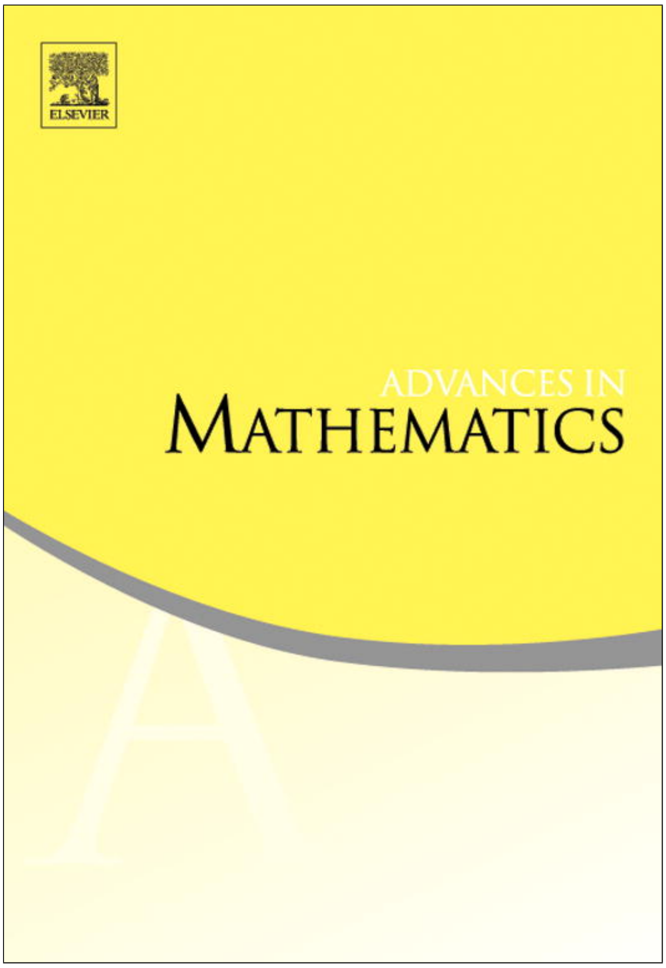

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Advances in Mathematics 230 (2012) 1780–1810
ADVANCES IN Mathematics
www.elsevier.com/locate/aim

Category equivalences involving graded modules over path algebras of quivers ☆

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Received 1 September 2011; accepted 28 March 2012

Communicated by Henning Krause

Abstract

Let Q be a finite quiver with vertex set I and arrow set Q_1, k a field, and kQ its path algebra with its standard grading. This paper proves some category equivalences involving the quotient category $\mathsf{QGr}(kQ) := \mathsf{Gr}(kQ)/\mathsf{Fdim}(kQ)$ of graded kQ-modules modulo those that are the sum of their finite dimensional submodules, namely

$$\mathsf{QGr}(kQ) \equiv \mathsf{Mod}S(Q) \equiv \mathsf{Gr}L(Q^\circ) \equiv \mathsf{Mod}L(Q^\circ)_0 \equiv \mathsf{QGr}(kQ^{(n)}).$$

Here $S(Q) = \varinjlim \operatorname{End}_{kI}(kQ_1^{\otimes n})$ is a direct limit of finite dimensional semisimple algebras; Q° is the quiver without sources or sinks that is obtained by repeatedly removing all sinks and sources from Q; $L(Q^\circ)$ is the Leavitt path algebra of Q°; $L(Q^\circ)_0$ is its degree zero component; and $Q^{(n)}$ is the quiver whose incidence matrix is the nth power of that for Q. It is also shown that all short exact sequences in $\mathsf{qgr}(kQ)$, the full subcategory of finitely presented objects in $\mathsf{QGr}(kQ)$, split. Consequently $\mathsf{qgr}(kQ)$ can be given the structure of a triangulated category with suspension functor the Serre degree twist (-1); this triangulated category is equivalent to the "singularity category" $\mathsf{D}^b(\Lambda)/\mathsf{D}^{\mathsf{perf}}(\Lambda)$ where Λ is the radical square zero algebra $kQ/kQ_{\geq 2}$, and $\mathsf{D}^b(\Lambda)$ is the bounded derived category of finite dimensional left Λ-modules.

