Thus the Gelfand-Kirillov

Received by the editors October 15, 1982 and, in revised form, April 11, 1983.
1980 *Mathematics Subject Classification*. Primary 16A55; Secondary 16A56, 16A72, 17B40.
Key words and phrases. Gelfand-Kirillov dimension, differential operators, derivations.

dimension of $A\langle D\rangle$ may be defined to be

$$\mathrm{GK}(A\langle D\rangle) = \limsup_{n\to\infty} (\log\dim W^n/\log n)$$

where W is a finite-dimensional subspace of $A\langle D\rangle$ generating $A\langle D\rangle$ as a k-algebra. We adopt the convention that any subspace generating a k-algebra actually contains the field k; W^n denotes the linear span of all words $w_1 \cdots w_n$ with each $w_j \in W$.

For any integral domain C with quotient field F, and any finitely generated C-module M, define the rank of M by $\mathrm{rk}\, M = \dim_F F \otimes_C M$.

The problem is to determine $\mathrm{GK}(A\langle D\rangle)$ in terms of other invariants of A and D. We prove

THEOREM A. *If D is a projective A-module and k is of characteristic* 0, *then*

$$\mathrm{GK}(A\langle D\rangle) = \max\{\mathrm{tr\,deg}_k(A/P) + \mathrm{rk}_{A/P}(D/PD) \mid P \text{ is a minimal prime of } A\}$$
$$= \mathrm{tr\,deg}_k\, S_A(D),$$

where $S_A(D)$ is the symmetric algebra of D.

The statement of the theorem simplifies when A is a domain or D is free: if A is a domain the assumption on the characteristic of k is unnecessary and $\mathrm{GK}(A\langle D\rangle) = \mathrm{tr\,deg}\, A + \mathrm{rk}\, D$; if D is free then $\mathrm{GK}(A\langle D\rangle) = \mathrm{tr\,deg}\, A + \mathrm{rk}\, D$. When D is free and the image of the generators of D in $\mathrm{Der}\, A$ span a finite dimensional solvable Lie algebra it follows easily from [**8**] that $\mathrm{GK}(A\langle D\rangle) = \mathrm{tr\,deg}\, A + \mathrm{rk}\, D$.

An earlier version of this paper benefitted from the referee's criticism and we would like to express our thanks.

2. Generalities concerning $A\langle D\rangle$. The natural map $A \oplus D \to A\langle D\rangle$ is an embedding and we identify A and D with their images in $A\langle D\rangle$. The ring $A\langle D\rangle$ is endowed with a natural filtration by the subspaces $R_n = (A + D)^n = A + D + D^2 + \cdots + D^n$. Using the Lie bracket on D it is easy to check that if $x \in R_n$ and $y \in R_m$ then $xy - yx \in R_{n+m-1}$. From this it follows that the associated graded algebra $\mathrm{gr}\, A\langle D\rangle = \sum_{n=0}^{\infty} R_n/R_{n-1}$ is commutative. Furthermore, $\mathrm{gr}\, A\langle D\rangle$ is generated as an algebra over A by the finitely generated A-module $R_1/R_0 \cong D$, so there is a canonical surjection $S_A(D) \to \mathrm{gr}\, A\langle D\rangle$, where $S_A(D)$ is the symmetric algebra of the A-module D.

PROPOSITION 2.1 [**9**, THEOREM 3.1]. *If D is projective, the canonical map $S_A(D) \to \mathrm{gr}\, A\langle D\rangle$ is an isomorphism.*

PROOF. Just observe that $A\langle D\rangle$ coincides with the ring $V(A, D)$ of [**9**] (or the ring V_A of [**7**]).

Recall the definition [**6**] of the ring D_X of differential operators on an affine algebraic variety X.

PROPOSITION 2.2 [**11**]. *Let X be a smooth affine algebraic variety with coordinate ring A. Then* $\mathrm{Der}\, A$ *is a projective A-module and D_X coincides with $A\langle \mathrm{Der}\, A\rangle$. Furthermore, the filtration on D_X given by the order of the differential operators coincides with the filtration on $A\langle \mathrm{Der}\, A\rangle$ given by the powers of the subspace* $A + \mathrm{Der}\, A$.

