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Krull dimension of factor rings of the enveloping algebra of a semi-simple Lie algebra

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Abstract

Let $\mathfrak{g}$ be a semi-simple complex Lie algebra with enveloping algebra $U(\mathfrak{g})$. It is shown that the Krull dimension of $U(\mathfrak{g})$ is bounded above by $\dim \mathfrak{g} - r$, where r is half the minimal dimension of a non-trivial G orbit in $\mathfrak{g}^*$ (G is the adjoint group of $\mathfrak{g}$).

1. *Introduction*

1·1. Let $\mathfrak{g}$ be a finite-dimensional semi-simple Lie algebra over the field $k = \mathbb{C}$. Let $U = U(\mathfrak{g})$ denote the enveloping algebra of $\mathfrak{g}$. This paper is motivated by the problem of determining the Krull dimension (in the sense of Rentschler and Gabriel (16)) of U. The problem is not solved but progress is made. Obvious lower and upper bounds for the Krull dimension are $\dim \mathbf{b}$ ($\mathbf{b}$ is a Borel sub-algebra of $\mathfrak{g}$) and $\dim \mathfrak{g}$; it is conjectured (in (15) and (20)) that the correct value for $K-\dim U(\mathfrak{g})$ is $\dim \mathbf{b}$. The problem is to obtain a better upper bound. We are able to obtain an upper bound on the Krull dimension of any factor ring of U which improves on the previously known bound. For example, we establish an upper bound of 6 for $K-\dim U(sl(3))$ compared with the previous bound of 8.

1·2. The adjoint group G of $\mathfrak{g}$ acts on the dual space $\mathfrak{g}^*$. For $x \in \mathfrak{g}^*$ let $\dim Gx$ denote the dimension of the orbit $G\cdot x$; put $r = \frac{1}{2}\inf\{\dim G\cdot x \mid x \in \mathfrak{g}\backslash\{0\}\}$. That is, $2r$ is the minimal dimension of a non-trivial G-orbit in $\mathfrak{g}^*$.

1·3. The techniques employed depend on finding the relationship between the Gelfand–Kirillov dimension (denoted $GK-\dim$ or GK) and the Krull dimension of certain U-modules. Specifically, we prove that

(*a*) If V is an irreducible representation of $\mathfrak{g}$ then either V is finite-dimensional (that is, $GK(V) = 0$) or $GK(V) \geqslant r$;

(*b*) If V is a finitely generated non-artinian U-module (that is, $K-\dim V \geqslant 1$) then $GK(V) \geqslant r+1$.

The result (*b*) enables us to show that, if R is an infinite-dimensional (k-vector space) factor ring of U, then $K-\dim R \leqslant GK(R) - r$. In particular $K-\dim U \leqslant \dim \mathfrak{g} - r$.

1·4. The results 1·3(*a*) and (*b*) are similar to Bernstein's theorem (1), that a simple module over the nth Weyl algebra A_n has $GK-\dim$ at least n. From this fact it is possible to determine the Krull dimension of A_n ((9) or (18)). The same idea is used in this paper to get from results about GK-dimension to results about Krull dimension.

1·5. In an earlier paper (18) it was established that $K-\dim U(sl(2)) = 2$. Subsequently, Levasseur (15), answering a question in the author's thesis (20), established that, when $\mathfrak{g}$ is a sum of n copies of $sl(2)$ the Krull dimension of $U(\mathfrak{g})$ is $2n$. These are the only semi-simple Lie algebras for which the Krull dimension of the enveloping algebra is known.

The result that $K\dim U(\mathfrak{g}) \leqslant \dim \mathfrak{g} - r$ appeared in the author's thesis, and has been obtained independently by Levasseur (14) using different techniques to those employed here. Gabber (6) has also pointed out that this upper bound follows from (5).

