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Irreducible Representations of the 4-Dimensional Sklyanin Algebra at Points of Infinite Order

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In 1982 Sklyanin (*Funct. Anal. Appl.* **16** (1982), 27–34) defined a family of graded algebras $A(E, \tau)$, depending on an elliptic curve E and a point $\tau \in E$ which is not 4-torsion. Basic properties of these algebras were established in Smith and Stafford (*Compositio Math.* **83** (1992), 259–289) and a study of their representation theory was begun in Levasseur and Smith (*Bull. Soc. Math. France* **121** (1993), 35–90). The present paper classifies the finite dimensional simple A-modules when τ is a point of infinite order. Sklyanin (*Funct. Anal. Appl.* **17** (1983), 273–284) defines for each $k \in \mathbb{N}$ a representation of A in a certain k-dimensional subspace of theta functions of order $2(k-1)$. We prove that these are irreducible representations, and that any other simple module is obtained by twisting one of these by an automorphism of A. The automorphism group of A is explicitly computed. The method of proof relies on results in Levasseur and Smith. In particular, it is proved that every finite dimensional simple module is a quotient of a line module. An important part of the analysis is a determination of the 1-critical A-modules, and the fact that such a module is (equivalent to) a quotient of a line module by a shifted line module.

INTRODUCTION

Fix $\eta \in \mathbb{C}$ with $\mathrm{Im}(\eta) > 0$, and write $\Lambda = \mathbb{Z} \oplus \mathbb{Z}\eta$. Let θ_{00}, θ_{01}, θ_{10}, θ_{11} be Jacobi's four theta functions associated to Λ, as defined in Weber's book [17, p. 71]. Fix $\tau \in \mathbb{C}$ such that τ is not of order 4 in $E = \mathbb{C}/\Lambda$. Whenever $\{ab, ij, kl\} = \{00, 01, 10\}$, define

$$\alpha_{ab} = (-1)^{a+b} \left[\frac{\theta_{11}(\tau)\,\theta_{ab}(\tau)}{\theta_{ij}(\tau)\,\theta_{kl}(\tau)}\right]^2,$$

and set $\alpha_1 = \alpha_{00}$, $\alpha_2 = \alpha_{01}$, $\alpha_3 = \alpha_{10}$. We remark that $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_1\alpha_2\alpha_3 = 0$. The 4-dimensional Sklyanin algebra is the graded algebra $A = \mathbb{C}[x_0, x_1, x_2, x_3]$ defined by the six relations

$$x_0 x_i - x_i x_0 = \alpha_i (x_j x_k + x_k x_j) \qquad x_0 x_i + x_i x_0 = x_j x_k - x_k x_j$$

where (i, j, k) is a cyclic permutation of $(1, 2, 3)$.

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This two parameter family of algebras was defined and first studied by E. K. Sklyanin in 1982 [13]. In that paper Sklyanin constructs a 2-dimensional and also a 3-dimensional simple A-module and poses the problem of finding all the simple A-modules. The only obvious finite dimensional simple A-module is the *trivial* module A/A^+, where $A^+ = \bigoplus_{n>0} A_n$ is the augmentation ideal of A. In a subsequent paper [14] Sklyanin defined for each $k \in \mathbb{N} \cup \{0\}$, an A-module $V_k = \Theta_{00}^{2k+}$ of dimension $k+1$ consisting of certain theta functions (see Section 3 for the precise definition of this module), and asked if these were all the finite dimensional irreducible representations. The present paper shows that when τ is of infinite order, then each V_k is irreducible, and all the finite dimensional irreducible representations can be constructed from the V_k as follows.

The automorphism group of A acts on the space of A-modules: if V is an A-module, and $\varphi \in \operatorname{Aut}(A)$ we write V^φ for V twisted by φ (see Section 2 for the definition). Our main theorem is that every non-trivial finite dimensional simple A-module is of the form V_k^φ. Furthermore, a precise description of $\operatorname{Aut}(A)$ is given: if $|\tau| = \infty$, then there is an exact sequence $1 \to \mathbb{C}^\times \to \operatorname{Aut}(A) \to E_4 \to 1$ where E_n denotes the points in E with n-torsion. If $\lambda \in \mathbb{C}^\times$, we write V^λ for the corresponding twisted module; if $\xi \in E_4$ we define (in Section 2) a representative $\Phi(\xi) \in \operatorname{Aut}(A)$, and write $V(2\xi + k\tau) = V_k^{\Phi(\xi)}$ for the corresponding twisted module. Our main result may be stated as follows (other notation will be defined later in the introduction).

MAIN THEOREM. *Suppose that* $|\tau| = \infty$. *For each* $\omega \in E_2$ *and for each* $k \in \mathbb{N} \cup \{0\}$ *there is a* $(k+1)$*-dimensional simple module* $V(\omega + k\tau)$. *The set of all the non-trivial finite dimensional simple A-modules is precisely the set of all the twisted modules* $V(\omega + k\tau)^\lambda$ *where* $\lambda \in \mathbb{C}^\times$. *There are no isomorphisms between these modules for distinct triples* (ω, k, λ). *Furthermore, each* $V(\omega + k\tau)^\lambda$ *is a quotient of the line module* $M(p, q)$ *for all* $p, q \in E$ *such that* $p + q = \omega + k\tau$.

The paper is organized as follows. Section 1 introduces notation, and gives a brief account of the main results from [7, 16] which are required in this paper. In particular, an embedding of E in $\mathbb{P}^3$ is described, and basic results on point modules and line modules are recalled. We also make use of the main result in [5], and this is recalled. Section 2 describes all the graded algebra automorphisms of A. There is (with one exception) an exact sequence $1 \to \mathbb{C}^\times \to \operatorname{Aut}(A) \to E_4 \to 1$ where $\mathbb{C}^\times$ is the "trivial" subgroup of automorphisms, namely $\lambda \in \mathbb{C}^\times$ acts on A_n as scalar multiplication by λ^n. There is a closely related linear action of E_4 on $\mathbb{P}^3$ which restricts to automorphisms of E in such a way that $\xi \in E_4$ acts as translation by ξ. The action of $\operatorname{Aut}(A)$ on point modules and line modules is also described.

Section 3 proves that each V_k is a quotient of a line module (in fact V_k is a quotient of the line module $M(p, q)$ whenever $p+q=k\tau$), and that V_k is simple. Section 3 also proves that the stabiliser for the $\mathrm{Aut}(A)$ action on V_k is a normal subgroup which is isomorphic to E_2. Section 4 proves that every finite dimensional simple A-module is a quotient of a line module, and describes all the line modules which can possibly have such a quotient. When τ is of infinite order, the only line modules which can have a non-trivial finite dimensional simple quotient are the line modules $M(p, q)$ where $p, q \in E$ satisfy $p+q=\omega+k\tau$ with $\omega \in E_2$ and $k \in \mathbb{N} \cup \{0\}$. Furthermore, if $p+q=\omega+k\tau$ with $\omega \in E_2$ then $M(p, q)$ has a 1-parameter family of $(k+1)$-dimensional non-trivial simple quotients, namely the twists $V(\omega+k\tau)^{\lambda}$ for $\lambda \in \mathbb{C}^{\times}$. The results in Section 4 give information about the "fat points" for A. Section 5 proves that every non-trivial finite dimensional simple A-module is of the form V_k^{φ}. It follows that the non-trivial finite dimensional simples are in bijection with the points of $\mathbb{N}\tau$ up to the action of $\mathrm{Aut}(A)$. This last formulation illustrates a certain similarity to the representation theory of $\mathfrak{gl}(2, \mathbb{C})$, which is satisfying because Sklyanin's original paper emphasises that A should be viewed as a deformation/quantization of the enveloping algebra $U(\mathfrak{gl}(2, \mathbb{C}))$. Section 5 also classifies the primitive ideals in A, showing that the primitive spectrum is analogous to that of $U(\mathfrak{gl}(2, \mathbb{C}))$.

The 4-dimensional Sklyanin algebra and higher dimensional analogues are also studied by Odesskii and Feigin in [11, 12].

1. Preliminaries

Let V be a 4-dimensional vector space with basis x_0, x_1, x_2, x_3. Define $A=T(V)/I$ to be the quotient of the tensor algebra $T(V)$ with defining relations as in the Introduction. Thus I is generated by its six dimensional subspace $I_2 \subset V \otimes V$. The algebraic properties of A are intimately related to certain subvarieties of $\mathbb{P}(V^*)$ and $\mathbb{P}(V^*) \times \mathbb{P}(V^*)$. To define these subvarieties, we first embed $\mathbb{C}/\Lambda$ in $\mathbb{P}(V^*)$.

Define holomorphic functions g_{ab} on $\mathbb{C}$ for each $ab \in \{00, 01, 10, 11\}$ as follows:

$$g_{ab}(z)=\gamma_{ab}\theta_{ab}(\tau)\,\theta_{ab}(2z) \qquad \text{where} \quad \gamma_{ab}=\begin{cases}\sqrt{-1} & ab=00, 11\\ 1 & ab=01, 10.\end{cases}$$

Define $E=j_\tau(\mathbb{C}/\Lambda)$ where $j_\tau\colon \mathbb{C}/\Lambda \to \mathbb{P}(V^*)=\mathbb{P}^3$ is given by

$$j_\tau(z)=(g_{11}(z), g_{00}(z), g_{01}(z), g_{10}(z))$$

with respect to the homogeneous coordinates x_0, x_1, x_2, x_3. Sometimes it will be convenient to label the basis for V as $X_{11}=x_0$, $X_{00}=x_1$, $X_{01}=x_2$, $X_{10}=x_3$. Now, we define

$$e_0=(1,0,0,0), \qquad e_1=(0,1,0,0), \qquad e_2=(0,0,1,0), \qquad e_3=(0,0,0,1)$$

$$\mathscr{S}=\{e_i \mid 0 \leqslant i \leqslant 3\}$$

$$\Delta_{\mathscr{S}}=\{(e_i, e_i) \mid 0 \leqslant i \leqslant 3\}$$

$$\Delta_\tau=\{(p, p+\tau) \mid p \in E\}$$

$$\Gamma=\Delta_{\mathscr{S}} \cup \Delta_\tau.$$

Thus Γ is the graph of the automorphism σ of $E \cup \mathscr{S}$ given by $\sigma(p)=p+\tau$ for $p \in E$, and $\sigma(e_i)=e_i$ for $i=0, 1, 2, 3$. By [16, Sects. 2, 3] the subvariety of $\mathbb{P}(V^*) \times \mathbb{P}(V^*)$ defined by I_2 is $\mathscr{V}(I_2)=\Gamma$, and by [7, 1.2], I_2 is precisely the subspace of $V \otimes V$ consisting of those forms which vanish on Γ.

It is convenient to label the points of E_2 as $\omega_0=0$, $\omega_1=\frac{1}{2}+\frac{1}{2}\eta$, $\omega_2=\frac{1}{2}\eta$, $\omega_3=\frac{1}{2}$. It is proved in [7, Sect. 3] that if $p, q \in E$, then the secant line l_{pq} passes through e_i if and only if $p+q=\omega_i$.

Most A-modules we consider will be finitely generated $\mathbb{Z}$-graded A-modules. If $M=\bigoplus_m M_m$ is such an A-module and $p \in \mathbb{Z}$, then the *shifted module* $M[p]$ is defined by setting $M[p]_m := M_{p+m}$. When not otherwise specified a map $\psi: M \to N$ between graded modules will be a graded map of degree zero; that is $\psi(M_m) \subset N_m$ for all m.

If M is an A-module, we write $E^j(M)=\operatorname{Ext}^j_A(M, A)$. We define the *j-number* of M to be the least j such that $E^j(M) \neq 0$; it is denoted by $j(M)$. We say that M is a *Cohen-Macaulay module* if $E^i(M)=0$ whenever $i \neq j(M)$.

If M is a finitely generated graded A-module then the Hilbert series of M is the formal series $H_M(t)=\sum_n (\dim M_n)\, t^n$. This series is always of the form $q_M(t)(1-t)^{-d}$ where $q_M(t) \in \mathbb{Z}[t, t^{-1}]$, $q_M(1) \neq 0$ and $d \in \{0, 1, 2, 3, 4\}$. Writing $H_M(t)$ in this form allows us to define the *Gelfand-Kirillov* dimension of M to be $d(M)=d$, and to define the *multiplicity* of M to be $e(M)=q_M(1)$. If $d(M)=d$, and $d(M/N)<d$ for all non-zero submodules N, then M is said to be *d-critical.*

We shall adopt the notation used in [7, 16]. Although we assume that the reader is already familiar with those papers, we will recall those results from [7, 16] which are relevant to the present paper.

It was proved in [16] that A is a noetherian domain, and has the same Hilbert series as the polynomial ring in 4 variables, namely $(1-t)^{-4}$. Furthermore, A is a Koszul algebra of global homological dimension 4, and is regular in the sense of Artin and Schelter [2].

The algebra A has other excellent homological properties; indeed in this respect it is as well-behaved as the polynomial ring. It is proved in [6] that A is *Auslander-regular*, which means that if M is a finitely generated A-module, and $i \geqslant 0$, then $j(N) \geqslant i$ for every submodule N of $E^i(M)$. It is also proved in [6] that A satisfies the *Cohen-Macaulay property* which means that $d(M)+j(M)=4$ for all finitely generated A-modules M. One useful consequence (see [7, 2.1e]) of these good homological properties is that the socle of a graded module M, which is the trace of A/A^+ in M, is zero if and only if $E^4(M)=0$, or equivalently, if and only if the projective dimension of M is strictly less than 4.