PROOF. By [**11**, Theorem 18.2], as X is smooth, D_X is the subalgebra of $\mathrm{End}_k A$ generated by A and $\mathrm{Der}_k A$, so there is a natural surjection $\phi\colon A\langle \mathrm{Der}\, A\rangle \to D_X$ and ϕ preserves the filtration. Hence, ϕ induces a map $\mathrm{gr}\,\phi\colon \mathrm{gr}\, A\langle \mathrm{Der}\, A\rangle \to \mathrm{gr}\, D_X$. But by Proposition 2.1, $\mathrm{gr}\, A\langle \mathrm{Der}\, A\rangle \cong S_A(\mathrm{Der}\, A)$ and by [**11**, Theorem 18.2], $\mathrm{gr}\, D_X \cong S_A(\mathrm{Der}\, A)$ also. Hence $\mathrm{gr}\,\phi$ is an isomorphism. The commutativity of the diagram

$$\begin{array}{ccc} A\langle \mathrm{Der}\, A\rangle & \xrightarrow{\phi} & D_X \\ \downarrow & & \downarrow \\ \mathrm{gr}\, A\langle \mathrm{Der}\, A\rangle & \xrightarrow[\mathrm{gr}\,\phi]{} & \mathrm{gr}\, D_X \end{array}$$

(where the vertical maps are the gradings) ensures that ϕ is injective and hence an isomorphism.

COROLLARY 2.3. *If X is a smooth affine algebraic variety then* $\mathrm{GK}(D_X) = 2\dim X$.

PROOF. Just apply Theorem A, together with the standard fact that $\mathrm{tr\,deg}_k A = \mathrm{rk}\,\mathrm{Der}\, A = \dim X$.

In [**2**, Chapter 2, §6] Bjork considers a noncommutative ring R equipped with a filtration such that the associated graded algebra is commutative and noetherian. An integer $d(R)$ is defined in terms of the properties of the associated graded algebra. The ring $A\langle D\rangle$ fits into this context and once Theorem A has been established it is easy to obtain the following

COROLLARY 2.4. *If D is projective, finitely generated and* $\mathrm{char}\, k = 0$, *then* $\mathrm{GK}(A\langle D\rangle) = d(A\langle D\rangle)$.

PROOF. The definition of d ensures that $d(A\langle D\rangle)$ equals the classical Krull dimension of $\mathrm{gr}\, A\langle D\rangle$. But $\mathrm{gr}\, A\langle D\rangle \cong S_A(D)$ by Proposition 2.1 and hence $d(A\langle D\rangle) = \mathrm{tr\,deg}\, S_A(D) = \mathrm{GK}(A\langle D\rangle)$ by Theorem A.

We do not know if $d(M)$ and $\mathrm{GK}(M)$ coincide for an arbitrary finitely generated $A\langle D\rangle$-module M. In the special case when $A\langle D\rangle$ is a Weyl algebra, Bjork [**2**, Chapter 3, §A.2.5] show that $d(M) = \mathrm{GK}(M)$ for all finitely generated M.

3. Two lemmas concerning polynomials. The following is useful in the context of Hilbert polynomials.

LEMMA 3.1. *Let $f \in \mathbf{Q}[x]$. Then f is a polynomial of degree d if and only if the polynomial $\tilde{f}$, defined by $\tilde{f}(x) = f(x+1) - f(x)$, is a polynomial of degree $d-1$.*

We will need another lemma concerning polynomials.

LEMMA 3.2. *Let $p, q \in \mathbf{Q}[x]$ be polynomials of degree r, t, respectively, and suppose $c \in \mathbf{N}$ is fixed. Define a function f on $\mathbf{N}$ by $f(n) = p(0)q(n) + p(1)q(n-1) + \cdots + p(n-c)q(c)$. Then f is a polynomial of degree $r + t + 1$.*

PROOF. By induction on the degree of q. If $\deg q = 0$ then q is a constant; say $q(n) = Q$. Thus, we have $f(n) = Q(p(0) + \cdots + p(n-c))$ and $f(n+1) - f(n) = Qp(n+1-c)$. This is a polynomial of degree r, so by Lemma 3.1, f is a polynomial of degree $r+1$.