MSC: 16W50; 16E50; 16G20; 16D90

Keywords: Path algebra; Quiver; Directed graph; Graded module; Quotient category; Equivalence of categories; Leavitt path algebra

☆ This work was partially supported by the U.S. National Science Foundation grant 0602347.
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0001-8708/$ - see front matter
doi:10.1016/j.aim.2012.03.031

1. Introduction

1.1

Throughout k is a field and Q a finite quiver (directed graph) with vertex set I. Loops and multiple arrows between vertices are allowed.

We write kQ for the path algebra of Q.

We make kQ an $\mathbb{N}$-graded algebra by declaring that a path is homogeneous of degree equal to its length. The category of $\mathbb{Z}$-graded left kQ-modules with degree-preserving homomorphisms is denoted by $\mathsf{Gr}(kQ)$ and we write $\mathsf{Fdim}(kQ)$ for its full subcategory of modules that are the sum of their finite-dimensional submodules. Since $\mathsf{Fdim}(kQ)$ is a localizing subcategory of $\mathsf{Gr}(kQ)$ we may form the quotient category

$$\mathsf{QGr}(kQ) := \frac{\mathsf{Gr}(kQ)}{\mathsf{Fdim}(kQ)}.$$

By [13, Proposition 4, p. 372], the quotient functor $\pi^* : \mathsf{Gr}(kQ) \to \mathsf{QGr}(kQ)$ has a right adjoint that we will denote by π_*. We define

$$\mathcal{O} := \pi^*(kQ).$$

The main result in this paper is the following theorem combined with an explicit description of the algebra $S(Q)$ that appears in its statement.

Theorem 1.1. *The endomorphism ring of $\mathcal{O}$ in $\mathsf{QGr}(kQ)$ is an ultramatricial k-algebra, $S(Q)$, and $\operatorname{Hom}_{\mathsf{QGr}(kQ)}(\mathcal{O}, -)$ is an equivalence*

$$\mathsf{QGr}(kQ) \equiv \mathsf{Mod}\, S(Q)$$

with the category of right $S(Q)$-modules.

1.2. Definition and description of $S(Q)$

We write Q_n for the set of paths of length n and kQ_n for the linear span of Q_n. With this notation

$$\begin{aligned} kQ &= kI \oplus kQ_1 \oplus kQ_2 \oplus \cdots \\ &= T_{kI}(kQ_1) \end{aligned}$$

where $T_{kI}(kQ_1)$ is the tensor algebra of the kI-bimodule kQ_1.

The ring of left kI-module endomorphisms of kQ_n is denoted by

$$S_n := \operatorname{End}_{kI}(kQ_n).$$

Since $kQ_{n+1} = kQ_1 \otimes_{kI} kQ_n$ the functor $kQ_1 \otimes_{kI}$-gives k-algebra homomorphisms

$$\theta_n : S_n \to S_{n+1}.$$

Explicitly, if $x_1, \ldots, x_{n+1} \in kQ_1$, $f \in S_n$, and $\otimes = \otimes_{kI}$, then

$$\theta_n(f)\big(x_1 \otimes \cdots \otimes x_{n+1}\big) := x_1 \otimes f(x_2 \otimes \cdots \otimes x_{n+1}).$$

The θ_ns give rise to a directed system $kI = S_0 \to S_1 \to \cdots$, and we define

$$S(Q) := \varinjlim S_n.$$

As a k-algebra, kI is isomorphic to a product of $|I|$ copies of k. Every left kI-module is therefore a direct sum of 1-dimensional kI-submodules and the endomorphism ring of a finite dimensional kI-module is therefore a direct sum of $\le |I|$ matrix algebras $M_r(k)$ where the rs that appear are determined by the multiplicities of the simple kI-modules.

Hence $S(Q)$ is a direct limit of products of matrix algebras. Such algebras are called ultramatricial.

Theorem 1.1 will follow from the following result.

Theorem 1.2. *The object $\mathcal{O}$ is a finitely generated projective generator in* $\mathsf{QGr}(kQ)$ *and*

$$\operatorname{End}_{\mathsf{QGr}(kQ)}\mathcal{O} \cong S(Q).$$

The functor implementing the equivalence in Theorem 1.1 is $\operatorname{Hom}_{\mathsf{QGr}(kQ)}(\mathcal{O}, -)$.

Ultramatricial algebras are described by Bratteli diagrams [4] (see also [16]). The Bratteli diagram for $S(Q)$, and hence $S(Q)$, is described explicitly in Proposition 5.1 in terms of the incidence matrix for Q.

