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HOMOGENIZED $\mathfrak{sl}(2)$

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ABSTRACT. This note studies a special case of Artin's projective geometry (*Geometry of quantum planes*, MIT, preprint, 1990) for noncommutative graded algebras. It is shown that (most of) the line modules over the homogenization of the enveloping algebra $U(\mathfrak{sl}(2, \mathbb{C}))$ are in bijection with the lines lying on the quadrics that are the (closures of the) conjugacy classes in $\mathfrak{sl}(2, \mathbb{C})$. Furthermore, these line modules are the homogenization of the Verma modules for $\mathfrak{sl}(2, \mathbb{C})$.

1. LEVEL QUADRICS FOR $\mathfrak{sl}(2, \mathbb{C})$

Throughout we will write $\mathfrak{g} = \mathbb{C}e \oplus \mathbb{C}f \oplus \mathbb{C}h$ and define a vector space isomorphism $\mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{g}$ by

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \to e, \qquad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \to f, \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \to h.$$

We transfer the Lie bracket on $\mathfrak{sl}(2, \mathbb{C})$ to $\mathfrak{g}$ giving

$$[e, f] = h, \qquad [h, e] = 2e, \qquad [h, f] = -2f.$$

The cone of nilpotent elements in $\mathfrak{sl}(2, \mathbb{C})$ is the variety defined by the quadratic relation $det = 0$ where *det* is the determinant function on $\mathfrak{sl}(2, \mathbb{C})$. The conjugacy classes of semisimple elements in $\mathfrak{sl}(2, \mathbb{C})$ are the level surfaces $det + \lambda^2 = 0$, where $\lambda \in \mathbb{C}^*$; in particular, this surface is the conjugacy class of the element $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & -\lambda \end{smallmatrix}\right)$.

Transferring this to $\mathfrak{g}$ via the above isomorphism, it follows that the determinant function on $\mathfrak{g}$ is given by $det = -h^{*2} - e^* f^*$, where e^*, f^*, h^* are the dual bases to e, f, h, respectively. Hence the nilpotent cone (resp. the conjugacy class of the element λh) is given by the quadric surface $-h^{*2} - e^* f^* = 0$ (resp. $= -\lambda^2$) in $\mathfrak{g}$.

We identify $\mathfrak{g}$ with $\mathfrak{g}^*$ through the Killing form induced by the nondegenerate pairing

$$\mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C}) \to \mathbb{C}, \qquad (x, y) \to \mathrm{Tr}(x.y).$$

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Transferring this to $\mathfrak{g}$ gives the identifications $e = f^*$, $f = e^*$, and $h = 2h^*$. Under this identification the nilpotent cone (resp. the conjugacy class of λh) is given by the equation $det = -\frac{1}{4}h^2 - ef = 0$ (resp. $= -\lambda^2$).

We may homogenize the defining equations with respect to a new variable t and thus consider the following pencil of quadrics in $\mathbb{P}^3$:

$$Q(\delta) = \mathscr{V}(det + \delta^2 t^2) \quad \text{for all } \delta \in \mathbb{P}^1.$$

The base locus of this pencil is the conic $\mathscr{V}(t^2, -h^2 - 4ef)$ in the plane at infinity. The only singular quadrics in this pencil are $Q(0)$ and $Q(\infty)$ (the plane at infinity twice). If we identify $\mathfrak{g}$ with the affine open piece $t = 1$, then the intersection of $Q(0)$ with $\mathfrak{g}$ is the cone of nilpotent elements. If $\delta \neq 0, \infty$ then $Q(\delta)$ is smooth and its intersection with $\mathfrak{g}$ is a conjugacy class of semisimple elements, the conjugacy class of δh.

If $Q(\delta)$ is smooth then $Q(\delta) \cong \mathbb{P}^1 \times \mathbb{P}^1$, and there are two families of lines lying on $Q(\delta)$. On the other hand, $Q(0)$ is singular and contains only one family of lines; see [4, Chapter II, §6]. Our first objective is to characterize in terms of $\mathfrak{sl}(2, \mathbb{C})$ those lines in $\mathbb{P}^3$ that lie on some quadric $Q(\delta)$.