Suppose now that the result is true for polynomials of degree $t-1$ and that $\deg q = t$. Then

$$f(n+1) - f(n) = \sum_{i=0}^{n-c} p(i)[q(n+1-i) - q(n-i)] + p(n+1-c)q(c).$$

The polynomial $q(n+1-i) - q(n-i)$ is of degree $t-1$, so the induction hypothesis applied to the function $g(n) = \sum_{i=0}^{n-c} p(i)[q(n+1-i) - q(n-i)]$ ensures that g is of degree $r+t$. As $p(n+1-c)q(c)$ is of degree r, $f(n+1) - f(n)$ is a polynomial of degree $r+t$. Another application of Lemma 3.1 shows $f(n)$ is of degree $r+t+1$.

4. Normal form of elements in $A\langle D\rangle$. Let C be a finite-dimensional generating subspace of D with basis $d_1, \ldots, d_r$. Choose a finite-dimensional generating subspace V of A with the property that $[C, C] \subset VC$ and $C(V) \subset V^2$. It is possible to find such a subspace: just pick any subspace U of A, generating A and satisfying $[C, C] \subset UC$; for some l, $C(U) \subset U^l$, hence $C(U^l) \subset U^{2l}$, and putting $V = U^l$ we have a subspace with the required properties.

Notice that $C(V^n) \subset \sum_{j=0}^{n-1} V^j C(V) V^{n-j-1} \subset V^{n+1}$ for all n. Another way of expressing this is that $CV^n \subset V^nC + V^{n+1}$; we shall make frequent use of this fact.

Let C_n denote the k-linear span of $\{d_1^{i_1} \cdots d_r^{i_r} \mid i_1 + \cdots + i_r = n\}$ with the convention that $C_0 = k$; notice that $C = C_1$. It is clear that $R_n = AC_0 + AC_1 + \cdots + AC_n$; if an element of $A\langle D\rangle$ is written in the form

$$\sum a_{i_1 \cdots i_r} d_1^{i_1} \cdots d_r^{i_r} \quad \left(\text{where each } a_{i_1 \cdots i_r} \in A\right),$$

we shall say that the element is expressed in *normal form*.

Put $W = V \oplus C$; W is a finite-dimensional generating subspace of $A\langle D\rangle$. It is necessary to study $\dim W^n$ and as a first step towards this we will show that $W^n = \sum_{t=0}^{n} V^{n-t} C_t$ (Theorem 4.4). This result may be thought of as a statement about the normal form of elements in W^n, or as a statement about the product of two elements in normal form.

LEMMA 4.1. $C^pV \subset \sum_{j=0}^{p} V^{j+1} C^{p-j}$.

PROOF. It is true for $p = 1$ by what has already been said. Suppose that the statement is true for p. We shall prove it true for $p+1$:

$$C^{p+1}V \subset \sum_{j=0}^{p} CV^{j+1}C^{p-j} \subset \sum_{j=0}^{p} (V^{j+1}C + V^{j+2})C^{p-j}$$

$$\subset \sum_{j=0}^{p} V^{j+1}C^{p-j+1} + V^{j+2}C^{p-j} \subset \sum_{j=0}^{p+1} V^{j+1}C^{p-j+1}.$$

LEMMA 4.2. $[C, C_m] \subset \sum_{j=0}^{m-1} V^{j+1} C^{m-j}$.

PROOF. Because $C_m \subset C^m$ we have

$$[C, C_m] \subset [C, C^m] \subset \sum_{i=0}^{m-1} C^i [C, C] C^{m-i-1}.$$

Hence $[C, C_m] \subset \sum_{i=0}^{m-1} C^i V C^{m-i}$ and the result follows at once from Lemma 4.1 applied to the term $C^i V$.

LEMMA 4.3. $C^p \subset \sum_{j=0}^{p-1} V^j C_{p-j}$.