In [7] a study of graded A-modules was begun. Attention was focused on the point, line and plane modules, these being the cyclic modules with Hilbert series $(1-t)^{-n}$ where $n=1, 2, 3$ respectively. It was already proved in [16] that the point modules are in bijection with the points of $E \cup \mathcal{S}$. If $p \in E \cup \mathcal{S}$, we write $M(p)$ for the corresponding point module. One of the main results in [7] is that the line modules are in bijection with the set of lines in $\mathbb{P}^3$ which are secant lines of E. If $p, q \in E$, the secant line through p and q is denoted by l_{pq}, and $M(p, q)$ denotes the corresponding line module. There is a short exact sequence $0 \to M(p+\tau, q-\tau)[-1] \to M(p, q) \to M(p) \to 0$. If $p+q \notin E_2$, then $M(p)$ and $M(q)$ are the only point modules which are quotients of $M(p, q)$. However, if $p+q=\omega_i$ then there is a short exact sequence $0 \to M(p-\tau, q-\tau)[-1] \to M(p, q) \to M(e_i) \to 0$. Point, line and plane modules can also be characterized by their homological properties. They are precisely the Cohen-Macaulay modules of multiplicity 1, and projective dimension 3, 2, 1 respectively.

In [13] Sklyanin found two central elements in A_2, namely

$$\Omega_1 = -x_0^2+x_1^2+x_2^2+x_3^2 \qquad \text{and} \qquad \Omega_2 = x_1^2+\left(\frac{1+\alpha_1}{1-\alpha_2}\right)x_2^2+\left(\frac{1-\alpha_1}{1+\alpha_3}\right)x_3^2.$$

If $|\tau|=\infty$ then the center of A is the polynomial ring $\mathbb{C}[\Omega_1, \Omega_2]$ by [7, 6.12]. We write Z_2 for the two dimensional space spanned by Ω_1 and Ω_2. There is a surjective map $\Omega: E \to \mathbb{P}(Z_2)$ with fibers $\{z, -z-2\tau\}$, having the property that $\Omega(p+q)$ annihilates $M(p, q)$. Furthermore, if $|\tau|=\infty$ then $\operatorname{Ann} M(p, q) = \langle \Omega(p+q) \rangle$.

Define $B := A/\langle Z_2 \rangle = A/A\Omega_1 + A\Omega_2$. We may write $B = T(V)/J$ where J is an ideal generated by its degree 2 component, namely J_2. It is proved in [16] that $\mathcal{V}(J_2) = \Delta_\tau$ and in [7] that J_2 is precisely the set of bilinear functions vanishing on Δ_τ. In [16] it is proved that if $p \in E$, then the point module $M(p)$ is annihilated by Z_2, so is a B-module. It is easy to see that the point modules $M(e_i)$ for $e_i \in \mathcal{S}$ are not B-modules.

The algebra B has a very explicit description in terms of E and τ. Let $\mathcal{L} = j_\tau^*(\mathcal{O}_{\mathbb{P}^3}(1))$ be the invertible $\mathcal{O}_E$-module of degree 4 on E determined by the embedding j_τ. In [3] it is explained how to construct a graded algebra $B(E, \sigma, \mathscr{L}) = \mathbb{C} \oplus B_1 \oplus B_2 \oplus \cdots$, which for $\sigma = \mathrm{Id}_E$ is the homogeneous coordinate ring of the projective embedding $E \subsetneq \mathbb{P}(H^0(E, \mathscr{L})^*)$. By [16, Sect. 3] $B \cong B(E, \sigma, \mathscr{L})$. It follows from this that B is a domain.

Algebras such as $B(E, \sigma, \mathscr{L})$ are studied in [5]. A corollary of their main result is that the 1-critical B-modules are precisely the point modules for B; of course these are just the $M(p)$ with $p \in E$.

2. Automorphisms of the Sklyanin Algebra

By an automorphism of A we always mean a $\mathbb{C}$-linear algebra automorphism which preserves the grading on A. Thus $\mathrm{Aut}(A)$ identifies with the subgroup of $GL(A_1) = GL(V)$ consisting of those φ such that $(\varphi \otimes \varphi)(I_2) \subset I_2$. There is an obvious normal subgroup of $\mathrm{Aut}(A)$, namely the subgroup of scalar matrices, which we denote by $\mathbb{C}^\times$. If $\lambda \in \mathbb{C}^\times$, then the corresponding automorphism will be denoted by φ_λ; that is $\varphi_\lambda(x) = \lambda^n x$ for all $x \in A_n$. Since I_2 may be characterised as those forms in $V \otimes V$ which vanish on $\Gamma = \Delta_\tau \cup \Delta_{\mathscr{S}}$, $\mathrm{Aut}(A)/\mathbb{C}^\times$ may be characterized as the subgroup of $PGL(V)$ such that the induced action on $\mathbb{P}(V^*) \times \mathbb{P}(V^*)$ leaves Γ stable. Warning: we shall adopt the convention that $\varphi \in GL(V)$ acts on V from the right, and on V^* from the left, so that $\langle \varphi(x), p \rangle = \langle x, \varphi(p) \rangle$, for $x \in V$ and $p \in V^*$. This convention "defines" an isomorphism $PGL(V) \to PGL(V^*) = \mathrm{Aut}\, \mathbb{P}(V^*)$ which we will use to identify these groups.

There is a natural action of $\mathrm{Aut}(A)$ on the category of A-modules. Let M be a left A-module, and let $\varphi \in \mathrm{Aut}(A)$. Define M^φ to be the A-module which is M as a $\mathbb{C}$-vector space, and with A-action given by $x * m = \varphi(x)m$ for all $x \in A$ and for all $m \in M$. We say that M^φ is obtained by *twisting* M by φ, and we refer to M^φ as a *twist* of M. It is clear that twisting by φ is an exact functor on the category of A-modules. If $\lambda \in \mathbb{C}^\times$, we shall write M^λ for the twist of M by φ_λ. If M is a graded module, then $M^\lambda \cong M$ for all $\lambda \in \mathbb{C}^\times$, so it is the action of $\mathrm{Aut}(A)/\mathbb{C}^\times$ on the graded A-modules which is important.

Our first goal is to show that (with one exception) there is an exact sequence

$$1 \to \mathbb{C}^\times \to \mathrm{Aut}(A) \to E_4 \to 1.$$

The group of group automorphisms of E is denoted by $\mathrm{Aut}_{\mathrm{group}}(E)$, whereas the automorphisms of the variety E is denoted $\mathrm{Aut}_{\mathrm{var}}(E)$.

PROPOSITION 2.1. *Suppose that φ belongs to the subgroup of $PGL(V^*)$ such that $(\varphi \times \varphi)(\Gamma) \subset \Gamma$. Then $\varphi \in \mathrm{Aut}_{\mathrm{var}}(E)$, and $\varphi(E_4) \subset E_4$. Furthermore, the restriction of φ to E is translation by a point of E_4 unless $E \cong \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\rho$ and $\tau \in \{\pm\frac{1}{3}(1+2\rho)+\Lambda\}$ where $\rho = e^{2\pi i/3}$. In this exceptional case it is also possible for φ to be of the form $\varphi(p) = \rho \cdot p + \xi$ or $\varphi(p) = \rho^2 \cdot p + \xi$ where $\xi \in E_4$.*

Proof. It follows from the definition of Γ that $\varphi(E) \subset E$ and $\varphi(\mathscr{S}) \subset \mathscr{S}$. Fix $p \in E_4$. By [7, 3.6] the tangent line to E at p passes through some $e_i \in \mathscr{S}$. Hence the tangent line to E at $\varphi(p)$ passes through $\varphi(e_i) \in \mathscr{S}$. By [7, 3.6] it follows that $\varphi(p) \in E_4$.

By [9, Corollary 1, p. 43] there exists $\xi \in E$ and a group automorphism $h: E \to E$ such that $\varphi(p) = h(p) + \xi$ for all $p \in E$. Since $\varphi(E_4) \subset E_4$, it follows that $h(E_4) + \xi \subset E_4$. But $h(E_4) = E_4$, so $\xi \in E_4$. If $p \in E$ then $\varphi(p+\tau) = \varphi(p) + \tau$ since $(\varphi \times \varphi)(p, p+\tau) \in \Gamma$. It follows that $h(\tau) = \tau$.

For most elliptic curves $\mathrm{Aut}_{\mathrm{group}}(E) = \{\pm 1\}$. If $h = 1$ then $\varphi(p) = p + \xi$ as required. On the other hand, if $h = -1$ then $\varphi(2\tau) = 0$ whence $2\tau = 0$; however, this possibility is excluded by our underlying hypothesis that $\tau \notin E_4$. Thus $h \neq -1$, so the result is true if $\mathrm{Aut}_{\mathrm{group}}(E) = \{\pm 1\}$. Since E is a complex curve, if $\mathrm{Aut}_{\mathrm{group}}(E) \neq \{\pm 1\}$ then either $E \cong \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}i$ or $E \cong \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\rho$ where $\rho = e^{2\pi i/3}$.

Suppose that $E \cong \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}i$. Then $\mathrm{Aut}_{\mathrm{group}}(E) \cong \mathbb{Z}/4\mathbb{Z}$, and is generated by multiplication by i. If h is not the identity, then a simple calculation shows that $h(\tau) = \tau$ implies that $2\tau = 0$. Since this possibility is excluded, the Proposition holds in this case.

Suppose that $E \cong \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\rho$. Then $\mathrm{Aut}_{\mathrm{group}}(E) \cong \mathbb{Z}/6\mathbb{Z}$, and is generated by multiplication by $-\rho$. If h is not the identity, then a simple calculation shows that the only possibilities are that h is multiplication by ρ or ρ^2, and $\tau = \pm\frac{1}{3}(1+2\rho) \pmod{\Lambda}$. ∎

Remark. The exceptional case in Proposition 2.1 can only occur when $3\tau = 0$. In particular, it does not occur when τ is of infinite order. These special cases arise when $\alpha_1 = \alpha_2 = \alpha_3 = \pm\sqrt{-3}$, in which case there exists $\varphi \in \mathrm{Aut}(A)$ such that $\varphi(x_0) = x_0$, $\varphi(x_1) = x_2$, $\varphi(x_2) = -x_3$, $\varphi(x_3) = -x_1$.

We now define a map $\Phi: E_4 \to \mathrm{Aut}(A)$ such that the composition $E_4 \to \mathrm{Aut}(A)/\mathbb{C}^\times$ is an injective group homomorphism. Recall our alternative notation for the generators of the algebra, namely $X_{11} = x_0$, $X_{00} = x_1$, $X_{01} = x_2$, $X_{10} = x_3$. Let (i, j, k) be a cyclic permutation of $(1, 2, 3)$. As stated in [14, Proposition 6] a map of the form $\varphi(x_0) = \lambda_0 x_i$, $\varphi(x_i) = \lambda_i x_0$, $\varphi(x_j) = \lambda_j x_k$, $\varphi(x_k) = \lambda_k x_j$ extends to an algebra automorphism of A if and only if

$$\frac{\lambda_0 \lambda_i}{\lambda_j \lambda_k} = -1, \qquad \frac{\lambda_0 \lambda_j}{\lambda_i \lambda_k} = -\alpha_j, \qquad \frac{\lambda_0 \lambda_k}{\lambda_i \lambda_j} = \alpha_k.$$

In particular, each of the following $\Phi(\xi) \in GL(V)$ extends to an automorphism of A:

$$\Phi(0)(X_{11}, X_{00}, X_{01}, X_{10}) = (X_{11}, X_{00}, X_{01}, X_{10}),$$

$$\Phi\left(\frac{1}{2}\right)(X_{11}, X_{00}, X_{01}, X_{10}) = (X_{11}, -X_{00}, -X_{01}, X_{10}),$$

$$\Phi\left(\frac{1}{2}\eta\right)(X_{11}, X_{00}, X_{01}, X_{10}) = (X_{11}, -X_{00}, X_{01}, -X_{10}),$$

$$\Phi\left(\frac{1}{2}+\frac{1}{2}\eta\right)(X_{11}, X_{00}, X_{01}, X_{10}) = (X_{11}, X_{00}, -X_{01}, -X_{10}),$$

$$\Phi\left(\frac{1}{4}\right)(X_{11}, X_{00}, X_{01}, X_{10}) = \left(i\frac{\theta_{11}(\tau)}{\theta_{10}(\tau)}X_{10}, i\frac{\theta_{01}(\tau)}{\theta_{01}(\tau)}X_{01}, -i\frac{\theta_{01}(\tau)}{\theta_{00}(\tau)}X_{00}, i\frac{\theta_{10}(\tau)}{\theta_{11}(\tau)}X_{11}\right),$$

$$\Phi\left(\frac{1}{4}\eta\right)(X_{11}, X_{00}, X_{01}, X_{10}) = \left(i\frac{\theta_{11}(\tau)}{\theta_{01}(\tau)}X_{01}, \frac{\theta_{00}(\tau)}{\theta_{10}(\tau)}X_{10}, -i\frac{\theta_{01}(\tau)}{\theta_{11}(\tau)}X_{11}, -\frac{\theta_{10}(\tau)}{\theta_{00}(\tau)}X_{00}\right),$$

$$\Phi\left(\frac{1}{4}+\frac{1}{4}\eta\right)(X_{11}, X_{00}, X_{01}, X_{10}) = \left(\frac{\theta_{11}(\tau)}{\theta_{00}(\tau)}X_{00}, i\frac{\theta_{00}(\tau)}{\theta_{11}(\tau)}X_{11}, \frac{\theta_{01}(\tau)}{\theta_{10}(\tau)}X_{10}, -i\frac{\theta_{10}(\tau)}{\theta_{01}(\tau)}X_{01}\right).$$

We define Φ on the other elements of E_4 by requiring that $\Phi(\xi+\omega) = \Phi(\xi) \circ \Phi(\omega)$ for each $\xi \in E_4$ and each $\omega \in E_2$. Notice that Φ is not a group homomorphism. However, it is easily checked that the restriction of Φ to E_2 is a group homomorphism and the composition $\Phi\colon E_4 \to \mathrm{Aut}(A)/\mathbb{C}^\times$ is a group homomorphism.