1·6. When $\mathfrak{g}$ is simple, the integer r defined in § 1·2 has another interpretation which will be more useful to us; r is the largest integer such that $\mathfrak{g}$ contains a Heisenberg subalgebra of dimension $2r-1$, and as such its values have been tabulated by Joseph (10). He uses the notation $k(\mathfrak{g})$ for r. If $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_n$ is a decomposition of $\mathfrak{g}$ into simple components then $r(\mathfrak{g}) = \min k(\mathfrak{g}_i)$. So that the reader can easily read off the upper bound which we obtain, we give for each simple Lie algebra the values of r. This table (Table 1) is taken from (10), with the two corrections noted in (11).

Table 1

Cartan label	$\dim \mathfrak{g}$	$\dim \mathbf{b}$	r	$\dim \mathfrak{g} - r$
A_n	$(n+1)^2-1$	$\frac{1}{2}n(n+3)$	n	n^2+n
B_n	$2n^2+n$	$n(n+1)$	$2n-2$	$2n^2-n+2$
C_n	$2n^2+n$	$n(n+1)$	n	$2n^2$
D_n	$2n^2-n$	n^2	$2n-3$	$2n^2-3n+3$
E_6	78	42	11	67
E_7	133	70	17	116
E_8	248	128	29	219
F_4	52	28	8	44
G_2	14	8	3	11

1·7. The author is grateful to his supervisor, J. C. McConnell, for bringing these questions to his attention, and more especially for the patience and insight he has shown in the course of many helpful conversations.

2. *Preliminaries and notation*

2·1. For a Cartan sub-algebra $\mathbf{h}$ of $\mathfrak{g}$, fix a root system Φ, with Φ^+ the set of positive roots; let δ be the half-sum of positive roots. For each root α let X_α denote the element of the Chevalley basis for $\mathfrak{g}$ of weight α. For $\lambda \in \mathbf{h}^*$, $M(\lambda)$ is the Verma module of highest weight $\lambda-\delta$, $L(\lambda)$ is the unique simple factor module of $M(\lambda)$, χ_λ is the central character of $M(\lambda)$. The Casimir central element of $U(\mathfrak{g})$ is denoted by Ω. The set of strongly dominant weights is denoted by Λ^{++}.

2·2. We require a number of results on Gelfand–Kirillov dimension for factor rings of enveloping algebras. We briefly list some of those we require; for details we refer the reader to (18), (19) and (20).

If R is a factor ring of an enveloping algebra and M is a finitely generated R-module, then $q(M)$ denotes the Hilbert polynomial, $d(M)$ the Gelfand–Kirillov dimension and $e(M) = d(M)! \times$ (leading coefficient of $q(M)$). When we want to be careful about the ring over which the module is defined we shall use the suffix R, viz. q_R, d_R, e_R. Denote the annihilator of an R-module by $\operatorname{ann} M$.

LEMMA 2.2.1 *Suppose that every finitely generated R-module M' with $K-\dim M' = \alpha$ satisfies $d(M') \geqslant \alpha + r(\alpha, r \in \mathbb{N})$. Let M be a finitely generated R-module with $K-\dim M \geqslant \alpha$. Then $d(M) \geqslant K-\dim M + r$.*

LEMMA 2.2.2 *If M is a finitely generated R-module with $d(M) = d$, and $M = M_0 \supset M_1 \supset \ldots$ is a chain of submodules satisfying $d(M_i/M_{i+1}) = d$, then the chain has at most $e(M)$ terms.*

LEMMA 2.2.3 *Let R have a subring S which is also a factor of an enveloping algebra. Let V and W be finite-dimensional generating subspaces of R and S respectively such that $W \subset V^t$ and the filtrations generated by the powers of W and V give commutative associated graded algebras. Let M be a finitely generated R-module and N a finitely generated S-submodule of M. Then*

(i) $d_S(N) \leqslant d_R(M)$.