PROOF. The result is true for $p = 1$. Suppose it is true for all integers less than or equal to p. Then

$$C^{p+1} \subset \sum_{j=0}^{p-1} C V^j C_{p-j} \subset \sum_{j=0}^{p-1} (V^j C + V^{j+1}) C_{p-j}.$$

We claim that for $m \leqslant p$, $CC_m \subset \sum_{i=0}^{m} V^i C_{m+1-i}$. This is true for $m = 0$, and we prove it by induction on m. Suppose it is true for $m - 1$. Let d_{j+1} be one of the basis elements of C, and pick $d_r^{i_1} \cdots d_r^{i_r} \in C_m$. It is enough to show that

$$x = d_{j+1}\left(d_1^{i_1} \cdots d_r^{i_r}\right) \in \sum_{i=0}^{m} V^i C_{m+1-i}.$$

Now

$$d_{j+1}\left(d_1^{i_1} \cdots d_r^{i_r}\right) = \left(\left[d_{j+1}, d_1^{i_1} \cdots d_j^{i_j}\right] - d_1^{i_1} \cdots d_j^{i_j} d_{j+1}\right) d_{j+1}^{i_{j+1}} \cdots d_r^{i_r}$$

is an element of $[C, C_t] C_{m-t} + C_{m+1}$ for some t, $m \geqslant t \geqslant 1$. By Lemma 4.2 we see that

$$x \in \sum_{i=0}^{t-1} V^{i+1} C^{t-i} C_{m-t} + C_{m+1} \subset \sum_{i=0}^{t-1} V^{i+1} C^{m-i} + C_{m+1}$$

$$\subset \sum_{i=0}^{m-1} V^{i+1} C^{m-i} + C_{m+1}.$$

If we now apply the induction hypothesis of the lemma to C^{m-i} for each i $(0 \leqslant i \leqslant m - 1)$, we have that

$$x \in \sum_{i=0}^{m-1} V^{i+1} \left(\sum_{j=0}^{m-i-1} V^j C_{m-i-j} \right) + C_{m+1}$$

$$= \sum_{i=0}^{m-1} \sum_{j=0}^{m-i-1} V^{i+j+1} C_{m-i-j} + C_{m+1}.$$

It is now clear that the claim is true for m.

Returning to the lemma, and applying the claim we have established that

$$C^{p+1} \subset \left(\sum_{j=0}^{p-1} V^j \sum_{i=0}^{p-j} V^i C_{p-j+1-i} \right) + \sum_{j=0}^{p-1} V^{j+1} C_{p-j}.$$

The truth of the lemma for $p + 1$ follows.

THEOREM 4.4. $W^n = \sum_{t=0}^{n} V^{n-t} C_t$.

PROOF. We proceed by induction, the case $n = 1$ being true from the definition of W. It is clear that $W^n \supset \sum_{t=0}^{n} V^{n-t} C_t$ as $C_t \subset W^t$, and $V^{n-t} \subset W^{n-t}$. We prove the reverse inclusion. Suppose it is true for n. Then $W^{n+1} = \sum_{t=0}^{n} V^{n+1-t}C_t + \sum_{t=0}^{n} CV^{n-t}C_t$. The first of these terms belongs to $\sum_{t=0}^{n+1} V^{n+1-t}C_t$, and it remains only to prove that the second also does. By Lemma 4.3, and the fact that $CC_t \subset C^{t+1}$ we have

$$\begin{aligned}\sum_{t=0}^{n} CV^{n-t}C_t &\subset \sum_{t=0}^{n} (V^{n-t}C + V^{n-t+1})C_t \\ &\subset \sum_{t=0}^{n} V^{n-t}\left(\sum_{j=0}^{t} V^jC_{t+1-j}\right) + \sum_{t=0}^{n+1} V^{n+1-t}C_t \\ &\subset \sum_{t=0}^{n+1} V^{n+1-t}C_t.\end{aligned}$$

5. Proof of the theorem.

LEMMA 5.1. *If* $\operatorname{gr} A\langle D\rangle \cong S_A(D)$ *then* $\operatorname{GK}(A\langle D\rangle) \geqslant \operatorname{GK}(S_A(D))$.