THEOREM 2.2. (a) *The map* $\Phi\colon E_4 \to \mathrm{Aut}(A)/\mathbb{C}^\times$ *defined above is a group homomorphism.*

(b) *If $\xi \in E_4$, then the induced action of $\Phi(\xi)$ on E is translation by ξ.*

(c) *Set $\rho = e^{2\pi i/3}$. Suppose that $E \cong \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\rho$ and $\tau = \pm\frac{1}{3}(1+2\rho)$ (mod Λ). Then* $\mathrm{Aut}(A)/\mathbb{C}^\times \cong E_4 \times (\mathbb{Z}/3\mathbb{Z})$.

(d) *If (E, τ) is not as in* (c) *then* $\mathrm{Aut}(A)/\mathbb{C}^\times \cong E_4$.

Proof. (a) This is a routine calculation.

(b) By the remarks at the beginning of this section, $\mathrm{Aut}(A)/\mathbb{C}^\times$ identifies with $\{\varphi \in PGL(V^*) \mid (\varphi \times \varphi)(\Gamma) \subset \Gamma\}$. It follows from (2.1) that the induced action of $\Phi(\xi)$ on E is translation by some point of E_4. In fact $\Phi(\xi)$ is translation by ξ itself; since this is easy to check we only give the details for $\xi = \frac{1}{4}$. Consider a point $z \in \mathbb{C}$ and $j_\tau(z) \in E$. Then

$$
\begin{aligned}
j_\tau\left(z+\frac{1}{4}\right) &= \Big(i\theta_{11}(\tau)\,\theta_{11}\left(2z+\frac{1}{2}\right), i\theta_{00}(\tau)\,\theta_{00}\left(2z+\frac{1}{2}\right),\\
&\qquad \theta_{01}(\tau)\,\theta_{01}\left(2z+\frac{1}{2}\right), \theta_{10}(\tau)\,\theta_{10}\left(2z+\frac{1}{2}\right)\Big)\\
&= (i\theta_{11}(\tau)\,\theta_{10}(2z), i\theta_{00}(\tau)\,\theta_{01}(2z), \theta_{01}(\tau)\,\theta_{00}(2z), -\theta_{10}(\tau)\,\theta_{11}(2z))\\
&= \left(i\frac{\theta_{11}(\tau)}{\theta_{10}(\tau)}g_{10}(z), i\frac{\theta_{00}(\tau)}{\theta_{01}(\tau)}g_{01}(z), -i\frac{\theta_{01}(\tau)}{\theta_{00}(\tau)}g_{00}(z), i\frac{\theta_{10}(\tau)}{\theta_{11}(\tau)}g_{11}(z)\right)\\
&= \Phi\left(\frac{1}{4}\right)(j_\tau(z))
\end{aligned}
$$

where the last equality makes use of the convention that $\mathrm{Aut}(A)$ acts on the right of V, and on the left of V^*.

(c, d) Since E is not contained in a hyperplane any automorphism of $\mathbb{P}(V^*)$ is determined by its action on E. Therefore the remarks at the beginning of this section, together with (2.1), imply that restriction gives an injective map $\mathrm{Aut}(A)/\mathbb{C}^\times \to \mathrm{Aut}_{\mathrm{var}}(E)$. Moreover (2.1) shows that the image is contained in either E_4 acting on E by translations, or in $E_4 \times (\mathbb{Z}/3\mathbb{Z})$ in the "exceptional case". The existence of Φ shows that the image of $\mathrm{Aut}(A)/\mathbb{C}^\times$ in $\mathrm{Aut}_{\mathrm{var}}(E)$ is at least as large as the group of translations by E_4, so this proves (d). To prove (c) first observe that multiplication by ρ is of order 3. As in the remark after (3.1), there exists $\varphi \in \mathrm{Aut}(A)$ of order 3, with $\varphi \notin \mathbb{C}^\times$. Thus $\mathrm{Aut}(A)/\mathbb{C}^\times$ contains a copy of $E \times (\mathbb{Z}/3\mathbb{Z})$. However, by (2.1) it can be no larger than this. ▮

Since our interest is in the case when $|\tau| = \infty$, the exceptional case (2.2c) cannot occur. Hence there is an exact sequence $1 \to \mathbb{C}^\times \to \mathrm{Aut}(A) \to E_4 \to 1$.

Twisting a graded module does not change its Hilbert series, so the twist of a point module (or a line module) is again a point module (respectively, a line module). Thus the action of $\mathrm{Aut}(A)$ on point modules induces an action of $\mathrm{Aut}(A)$ on $E \cup \mathcal{S}$ (the variety which parametrises the point modules), and an action on those lines in $\mathbb{P}(V^*)$ which are secant lines of E. As the result shows, it is easy to check that this coincides with the restriction of the action of $\mathrm{Aut}(A)/\mathbb{C}^\times \subset PGL(V^*)$ on $\mathbb{P}(V^*)$.

PROPOSITION 2.3. *The action of* $\mathrm{Aut}(A)/\mathbb{C}^\times$ *on* $\mathbb{P}(V^*)$ *is such that if* $\varphi \in \mathrm{Aut}(A)$ *then*

(a) *for all* $p \in E \cup \mathcal{S}$, $M(p)^\varphi \cong M(\varphi(p))$;

(b) *for all* $p, q \in E$, $M(p, q)^\varphi \cong M(\varphi(p), \varphi(q))$;

(c) *if* $\varphi \in \mathrm{Aut}(A)$, *and* $\varphi \equiv \Phi(\xi)(\mathrm{mod}\, \mathbb{C}^\times)$, *then* $M(p, q)^\varphi \cong M(p+\xi, q+\xi)$.

Proof. Let M be either $M(p)$ or $M(p, q)$. The action of $a \in A$ on $m \in M^\varphi$ is such that $a * m = 0 \Leftrightarrow \varphi(a) \cdot m = 0$. Hence if $M = A/AU$ where U is a subspace of A_1, then $M^\varphi \cong A/A\varphi(U)$. Furthermore, our convention regarding the action of φ on V and V^* is such that if $U \subset A_1$ then $\mathcal{V}(\varphi(U)) = \varphi(\mathcal{V}(U))$. The result follows. ∎

3. Sklyanin's Representations

The first task in this section is to define Sklyanin's finite dimensional modules V_k. We do this after recalling some preliminary results (3.1)-(3.3) which appear in his paper [14]. The proofs of these results are fairly straightforward calculations using the addition theorems for theta functions in [17, Sect. 22]. Nevertheless, it seems to us that remarkable ingenuity was required for Sklyanin to find the action on $\mathcal{M}(\mathbb{C})$ described in (3.1). Indeed, we do not understand the real reason for the existence of this A-module. Nor do we fully understand the real reason for the existence of the modules V_k.

Although our main interest here is when τ is of infinite order, a number of our results are also valid when τ is of finite order, so we will often work in that generality. If τ is of finite order, we will denote by s the smallest positive integer such that $2s\tau = 0$. If τ is of infinite order we declare that $s = \infty$.

Having defined the modules V_k, we will prove in (3.6) that V_k is simple whenever $k < s$ (this includes the case where τ is of infinite order). A preliminary result is that V_k is a quotient of a line module. The last part of this section discusses the twists of the modules V_k, and the action of the center on them. Finally Theorem 3.10 gives a complete list of all the modules obtained by twisting the various V_k, and proves that there are no isomorphisms between these twists. Of course, the main goal of the paper is to prove that these are all the finite dimensional modules.

The space of meromorphic functions on $\mathbb{C}$ is denoted by $\mathcal{M}(\mathbb{C})$. Recall that A_1 has basis X_{00}, X_{01}, X_{10}, X_{11}. If $X = \sum_{ab} \lambda_{ab} X_{ab}$, then we consider X as a function on $\mathbb{C}$ by defining $X(z) = \sum_{ab} \lambda_{ab} g_{ab}(z)$, where the g_{ab} are defined in Section 1.

THEOREM 3.1 [14]. *For each $k \in \mathbb{N} \cup \{0\}$, $\mathcal{M}(\mathbb{C})$ is an A-module with the action of $X \in A_1$ on $f \in \mathcal{M}(\mathbb{C})$ given by*

$$(X \cdot f)(z) = \frac{X(z - (1/2)\,k\tau)}{\theta_{11}(2z)} f(z+\tau) - \frac{X(-z-(1/2)\,k\tau)}{\theta_{11}(2z)} f(z-\tau).$$

The central elements Ω_1 and Ω_2 act by scalar multiplication on this module. Indeed $\mathcal{M}(\mathbb{C})$ is annihilated by

$$\Omega_1 - 4\theta_{11}((k+1)\tau)^2 \qquad \text{and} \qquad \Omega_2 - 4\,\frac{\theta_{00}(\tau)^2}{\theta_{00}(0)\,\theta_{00}(2\tau)}\,\theta_{11}((k+2)\tau)\,\theta_{11}(k\tau).$$

Remark. Since $4\tau \neq 0$, it follows that $\mathcal{M}(\mathbb{C})$ is not annihilated by both Ω_1 and Ω_2. In particular, $\mathcal{M}(\mathbb{C})$ is not a B-module. In fact, $\mathcal{M}(\mathbb{C})$ can not have a non-zero subquotient which is a B-module.

DEFINITION. For each $p \in \mathbb{N}$, and for each $ab \in \{00, 01, 10, 11\}$ let Θ^p_{ab} denote the space of holomorphic functions on $\mathbb{C}$ satisfying

$$f(z+1) = (-1)^a f(z)$$

and

$$f(z+\eta) = e^{-\pi i b - p\pi i(2z+\eta)} f(z) \qquad \text{for all} \quad z \in \mathbb{C}.$$

Such functions are called *theta functions* of *weight p* and *characteristics ab*. It is convenient to define Θ^0_{00} to be $\mathbb{C}$, the space of constant functions. The subspace of Θ^p_{ab} consisting of the even functions is denoted by Θ^{p+}_{ab}.

PROPOSITION 3.2. (a) *For each $p \in \mathbb{N}$, $\dim \Theta^p_{00} = p$ and each non-zero $f \in \Theta^p_{00}$ has exactly p zeroes (counted with multiplicities) in the fundamental parallelogram.*

(b) *Given any p points in the fundamental parallelogram whose sum is $p/2 - (p/2)\eta$, there exists some $0 \neq f \in \Theta^p_{00}$ with zero locus precisely that set of p points. Furthermore, f is determined up to scalar multiples by the location of its zeroes.*

(c) *For any* $k \in \mathbb{N} \cup \{0\}$, $\dim \Theta_{00}^{2k+} = k+1$, *and for any* $2k$ *points* $\{\pm z_j \mid 0 \leqslant j \leqslant k-1\}$, *there exists some* $0 \neq f \in \Theta_{00}^{2k+}$ *with zero locus precisely that set.*

PROPOSITION 3.3. *For each* $k \in \mathbb{N} \cup \{0\}$, *the space* Θ_{00}^{2k+} *is stable under the* A*-module action on* $\mathcal{M}(\mathbb{C})$ *defined in* (3.1).

DEFINITION. For each $k \in \mathbb{N} \cup \{0\}$, define V_k to be the A-module $V_k := \Theta_{00}^{2k+}$ with the A-action given in (3.1). Thus $\dim V_k = k+1$.

Remark. In some of the later proofs $k=0$ is a special case. The module V_0 is just the space of constant functions, namely $V_0 = \mathbb{C}$. The A-action is given by (3.1), namely $X \cdot 1 = X(z)/\theta_{11}(2z) - X(-z)/\theta_{11}(2z)$ for $X \in A_1$. Since x_1, x_2, x_3 are even functions, they all annihilate $1 \in V_0$. On the other hand $x_0 \cdot 1 = 2i\theta_{11}(\tau) \neq 0$. Thus V_0 is a quotient of the point module $A/Ax_1 + Ax_2 + Ax_3 \cong M(e_0)$ corresponding to $e_0 \in \mathcal{S}$. By [7, 5.7] $M(e_0)$ and hence V_0 is a quotient of every line module $M(p, q)$ such that $p+q = \omega_0 = 0$. That is, V_0 is a quotient of $M(p, -p)$ for every $p \in E$. This proves (3.4), (3.5) and (3.6) for V_0.