1.3. Relation to Leavitt path algebras and Cuntz–Krieger algebras

Apart from the path algebra kQ two other algebras are commonly associated to Q, the Leavitt path algebra $L(Q)$ and the Cuntz–Krieger algebra $\mathcal{O}_Q$. The algebra $L(Q)$, which can be defined over any commutative ring, is an algebraic analogue of the C^*-algebra $\mathcal{O}_Q$ because (when the base field is $\mathbb{C}$) $\mathcal{O}_Q$ contains $L(Q)$ as a dense subalgebra.

Theorem 1.3. *Let Q° be the quiver without sources or sinks that is obtained by repeatedly removing all sinks and sources from Q. Then*

(1) $\mathsf{QGr}(kQ) \equiv \mathsf{QGr}(kQ^\circ)$;
(2) $S(Q^\circ) \cong L(Q^\circ)_0$;
(3) $L(Q^\circ)$ *is a strongly graded ring;*
(4) $\mathsf{QGr}(kQ) \equiv \mathsf{Mod}\, S(Q) \equiv \mathsf{Gr} L(Q^\circ) \equiv \mathsf{Mod}\, L(Q^\circ)_0$.

After proving this theorem the author learned that Roozbeh Hazrat had previously given necessary and sufficient conditions for $L(Q^\circ)$ to be a strongly graded ring [18, Theorem 3.15]. The idea in our proof of (3) differs from that in Hazrat's paper.

1.4. Coherence

A ring R is left coherent if the kernel of every homomorphism $f : R^m \to R^n$ between finitely generated free left R-modules is finitely generated. If R is left coherent we write $\mathsf{mod}\, R$ for the full subcategory of $\mathsf{Mod}\, R$ consisting of finitely presented modules; $\mathsf{mod}\, R$ is then an abelian category.

To prove R is left coherent it suffices to show that every finitely generated left ideal is finitely presented.

A ring in which every left ideal is projective is left coherent so kQ is left coherent. A direct limit of left coherent rings is left coherent so $S(Q)$ is left coherent.

Because kQ is left coherent the full subcategory

$$\mathsf{gr}(kQ) \subset \mathsf{Gr}(kQ)$$

consisting of finitely presented graded left kQ-modules is abelian. The category $\mathsf{fdim}(kQ) := (\mathsf{gr}(kQ)) \cap (\mathsf{Fdim}(kQ))$ is the full subcategory of $\mathsf{gr}(kQ)$ consisting of finite dimensional modules. We now define

$$\mathsf{qgr}(kQ) := \frac{\mathsf{gr}(kQ)}{\mathsf{fdim}(kQ)} \subset \mathsf{QGr}(kQ).$$

Proposition 1.4. *The equivalence in Theorem 1.1 restricts to an equivalence*

$$\mathsf{qgr}(kQ) \equiv \mathsf{mod}\, S(Q).$$

By [20, Proposition A.5, p. 113], $\mathsf{qgr}\, R$ consists of the finitely presented objects in $\mathsf{QGr}\, R$ and every object in $\mathsf{QGr}\, R$ is a direct limit of objects in $\mathsf{qgr}\, R$.[1]

1.5. $\mathsf{qgr}(kQ)$ *as a triangulated category*

One of the main steps in proving Theorems 1.1 and 1.2 is to prove the following.

Proposition 1.5. *Every short exact sequence in* $\mathsf{qgr}(kQ)$ *splits.*

If Σ is an auto-equivalence of an abelian category A in which every short exact sequence splits, then A can be given the structure of a triangulated category with Σ being the translation: one declares that the distinguished triangles are all direct sums of the following triangles:

$$\mathcal{M} \to 0 \to \Sigma\mathcal{M} \xrightarrow{\mathrm{id}} \Sigma\mathcal{M},$$
$$\mathcal{M} \xrightarrow{\mathrm{id}} \mathcal{M} \to 0 \to \Sigma\mathcal{M},$$
$$0 \to \mathcal{M} \xrightarrow{\mathrm{id}} \mathcal{M} \to 0,$$

as $\mathcal{M}$ ranges over the objects of A. Hence $\mathsf{qgr}(kQ)$ endowed with the Serre twist (-1) is a triangulated category. In Section 7 we use a result of Xiao-Wu Chen to prove the following.