(ii) *If $d = d_S(N) = d_R(M)$ then $e_S(N) \leqslant t^d e_R(M)$.*

LEMMA 2.2.4 ((19), § 3·1). *Let R be a domain and a factor of the enveloping algebra of a finite-dimensional Lie algebra over k. Let E denote the centre of R, K the quotient field of E, and R_E the localisation of R at E. Suppose that $R_E \cong A_m(K)$, the mth Weyl algebra over K. If M is an R-module satisfying $d(M) < m + \mathrm{tr}\cdot\deg_k K$ then $E \cap \operatorname{ann} M \neq 0$.*

2·3. The reader is referred to (7) for basic results on Krull dimension. We remind those unfamiliar with Krull dimension that a 1-critical module is a non-artinian module, every proper factor of which is artinian.

3. *Simple Lie algebras*

3·1. Throughout this section $\mathfrak{g}$ will denote a simple Lie algebra. We will establish the main results of the paper for $\mathfrak{g}$ simple in this section, and in Section 4 deal with $\mathfrak{g}$ semi-simple.

The starting-point is to note that, given a long root β, then there is a Heisenberg sub-algebra $\mathbf{a}$ of $\mathfrak{g}$ such that $\dim \mathbf{a} = 2r-1$ and the centre of $\mathbf{a}$ is generated by X_β ((11), corollary 2·3).

3·2. LEMMA. *Let I be a left ideal of U, and let β be a long root.*

(1) *If $d(U/I) < r$ then $I \cap k[X_\beta] \neq 0$.*

(2) *If $d(U/I) \leqslant r$ then $I \cap k[X_\beta, \Omega] \neq 0$.*

Proof. Pick a Heisenberg sub-algebra $\mathbf{a}$ of $\mathfrak{g}$ as in § 3·1.

(1) Put $R = U(\mathbf{a}) \subset U$. R may be localized at its central sub-algebra $k[X_\beta]$, the localization being isomorphic to the Weyl algebra $A_{r-1}(k(X_\beta))$. Looking at U/I as an R-module it has a submodule isomorphic to R/J where $J = R \cap I$. As $d(U/I) < r$, so too is $d_R(R/J) < r$. Hence by Lemma 2·2·4 $\operatorname{ann}_R(R/J) \cap k[X_\beta] \neq 0$; it follows that $I \cap k[X_\beta] \neq 0$.

(2) Put $R = U(\mathbf{a})[\Omega]$; R may be localized at its central sub-algebra $k[X_\beta, \Omega]$ (which is of transcendence degree 2), the localized ring being isomorphic to $A_{r-1}(k(X_\beta, \Omega))$. Now apply the same analysis as used for the first part of the lemma.

3·3. Put $A = U(sl(2,k))$ and take as a basis for $sl(2,k)$ the elements e, f, h subject to the relations $[e,f] = h, [h,e] = 2e, [h,f] = -2f$. Put $q = 4ef + h^2 - 2h$ the Casimir central element of A. Let I be a left ideal of A such that $I \cap k[e] \neq 0$ and $I \cap K[f] \neq 0$. It is implicit in ((17), corollaire 1) that, if I contains the ideal $\langle q - \lambda\rangle$ for some scalar $\lambda \in k$, then A/I is finite-dimensional. This result holds for any field k and is easiest to see by passing to the associated graded algebra of $A/\langle q-\lambda\rangle$, which is isomorphic to the commutative ring $k[X, Y, Z]/\langle 4XY - Z^2\rangle$; the hypothesis on I forces $(\mathrm{gr}\, I) \cap k[X] \neq 0$ and $(\mathrm{gr}\, I) \cap k[Y] \neq 0$ (where $\mathrm{gr}\, I$ is the associated graded ideal of $I/\langle q-\lambda\rangle$) and so $\mathrm{gr}\, A/\mathrm{gr}\, I$ is finite-dimensional, hence so is A/I.

Our immediate goal is to prove the following strengthening of Roos' result.