PROPOSITION 3.4. *Fix* $k \in \mathbb{N}$ *such that* $k < s$. *Suppose that* $p, q \in E$ *are such that* $p+q = k\tau$, *and* $p-q \notin \mathbb{Z}\tau$. *Then* V_k *has a basis* $\{f_i \mid 0 \leqslant i \leqslant k\}$ *such that the zero locus of* f_i *is*

$$\begin{aligned}\mathcal{Z}(f_i) &= \{\pm(p + (\tfrac{1}{2}k - 1)\tau - 2(i+j)\tau) \mid 0 \leqslant j \leqslant k-1\} \\ &= \{\pm(q + (\tfrac{1}{2}k - 1)\tau + 2(i-j)\tau) \mid 0 \leqslant j \leqslant k-1\}.\end{aligned}$$

Furthermore, if $u_i, v_i \in A_1$ *satisfy* $\mathcal{V}(u_i, v_i) = l_{p-2i\tau, q+2i\tau}$ *then* $u_i \cdot f_i = v_i \cdot f_i = 0$.

Proof. The hypothesis on k ensures that this set of potential zeroes consists on $2k$ distinct points. By (3.2c) there is a (unique up to scalar multiples) $0 \neq f_i \in V_k$ with this set of zeroes. It remains to show that the f_i are linearly independent. Suppose that $\sum_{0 \leqslant m \leqslant k} \lambda_m f_m = 0$.

Set $r = p + (\frac{1}{2}k - 1)\tau$. Thus $\mathcal{Z}(f_i) = \{\pm(r - 2(i+j)\tau) \mid 0 \leqslant j \leqslant k-1\}$. Set $\xi_0 = r - 2k\tau = r - 2(m + (k-m))\tau$. Clearly $f_m(\xi_0) = 0$ if $1 \leqslant m \leqslant k$. However, if $f_0(\xi_0) = 0$, then either $\xi_0 = r - 2j\tau$ or $\xi_0 = -r + 2j\tau$ for some j, with $0 \leqslant j \leqslant k-1$. The first possibility can not occur because $k < s$, and the second can not occur because $2r \notin \mathbb{Z} \cdot 2\tau$. Hence $f_0(\xi_0) \neq 0$. By evaluating $\sum_{0 \leqslant m \leqslant k} \lambda_m f_m$ at ξ_0 it follows that $\lambda_0 = 0$.

Suppose that $\lambda_0 = \lambda_1 = \cdots = \lambda_{n-1} = 0$. Thus $\sum_{n \leqslant m \leqslant k} \lambda_m f_m = 0$. If $n = k$ then the proof is complete, so suppose that $n < k$. Set $\xi_n = r - 2(k+n)\tau = r - 2(m + (k+n-m))\tau$. Clearly $f_m(\xi_n) = 0$ if $n+1 \leqslant m \leqslant k$ and a similar argument to the above shows that $f_n(\xi_n) \neq 0$, from which it follows that $\lambda_n = 0$. Thus the f_i form a basis for V_k.

To prove that f_i is annihilated by u_i and v_i it is enough to do this when $i=0$, since the general case is then obtained by replacing (p, q) by $(p-2i\tau, q+2i\tau)$. Set $z_0 = p + (\frac{1}{2}k - 1)\tau = -q + (\frac{3}{2}k - 1)\tau$. Since $\mathscr{Z}(f_0) = \{\pm(z_0 - 2j\tau) \mid 0 \leqslant j \leqslant k-1\}$ it follows that $f_0(z+\tau)$ is zero at $z \in \{z_0 - (2j+1)\tau, -z_0 + (2j-1)\tau \mid 0 \leqslant j \leqslant k-1\}$ and $f_0(z-\tau)$ is zero at $z \in \{z_0 - (2j-1)\tau, -z_0 + (2j+1)\tau \mid 0 \leqslant j \leqslant k-1\}$. In particular, both $f_0(z+\tau)$ and $f_0(z-\tau)$ are zero at $z \in \{z_0 - (2j-1)\tau, -z_0 + (2j-1)\tau \mid 1 \leqslant j \leqslant k-1\}$. Furthermore $f_0(z+\tau)$ is zero at $z \in \{-z_0 - \tau, z_0 - (2k-1)\tau\}$ and $f_0(z-\tau)$ is zero at $z \in \{z_0 + \tau, -z_0 + (2k-1)\tau\}$.

Suppose that $X \in A_1$ vanishes at $\{z_0 - (\frac{1}{2}k - 1)\tau, -z_0 + (\frac{3}{2}k - 1)\tau\} = \{p, q\}$. Then $X(z - \frac{1}{2}k\tau)$ is zero at $z \in \{z_0 + \tau, -z_0 + (2k-1)\tau\}$, and $X(-z - \frac{1}{2}k\tau)$ is zero at $z \in \{-z_0 - \tau, z_0 - (2k-1)\tau\}$. It follows that both $X(z - \frac{1}{2}k\tau) f_0(z+\tau)$ and $X(-z - \frac{1}{2}k\tau) f_0(z-\tau)$ are zero whenever $z \in \{\pm(z_0 - (2j-1)\tau) \mid 0 \leqslant j \leqslant k\}$. Thus $X \cdot f_0$ has $2k+2$ zeroes, so is identically zero. That is $u_0 \cdot f_0 = v_0 \cdot f_0 = 0$. ∎

Remark. Part of (3.4) holds under weaker hypotheses. Let $k \in \mathbb{N}$ and p, q be arbitrary, and suppose that $p + q = k\tau$. Then there exist elements $f_i \in V_k$ such that $\mathscr{Z}(f_i)$ is as stated in (3.4) (possibly $k \geqslant s$ now so the zeroes must be counted with multiplicity), and $u_i \cdot f_i = v_i \cdot f_i = 0$ as before. In particular

$$\operatorname{Hom}(M(p, q), V_k) \neq 0$$

for all p, q such that $p + q = k\tau$.

The proofs of (3.5) and (3.6) require a result from [7, Sect. 5] which we briefly recall. Suppose that $p, q \in E$ and that $p - q \notin \mathbb{Z} \cdot 2\tau$. Then $M(p, q)$ has a basis e_{ij} such that $e_{ij} \in M(p, q)_{i+j}$, $Ae_{ij} \cong M(p + (j-i)\tau, q + (i-j)\tau)$, and if $X \in A_1$ then $X \cdot e_{ij} \in \mathbb{C}e_{i+1,j} \oplus \mathbb{C}e_{i,j+1}$. Furthermore, $X \cdot e_{ij} \in \mathbb{C}e_{i+1,j}$ if and only if $X(q + (i-j)\tau) = 0$ and $X \cdot e_{ij} \in \mathbb{C}e_{i,j+1}$ if and only if $X(p + (j-i)\tau) = 0$.

THEOREM 3.5. *If $k < s$ and $p + q = k\tau$ and $p - q \notin \mathbb{Z} \cdot 2\tau$, then V_k is a quotient of $M(p, q)$.*

Proof. By the last part of (3.4) if $u, v \in A_1$ satisfy $\mathscr{V}(u, v) = l_{pq}$ then $u \cdot f_0 = v \cdot f_0 = 0$. It follows from this that there is an A-module map $\Psi\colon M(p, q) \to V_k$ such that $\Psi(e_{00}) = f_0$. The surjectivity of Ψ will be proved by showing that $f_n = a_n \cdot f_0$ for some $a_n \in A_{2n}$, where $\{f_n \mid 0 \leqslant n \leqslant k\}$ is the basis for V_k described in (3.4).

If $0 \leqslant i \leqslant 2k-1$ then there exists $Y_i \in A_1$ such that $Y_i(q + i\tau) = 0$ and $Y_i(p - i\tau) \neq 0$; notice that $p - i\tau \neq q + i\tau$ by hypothesis on $p - q$. If $0 \leqslant n \leqslant k$ define $a_n = Y_{2n-1} Y_{2n-2} \cdots Y_1 Y_0 \in A_{2n}$. We will show that $a_n \cdot f_0$ is non-zero and has the same zeroes as f_n, hence is a non-zero scalar multiple of f_n. To do this it suffices to prove for each i that $Y_{2i-1}Y_{2i-2}\cdot f_{i-1}$ is non-zero and has the same zeroes as f_i.

Set $r=p+(\frac{1}{2}k-1)\tau$ so that $\mathscr{Z}(f_{i-1})=\{\pm(r-2(i+j-1)\tau)\mid 0<j\leqslant k-1\}$. Now

$$(Y_{2i-2}\cdot f_{i-1})(r-(2i-3)\tau)=\frac{Y_{2i-2}(p-2(i-1)\tau)}{\theta_{11}(2r-2(2i-3)\tau)}f_{i-1}(r-2(i-2)\tau).$$

But $f_{i-1}(r-2(i-2)\tau)\neq 0$ since $2p+k\tau\notin\mathbb{Z}\cdot 2\tau$ and $k<s$. Also $Y_{2i-2}(p-2(i-1)\tau)\neq 0$, so $Y_{2i-2}\cdot f_{i-1}\neq 0$. Both $f_{i-1}(z+\tau)$ and $f_{i-1}(z-\tau)$ are zero for $z\in\{\pm(r-2(i+j)\tau+\tau)\mid 0\leqslant j\leqslant k-2\}$. Furthermore if $z_0=r-2(i+k-1)\tau+\tau$ then $f_{i-1}(z_0+\tau)=0$ and $Y_{2i-2}(-z_0-\frac{1}{2}k\tau)=Y_{2i-2}(q+2(i-1)\tau)=0$. Finally, since $f_{i-1}(-z_0-\tau)=0$ it follows that

$$\mathscr{Z}(Y_{2i-2}\cdot f_{i-1})=\{\pm(r-2(i+j-1)\tau+\tau)\mid 0\leqslant j\leqslant k-1\}.$$

By repeating the argument in this paragraph with $Y_{2i-2}\cdot f_{i-1}$ in place of f_{i-1} and $r-\tau$ in place of r it follows that $Y_{2i-1}\cdot Y_{2i-2}f_{i-1}\neq 0$ and

$$\mathscr{Z}(Y_{2i-1}Y_{2i-2}\cdot f_{i-1})=\{\pm(r-2(i+j-1)\tau-2\tau)\mid 0\leqslant j\leqslant k-1\}=\mathscr{Z}(f_i).$$

Hence $Y_{2i-1}Y_{2i-2}\cdot f_{i-1}$ is a non-zero scalar multiple of f_i. Thus Ψ is surjective. ∎

Remarks. 1. With the notation of (3.5) it is easy to see that $0\neq Y_i\cdot e_{i,0}\in\mathbb{C}e_{i+1,0}$ so it follows by induction that $Y_{2n-1}Y_{2n-2}\cdots Y_1Y_0\cdot e_{00}$ is a non-zero scalar multiple of $e_{2n,0}$ and hence that $\Psi(\mathbb{C}e_{2i,0})=\mathbb{C}f_i$ for all i.

2. If (3.5) is combined with (2.3) the following result is obtained. If $\varphi\in\operatorname{Aut}(A)$, and $\xi\in E_4$ is such that $\varphi\equiv\Phi(\xi)(\operatorname{mod}\mathbb{C}^\times)$, then V_k^φ is a homomorphic image of $M(p+\xi,k\tau-p+\xi)$ for all $p\in E$ such that $2p\notin E_2+\mathbb{Z}\tau$.

THEOREM 3.6. *If $k<s$ then V_k is simple.*

Proof. The result is obviously true for $k=0$ so we suppose that $k\geqslant 1$. Choose $p,q\in E$ such that $p-q\notin\mathbb{Z}\cdot 2\tau$ and $p+q=k\tau$, and let $\{f_i\mid 0\leqslant i\leqslant k\}$ be the basis for V_k obtained in (3.4). Let $\Psi\colon M(p,q)\to V_k$ be as in the proof of (3.5).

Suppose that $0\neq f=\sum_{0\leqslant i\leqslant k}\delta_i f_i\in V_k$ and let m be maximal such that $\delta_m\neq 0$. We must show that f generates V_k. If $m=0$ then (3.5) shows that $A\cdot f=A\cdot f_0=V_k$ so we may suppose that $m\geqslant 1$. By the previous remark f is the image of an element $e=\sum_{0\leqslant i\leqslant m}\varepsilon_i e_{2i,0}\in M(p,q)$ with $\varepsilon_m\neq 0$.

For $0 \leqslant i \leqslant m-1$ choose $0 \neq Y_i \in A_1$ such that its divisor of zeroes on E is

$$(Y_i)_0 = (q+i\tau) + (p-(2m-i)\tau) + (p-(2m-i-2)\tau) + (-p+(4m-2-3i-k)\tau).$$

It follows from the location of the zeroes, and [7, 5.6] that

(i) $Y_i \cdot e_{i+r,r} \in \mathbb{C}e_{i+r+1,r}$ *for all* $r \geqslant 0$;

(ii) $Y_i \cdot e_{2m-r,i-r} \in \mathbb{C}e_{2m-r,i-r+1}$ for all $0 \leqslant r \leqslant i$;

(iii) $Y_i \cdot e_{2m-r,i-r+2} \in \mathbb{C}e_{2m-r,i-r+3}$ for all $0 \leqslant r \leqslant i$.

CLAIM. $Y_{m-1} \cdots Y_1 Y_0 \cdot (\sum_{0 \leqslant i \leqslant m} \varepsilon_i e_{2i,0}) = 0$.