Any two-dimensional Lie subalgebra of $\mathfrak{g}$ is a Borel subalgebra. A *standard basis* for a Borel subalgebra is an ordered basis (E, H) such that $[H, E] = 2E$. Any standard basis for a Borel subalgebra may be extended to a standard triple (E, F, H) that is a triple of linearly independent elements such that

$$[E, F] = H, \qquad [H, E] = 2E, \qquad [H, F] = -2F.$$

The standard triples form a single orbit under conjugation by $\mathrm{GL}(2)$. If $\mathfrak{b}$ is a Borel subalgebra with standard basis (E, H) then $\lambda \in \mathbb{C}$ determines $f_\lambda \in \mathfrak{b}^*$ by requiring that $f_\lambda(E) = 0$ and $f_\lambda(H) = \lambda$. Notice that f_λ depends only on λ and not on the choice of the standard basis, since any other standard basis for $\mathfrak{b}$ is of the form $\nu E, H + \mu E$.

For each pair $(\mathfrak{b}, \lambda)$ we define a line in $\mathbb{P}^3$:

$$l_{\mathfrak{b}, \lambda} = \mathscr{V}(E, H - \lambda t),$$

where (E, H) is standard basis for $\mathfrak{b}$. Notice that this line does not depend on the choice of a standard basis for $\mathfrak{b}$. If we identify the plane at infinity with $\mathbb{P}(\mathfrak{g})$ and the affine open piece as before with $\mathfrak{g}$, then the following two facts are easily verified:

(1) $l_{\mathfrak{b}, \lambda} \cap \mathbb{P}(\mathfrak{g}) = [\mathfrak{b}, \mathfrak{b}] = \mathbb{C}E$,

(2) $l_{\mathfrak{b}, \lambda} \cap \mathfrak{g} = \frac{1}{2}\lambda H + \mathbb{C}E$, where (E, H) is a standard basis for $\mathfrak{b}$.

In particular, $l_{\mathfrak{b}, \lambda}$ is a line on the quadric $Q(\frac{1}{2}\lambda)$. In fact,

Theorem 1. *The lines that lie on the quadrics* $Q(\delta) = \mathscr{V}(det + \delta^2 t^2)$ *for some* $\delta \in \mathbb{P}^1$ *are*

1. *the lines at infinity and*
2. *lines* $l_{\mathfrak{b}, \lambda}$ *for a Borel subalgebra* $\mathfrak{b}$ *and* $\lambda \in \mathbb{C}$.

Proof. The first case corresponds to lines on $Q(\infty)$. Suppose $\delta \neq 0$. Then $Q(\delta) \cap \mathfrak{g}$ is the conjugacy class of $\left(\begin{smallmatrix} \delta & 0 \\ 0 & -\delta \end{smallmatrix}\right)$. If l is a line lying on $Q(\delta)$, then we can choose a basis for $\mathfrak{sl}(2, \mathbb{C})$ such that $l = \{x + \nu y | \nu \in \mathbb{C}\}$, where $x = \left(\begin{smallmatrix} \delta & 0 \\ 0 & -\delta \end{smallmatrix}\right)$ and y is some other element of $\mathfrak{sl}(2, \mathbb{C})$. Write $y = \left(\begin{smallmatrix} 1 & b \\ c & d \end{smallmatrix}\right)$. Since $\det(x + \nu y) = \det(x)$ for all ν, a calculation shows that $\det(y) = 0$ and $a = 0$.

Thus y is (up to scalar multiples) either $\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$ of $\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)$. In the first case $l = l_{\mathfrak{b},2\delta}$ with $\mathfrak{b}$ having standard basis y, x. In the second case $l = l_{\mathfrak{b},-2\delta}$ with $\mathfrak{b}$ having standard basis $y, -x$.