Theorem 1.6. *Let Q be a quiver and Λ the finite dimensional algebra $kQ/kQ_{\geq 2}$. There is an equivalence of triangulated categories*

$$\big(\mathsf{qgr}(kQ), (-1)\big) \equiv \mathsf{D}^b(\mathsf{mod}\,\Lambda)/\mathsf{D}^b_{\mathrm{perf}}(\mathsf{mod}\,\Lambda).$$

1.6. Equivalences of categories

It can happen that $\mathsf{QGr}(kQ)$ is equivalent to $\mathsf{QGr}(kQ')$ with Q and Q' being non-isomorphic quivers.

Theorem 1.7 (*See Section 4*)**.** *If Q and Q' become the same after repeatedly removing vertices that are sources or sinks, then* $\mathsf{QGr}(kQ) \equiv \mathsf{QGr}(kQ')$.

Let A be a $\mathbb{Z}$-graded algebra. If m is a positive integer the algebra $A^{(m)} = \oplus_{i\in\mathbb{Z}} A_{im}$ is called the m^{th} Veronese subalgebra of A. When A is a commutative $\mathbb{N}$-graded algebra the schemes

[1] An object $\mathcal{M}$ in an additive category A is finitely presented if $\mathrm{Hom}_{\mathsf{A}}(\mathcal{M}, -)$ commutes with direct limits; is finitely generated if whenever $\mathcal{M} = \sum \mathcal{M}_i$ for some directed family of subobjects $\mathcal{M}_i$ there is an index j such that $\mathcal{M} = \mathcal{M}_j$; is coherent if it is finitely presented and all its finitely generated subobjects are finitely presented.

Proj A and Proj $A^{(m)}$ are isomorphic. Verevkin proved a non-commutative version of that result: $\mathsf{QGr}\, A \equiv \mathsf{QGr}\, A^{(m)}$ if A is an $\mathbb{N}$-graded ring generated by A_0 and A_1 [29, Theorem 4.4].

Theorem 1.8. *Let Q be a quiver with incidence matrix C. Let $Q^{(m)}$ be the quiver with incidence matrix C^m, $m \geq 1$; i.e., $Q^{(m)}$ has the same vertices as Q but the arrows in $Q^{(m)}$ are the paths of length m in Q. Then*

$$\mathsf{QGr}(kQ) \equiv \mathsf{QGr}(kQ^{(m)}).$$

Proof. This follows from Verevkin's result because $kQ^{(m)} = (kQ)^{(m)}$. $\square$

The referee pointed out the following alternative proof of Theorem 1.8. First, $S_n(Q^{(m)}) = S_{nm}(Q)$ so the directed system used to define $S(Q^{(m)})$ is equal to the directed system obtained by taking every m^{th} term of the directed system for $S(Q)$. Hence $S(Q) = S(Q^{(m)})$. Therefore

$$\mathsf{QGr}(kQ) \equiv \mathsf{Mod}\, S(Q) = \mathsf{Mod}\, S(Q^{(m)}) \equiv \mathsf{QGr}(kQ^{(m)}).$$

We call $Q^{(n)}$ the nth Veronese of Q. In symbolic dynamics $Q^{(n)}$ is called the nth higher power graph of Q [21, Definition 2.3.10].

Other equivalences involving strong shift equivalence of incidence matrices, a notion from symbolic dynamics, appear in [24].

2. The endomorphism ring of $\mathcal{O}$

Recall that $\mathcal{O}$ denotes $\pi^*(kQ)$, the image of the graded left module kQ in the quotient category $\mathsf{QGr}(kQ)$.

Notation. In addition to the notation set out at the beginning of Section 1.2 we write $Q_{\geq n}$ for the set of paths of length $\geq n$ and $kQ_{\geq n}$ for its linear span. We note that $kQ_{\geq n}$ is a graded two-sided ideal in kQ.

We write e_i for the trivial path at vertex i, E_i for the simple module at vertex i, and $P_i = (kQ)e_i$.

If p is a path in Q we write $s(p)$ for its starting point and $t(p)$ for the vertex at which it terminates.

We write pq to denote the path q followed by the path p.