Proof. Write $e' = \sum_{0 \leqslant i \leqslant m-1} \varepsilon_i e_{2i,0}$. Define a new degree function on $M(p,q)$ by defining $\deg(e_{ij}) = i-j$, and write M_i for the degree i component of $M(p,q)$. Thus $e' \in M_0 \oplus M_2 \cdots M_{2m-4} \oplus M_{2m-2}$. It is clear that if $X \in A_1$ then $X \cdot M_r \subset M_{r-1} \oplus M_{r+1}$. However, it follows from (i) that $Y_i \cdot M_i \subset M_{i+1}$ and from (ii) that $Y_i \cdot M_{2m-i} \subset M_{2m-i-1}$, and from (iii) that $Y_i \cdot M_{2m-i+2} \subset M_{2m-i-3}$. In particular, $Y_{m-1} \cdot M_{m-1} = 0$ because $Y_{m-1}(q+(m-1)\tau) = Y_{m-1}(p-(m-1)\tau) = 0$. Thus

$$\begin{aligned}
&Y_0 \cdot (M_0 \oplus M_2 \cdots M_{2m-4} \oplus M_{2m-2}) \\
&\qquad \subset M_1 \oplus M_3 \cdots M_{2m-5} \oplus M_{2m-3} \\
&Y_1 Y_0 \cdot (M_0 \oplus M_2 \cdots M_{2m-4} \oplus M_{2m-2}) \\
&\qquad \subset M_2 \oplus M_4 \cdots M_{2m-6} \oplus M_{2m-4} \\
&Y_{m-2} \cdots Y_1 Y_0 \cdot (M_0 \oplus M_2 \cdots M_{2m-4} \oplus M_{2m-2}) \subset M_{m-1} \\
&Y_{m-1} \cdots Y_1 Y_0 \cdot (M_0 \oplus M_2 \cdots M_{2m-4} \oplus M_{2m-2}) = 0. \quad \blacksquare
\end{aligned}$$

The hypothesis on $p-q$ and the choice of k ensures that $Y_i(q+(2m-i)\tau) \neq 0$, and therefore $0 \neq Y_i \cdot e_{2m,i} \in \mathbb{C}e_{2m,i+1}$ by (ii) above. Hence $Y_{m-1} \cdots Y_1 Y_0 \cdot e_{2m,0}$ is a non-zero scalar multiple of $e_{2m,m}$. It follows from this fact and the claim that $e_{2m,m}$ is in the submodule of $M(p,q)$ generated by the element e. Thus $e_{2m,2m}$ is also in the submodule generated by e, whence $\Psi(e_{2m,2m})$ is in the submodule of V_k generated by $\Psi(e) = f$.

Now let $\Omega \in \mathbb{C}\Omega_1 \oplus \mathbb{C}\Omega_2$ be such that $\Omega \cdot M(p,q) \neq 0$. Then $\Omega^{2m} \cdot e_{00} \neq 0$ since $M(p,q)$ is critical. By the remark after [7, 5.6] $\Omega^{2m} \cdot e_{00} \in \mathbb{C}e_{2m,2m}$. Since V_k is contained in $\mathcal{M}(\mathbb{C})$ it follows that Ω acts on V_k by scalar multiplication; since V_k is a quotient of $M(p,q)$ it is killed by $\Omega(p+q)$, but by (3.1) V_k is not killed by both Ω_1 and Ω_2. Thus Ω acts on V_k as a non-zero scalar. Hence $\Omega^{2m} \cdot f_0$ is a non-zero scalar multiple of f_0. Therefore $\Psi(e_{2m,2m})$ is a non-zero scalar. It follows that f_0 is in the submodule of V_k generated by f. Since f_0 generates V_k, so too does f, and it follows that V_k is simple. ∎

Remark. We have actually proved a somewhat stronger result than (3.6): if $k < s$ then V_k is simple over the (Veronese) subalgebra of A given by $A^{(2)} = \mathbb{C}[A_2] = \bigoplus_{i \geqslant 0} A_{2i}$. To see this first observe that (3.5) proves that $f_i \in A_{2i} \cdot f_0$, so V_k is generated by f_0 as an $A^{(2)}$-module. Secondly in (3.6) we prove that $e_{2m,2m} \in A \cdot e$ and since every component of e is of even degree, in fact $e_{2m,2m} \in A^{(2)} \cdot e$. Hence (3.6) proves that $f_0 \in A^{(2)} \cdot f$ showing that $V_k = A^{(2)} \cdot f$.

Notation. In the next proof we will write $a^{\Phi(\xi)}$ in place of $\Phi(\xi)(a)$ whenever $a \in A$ and $\Phi(\xi)$ is one of the automorphisms in Section 2.

PROPOSITION 3.7. *If $\xi \in E_2$ then $V_k^{\Phi(\xi)} \cong V_k$.*

Proof. Suppose that $\xi = l/2 + (m/2)\eta$ where $l, m \in \{0, 1\}$. Let $X \in A_1$. The reader can verify the following three identitities:

$$X^{\Phi(\xi)}(z) = (-1)^{l+m} e^{m\pi i(4z+\eta)} X(z+\xi)$$

$$\theta_{11}(2z + 2\xi) = (-1)^{l+m} e^{-m\pi i(4z+\eta)} \theta_{11}(2z)$$

$$X(-z - \tfrac{1}{2}k\tau - \xi) = e^{-m\pi i(8z+4k\tau)} X(-z - \tfrac{1}{2}k\tau + \xi).$$

These identities are not true for arbitrary l, m but only for those in $\{0, 1\}$. The first identity is proved by using [17, (1), p. 69]; this could have been used in Section 2 as the definition of $\Phi(\xi)$ for $\xi \in E_2$. The second identity follows from the very definition of θ_{11} viz. [17, (1), p. 69]. The third identity is a simple consequence of the useful fact that if $X \in A_1$, then $X \in \Theta_{00}^4$.

Define a linear map $\psi : V_k \to V_k$ by

$$\psi(f)(z) := e^{2km\pi i z} f(z + \xi).$$

A calculation is required to check that the image really is in V_k. Having checked this, it follows that ψ is a linear isomorphism. The proposition follows from the fact that ψ is an A-module map from V_k to $V_k^{\Phi(\xi)}$. This is proved by showing that $\psi(X \cdot f)(z) = (X^{\Phi(\xi)} \cdot \psi(f))(z)$, which is a straightforward (although potentially error prone) calculation using the identities in the first paragraph of the proof. ∎

DEFINITION. Let $\omega \in E_2$, and let $k \in \mathbb{N} \cup \{0\}$. Choose any $\xi \in E_4$ such that $2\xi = \omega$. Define $V(\omega + k\tau) := V_k^{\Phi(\xi)}$. By (3.7) this is independent of the choice of ξ.

THEOREM 3.8. *Suppose that* $|\tau| = \infty$. *Let* $\omega \in E_2$, $k \in \mathbb{N} \cup \{0\}$ *and* $\lambda \in \mathbb{C}^\times$.

(a) $V(\omega + k\tau)^\lambda$ *is a simple A-module of dimension* $k+1$.

(b) *If* $p, q \in E$ *satisfy* $p + q = \omega + k\tau$, *then* $V(\omega + k\tau)^\lambda$ *is a quotient of* $M(p, q)$.

Proof. It suffices to prove this when $\omega = 0$ and $\lambda = 1$.

(a) This is already proved in (3.6).

(b) Now let $p, q \in E$ be such that $p + q = k\tau$. By the remark after (3.4) there exists $0 \neq f \in V_k$ such that Af is a quotient of $M(p, q)$. However, by (a) $Af = V_k$. ∎

Our next goal is to show that for distinct triples (ω, k, λ) the corresponding modules are non-isomorphic (it is obvious that $V(\omega + k\tau)^\lambda \not\cong V(\omega' + m\tau)$ if $k \neq m$ because the dimensions differ). This is achieved in (3.10), but prior to that, we need to understand the action of the center on these modules.

PROPOSITION 3.9. *Let* $\omega = (1-b)/2 + ((1-a)/2)\eta \in E_2$ *where* $ab \in \{11, 00, 01, 10\}$. *The central elements* Ω_1 *and* Ω_2 *act by scalar multiplication on* $V(\omega + k\tau)$. *More precisely* $V(\omega + k\tau)$ *is annihilated by*

$$\Omega_1 - 4\theta_{ab}((k+1)\tau)^2 \qquad \text{and} \qquad \Omega_2 - 4\frac{\theta_{00}(\tau)^2}{\theta_{00}(0)\,\theta_{00}(2\tau)}\theta_{ab}((k+2)\tau)\,\theta_{ab}(k\tau).$$

Proof. Define $\xi = (1-b)/4 + ((1-a)/4)\eta$. The explicit form of $\Phi(\xi)$ is given in Section 2. Before proving that $\Omega_1 - 4\theta_{ab}((k+1)\tau)^2$ annihilates $V(2\xi + k\tau)$ we make the useful observation that $X^{\Phi(\xi)}(z) = e^{-(1/2)\pi i b}e^{\pi i(1-a)(2z+\eta/4)}X(z+\xi)$. This can be checked by using the formulae in [17, (8), p. 73].

Let $ij \in \{00, 01, 10, 11\}$, and $f \in V_k = V(k\tau)$. Then

$$\begin{aligned}
&((X_{ij}^2)^{\Phi(\xi)} \cdot f)(z)\\
&\quad= \left(\frac{X_{ij}(z+\xi-(k/2)\tau)\,X_{ij}(z+\xi-(k/2-1)\tau)}{\theta_{11}(2z)\,\theta_{11}(2z+2\tau)}\right)e^{P(z)}\cdot f(z+2\tau)\\
&\quad\quad-\left(\frac{X_{ij}(z+\xi-(k/2)\tau)\,X_{ij}(-z+\xi-(k/2+1)\tau)}{\theta_{11}(2z)\,\theta_{11}(2z+2\tau)}\right.\\
&\quad\quad+\left.\frac{X_{ij}(-z+\xi-(k/2)\tau)\,X_{ij}(z+\xi-(k/2+1)\tau)}{\theta_{11}(2z)\,\theta_{11}(2z-2\tau)}\right)e^{Q}\cdot f(z)\\
&\quad\quad+\left(\frac{X_{ij}(-z+\xi-(k/2)\tau)\,X_{ij}(-z+\xi-(k/2-1)\tau)}{\theta_{11}(2z)\,\theta_{11}(2z-2\tau)}\right)e^{P(-z)}\cdot f(z-2\tau)
\end{aligned}$$

where $P(z)=\pi i(1-a)(4z+\eta/2-(2(k+1)\tau)+\pi ib$ and $Q=\pi i(1-a)(\eta/2-(2(k+1)\tau)+\pi ib$. Hence

$$(\Omega_1^{\Phi(\xi)}\cdot f)(z)=(A(z)\,e^{P(z)}f(z+2\tau)-(B(z)+B(-z))\,e^{Q}f(z)\\ +A(-z)\,e^{P(-z)}f(z-2\tau))$$

where

$$A(z)=\sum_{ij}\gamma_{ij}^2\frac{\theta_{ij}(\tau)\,\theta_{ij}(-\tau)\,\theta_{ij}(2z+2\xi-k\tau)\,\theta_{ij}(2z+2\xi-(k-2)\tau)}{\theta_{11}(2z)\,\theta_{11}(2z+2\tau)}$$

and

$$B(z)=\sum_{ij}\gamma_{ij}^2\frac{\theta_{ij}(\tau)\,\theta_{ij}(-\tau)\,\theta_{ij}(2z+2\xi-k\tau)\,\theta_{ij}(-2z+2\xi-(k+2)\tau)}{\theta_{11}(2z)\,\theta_{11}(2z+2\tau)}.$$

By [10, (R5), p. 20]

$$A(z)=\frac{-2\theta_{11}(2z+2\xi-(k-1)\tau)\,\theta_{11}(-2z-2\xi+(k-1)\tau)\,\theta_{11}(0)\,\theta_{11}(2\tau)}{\theta_{11}(2z)\,\theta_{11}(2z+2\tau)}\\ =0.$$

Again, by [10, (R5), p. 20] we have

$$B(z)=\frac{-2\theta_{11}(2\xi-(k+1)\tau)\,\theta_{11}(-2\xi+(k+1)\tau)\,\theta_{11}(2z+2\tau)\,\theta_{11}(-2z)}{\theta_{11}(2z)\,\theta_{11}(2z+2\tau)}\\ =-2\theta_{11}(2\xi-(k+1)\tau)^2\\ =B(-z).$$

Thus

$$(B(z)+B(-z))\,e^{Q}=-4\theta_{11}((k+1)\tau-2\xi)^2\,e^{-\pi ib}e^{\pi i(1-a)(\eta/2-(2(k+1)\tau))}\\ =-4\theta_{ab}((k+1)\tau)^2.$$

Thus, $\Omega_1^{\Phi(\xi)}$ acts on V_k as multiplication by $4\theta_{ab}((k+1)\tau)^2$. Hence $\Omega_1-4\theta_{ab}((k+1)\tau)^2$ annihilates $V(2\xi+k\tau)$.

Now we look at the action of $\Omega_2^{\Phi(\xi)}$ on V_k. We do this by writing $\Omega_2^{\Phi(\xi)}$ as a linear combination of Ω_1 and $\Omega_1^{\Phi(\xi)}$, and using the first part of the proof which shows that both these act on V_k as explicitly determined scalars. This method is not effective when $ab=11$, so we do that case separately.