Lemma. *Let I be a left ideal of A such that $I \cap k[e] \neq 0$ and $I \cap k[f] \neq 0$. Then A/I is finite-dimensional.*

Proof. It can be seen by passing to the associated graded algebra, $\mathrm{gr}\, A$, that $d(\mathrm{gr}\, A/\mathrm{gr}\, I) \leqslant 1$, hence $d(A/I) \leqslant 1$. By ((18), lemma 3·2), A/I is artinian. We shall prove by induction on the length of A/I that every composition factor of A/I is finite-dimensional. If A/I is simple then $\langle q-\lambda\rangle \subset I$ for some scalar λ (because q is central, q acts on A/I as an A-module endomorphism, and so, by Quillen's Lemma, as a scalar). It follows from the comments prior to the Lemma that A/I is finite-dimensional in this case. Suppose now that A/I is not simple, and let J be a left ideal containing I such that J/I is simple. The induction hypothesis implies that A/J is finite-dimensional. Pick $\mu \in k$ such that $(q-\mu)J \subset I$, and let θ denote the A-module endomorphism of A/I induced by multiplication by $q-\mu$. Thus J/I is in the kernel of θ, and so the image of θ is a homomorphic image of A/J (certainly finite-dimensional). If $\theta(A/I) = 0$ then $q-\mu \in I$ and A/I is finite-dimensional by Roos. If $\theta(A/I) \neq 0$ then there is a left ideal J' properly containing I such that J'/I is finite-dimensional ($J'/I = \theta(A/I)$). The induction hypothesis applied to A/J' shows that A/J' is finite-dimensional. Hence A/I is finite-dimensional.

3·4. Theorem. *Let V be an irreducible representation of $\mathfrak{g}$. Then either V is finite-dimensional or $d(V) \geqslant r$.*

Proof. Let I be a left ideal of U with $V = U/I$ and $d(U/I) < r$. We show that V is finite-dimensional. By Lemma 3·2, it follows that $I \cap k[X_\beta] \neq 0$ for every long root β. In particular, if β is long then $I \cap k[X_\beta] \neq 0$ and $I \cap k[X_{-\beta}] \neq 0$, so by Lemma 3·3 applied to the sub-algebra of $\mathfrak{g}$ spanned by $X_\beta, X_{-\beta}, H_\beta$ it follows that $I \cap k[H_\beta] \neq 0$. As the set $\{H_\beta \mid \beta \text{ is long}\}$ spans $\mathbf{h}$ it follows that $I \cap k[H] \neq 0$ for every non-zero $H \in \mathbf{h}$.

However, the Cartan sub-algebra $\mathbf{h}$ was arbitrary, and so the same argument shows that if $\mathbf{h}'$ is another Cartan sub-algebra of $\mathfrak{g}$ and $0 \neq H' \in \mathbf{h}'$ then $I \cap k[H'] \neq 0$. In particular, if α is any root and σ denotes the automorphism $\exp(adX_\alpha)$ of $\mathfrak{g}$ then $H_\alpha - 2X_\alpha \in \sigma(\mathbf{h})$, and so $I \cap k[H_\alpha - 2X_\alpha] \neq 0$. This, together with the fact that $I \cap k[H_\alpha] \neq 0$, implies that $I \cap k[X_\alpha] \neq 0$. Hence for every root α, $I \cap k[H_\alpha] \neq 0$ and $I \cap k[X_\alpha] \neq 0$. Because $\mathfrak{g}$ is spanned by the H_α and X_α (by passing to the associated graded algebra of U) it follows that U/I is finite-dimensional.

Since establishing this theorem it has been brought to our attention that the result has been established more generally by Gabber. He has proved that, for any algebraic $\mathfrak{g}$ and any finitely generated $U(\mathfrak{g})$-module M, $d(M) \geqslant \frac{1}{2}d(U(\mathfrak{g})/\mathrm{ann}\, M)$ (for a proof see (13).) That his result implies ours is clear because any primitive factor ring of $U(\mathfrak{g})$ which is not finite-dimensional has Gelfand–Kirillov dimension at least $2r$.

3·5. We now proceed to establish the second of our main results in this section: namely, if M is a finitely generated non-artinian U-module, then $d(M) \geqslant r+1$. This result forms Theorem 3·9 and as explained there it is necessary to examine a 1-critical cyclic U-module, U/I say, with the property that every proper factor module of U/I is finite-dimensional. So let I denote such a left ideal; the next few lemmas give some information about U/I which will be required in the proof of Theorem 3·9.