Lemma 2.1. *Let $I^0 = \{i \in I \mid$ the number of paths starting at i is finite$\}$. Let $I^\infty := I - I^0$ and let Q^∞ be the subquiver of Q consisting of the vertices in I^∞ and all arrows that begin and end at points in I^∞. Let T be the sum of all finite-dimensional left ideals in kQ. Then*

(1) *T is a two sided ideal;*
(2) *$T = (e_i \mid i \in I^0)$;*
(3) *$kQ/T \cong kQ^\infty$;*
(4) *the only finite-dimensional left ideal in kQ^∞ is $\{0\}$;*
(5) *if $f : kQ_{\geq n} \to T$ is a homomorphism of graded left kQ-modules, then $f(kQ_{\geq n+r}) = 0$ for $r \gg 0$.*

Proof. The result is obviously true if $\dim_k kQ < \infty$ so we assume this is not the case, i.e., Q has arbitrarily long paths; equivalently, $Q^\infty \neq \varnothing$.

(1) If L is a finite-dimensional left ideal in kQ so is Lx for all $x \in kQ$, whence T is a two-sided ideal.

(2), (3), (4). Since the paths beginning at a vertex i are a basis for $(kQ)e_i$, $\dim_k(kQ)e_i < \infty$ if and only if $i \in I^0$. Hence T contains $\{e_i \mid i \in I^0\}$. It is clear that

$$\frac{kQ}{(e_i \mid i \in I^0)} \cong kQ^\infty.$$

Let p be a path in Q^∞. Then there is an arrow $a \in Q^\infty$ such that $ap \neq 0$. It follows that $\dim_k(kQ^\infty)p = \infty$. It follows that the only finite-dimensional left ideal in kQ^∞ is $\{0\}$. Therefore $T/(e_i \mid i \in I^0) = 0$.

(5) Let $f : kQ_{\geq n} \to T$ be a homomorphism of graded left kQ-modules. Every finitely generated left ideal contained in T has finite dimension so $f(kQ_{\geq n})$ has finite dimension. Hence $kQ_{\geq n}/\ker f$ is annihilated by $kQ_{\geq r}$ for $r \gg 0$. In other words, $\ker f \supset (kQ_{\geq r}) \cdot (kQ_{\geq n}) = kQ_{\geq n+r}$. $\square$

The ideal T in Lemma 2.1 need not have finite dimension; for example, if Q is the quiver in Proposition 6.3, $\dim_k T = \infty$.

Lemma 2.2. *Let I be a graded left ideal of kQ. If kQ/I is the sum of its finite dimensional submodules, then $I \supset kQ_{\geq n}$ for $n \gg 0$.*

Proof. The image of 1 in kQ/I belongs to a *finite* sum of finite dimensional submodules of kQ/I so the submodule it generates is finite dimensional. Hence $\dim_k(kQ/I) < \infty$. Therefore kQ/I is non-zero in only finitely many degrees; thus I contains $kQ_{\geq n}$ for $n \gg 0$. $\square$

By definition, the objects in $\mathsf{QGr}(kQ)$ are the same as those in the $\mathsf{Gr}(kQ)$ and the morphisms are

$$\operatorname{Hom}_{\mathsf{QGr}(kQ)}(\pi^*M, \pi^*N) = \varinjlim \operatorname{Hom}_{\mathsf{Gr}(kQ)}(M', N/N')$$

where the direct limit is taken as M' and N' range over all graded submodules of M and N such that M/M' and N' belong to $\mathsf{Fdim}(kQ)$.

Proposition 2.3. *If $N \in \mathsf{Gr}(kQ)$, then*

$$\operatorname{Hom}_{\mathsf{QGr}(kQ)}(\mathcal{O}, \pi^*N) = \varinjlim \operatorname{Hom}_{\mathsf{Gr}(kQ)}(kQ_{\geq n}, N/N')$$

where the direct limit is taken over all integers $n \geq 0$ and all submodules N' of N such that N' is the sum of its finite dimensional submodules.

Proof. This follows from Lemma 2.2. $\square$

Lemma 2.4. *Consider kQ_n as a left kI-module. The restriction map*

$$\Phi : \operatorname{End}_{\mathsf{Gr}(kQ)}(kQ_{\geq n}) \longrightarrow \operatorname{End}_{kI}(kQ_n), \qquad \Phi(f) = f|_{kQ_n},$$

is a k-algebra isomorphism with inverse given by applying the functor $kQ \otimes_{kI}$-to each kI-module endomorphism of kQ_n.