It follows from the definition of $\Phi(\xi)$ in Section 2 that

$$\Omega_1^{\Phi(\xi)} = -(-1)^{a+b}\frac{\theta_{11}(\tau)^2}{\theta_{ab}(\tau)^2}X_{ab}^2 + (-1)^{b+1}\frac{\theta_{10}(\tau)^2}{\theta_{a,b+1}(\tau)^2}X_{a,b+1}^2 + (-1)^{a+b}\frac{\theta_{01}(\tau)^2}{\theta_{a+1,b}(\tau)^2}X_{a+1,b}^2 + (-1)^{b+1}\frac{\theta_{00}(\tau)^2}{\theta_{a+1,b+1}(\tau)^2}X_{a+1,b+1}^2$$

and

$$\Omega_2^{\Phi(\xi)} = (-1)^{b+1}\frac{\theta_{10}(2\tau)\,\theta_{10}(0)\,\times_{00}(\tau)^2}{\theta_{00}(2\tau)\,\theta_{00}(0)\,\theta_{a,b+1}(\tau)^2}X_{a,b+1}^2 + (-1)^{a+b}\frac{\theta_{01}(2\tau)\,\theta_{01}(0)\,\theta_{00}(\tau)^2}{\theta_{00}(2\tau)\,\theta_{00}(0)\,\theta_{a+1,b}(\tau)^2}X_{a+1,b}^2 + (-1)^{b+1}\frac{\theta_{00}(\tau)^2}{\theta_{a+1,b+1}(\tau)^2}X_{a+1,b+1}^2.$$

We now do the special case when $ab = 11$; that is, we describe the action of Ω_2 on V_k. Combining the expression above for $\xi = \eta/4$, with [17, (9), p. 77] we obtain

$$\Omega_1^{\Phi(\eta/4)} = \frac{\theta_{01}(\tau)^2}{\theta_{11}(\tau)^2}\left(\Omega_1 - \frac{\theta_{00}(0)\,\theta_{01}(0)^2\,\theta_{00}(2\tau)}{\theta_{00}(\tau)^2\,\theta_{01}(\tau)^2}\Omega_2\right)$$

The description of the action of Ω_1 and $\Omega_1^{\Phi(\eta/4)}$ on V_k in the first part of the proof, together with a calculation using [17, (4), p. 77] shows that Ω_2 acts on V_k as scalar multiplication by $4(\theta_{00}(\tau)^2/\theta_{00}(0)\,\theta_{00}(2\tau))\,\theta_{11}((k+2)\tau)\,\theta_{11}(k\tau)$. Hence the Proposition is true when $ab = 11$.

Now suppose that $ab \neq 11$. Then, we claim that

$$\Omega_2^{\Phi(\xi)} = \frac{\theta_{11}(\tau)^2\,\theta_{00}(\tau)^2}{\theta_{00}(0)\,\theta_{00}(2\tau)\,\theta_{ab}(0)^2}\left((-1)^{a+b}\,\Omega_1 + \frac{\theta_{ab}(\tau)^2}{\theta_{11}(\tau)^2}\Omega_1^{\Phi(\xi)}\right). \qquad (\dagger)$$

This is proved by comparing coefficients of the various X_{ij}^2. It is clear that the coefficient of X_{ab}^2 on both sides is zero. Comparing the coefficients of $X_{a+1,b+1}^2$ involves showing that

$$\theta_{00}(0)\,\theta_{00}(2\tau)\,\theta_{ab}(0)^2 = (-1)^b\,\theta_{11}(\tau)^2\,\theta_{a+1,b+1}(\tau)^2 + \theta_{ab}(\tau)^2\,\theta_{00}(\tau)^2.$$

This is seen to be true by using the identities [17, (1), (9),(10), pp. 76–77] at $u = v = \tau$. Thus the coefficient of $X_{a+1,b+1}^2$ is the same on both sides of ($\dagger$). It is not necessary to compare any more coefficients because $\Phi(\xi)$ preserves the center of A, so in particular it leaves $\mathbb{C}\Omega_1 \oplus \mathbb{C}\Omega_2$ stable.

Hence both sides of (†) belong to this 2-dimensional space and it follows that the claim is true.

It follows from (†), and the first part of the proof that $\Omega_2^{\Phi(\xi)}$ acts on V_k as scalar multiplication by

$$4\frac{\theta_{11}(\tau)^2\,\theta_{00}(\tau)^2}{\theta_{00}(0)\,\theta_{00}(2\tau)\,\theta_{ab}(0)^2}\left((-1)^{a+b}\,\theta_{11}((k+1)\tau)^2+\frac{\theta_{ab}(\tau)^2}{\theta_{11}(\tau)^2}\,\theta_{ab}((k+1)\tau)^2\right).$$

By [17, (1)–(3), pp. 76–77] this equals $4(\theta_{00}(\tau)^2/\theta_{00}(0)\,\theta_{00}(2\tau))\,\theta_{ab}((k+2)\tau)\,\theta_{ab}(k\tau)$ as required. ∎

Remark. If $|\tau|=\infty$, then neither Ω_1 nor Ω_2 can annihilate $V(\omega+k\tau)$, and hence cannot annihilate $V(\omega+k\tau)^\lambda$ for any $\lambda\in\mathbb{C}^\times$. We shall see later that B actually has no finite dimensional simple modules, apart from the trivial module. This also follows from (4.1) below, together with the main theorem of [5] and [7, 5.9].

THEOREM 3.10. *Suppose that $|\tau|=\infty$. There are no isomorphisms among the modules $V(\omega+k\tau)^\lambda$ for $(\omega,k,\lambda)\in E_2\times(\mathbb{N}\cup\{0\})\times\mathbb{C}^\times$.*

Proof. Suppose that $V(\omega+k\tau)^\lambda\cong V_k$ where $\omega=(1-b)/2+((1-a)/2)\eta$ for some $ab\in\{00, 01, 10, 11\}$. It is sufficient to show that $\omega=0$ and $\lambda=1$. These two modules have the same annihilators in the central subalgebra $\mathbb{C}[\Omega_1,\Omega_2]$, so it follows from (3.1) and (3.9) that

$$\lambda^2\theta_{ab}((k+1)\tau)^2=\theta_{11}((k+1)\tau)^2$$

and

$$\lambda^2\theta_{ab}((k+2)\tau)\,\theta_{ab}(k\tau)=\theta_{11}((k+2)\tau)\,\theta_{11}(k\tau).$$

In particular

$$\theta_{11}((k+1)\tau)^2\,\theta_{ab}((k+2)\tau)\,\theta_{ab}(k\tau)=\theta_{ab}((k+1)\tau)^2\,\theta_{11}((k+2)\tau)\,\theta_{11}(k\tau).$$

By [17, (1)–(4), p. 76], this is equivalent to

$$\begin{aligned}\theta_{ab}((k+1)\tau)^2\,\theta_{11}((k+1)\tau)^2\,&[\theta_{ab}(\tau)^2\,\theta_{01}(0)^2-\theta_{01}(\tau)^2\,\theta_{ab}(0)^2]\\ =\theta_{11}(\tau)^2\,&[(-1)^{a+b+1}\,\theta_{11}((k+1)\tau)^4\,\theta_{01}(0)^2\\ &-\theta_{ab}((k+1)\tau)^2\,\theta_{01}((k+1)\tau)^2\,\theta_{ab}(0)^2].\end{aligned}$$

The next part of the proof shows that for each $ab\in\{00, 01, 10\}$ this implies that τ is of finite order. Since this is not the case, we conclude that $ab=11$.

Suppose that $ab=01$. Then $\theta_{11}((k+1)\tau)^4-\theta_{01}((k+1)\tau)^4=0$; by [10, (A2), p. 22] $\theta_{01}(2(k+1)\tau)=0$ whence τ is of finite order.

Suppose that $ab=10$. By [10, (A10), p. 22] we have $\theta_{10}(\tau)^2\,\theta_{01}(0)^2 - \theta_{01}(\tau)^2\,\theta_{10}(0)^2 = -\theta_{11}(\tau)^2\,\theta_{00}(0)^2$. Therefore

$$\begin{aligned}&\theta_{10}((k+1)\tau)^2\,\theta_{11}((k+1)\tau)^2\,\theta_{00}(0)^2\\&\quad= -\theta_{11}((k+1)\tau)^4\,\theta_{01}(0)^2 + \theta_{10}((k+1)\tau)^2\,\theta_{01}((k+1)\tau)^2\,\theta_{10}(0)^2.\end{aligned}$$

By [10, (A10), p. 22] $\theta_{11}((k+1)\tau)^2\,\theta_{00}(0)^2 = \theta_{01}((k+1)\tau)^2\,\theta_{10}(0)^2 - \theta_{10}((k+1)\tau)^2\,\theta_{01}(0)^2$. Therefore $\theta_{10}((k+1)\tau)^4 = \theta_{11}((k+1)\tau)^4$. Hence $\theta_{10}(2(k+1)\tau)=0$ by [10, (A3), p. 22]. It follows that τ is of finite order.

Suppose that $ab=00$. A similar argument, using [10, (A1), p. 22] and the identity $\theta_{11}(x+u)\,\theta_{11}(x-u)\,\theta_{10}(0)^2 = \theta_{01}(x)^2\,\theta_{00}(u)^2 - \theta_{00}(x)^2\,\theta_{01}(u)^2$ shows that τ is of finite order.

Thus $ab=11$. It follows that $\lambda^2=1$, so it remains to show that $V_k^{-1} \not\cong V_k$. Suppose to the contrary that $\psi\colon V_k \to V_k^{-1}$ is an A-module isomorphism.

If $f\in V_k$ and $a\in A$, then $\psi(a\cdot f) = a * f = -a\cdot f$. In particular, $a\cdot f = 0 \Leftrightarrow a * f = 0$. Choose $p\in E$ such that $2p\notin \mathbb{Z}\tau$, and let $\mathscr{B} = \{f_0, ..., f_k\}$ be the basis for V_k as in (3.4). Set $q = k\tau - p$ and $z_0 = p + (\frac{1}{2}k - 1)\tau$. Choose u_i, $v_i \in A_1$ such that $\mathscr{V}(u_i, v_i) = l_{p-2i\tau, q+2i\tau}$. Since f_i is the unique element (up to scalar multiples) which is annihilated by u_i and v_i, it follows that $\psi(f_i) = \gamma_i f_i$ for some $0\neq \gamma_i \in \mathbb{C}$.

Let $X\in A_1$ be such that $X(p-(2k+1)\tau)\neq 0$ and $X(q+\tau)=0$. Recall that $\mathscr{Z}(f_i) = \{\pm(z_0 - 2(i+j)\tau) \mid 0\leqslant j\leqslant k-1\}$. By choice of p, it follows that $(X\cdot f_0)(z_0 - 2k\tau)\neq 0$ and $f_i(z_0-2k\tau)=0$ for all $i=1, ..., k$. Hence if $X\cdot f_0 = \sum_{0\leqslant i\leqslant k} \lambda_i f_i$, then $\lambda_0\neq 0$. But $\psi(X\cdot f_0) = X * \psi(f_0) = -X\cdot \gamma_0 f_0$, so $\sum_{0\leqslant i\leqslant k} \gamma_i\lambda_i f_i = -\gamma_0 \sum_{0\leqslant i\leqslant k} \lambda_i f_i$. Hence $(\gamma_i+\gamma_0)\,\lambda_i = 0$ for all i. In particular, $2\gamma_0\lambda_0 = 0$. This contradiction forces us to conclude that $V_k^{-1}\not\cong V_k$. ∎

Remark. The key step in (3.10) shows that $V(\omega+k\tau)^\lambda$ and $V(\omega'+k'\tau)^\mu$ have different central characters if $(\omega, k)\neq(\omega', k')$. It follows that $\operatorname{Ext}^1_A(V(\omega+k\tau)^\lambda, V(\omega'+k'\tau)^\mu) = 0$ if $(\omega, k)\neq(\omega', k')$. This is similar to what occurs for the finite dimensional simple modules over $\mathfrak{gl}(2, \mathbb{C})$.

4. Finite Dimensional Simple Modules as Quotients of Line Modules

This section completes the proof of the main theorem by showing that there are no finite dimensional simple modules other then those found in Section 3. The proof of this depends on some preliminary results which are of independent interest.

The first goal of the preliminary results is to relate a finite dimensional module which is not graded to a graded module. Thus (4.1) proves that a finite dimensional simple A-module is a quotient of a 1-critical graded module, and (4.2) proves that we can find such a graded module which is a quotient of a line module. A rather simple illustration of this phenomenon occurs for the 1-dimensional A-modules: a 1-dimensional A-module is necessarily a quotient of one of the four point modules $M(e_i)$ where $e_i \in \mathcal{S}$.

Proposition 4.4 proves that only "special" line modules can possibly have a 1-critical quotient which is not a point module. Indeed if $|\tau| = \infty$ then $M(p, q)$ can not have such a quotient unless $p + q \in E_2 + d\tau$ for some $d \in \mathbb{N} \cup \{0\}$. By using the modules $V(\omega + k\tau)^{\lambda}$ we can show that all these "special" line modules do have such a quotient, and hence have a non-trivial finite-dimensional simple quotient. Such quotients exist because there is an injective map $M(p - d\tau, q - d\tau)[-d] \to M(p, q)$; in fact, up to scalar multiples there is a unique such map of degree zero, and any finite dimensional simple quotient of $M(p, q)$ is actually a quotient of the cokernel of this map.