PROOF. Pick V, C, W as in §4, so that $W^n = \sum_{t=0}^{n} V^{n-t}C_t$. Now $S_A(D)$ is generated as a k-algebra by the subspace $V + \bar{C}$, where $\bar{C} = C + R_0/R_0$. So to evaluate $\operatorname{GK}(S_A(D))$ we must examine $\dim(V + \bar{C})^n$. But

$$\begin{aligned}\dim(V + \bar{C})^n &= \dim \sum_{t=0}^{n} V^{n-t}(\bar{C})^t = \dim \sum_{t=0}^{n} V^{n-t}(C_t + R_{t-1}/R_{t-1}) \\ &= \dim \sum_{t=0}^{n} \left(V^{n-t}C_t + R_{t-1}/R_{t-1}\right) \\ &\leqslant \dim \sum_{t=0}^{n} V^{n-t}C_t = \dim W^n\end{aligned}$$

(by Theorem 4.4), and the lemma follows from the inequality $\dim(V + \bar{C})^n \leqslant \dim W^n$.

We shall first obtain $\operatorname{GK}(A\langle D\rangle)$ under the assumption that A is a domain. The general case will be reduced to the domain case by considering A/P for the minimal primes P of A.

LEMMA 5.2. *If A is a domain and D a torsion-free A-module then*

$$\operatorname{GK}(S_A(D)) = \operatorname{tr\,deg}_k A + \operatorname{rk} D.$$

PROOF. As remarked in **[10]**, under the hypotheses of the lemma, $S_A(D)$ is a torsion-free A-module; in particular, regular elements of A remain regular as elements of $S_A(D)$. Hence, the natural map $S_A(D) \to F \otimes_A S_A(D)$ is an embedding and the latter may be considered as lying in the quotient field of $S_A(D)$. In a commutative ring the GK-dimension coincides with the transcendence degree, so $\operatorname{GK}(S_A(D)) = \operatorname{tr\,deg}_k(F \otimes_A S_A(D))$. It is standard that $F \otimes_A S_A(D) \cong S_F(F \otimes_A D)$; but $F \otimes_A D$ is just a free F-module on $r = \operatorname{rk} D$ generators, so $S_F(F \otimes_A D) \cong F[X_1, \ldots, X_r]$, the polynomial extension in r indeterminates. The transcendence degree of $F[X_1, \ldots, X_r]$ is simply $\operatorname{tr\,deg}_k F + r$; whence the result.

Combining these two lemmas gives half of Theorem A (at least for A a domain) as any projective module is certainly torsion free.

PROPOSITION 5.3. *If A is a domain and D torsion free, then*

$$\mathrm{GK}(A\langle D\rangle) \leqslant \mathrm{tr\,deg}\, A + \mathrm{rk}\, D.$$

PROOF. Pick $0 \neq x \in A$ such that, if $B = A[x^{-1}]$ then $E = B \otimes_A D$ is a free B-module of rank $r = \mathrm{rk}\, D$. As D is torsion free, so too is $S_A(D)$, and hence $A\langle D\rangle$ itself is a torsion-free A-module. In particular, x is a regular element of $A\langle D\rangle$ so the natural map $A\langle D\rangle \to B\langle E\rangle$ is injective, and it is enough to prove the proposition for $B\langle E\rangle$. So assume D is a free A-module.

Pick W, V, C as in §4, but with the extra condition that C is a vector space of dimension $r = \mathrm{rk}\, D$. As $\mathrm{gr}\, A\langle D\rangle \cong S_A(D) \cong A[X_1,\ldots,X_r]$, one has for all t that $\dim C_t = \binom{t+r}{r}$. Because $W^n = \sum_{t=0}^n V^{n-t}C_t$,

$$\dim W^n \leqslant \sum_{t=0}^{n} (\dim V^{n-t})(\dim C_t) = \sum_{t=0}^{n} q(n-t)p(t),$$

say, where $q(n-t) = \dim V^{n-t}$, $p(t) = \dim C_t$. However, $p(t)$ is a polynomial of degree r, and $q(n-t)$ is a polynomial of degree $d = \mathrm{tr\,deg}\, A$. Hence by Lemma 3.2, $\sum_{t=0}^n q(n-t)p(t)$ is a polynomial of degree $d + r$. Because $\dim W^n$ is bounded above by a polynomial of degree $\mathrm{tr\,deg}\, A = \mathrm{rk}\, D$, the result follows.