Proof. Each $f \in \operatorname{End}_{\mathsf{Gr}(kQ)}(kQ_{\geq n})$ sends kQ_n to itself. Since f is a left kQ-module homomorphism it is a left kI-module homomorphism. Hence Φ is a well-defined algebra homomorphism. Since $kQ_{\geq n}$ is generated by kQ_n as a left kQ-module, Φ is injective. Since $kQ_{\geq n} \cong kQ \otimes_{kI} kQ_n$ every left kI-module homomorphism $kQ_n \to kQ_n$ extends in a unique way to a kQ-module homomorphism $kQ_{\geq n} \to kQ_{\geq n}$ (by applying the functor $kQ \otimes_{kI} -$). Hence Φ is surjective. $\square$

Theorem 2.5. *There is a k-algebra isomorphism*

$$\operatorname{End}_{\mathsf{QGr}(kQ)}\mathcal{O} \cong \varinjlim \operatorname{Hom}_{\mathsf{Gr}(kQ)}(kQ_{\geq n}, kQ_{\geq n}) = S(Q).$$

Proof. By the definition of morphisms in a quotient category,

$$\operatorname{End}_{\mathsf{QGr}(kQ)}\mathcal{O} = \varinjlim \operatorname{Hom}_{\mathsf{Gr}(kQ)}(I, kQ/T') \tag{2.1}$$

where I runs over all graded left ideals such that $\dim_k(kQ/I) < \infty$ and T' runs over all graded left ideals such that $\dim_k T' < \infty$.

If T' is a graded left ideal of finite dimension it is contained in the ideal T that appears in Lemma 2.1. The system of graded left ideals of finite codimension in kQ is cofinal with the system of left ideals $kQ_{\geq n}$. These two facts imply that

$$\operatorname{End}_{\mathsf{QGr}(kQ)}\mathcal{O} = \varinjlim_n \operatorname{Hom}_{\mathsf{Gr}(kQ)}(kQ_{\geq n}, kQ/T).$$

Since $kQ_{\geq n}$ is a projective left kQ-module the map

$$\operatorname{Hom}_{\mathsf{Gr}(kQ)}(kQ_{\geq n}, kQ) \to \operatorname{Hom}_{\mathsf{Gr}(kQ)}(kQ_{\geq n}, kQ/T)$$

is surjective. This leads to a surjective map

$$\varinjlim_n \operatorname{Hom}_{\mathsf{Gr}(kQ)}(kQ_{\geq n}, kQ) \to \varinjlim_n \operatorname{Hom}_{\mathsf{Gr}(kQ)}(kQ_{\geq n}, kQ/T). \tag{2.2}$$

Suppose the image of a map $f \in \operatorname{Hom}_{\mathsf{Gr}(kQ)}(kQ_{\geq n}, kQ)$ is contained in T. By Lemma 2.1(5), the restriction of f to $kQ_{\geq n+r}$ is zero for $r \gg 0$. The map in (2.2) is therefore injective and hence an isomorphism.

Since morphisms in $\operatorname{Hom}_{\mathsf{Gr}(kQ)}(kQ_{\geq n}, kQ)$ preserve degree the natural map $\operatorname{Hom}_{\mathsf{Gr}(kQ)}(kQ_{\geq n}, kQ_{\geq n}) \to \operatorname{Hom}_{\mathsf{Gr}(kQ)}(kQ_{\geq n}, kQ)$ is an isomorphism. It follows that

$$\operatorname{End}_{\mathsf{QGr}(kQ)}\mathcal{O} = \varinjlim_n \operatorname{Hom}_{\mathsf{Gr}(kQ)}(kQ_{\geq n}, kQ_{\geq n}).$$

However, $\operatorname{Hom}_{\mathsf{Gr}(kQ)}(kQ_{\geq n}, kQ_{\geq n}) \cong \operatorname{End}_{kI}(kQ_n)$ by Lemma 2.4 so the result follows from the definition of $S(Q)$. $\square$

3. Proof that $\mathcal{O}$ is a progenerator in $\mathsf{QGr}(kQ)$

Each $M \in \mathsf{Gr}(kQ)$ has a largest submodule belonging to $\mathsf{Fdim}(kQ)$, namely

$$\tau M := \text{ the sum of all finite-dimensional graded submodules of } M.$$

3.1

Up to isomorphism and degree shift, the indecomposable projective graded left kQ-modules are

$$P_i = (kQ)e_i \cong kQ \otimes_{kI} ke_i, \quad i \in I,$$

where ke_i is the simple left kI-module at vertex i. It follows that every projective module in $\mathsf{Gr}(kQ)$ is isomorphic to $kQ \otimes_{kI} V$ for a suitable graded kI-module V.