The preliminary results in this section can be phrased in the language of "fat" points, a notion introduced by Artin in [1].

Let M and N be graded A-modules. Then M and N are *equivalent* if they contain submodules M' and N' of finite codimension such that $M' \cong N'$ via a graded map of degree zero. If M and N are equivalent, we write $M \sim N$. This is indeed an equivalent relation. The modules M and N are equivalent if and only if they give isomorphic objects in the quotient category $\operatorname{Proj}(A) := \operatorname{GrMod}(A)/\text{tors}$ where $\operatorname{GrMod}(A)$ is the category of finitely generated graded A-modules and morphisms being the A-module maps of degree zero, and tors is the full subcategory consisting of those modules which are finite dimensional. Equivalent modules have the same GK-dimension, and if they are not finite dimensional, they also have the same multiplicity.

A *fat point* is an equivalence class of 1-critical modules of multiplicity >1. Thus the fat points are irreducible objects in $\operatorname{Proj}(A)$. In addition, the point modules give irreducible objects of $\operatorname{Proj}(A)$.

If l_{pq} is a secant line, and N is a 1-critical A-module such that there is a non-zero map $M(p, q) \to N$ of degree zero, then we say that the corresponding fat point is *contained in the line* l_p.

LEMMA 4.1. *Let S be a finite dimensional simple A-module. Then S is a quotient of some* 1*-critical graded module.*

Proof. If S is the trivial module, then S is a quotient of every point module, so the result is true. Henceforth, suppose that S is not trivial. It is clear that S is a quotient of some graded module, namely A itself.

Suppose that S is a quotient of a graded module M of GK-dimension $\leqslant d$. By [8, 6.2.19] there is a filtration $M = M^0 \supset M^1 \supset \cdots \supset M^k = 0$ by graded submodules such that each factor M^i/M^{i+1} is critical; actually [8] does this for non-graded modules, but the same proof will give the result we require. Clearly S must be a quotient of one of these factors. Thus S is a quotient of a *critical* module of GK-dimension $\leqslant d$.

Now choose $d \in \mathbb{N}$ minimal such that S is a quotient of a d-critical graded module M, say. Write $S = M/N$. Such an M exists by the previous paragraph. Suppose that $d \geqslant 2$. Then $\dim(M_n) \to \infty$ as $n \to \infty$. However, $\dim(S) < \infty$ so there exists $0 \neq m \in N \cap M_n$ for some n. Thus S is a quotient of the graded module M/Am which is of GK-dimension $< d$. This contradicts the minimality of d, so we conclude that $d \leqslant 1$. Since the only 0-critical module is the trivial module, it follows that $d = 1$. Thus S is a quotient of a 1-critical graded module.

Remarks. 1. We will make frequent use of the observation that if S is a simple quotient of a 1-critical graded module N, then S is also a quotient of every non-zero submodule of N.

2. One can be rather more explicit about the 1-critical graded module N which maps onto S. Define a new A-module $\tilde{S} := S \otimes \mathbb{C}[t]$ with $a \in A_n$ acting by $a \cdot (s \otimes t^i) = (a \cdot s) \otimes t^{i+n}$. Thus $\tilde{S}$ becomes a graded A-module with degree n component $\tilde{S}_n = S \otimes \mathbb{C}t^n$. Since $\dim S < \infty$, $\tilde{S}$ is finitely generated. For each $\lambda \in \mathbb{C}^\times$, $\tilde{S}(t - \lambda)$ is a submodule, and the quotient is isomorphic to S^λ, the twist of S by $\lambda \in \operatorname{Aut}(A)$. We write $\pi : \tilde{S} \to S$ for the map with $\ker \pi = \tilde{S}(t-1)$.

$\tilde{S}$ has the following universal property. If M is any graded A-module such that $M_n = 0$ for $n < 0$, and $\psi : M \to S$ is an A-module map, then there exists a unique degree 0 map $\bar{\psi} : M \to \tilde{S}$ such that $\psi = \pi \circ \bar{\psi}$. It follows that if S is a quotient of a 1-critical graded module N, then $N[k]$ is isomorphic to a submodule of $\tilde{S}$ for some $k \in \mathbb{Z}$.

3. One can use an argument like that in (4.1) to show that if S is a finite dimensional non-trivial simple A-module, then there exists a homogeneous prime ideal P such that S is an A/P-module and $d(A/P) = 1$. In fact, P is the annihilator of any 1-critical module which maps onto S.

4. If $|\tau| = \infty$, then $B = A/\langle \Omega_1, \Omega_2 \rangle$ has no non-trivial finite dimensional simple modules. To see this, suppose that S were such a module. By (4.1) S is a quotient of a 1-critical B-module. However, by [5] such a B-module is equivalent to a point module. By [7, 5.8] the only point modules having a non-trivial simple quotient are the modules $M(e_i)$ where $e_i \in \mathscr{S}$. However, none of these is a B-module.

THEOREM 4.2. *Let N be a 1-critical graded A-module. Then N contains a non-zero graded submodule which is a quotient of a line module.*

Proof. Let $P = \mathrm{Ann}(N)$. Since N is critical, P is a prime ideal. If $0 \neq m \in N$ then $\mathbb{C}[\Omega_1, \Omega_2] \cdot m$ is of GK-dimension 1, so $P \cap \mathbb{C}[\Omega_1, \Omega_2] \neq 0$. This is a homogeneous prime ideal of $\mathbb{C}[\Omega_1, \Omega_2]$ so it must contain some $0 \neq \Omega \in \mathbb{C}\Omega_1 \oplus \mathbb{C}\Omega_2 = Z_2$. Thus $\Omega \cdot N = 0$.

Set $e = e(N)$. There exists n such that $\dim(N_k) = e$ for all $k \geqslant n$.

By [7, 6.12] there is a line module, M say, such that $\Omega \cdot M = 0$. Let $U \subset A_1$ be the 2-dimensional subspace such that $M \cong A/AU$. The homogeneous polynomial function $U \to \mathrm{Hom}_{\mathbb{C}}(N_k, N_{k+1}) \cong \mathrm{End}_{\mathbb{C}}(\mathbb{C}^e) \xrightarrow{\det} \mathbb{C}$ has a non-trivial zero so there exists $0 \neq u \in U$ and $0 \neq m \in N_k$ such that $u \cdot m = 0$. But $\Omega \cdot m = 0$ also, so there is a non-zero map $\varphi: A/Au + A\Omega \to N$. Consider the diagram

$$\begin{array}{c} 0 \to L \to A/Au + A\Omega \to M \to 0 \\ \qquad\quad\downarrow \\ \qquad\quad N \end{array}$$

where the sequence is exact.

By [7, 6.2] $A/\langle \Omega \rangle$ is a domain. Since A is a domain $H_{A/\langle \Omega \rangle}(t) = (1-t^2)(1-t)^{-4}$, and $H_{A/Au + A\Omega}(t) = (1-t)(1-t^2)(1-t)^{-4}$. It follows that $H_L(t) = t(1-t)^{-2}$. But L is cyclic, so L is a shifted line module.

If $L \subset \ker(\varphi)$, then there is an induced map $\bar{\varphi}: M \to N$ which is non-zero. If $L \not\subset \ker(\varphi)$, then $\varphi(L) \neq 0$, so in either case N contains a non-zero submodule which is a quotient of a line module. ▮

COROLLARY 4.3. *If S is a finite dimensional simple A-module, then S is a quotient of a 1-critical graded module which is a quotient of a line module.*

Proof. By (4.1) and (4.2), S is a quotient of some 1-critical graded module N which contains a graded submodule N' such that N' is a quotient of a line module. Since $\dim(N/N') < \infty$ all its composition factors are isomorphic to the trivial module. But S must be a quotient of either N/N' or N' so we conclude that S is a quotient of N'. ▮

PROPOSITION 4.4. *If N is a 1-critical graded quotient of $M(p, q)$ of multiplicity d, then there is an exact sequence*

$$0 \to M(l')[-d] \to M(p, q) \to N \to 0$$

where $M(l')[-d]$ is a shifted line module. Furthermore, either

(a) *N is a point module (equivalently $d = 1$), or*

(b) *$d \geqslant 2$, and $M(l') \cong M(p - d\tau, q - d\tau)$ and either $2d\tau = 0$ or $p + q \in E_2 + (d-1)\tau$.*

Proof. Apply $\mathrm{Hom}_A(-, A)$ to the short exact sequence $0 \to K \to M(p, q) \to N \to 0$ and take cohomology. Since $d(N) = 1$, we have $E^1(N) = E^2(N) = 0$. Since the socle of N is zero, $E^4(N) = 0$ by [7, 2.1e]. Since $M(p, q)$ is Cohen–Macaulay, $E^i(M(p, q)) = 0$ for $i \neq 2$. Thus $E^i(K) = 0$ for $i \neq 2$, so K is also Cohen–Macaulay of GK-dimension 2. Furthermore, $e(K) = 1$ since $e(M(p, q)) = 1$, so [7, Theorem 2.2] implies that K is also a shifted line module. If K is generated in degree k then $H_N(t) = (1 - t^k)(1 - t)^{-2}$ and $e(N) = k$. However, $e(N) = d$ by hypothesis so K is generated in degree d, and $K \cong M(l')[-d]$. Examination of the Hilbert series shows that $d = 1$ if and only if N is a point module.

Now let $S = l_{pq} \cap E$ be the scheme theoretic intersection, and let $M(S)$ be the point module with values in S as defined in [3, Sect. 3]. In the terminology of [5], $M(S) = (\Gamma^*(\mathcal{O}_S))_{\geqslant 0}$. In particular, $M(S)$ is a B-module. As in [4, 6.24] there is an A-module map $\psi: M(p, q) \to M(S)$ which has finite dimensional cokernel. Consider the diagram:

$$\begin{array}{ccc} 0 \to M(l')[-d] \cong K \to & M(p, q) & \to N \to 0 \\ & \downarrow{\scriptstyle\psi} & \\ & M(S) & \to M(p) \to 0. \end{array}$$

Suppose that $\mathrm{Ker}\, \psi \subset K$. Then N is isomorphic to a subquotient of $M(S)$ so is a B-module. Since every 1-critical B-module is a point module [5], this gives alternative (a).

Suppose that $\ker(\psi) \not\subset K$. Then $M(p, q)/\ker(\psi) + K$ is finite dimensional since N is 1-critical. Thus $\psi(K)$ is of finite codimension in $M(S)$. Shifting, this gives a map $M(l') \to M(S^{\sigma^{-d}})$ with finite dimensional cokernel. Hence by [4, 6.23] $l' \cap E = S^{\sigma^{-d}}$ scheme theoretically. Since $\sigma^{-d}(p) = p - d\tau$, it follows that $l' = l_{p - d\tau, q - d\tau}$. Furthermore, since $M(p - d\tau, q - d\tau)$ embeds in $M(p, q)$, it follows that $\Omega(p + q) = \Omega(p + q - 2d\tau)$. Hence by [6, 6.9] either $2d\tau = 0$ or $(p + q) + (p + q - 2d\tau) = -2\tau$. This completes the proof of (b). ∎

Remark. Since line modules are Cohen–Macaulay modules, it follows that the module N in (4.4) is Cohen–Macaulay. By (4.2) it follows that every fat point has a representative which is a Cohen–Macaulay module.

The result in (4.4) allows us to give a more precise version of (4.3).

LEMMA 4.5. *Suppose that* $|\tau| = \infty$. *Let* $\omega \in E_2$, $k \in \mathbb{N} \cup \{0\}$ *and suppose that* $p, q \in E$ *satisfy* $p + q = \omega + k\tau$. *Let* S *be a non-trivial finite dimensional simple quotient of* $M(p, q)$. *Then* S *is a quotient of a* 1-*critical graded module which is a quotient of* $M(p, q)$.

Proof. Let N be a 1-critical module mapping onto S. Set $\Omega = \Omega(p+q)$. Thus $\Omega \cdot S = 0$, and $\Omega \cdot N = 0$ by Remark (2) after (4.1). Let $U \subset A_1$ be such that $M(p, q) \cong A/AU$. The argument in the proof of (4.2) shows that for some $0 \neq u \in U$ there is a diagram

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r] & A/Au + A\Omega \arrow[r] \arrow[d, "\psi"] & M(p,q) \arrow[r] & 0 \\ & & N & & \end{tikzcd}$$

where the sequence is exact, and L is a line module.

We now determine which line module L is. Choose $v \in A_1$ such that $\mathscr{V}(u, v) = l_{pq}$. Then $L = A \cdot \bar{v}$, so $L \cong A/Aa + Aa'$ where $a, a' \in A_1$ are such that av, $a'v \in A_1 u + \mathbb{C}\Omega$. By [7, 4.2], $\mathscr{V}(a, a') = l_{p'-\tau, q'-\tau}$ where $(u)_0 = p + q + p' + q'$. Thus $L \cong M(p' - \tau, q' - \tau)$.