3·6. LEMMA. *Put* $\Lambda = \{\lambda \in \Lambda^{++} \mid L(\lambda+\delta)$ is a factor of $U/I\}$. *The set* Λ *is infinite.*

Proof. Because $\mathfrak{g}$ is simple every proper factor of U/I, being finite-dimensional, splits into a direct sum of simple modules. Suppose Λ were finite. Then given any infinite chain of submodules $U/I = M_0 \supset M_1 \supset \dots$ there exists $\lambda \in \Lambda$ such that M_j/M_{j+1} is isomorphic to $L(\lambda+\delta)$ for infinitely many j. By using the semi-simplicity of the proper factors of U/I, given any integer n there is a chain of submodules $U/I = M_0 \supset M_1 \dots \supset M_n$, such that $M_j/M_{j+1} \simeq L(\lambda+\delta)$ for all j. But if $P = \operatorname{ann} L(\lambda+\delta)$ then $P(M_0/M_n) = 0$ as M_0/M_n is semi-simple, and it follows that $\dim M_0/M_n \leqslant \dim U/P = (\dim L(\lambda+\delta))^2$, as U/P is simple artinian with simple module $L(\lambda+\delta)$. However, n was arbitrary, so $\dim M_0/M_n$ may be made arbitrarily large. This contradiction shows that Λ must be infinite.

3·7. LEMMA. *Let* $\Lambda \subset \Lambda^{++}$ *be infinite and let* Ω *be the Casimir central element of* U. *Then* $\{\chi_\lambda(\Omega) \mid \lambda \in \Lambda\}$ *is infinite.*

Proof. After ((8), p. 143) $\chi_\lambda(\Omega) = (\lambda,\lambda) + 2(\lambda,\delta)$. Now $(\lambda,\lambda) \geqslant 0$ for any $\lambda \in \mathbf{h}_R^*$, and it is easy to see that (λ,δ) becomes arbitrarily large as λ runs through the infinite set $\Lambda \subset \Lambda^{++}$ (this is easiest to see by writing each $\lambda \in \Lambda$ in terms of the fundamental dominant weights). The result is immediate.

3·8. The reader will notice that the previous two lemmas together imply that $I \cap k[\Omega] = 0$ (or, equivalently, that there are infinitely many non-isomorphic simple factor modules of U/I).

LEMMA. *Let* $\Lambda \subset \Lambda^{++}$ *be infinite. Then there exists a long root* $\beta \in \Phi^+$ *such that* $\{\langle\lambda,\beta\rangle \mid \lambda \in \Lambda\}$ *is unbounded.*

Proof. Given any integer N there certainly exists $\alpha \in \Phi^+$ and some $\lambda \in \Lambda$ such that $\langle\lambda,\alpha\rangle \geqslant 2N$ (this is clear when the elements of Λ are expressed as linear combinations of the fundamental dominant weights). If α is short there exists $\gamma \in \Phi^+$ such that $\alpha+\gamma = \beta$ is a long root. Now $\langle\lambda,\beta\rangle = ((\lambda,\alpha)+(\lambda,\gamma))/2(\beta,\beta) \geqslant \frac{1}{2}\langle\lambda,\alpha\rangle \geqslant N$. So we can certainly find a long root $\beta \in \Phi^+$ such that $\langle\lambda,\beta\rangle \geqslant N$. Hence $\{\langle\lambda,\beta\rangle \mid \lambda \in \Lambda, \beta \in \Phi^+ \text{ is long}\}$ is unbounded. As Φ^+ is a finite set the result follows.

3·9. THEOREM. *Let* M *be a finitely generated* U-*module. If* M *is not artinian then* $d(M) \geqslant r+1$.