Suppose that $\delta = 0$, so that $Q(\delta)$ is the nilpotent cone. We can choose a basis for $\mathfrak{sl}(2, \mathbb{C})$ such that $l = \{x + \nu y | \nu \in \mathbb{C}\}$, where $x = \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$ and $y = \left(\begin{smallmatrix}1&b\\c&d\end{smallmatrix}\right)$. Since $\det(x+\nu y) = 0$ for all ν, a calculation shows that $\det(y) = 0$ and $c = 0$ (whence $a = 0$ also). Thus y is a scalar multiple of x, so the line is $l = \mathbb{C}x = l_{\mathfrak{b},\lambda}$ having standard basis x, H, where H satisfies $[H, x] = 2x$ (such an H does exist since every nonzero nilpotent element belongs to some standard triple).

Conversely, we have already seen that $l_{\mathfrak{b},\lambda}$ lies on the quadric $Q(\frac{1}{2}\lambda)$. $\square$

If $\delta \neq 0, \infty$ then the two rulings on $Q(\delta)$ are given by $\{l_{\mathfrak{b},2\delta} | \mathfrak{b}$ is a Borel$\}$ and $\{l_{\mathfrak{b},-2\delta} | \mathfrak{b}$ is a Borel$\}$.

2. The quantum space of $\mathfrak{sl}(2, \mathbb{C})$

Let A denote the homogenization of the enveloping algebra of $\mathfrak{g}$ with respect to a central variable t. That is, $A = \mathbb{C}[e, f, h, t]$ with defining equations

$$ef - fe = ht, \qquad he - eh = 2et, \qquad hf - fh = -2ft;$$
$$et - te = ft - tf = ht - th = 0.$$

Notice that $A/A(t-1) \simeq U(\mathfrak{g})$, the enveloping algebra of $\mathfrak{g}$, and $A/At \simeq \mathbb{C}[e, f, h] = S(\mathfrak{g})$, the symmetric algebra on $\mathfrak{g}$. From these facts one deduces that A has Hilbert series $(1-t)^{-4}$, is a positively graded Noetherian domain, a maximal order, Auslander-regular of dimension 4, and satisfies the Cohen-Macaulay property, see, e.g., [5–7]. Moreover, the center of A is $\mathbb{C}[\Omega, t]$, where $\Omega = h^2 + 2ef + 2fe$ is the Casimir element.

Artin [1] associates to any regular algebra R its quantum space $\operatorname{Proj}(R)$, which is by definition the quotient-category of all finitely generated graded left R-modules by the full Serre subcategory of the finite length modules. We will denote the quantum space associated to A by $Q(\mathfrak{g})$. We want to characterize the linear subspaces in $Q(\mathfrak{g})$. There are three types to consider:

(1) A *plane module* $\mathscr{S}$ is a cyclic module with Hilbert series $(1-t)^{-3}$.
(2) A *line module* $\mathscr{L}$ is a cyclic module with Hilbert series $(1-t)^{-2}$.
(3) A *point module* $\mathscr{P}$ is a cyclic module with Hilbert series $(1-t)^{-1}$.

As in [6] these modules (together with A and the trivial module $A/Ae + Af + Ah + At$) are precisely the Cohen-Macaulay modules of multiplicity one. We will associate to each of them a linear subspace in ordinary $\mathbb{P}^3$.

As $\dim(A_1) = 4$ and the homogeneous degree 1 component of a plane (resp. line, point) module is 3 (resp. 2, 1) one can find $a \in A_1$ (resp. $a, b \in A_1$, resp. $a, b, c \in A_1$) and surjections

$$A/Aa \to \mathscr{S} \quad (\text{resp. } A/(Aa + Ab) \to \mathscr{L}, \quad \text{resp. } A/(Aa + Ab + Ac) \to \mathscr{P})$$

(in fact we will see shortly that these maps have to be isomorphisms). Hence to each plane (resp. line, point) module we can associate a plane (resp. line, point) in $\mathbb{P}^3 = \mathbb{P}(A_1^*)$, namely, $\mathscr{V}(a)$ (resp. $\mathscr{V}(a, b)$, resp. $\mathscr{V}(a, b, c)$). We will now determine which linear subspaces of $\mathbb{P}^3$ arise in this way.