Suppose that $\psi(L) \neq 0$; thus (4.4) applies to $L \to \psi(L) \to 0$. Since $(p' - \tau) + (q' - \tau) = \omega - (k+2)\tau$, it follows from (4.4) that $\psi(L)$ is a point module. Since S is a quotient of $\psi(L)$, it follows from [7, 5.8d] that $\psi(L) \cong M(e_i)$ for some $e_i \in \mathscr{S}$. This implies that $(p' - \tau) + (q' - \tau) = \omega_i \in E_2$. This is impossible since $k \geqslant 0$, so we conclude that $\psi(L) = 0$. Hence there is an induced map $M(p, q) \to N$, and S is a quotient of this image. ∎

LEMMA 4.6. *Let L_1 and L_2 be submodules of $M(p, q)$ which are shifted modules. Then $L_1 \cap L_2$ is also a shifted line module.*

Proof. Consider the long exact sequence obtained by applying $\mathrm{Hom}_A(-, A)$ to the sequence $0 \to L_1 \cap L_2 \to L_1 \oplus L_2 \to L_1 + L_2 \to 0$. Because $d(L_1 \cap L_2) = 2$, it follows that $E^0(L_1 \cap L_2) = E^1(L_1 \cap L_2) = 0$. Because $\mathrm{socle}(L_1 \cap L_2) = 0$, we have $E^4(L_1 \cap L_2) = 0$. Similarly $E^4(L_1 + L_2) = 0$. Since both L_1 and L_2 are Cohen–Macaulay, so is $L_1 \oplus L_2$. Hence $E^3(L_1 \oplus L_2) = 0$. It follows that $E^3(L_1 \cap L_2) = 0$, whence $L_1 \cap L_2$ is Cohen–Macaulay. But $e(L_1 \cap L_2) = 1$, so by [7, 2.2] $L_1 \cap L_2$ is a shifted line module. ∎

PROPOSITION 4.7. *Suppose that $|\tau| = \infty$. Let $k \in \mathbb{N}$, and let $\omega \in E_2$. If $p + q = \omega + (k-1)\tau$ then*

(a) *$M(p, q)$ has a non-trivial finite dimensional simple quotient (this also holds if $|\tau| < \infty$);*

(b) $\dim_{\mathbb{C}} \mathrm{Hom}_A(M(p - k\tau, q - k\tau)[-k], M(p, q)) = 1$; *we will write $K(p, q)$ for the unique submodule of $M(p, q)$ which is isomorphic to $M(p - k\tau, q - k\tau)[-k]$;*

(c) *$M(p, q)/K(p, q)$ is a 1-critical module; it is a point module if and only if $k=1$, in which case it is isomorphic to some $M(e_i)$;*

(d) *if S is any non-trivial finite dimensional simple quotient of $M(p, q)$, then S is a quotient of $M(p, q)/K(p, q)$.*

(e) *$M(p, q)/K(p, q)$ is not a B-module.*

Proof. Write $M = M(p, q)$.

(a) This is already proved in (3.8) for $|\tau| = \infty$. If $|\tau| < \infty$ then by the remarks after (3.4) and (3.5), there exists $0 \neq f \in V(\omega + (k-1)\tau)$ such that Af is a quotient of M. By (3.8) there exists $0 \neq \Omega \in Z_2$ and $0 \neq \nu \in \mathbb{C}$ such that $(\Omega - \nu) \cdot Af = 0$. Thus no simple quotient of Af can be the trivial module, so M has a non-trivial simple quotient, S say.

(b) By (4.5) S is a quotient of a 1-critical graded module N which is a quotient of $M(p, q)$. If N is a point module, then $N \cong M(e_i)$ and $p+q = \omega_i$, whence $k=1$ and $N \cong M(p, q)/M(p-\tau, q-\tau)[-1]$ by [7, 5.7]. If N is not a point module, then (4.4) shows that $N \cong M(p, q)/M(p-d\tau, q-d\tau)[-d]$ and $p+q \in E_2 + (d-1)\tau$. It follows that $2(d-1)\tau = 2(k-1)\tau$, whence $d=k$.

In either case there exists $0 \neq \delta \in \operatorname{Hom}_A(M(p-k\tau, q-k\tau)[-k], M(p, q))$ and any non-trivial finite dimensional simple quotient of M is actually a quotient of $M/\operatorname{Im}(\delta)$. Furthermore, it is implicit in the above that $M/\operatorname{Im}(\delta)$ is 1-critical, and is a point module only when $k=1$. This proves (c) and (d), and we now complete the proof of (b).

Set $L_1 = \operatorname{Im}(\delta)$. Suppose that L_2 is another submodule of M which is isomorphic to $M(p-k\tau, q-k\tau)[-k]$. Notice that $H_{M/L_1}(t) = H_{M/L_2}(t)$. By (4.6) $L_1 \cap L_2$ is a shifted line module, and we may consider the inclusion $L_1 \cap L_2 \subset L_2$. There are two consequences of the fact that M/L_1 is 1-critical. Firstly $L_2/L_1 \cap L_2$ is 1-critical, and secondly $e(L_2/L_1 \cap L_2) = e(M/L_1)$. In particular, M/L_1 is a point module if and only if $L_2/L_1 \cap L_2$ is a point module.

Suppose that $k=1$. Then $M/L_1 \cong M(e_i)$ for some $e_i \in \mathcal{S}$ by the earlier part of the proof. Hence M/L_2 is also a point module. By [7, Sect. 5] there are only three possibilities for M/L_2, and since $L_2 \cong M(p-\tau, q-\tau)[-1]$, it follows that $M/L_2 \cong M(e_i)$ and $p+q = \omega_i$ for some i. Similarly $M/L_1 \cong M(e_i)$. However, $\dim_{\mathbb{C}} \operatorname{Hom}_A(M(p, q), M(e_i)) = 1$, so $L_1 = L_2$.

Suppose that $k > 1$. Then $L_2/L_1 \cap L_2$ is not a point module, so (4.4b) applies to $L_2/L_1 \cap L_2$ as a quotient of L_2. Therefore $(p-k\tau)+(q-k\tau) \in E_2 + (d-1)\tau$ for some $d \in \mathbb{N}$. Since $p+q \in E_2 + (k-1)\tau$, it follows that $2(d+k)\tau = 0$. This is impossible, so we conclude that no such L_2 exists.

(e) By [5, Theorem 1.3] the only 1-critical B-modules are the point modules $M(p)$ where $p \in E$. ▮

The Proof of the Main Theorem. We must prove that the only finite dimensional simple A-modules are the trivial module, and the various $V(\omega+k\tau)^{\lambda}$.

Let S be a finite dimensional simple A-module. By (4.3) and (4.4) there exist $p, q \in E$, $\omega \in E_2$, and $k \in \mathbb{N} \cup \{0\}$ such that S is a quotient of $M(p, q)$ and $p+q=\omega+k\tau$. By (4.7d) S is actually a quotient of $M(p, q)/K(p, q)$. Let $\Omega \in Z_2$ be such that $\Omega \cdot M(p, q) \neq 0$. Since S is simple, there exists $\nu \in \mathbb{C}$ such that $(\Omega-\nu)\cdot S=0$. Hence S is a quotient of $M(p, q)/K(p, q)+(\Omega-\nu)\, M(p, q)$.

By (3.8) and (4.7d) $V(\omega+k\tau)^{\lambda}$ is a quotient of $M(p, q)/K(p, q)$ for all λ. Since $\Omega(p+q)\cdot V(\omega+k\tau)=0$ and $V(\omega+k\tau)$ is not a B-module, we have $\Omega \cdot V(\omega+k\tau)\neq 0$. Hence $(\Omega-\mu)\cdot V(\omega+k\tau)=0$ for some $0\neq\mu\in\mathbb{C}$. If $\lambda^2=\nu\mu^{-1}$ then $(\Omega-\nu)\cdot V(\omega+k\tau)^{\pm\lambda}=0$. In particular, there exists $\lambda\in\mathbb{C}^{\times}$ such that both $V(\omega+k\tau)^{\lambda}$ and $V(\omega+k\tau)^{-\lambda}$ are quotients of $M(p, q)/K(p, q)+(\Omega-\nu)\, M(p, q)$.

Since $M(p, q)/K(p, q)$ is a 1-critical module of multiplicity $k+1$ which is annihilated by Ω, it follows that as a $\mathbb{C}[\Omega]$-module it is free of rank $2(k+1)$ (since $\Omega \in A_2$). Therefore $\dim(M(p, q)/K(p, q)+(\Omega-\nu)\, M(p, q))=2(k+1)$. Since the non-isomorphic $(k+1)$-dimensional simple modules $V(\omega+k\tau)^{\lambda}$ and $V(\omega+k\tau)^{-\lambda}$ are both quotients of this module, it follows that $M(p, q)/K(p, q)+(\Omega-\nu)\, M(p, q)\cong V(\omega+k\tau)^{\lambda}\oplus V(\omega+k\tau)^{-\lambda}$. Thus S is isomorphic to either $V(\omega+k\tau)^{\lambda}$ or $V(\omega+k\tau)^{-\lambda}$. ∎

Remark. Fix $\omega\in E_2$ and $k\in\mathbb{N}\cup\{0\}$. Consider the lines $\{l_{pq} \mid p+q=\omega+k\tau\}$. These lines all lie on a common quadric by [7, 3.11]. If $k=0$ this quadric has a unique singular point, and all these lines pass through this point; if $\omega=\omega_i$ then this singular point is $e_i\in\mathscr{S}$ and $M(e_i)$ has the 1-dimensional quotient modules $V(\omega_i)^{\lambda}$. In some sense the singularity is being recognised by these finite dimensional simple modules (or vice versa). Now suppose that $k\neq 0$. Then the quadric is smooth, and the lines l_{pq} never intersect one another by [7, 3.10c]. However, $V(\omega+k\tau)^{\lambda}$ is a quotient of all the line modules $M(p, q)$. If N is a 1-critical graded A-module mapping onto $V(\omega+k\tau)$ then (by the proof of (4.5)) there is a non-zero map $M(p, q)\to N$. Thus, in the terminology of [1], the fat point N is contained in all the lines l_{pq}. Thus from the algebraic point of view these lines behave as if they were on a singular quadric, with the singular point being created by the existence of the simple module $V(\omega+k\tau)$. This is reminiscent of the situation for semisimple Lie algebras where the finite dimensional simple modules are associated to the singular point $\{0\}$ of the nilpotent cone.

The results in this section classify all the fat point for A when $|\tau|=\infty$. They are precisely the quotient $M(p, q)/K(p, q)$ given in (4.7c) where $p+q=\omega+k\tau$ for some $\omega\in E_2$ and $k\in\mathbb{N}$. This fat point has multiplicity $k+1$.

5. Classification of the Primitive Ideals

As a consequence of the main theorem we classify all the primitive ideals in A.

Before doing this, recall that if $|\tau| = \infty$, then the only finite dimensional simple B-module is the trivial module. This follows from [5] and Lemma 4.1 (see Remark 4 after (4.1)). It also follows from the Main Theorem and the fact that none of the V_k^φ is annihilated by both Ω_1 and Ω_2 (3.9).

For each $\nu_1, \nu_2 \in \mathbb{C}$ define $J(\nu_1, \nu_2) = \langle \Omega_1 - \nu_1, \Omega_2 - \nu_2 \rangle$.

THEOREM 5.1. *Suppose that $|\tau| = \infty$. The primitive ideals in A consist of the ideals $J(\nu_1, \nu_2)$ where $\nu_1, \nu_2 \in \mathbb{C}$, the annihilators of the modules $V(\omega + k\tau)^\lambda$, and the augmentation ideal. The completely prime primitive ideals are all the $J(\nu_1, \nu_2)$, and also the annihilators of the 1-dimensional modules, namely $A(x_i - \mu) + Ax_j + Ax_k + Ax_l$ where $\{i, j, k, l\} = \{0, 1, 2, 3\}$ and $\mu \in \mathbb{C}$.*

Proof. A primitive ideal of finite codimension in A is the annihilator of a finite dimensional simple module, so is either the augmentation ideal or is of the form Ann $V(\omega + k\tau)^\lambda$. Furthermore, the 1-dimensional simple modules are the quotients of point modules $M(e_i)$. So it only remains to prove that a primitive ideal of infinite codimension is one of the $J(\nu_1, \nu_2)$, and the quotient by this ideal is a domain. The latter is proved as follows. If A is made into a filtered algebra by defining $F^iA := \bigoplus_{j \leqslant i} A_j$, then A is its own associated graded algebra, and the associated graded ideal of $J(\nu_1, \nu_2)$ is the ideal $J(0, 0)$. Since $A/J(0, 0) = B$ is a domain, $A/J(\nu_1, \nu_2)$ is also a domain by [8, 1.6.6].

Suppose that J is a primitive ideal of infinite codimension. Since A is a noetherian algebra of countable dimension over an uncountable field, J meets the center of A in a maximal ideal. Hence J contains some $J(\nu_1, \nu_2)$. Now $A/J(\nu_1, \nu_2)$ is a domain of GK-dimension 2, so any proper factor is of GK-dimension at most 1. However, as is well-known (see e.g. [15, 3.2]) an algebra such as A cannot have a primitive quotient of GK-dimension 1. Therefore $J = J(\nu_1, \nu_2)$.

It remains to show that each $J(\nu_1, \nu_2)$ is a primitive ideal. Recall that every prime ideal of A is an intersection of primitive ideals (see [8, Sect. 1]). Hence if $J(\nu_1, \nu_2)$ is not primitive, then it must be contained in infinitely many primitive ideals which are necessarily of finite codimension in A. Thus $J(\nu_1, \nu_2)$ annihilates infinitely many of the $V(\omega + k\tau)^\lambda$. It is an easy consequence of (3.9) that this is impossible. ∎

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