Proof. Suppose the result is false – that is, suppose $d(M) \leqslant r$. M has a factor module which is 1-critical, and so it is enough to prove the result when M is 1-critical. Suppose this is the case. If there exists an infinite chain $M = M_0 \supset M_1 \supset \dots$ of submodules of M such that $d(M_j/M_{j+1}) \geqslant r$ for all j then it would follow that $d(M) > r$. Consequently there is a non-zero submodule N of M such that every proper factor of N has GK-dimension strictly less than r. Any proper factor of N is artinian, as N is 1-critical, so by Theorem 3·4 is finite-dimensional. Furthermore, we may suppose that N is cyclic. So M has a 1-critical submodule, U/I say, every proper factor of which is finite-dimensional.

Because $d(U/I) \leqslant r$, it follows from Lemma 3·2 that $I \cap k[X_\beta, \Omega] \neq 0$ for every long root β. Set $\Lambda = \{\lambda \in \Lambda^{++} \mid L(\lambda+\delta)$ is a factor of $U/I\}$. By Lemma 3·6, Λ is an infinite set, and after Lemma 3·8 we may pick a long root $\beta \in \Phi^+$ such that $\{\langle \lambda, \beta\rangle \mid \lambda \in \Lambda\}$ is unbounded. Pick $0 \neq p_+ \in I \cap k[X_\beta, \Omega]$ and $0 \neq p_- \in I \cap k[X_{-\beta}, \Omega]$, and put $N = \max\{\deg p_+, \deg p_-\}$ where this degree is defined by looking at p_+ and p_- as polynomials in X_β and $X_{-\beta}$ respectively with coefficients in $k[\Omega]$. Notice that p_+ and p_- can lie in the ideals $\langle \Omega - \chi_\lambda(\Omega)\rangle$ for at most finitely many $\lambda \in \Lambda$. So by Lemma 3·7, there is a set $\Lambda' \subset \Lambda$ such that if $\lambda \in \Lambda'$ then $p_+ \notin \langle \Omega - \chi_\lambda(\Omega)\rangle$, $p_- \notin \langle \Omega - \chi_\lambda(\Omega)\rangle$ and $\{\langle \lambda, \beta\rangle \mid \lambda \in \Lambda'\}$ is unbounded.

Let $\lambda \in \Lambda'$ and suppose $L(\lambda+\delta) \simeq U/J$, where J is a left ideal containing I. Let A denote the enveloping algebra of the $sl(2)$ sub-algebra of $\mathfrak{g}$ generated by X_β and $X_{-\beta}$. If $\nu \neq 0$ is a highest weight vector of $L(\lambda+\delta)$, let V denote the A-submodule of $L(\lambda+\delta)$ generated by ν. Then V is a simple A-module of dimension $\langle \lambda, \beta\rangle + 1$ (see, for example, ((2), chapter VIII, 7·2)). If L is an A-submodule of U containing J such that $V \simeq U/L$, then it follows from the simplicity of V that $V \simeq A/A \cap L$. Recall that $p_+, p_- \in J$ and also $\langle \Omega - \chi_\lambda(\Omega)\rangle \subset J$. The choice of λ ensures that the elements of q_+ and q_- defined by $q_+ = p_+(X_\beta, \chi_\lambda(\Omega))$ and $q_- = p_-(X_{-\beta}, \chi_\lambda(\Omega))$ are non-zero elements of $J \cap k[X_\beta]$ and $J \cap k[X_{-\beta}]$ respectively. Hence $q_+, q_- \in A \cap L$, and so there is a non-zero element of V which is annihilated by both q_+ and q_-. The same argument as in ((18), lemma 3·2) shows that $\deg q_+ + \deg q_- \geqslant \langle \lambda, \beta\rangle + 1$, and hence $2N \geqslant \langle \lambda, \beta\rangle + 1$. But N is fixed and as λ runs through Λ', $\langle \lambda, \beta\rangle$ becomes arbitrarily large – this contradiction completes the proof.