We observe that there is a dichotomy in the problem. As a linear subspace module is critical (because it is of multiplicity 1) it follows from [6] that t either kills the module or acts faithfully. The first case gives a linear subspace module over the commutative polynomial ring $A/At \simeq S(\mathfrak{g})$, i.e., we get a linear subspace in $\mathbb{P}^2 = \mathscr{V}(t) = \mathbb{P}(\mathfrak{g})$. In the second case we can form $\overline{M} = M/(t-1)M$, which is a filtered $U(\mathfrak{g})$-module $\overline{M}_0 \subset \overline{M}_1 \subset \cdots$, where $\dim(\overline{M}_i)$ is equal to a (if M is a point module), $i+1$ (if M is a line module), or $\frac{1}{2}(i+1)(i+2)$ (if M is a plane module). Moreover, this process can be reversed, namely, $M \simeq \bigoplus \overline{M}_i t^i$ (see [3] or [7]).

Proposition 1. 1. *Every plane $\mathscr{V}(a)$ in $\mathbb{P}^3$ determines a plane module S.*

2. *The points at infinity and the origin $(0, 0, 0, 1)$ are the only points in $\mathbb{P}^3$ that determine a point module $\mathscr{P}$.*

Proof. Since A is a domain, A/Aa has Hilbert series $(1-t)^{-3}$ for every nonzero $a \in A_1$. Hence the surjection $A/Aa \to \mathscr{S}$ is an isomorphism.

From the dichotomy remark it follows that a point-module either corresponds to a point on the plane at infinity (when it is killed by t) or corresponds to the 1-dimensional $\mathfrak{g}$-module, whence it corresponds to the origin in $\mathfrak{g}$ (which is bing identified with the complement to the plane at infinity). In the first case we can take $a = t$ and let b, c determine the point in $\mathrm{Proj}(A/At) = \mathbb{P}^2$. Hence the Hilbert series of $A/At + Ab + Ac$ is $(1-t)^{-1}$ so $\mathscr{P} = A/At + Ab + Ac$. In the second case $A/Ae + Af + Ah \simeq \mathbb{C}[t]$, which also has the right Hilbert series. $\square$

Still the situation concerning point-modules is slightly more subtle. The set of points obtained from point modules is called the *point variety*, and it may be described by multilinearizing the defining relations as in [2] or [8]; that is, the point-variety is the zero set of the 4×4 minors of either of the following two matrices:

$$\begin{pmatrix} -f & e & 0 & -h \\ h & 0 & -e & -2e \\ 0 & h & -f & 2f \\ -t & 0 & 0 & e \\ 0 & -t & 0 & f \\ 0 & 0 & -t & h \end{pmatrix}$$

or

$$\begin{pmatrix} f & -h-2t & 0 & t & 0 & 0 \\ -e & 0 & -h+2t & 0 & t & 0 \\ -t & e & f & 0 & 0 & t \\ 0 & 0 & 0 & -e & -f & -h \end{pmatrix}.$$

Proposition 2. *The ideal determining the point-modules of A in $\mathbb{P}^3$ is*

$$t\langle (h^2+4ef)(e, f, h), te, tf, th\rangle.$$

Hence the conic at infinity $\mathscr{V}(t, h^2+4ef)$ is an embedded component of the point-variety.

So the base locus of the pencil of quadrics described in §1 appears here as the embedded component of the point-variety for A. Let us now describe the line-modules of $Q(\mathfrak{g})$.

Theorem 2. *The lines in* $\mathbb{P}^3$ *determining line modules are precisely the lines in the pencil of quadrics* $Q(\delta) = \mathscr{V}(det + \delta^2 t^2)$ *for* $\delta \in \mathbb{P}^1$.

Proof. As in the case of point-modules, the lines in the plane at infinity are already accounted for. Hence we have to prove that any line module of A is of the form

$$\mathscr{L} \simeq A/(AE + A(H - \lambda t)),$$

where E, H is a standard basis for a Borel subalgebra $\mathfrak{b}$ and $\lambda \in \mathbb{C}$.