Now we have the upper bound for $\mathrm{GK}(A\langle D\rangle)$, and combining the previous three lemmas proves Theorem A for A a domain. Notice that to prove the theorem for a domain no assumption on $\mathrm{char}\, k$ was required. The necessity of the condition becomes clear in the following (where we no longer assume A is a domain).

PROPOSITION 5.4. *Let A be an algebra over a field of characteristic zero. Let D be a projective (k, A)-Lie algebra. Then*

$$\mathrm{GK}(A\langle D\rangle) = \max\left\{\mathrm{GK}\left(\frac{A}{P}\langle D/DP\rangle\right) \,\middle|\, P \text{ is a minimal prime of } A\right\}.$$

PROOF. When A is an algebra over a field of characteristic zero then the minimal primes of A are invariant under every derivation on A. There are minimal primes $P_1,\ldots,P_n$ of A with $P_1 \cdots P_n = 0$. Putting $R = A\langle D\rangle$ and using the fact that $RP_i = P_iR$ is an ideal of R, one has the product $(P_1R)(P_2R)\cdots(P_nR)$ equal to zero. Consequently, $\mathrm{GK}(A\langle D\rangle) = \max\{\mathrm{GK}(R/P_iR) \mid i = 1,\ldots,n\}$. However, given an ideal I of A, invariant under any derivation, $R/IR \cong (A/I)\langle D/DI\rangle$. This follows from the fact that the diagram

$$\begin{array}{ccc} A\langle D\rangle & \to & (A/I)\langle D/DI\rangle \\ \downarrow & & \downarrow \\ S_A(D) & \to & S_{A/I}(D/DI) \end{array}$$

(with the vertical maps being the gradings and the horizontal maps being those induced by the natural surjections $A \to A/I$, $D \to D/DI$) is commutative, and the kernel of the lower map is the ideal of $S_A(D)$ generated by I.

COROLLARY 5.5. *Let A and D be as above. Then*

$$\mathrm{GK}(A\langle D\rangle) = \max\{\mathrm{tr}\deg_k A/P + \mathrm{rk}_{A/P}(D/DP) \mid P \text{ is a minimal prime of } A\}.$$

PROOF. Just use the above propositions, and note that if D is projective as an A-module then D/DP is projective as an A/P-module. This completes the proof of Theorem A.

REFERENCES

1. J. E. Bjork, *The global homological dimension of some algebras of differential operators*, Invent. Math. **17** (1972), 67–78.

2. ______, *Rings of differential operators*, North-Holland Math. Library, North Holland, New York, 1979.

3. K. R. Goodearl, *Global dimension of differential operator rings*, Proc. Amer. Math. Soc. **45** (1974), 315–322.

4. ______, *Global dimension of differential operator rings*. II, Trans. Amer. Math. Soc. **209** (1975), 65–85.

5. ______, *Global dimension of differential operator rings*. III, J. London Math. Soc. **17** (1978), 397–409.

6. A. Grothendieck, *Eléments de géométrique algébrique*. IV, Inst. Hautes Études Sci. Publ. Math. No. 32 (1967).

7. G. Hochschild, B. Kostant and A. Rosenberg, *Differential forms on regular affine algebras*, Trans. Amer. Math. Soc. **102** (1962), 383–408.

8. M. Lorenz, *On the Gelfand-Kirillov dimension of skew polynomial rings*, preprint.

9. G. S. Rinehart, *Differential forms on general commutative algebras*, Trans. Amer. Math. Soc. **108** (1963), 195–222.

10. P. Samuel, *Anneaux gradues factoriels et modules reflexifs*, Bull. Soc. Math. France **92** (1964), 234–249.

11. M. Sweedler, *Groups of simple algebras*, Inst. Hautes Études Sci. Publ. Math. **44** (1974).

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTHERN CALIFORNIA, LOS ANGELES, CALIFORNIA 90089

Current address: Mathematics Institute, University of Warwick, Coventry CV4 7AL, England