3·10. Theorems 3·4 and 3·9 are both best possible in the sense that the bounds are actually obtained. First, there does exist an irreducible representation, V, of $\mathfrak{g}$ with $d(V) = r$. To see this, begin with Joseph's result ((11), theorem 4·1) that there is a primitive ideal P of U such that $d(U/P) = 2r$; by Duflo's theorem (4) there is a simple module $L(\lambda)$ such that $P = \operatorname{ann} L(\lambda)$; finally the result due to Joseph (12) that $d(L(\lambda)) = \frac{1}{2} d(U/\operatorname{ann} L(\lambda))$ ensures that $d(L(\lambda)) = r$. We explain in 3·13 why Theorem 3·9 is best possible.

3·11. COROLLARY. *Let $\mathfrak{g}$ be a simple Lie algebra. If R is a non-artinian factor ring of $U(\mathfrak{g})$ then $K-\dim R \leqslant d(R) - r$.*

Proof. This is immediate from Theorem 3·9 and Lemma 2·2·1.

3·12. THEOREM. *Let $\mathfrak{g}$ be simple and let P be a primitive ideal of $U = U(\mathfrak{g})$, with $d(U/P) = 2r$. Then $K-\dim(U/P) = r$.*

Proof. From the above Corollary, $K-\dim(U/P) \leqslant r$. Let $\mathfrak{a}$ be as in §3·1 with centre spanned by X_β for some long root β. If $P \cap U(\mathfrak{a}) = 0$ then we may consider $U(\mathfrak{a})$ as a sub-algebra of U/P, and, because $U(\mathfrak{a})$ has a commutative sub-algebra of transcendence degree r, the generalization of Quillen's lemma (9) completes the proof.

Suppose then that $P \cap U(\mathfrak{a}) \neq 0$. By ((3), 4·4·1) P must have non-zero intersection with $k[X_\beta]$, the centre of $U(\mathfrak{a})$. But if this is true for every long root β then the argument of Theorem 3·4 will show that $d(U/P) = 0$. Hence the theorem.

3·13. To see that Theorem 3·9 is best possible let P be as in Theorem 3·12. Bearing Lemma 2·2·1 in mind, it is clear that some non-artinian factor module M, of U/P, satisfies $d(M) \leqslant r+1$ (and hence $d(M) = r+1$).

4. *Semi-simple Lie algebras*

4·1. We now proceed to extend the results for $\mathfrak{g}$ simple to the semi-simple case. This is just a technicality and is essentially done in the following lemma.

We remark that for $\mathfrak{g}$ semi-simple $r = r(\mathfrak{g})$ is just the minimum of $r(\mathfrak{g}_i)$ where $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_n$ is a decomposition of $\mathfrak{g}$ as a direct sum of simple sub-algebras.

LEMMA. *Let A and B be factor rings of enveloping algebras. Suppose that both A and B satisfy (for some $r \in \mathbb{N}$):*

(1) *If V is a simple module with $d(V) < r$, then V is finite-dimensional.*

(2) *If M is finitely generated non-artinian module, then $d(M) \geqslant r+1$.*

Then the ring $R = A \otimes_k B$ has the same properties.

Proof. (1) Let V be a simple R-module with $d_R(V) < r$. Any finitely generated A-submodule, N say, of V satisfies $d_A(N) < r$ (Lemma 2·3), so is finite-dimensional. Hence V has a simple A-submodule, S say. Because V is simple, $V = (A \otimes B)S = (1 \otimes B)S$, and because the elements of $1 \otimes B$ commute with the elements of $A \otimes 1$, $1 \otimes B$ acts on V as A-endomorphisms. Hence V (as an A-module) is a sum of copies of S; in fact V is a direct sum of copies of S. So if $P = \operatorname{ann}_A(S)$ then $P \otimes B \subset \operatorname{ann}_R(V)$. Similarly, V has a finite-dimensional simple B-submodule, T say, and V is a direct sum of copies of T; so if $Q = \operatorname{ann}_B(T)$ then $A \otimes Q \subset \operatorname{ann}_R(V)$. Thus V becomes a simple module over the ring $\bar{R} = R/(P \otimes B + A \otimes Q) \simeq (A/P) \otimes (B/Q)$. But, A/P and B/Q are finite-dimensional, hence so is $\bar{R}$ and, in particular, V is finite-dimensional.