There is a surjection $A/Aa+Ab \to \mathscr{L}$ for some $a, b \in A_1$ since $\dim L_1 = 2$. Clearly, we can change a, b if necessary so that $a = y$, $b = z + \lambda t$ with $y, z \in \mathfrak{g}$. Let x, y, z be a basis of $\mathfrak{g}$. Then the following seven linearly independent elements in A_2 belong to $Aa + Ab$:

$$xy, yy, ty = yt, x(z + \lambda t), y(z + \lambda t), z(z + \lambda t), t(z + \lambda t) = (z + \lambda t)t.$$

As $\dim \mathscr{L}_2 = 3$ and $\dim(A_2) = 10$ these elements must be a basis for $(Aa + Ab)_2$. Note that both $yz = y(z + \lambda t) - \lambda yt$ and zy belong to this space. Hence $yz - zy$ can be written as a linear combination of the seven elements. If $\lambda \neq 0$ then only ty can occur with nonzero coefficient, and if $\lambda = 0$ so might tz. At any rate $\mathfrak{b} = \mathbb{C}y \oplus \mathbb{C}z$ is a two-dimensional Lie subalgebra of $\mathfrak{g}$ and hence is a Borel subalgebra.

But then $A/(Aa + Ab)$ is the homogenization of $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_{f|_\mathfrak{b}}$ for some $f \in \mathfrak{g}^*$ such that $f|_{[\mathfrak{b},\mathfrak{b}]} = 0$ and hence has as its Hilbert series $(1 - t)^{-2}$. Therefore, $A/Aa + Ab = \mathscr{L}$ and $\mathscr{V}(a, b) = l_{\mathfrak{b},\lambda}$ as claimed. □

3. Some comments

The line module associated to $l_{\mathfrak{b},\lambda}$ will be denoted by $M(\mathfrak{b}, \lambda)$. By Theorem 2 we have $M(\mathfrak{b}, \lambda) \cong A/AE + A(H - \lambda t)$, and hence $M(\mathfrak{b}, \lambda)$ is the homogenization of the Verma module $M_\mathfrak{b}(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$ of highest weight λ. The $(n + 1)$-dimensional simple $U(\mathfrak{sl}(2, \mathbb{C}))$-modules will be denoted $V(n)$. For each Borel $\mathfrak{b}$ there is a short exact sequence $0 \to M_\mathfrak{b}(-n - 2) \to M_\mathfrak{b}(n) \to V(n) \to 0$. Taking homogenized modules, there is a corresponding short exact sequence $0 \to M(\mathfrak{b}, -n - 2) \to M(\mathfrak{b}, n) \to F(n) \to 0$ where $F(n)$ is a certain fat point of multiplicity $n + 1$; a *fat point module* [1] is a 1-critical module generated in degree 0, having constant Hilbert series, and a *fat point* is a module that is an A-module that is equivalent to a fat point module in the sense that they give isomorphic objects of $\mathrm{Proj}(A)$.

This is reminiscent of some of the results on the Sklyanin algebra in [9]. Homogenized $\mathfrak{sl}(2, \mathbb{C})$ shares some other common features with the Sklyanin algebra; for example annihilators of line modules behave in a similar way. We leave the details to the interested reader.

The quantum space of any three-dimensional Lie algebra has similar properties. Let us briefly sketch the case of the three-dimensional Lie algebra $\mathfrak{h} = \mathbb{C}x \oplus \mathbb{C}y \oplus \mathbb{C}z$ with z central and $[x, y] = z$. The corresponding algebra $H(\mathfrak{h})$ is $\mathbb{C}[x, y, z, t]$ with relations $xy - yx = zt$ and z and t central; the point variety in $Q(\mathfrak{h}) = \mathrm{Proj}(H(\mathfrak{h}))$ is determined by the ideal $tz(z, t)$ and so consists of the union of the plane $\mathscr{V}(z)$ and the plane at infinity $\mathscr{V}(t)$; their intersection is an embedded component.

The line modules of $H(\mathfrak{h})$ are precisely the lines in the pencil of planes $P(\delta) = \mathscr{V}(z + \delta t)$ for $\delta \in \mathbb{P}^1$, which has as its base locus the embedded component (z, t).

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