(2) It is enough to show that if M is a finitely generated 1-critical R-module then $d_R(M) \geqslant r+1$. So suppose that M is finitely generated, 1-critical and that $d_R(M) \leqslant r$. Any finitely generated A-submodule, N, of M must have $d_A(N) \leqslant r$, so is artinian by the hypothesis on A. Hence M has a simple A-submodule S. As M is 1-critical, RS is 1-critical too. Replace M by the module RS; M still satisfies the same hypotheses, but in addition is a direct sum of copies of S as an A-module (for the same reason as in (1) above).

Suppose for the moment that S is not finite-dimensional. Then the only possibility is that $d_A(S) = d_R(M) = r$. Now any finitely generated A-submodule, N, of M is a direct sum of copies of S and is artinian. The length of a composition series for N, $l(N)$, satisfies $l(N) \leqslant e_A(N)$ by Lemma 2·2·1. By Lemma 2·3, $e_A(N)$ is finite and we conclude that M itself, as an A-module, has finite length. This cannot happen though, as then M would be artinian as an R-module. Consequently S must be finite-dimensional.

A similar argument to that used above will show that M has a simple B-submodule, T say, and replacing $M = RS$ by RT (which is still a direct sum of copies of S), this new M has the extra property that it is a direct sum of copies of T. Again we can show that T is finite-dimensional. Now let $P = \operatorname{ann}_A(S)$, $Q = \operatorname{ann}_B(T)$, so $P \otimes B + A \otimes Q \subset \operatorname{ann}_R(M)$ and M becomes a finitely generated module over the finite-dimensional ring $R/(P \otimes B + A \otimes Q)$. This is absurd since $K\text{-}dim\ M = 1$.

4·2. THEOREM. *Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_n$ be a decomposition of the semi-simple Lie algebra $\mathfrak{g}$ as a sum of simple Lie algebras.*

(*a*) *If V is an irreducible representation of $\mathfrak{g}$ then either V is finite-dimensional or $d(V) \geqslant r$;*

(*b*) *If M is a finitely generated non-artinian $U(\mathfrak{g})$-module then $d(M) \geqslant r+1$.*

Proof. This follows from the corresponding results for $\mathfrak{g}$ simple (Theorem 3·4 and 3·9), and by repeatedly applying Lemma 4·1 to $U(\mathfrak{g}) = U(\mathfrak{g}_1) \otimes \ldots \otimes U(\mathfrak{g}_n)$.

4·3. Corollary ($\mathfrak{g}$ *semi-simple*)

(1) *If R is a non-artinian factor ring of $U(\mathfrak{g})$ then K-dim $R \leqslant d(R) - r$.*

(2) *If P is a primitive ideal of $U = U(\mathfrak{g})$ with $d(U/P) = 2r$ then K-dim $U/P = r$.*

Proof. Just as in 3·11 and 3·12 there is a primitive ideal P of $U(g)$ with $d(U/P) = 2r$. Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_n$ be a decomposition of $\mathfrak{g}$ as a sum of simple subalgebras with $r = r(\mathfrak{g}_1)$; pick P_1 a primitive ideal of $U(\mathfrak{g}_1)$ with $d(U(\mathfrak{g}_1)/P_1) = 2r$ (this we can do by the comment in § 3·10), and for $j > 1$ let P_j be the ideal of $U(\mathfrak{g}_j)$ generated by $\mathfrak{g}_j$; if P is the ideal of $U(\mathfrak{g})$ generated by $P_1, P_2, \ldots, P_n$ then $U/P \simeq U(\mathfrak{g}_1)/P_1 \otimes U(\mathfrak{g}_2)/P_2 \otimes \cdots \otimes U(\mathfrak{g}_n)/P_n \simeq U(\mathfrak{g}_1)/P_1$ since $U(\mathfrak{g}_j)/P_j \simeq k$ for $j > 1$.

Both parts of Theorem 4·2 are best possible – the arguments are essentially the same as for $\mathfrak{g}$ simple (§§ 3·10 and 3·13) and the details are omitted.

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