Lemma 3.1. *Let $P, P' \in \mathsf{Gr}(kQ)$ be graded projective modules generated by their degree n components. Every injective degree-preserving homomorphism $f : P \to P'$ splits.*

Proof. Without loss of generality we can assume $n = 0$, $P = kQ \otimes_{kI} U$, and $P' = kQ \otimes_{kI} V$. The natural map

$$kQ \otimes_{kI} - : \mathrm{Hom}_{kI}(U, V) \to \mathrm{Hom}_{\mathsf{Gr}(kQ)}(kQ \otimes_{kI} U, kQ \otimes_{kI} V)$$

is an isomorphism with inverse given by restricting a kQ-module homomorphism to the degree-zero components. An injective homomorphism $f : kQ \otimes_{kI} U \to kQ \otimes_{kI} V$ in $\mathsf{Gr}(kQ)$ restricts to an injective kI-module homomorphism $U \to V$ which splits because kI is a semisimple ring. $\square$

Part (1) of the next result is implied by [2, Theorem 3.14] but because we only prove it for graded modules a simpler proof is possible.

Proposition 3.2.

(1) *Let M be a finitely generated graded left kQ-module. Then M is finitely presented if and only if for all $n \gg 0$*

$$M_{\geq n} \cong \bigoplus_{i \in I} P_i(-n)^{\oplus m_i}$$

for some integers m_i depending on M and n.

(2) *If $0 \to L \to M \to N \to 0$ is an exact sequence in $\mathsf{gr}(kQ)$, then $0 \to L_{\geq n} \to M_{\geq n} \to N_{\geq n} \to 0$ splits for $n \gg 0$.*

(3) *If $\mathcal{M} \in \mathsf{qgr}(kQ)$, there is a projective $M \in \mathsf{gr}(kQ)$ such that $\mathcal{M} \cong \pi^* M$.*

(4) *Every short exact sequence in $\mathsf{qgr}(kQ)$ splits.*

(5) *All objects in $\mathsf{qgr}(kQ)$ are injective and projective.*

Proof. (1) ($\Leftarrow$) Each P_i is finitely presented because it is a finitely generated left ideal of the coherent ring kQ. Hence every $P_i(-n)$ is in $\mathsf{gr}(kQ)$. Therefore, if there is an integer n such that $M_{\geq n}$ is a finite direct sum of various $P_i(-n)$s, then $M_{\geq n}$ is in $\mathsf{gr}(kQ)$ too. The hypothesis that M is finitely generated implies that $M/M_{\geq n}$ is finite dimensional. But every finite dimensional graded kQ-module is a quotient of a direct sum of twists of the finitely presented finite dimensional module $kQ/kQ_{\geq 1}$ and is therefore in $\mathsf{gr}(kQ)$. In particular, $M/M_{\geq n} \in \mathsf{gr}(kQ)$. Since $\mathsf{gr}(kQ)$ is closed under extensions, M is in $\mathsf{gr}(kQ)$ too.

($\Rightarrow$) Let $M \in \mathsf{gr}(kQ)$. Then there is an exact sequence $0 \to F' \xrightarrow{f} F \to M \to 0$ in $\mathsf{gr}(kQ)$ with F and F' finitely generated graded projective kQ-modules. Since F', F, and M, are finitely generated, for all sufficiently large n the modules $F'_{\geq n}$, $F_{\geq n}$, and $M_{\geq n}$, are generated as kQ-modules by F'_n, F_n, and M_n, respectively. But kQ is hereditary so $F_{\geq n}$ and $F'_{\geq n}$ are graded projective. Now Lemma 3.1 implies that the restriction $f : F'_{\geq n} \to F_{\geq n}$ splits. Hence $M_{\geq n}$ is a direct summand of $F_{\geq n}$. The result follows.

(2) By (1), $N_{\geq n}$ is projective for $n \gg 0$, hence the splitting.

(3) There is some M in $\mathsf{gr}(kQ)$ such that $\mathcal{M} \cong \pi^* M$. But $\pi^* M \cong \pi^*(M_{\geq n})$ for all n so (3) follows from (1).

(4) By [13, Corollary 1, p. 368], every short exact sequence in $\mathsf{qgr}(kQ)$ is of the form

$$0 \longrightarrow \pi^* L \xrightarrow{\pi^* f} \pi^* M \xrightarrow{\pi^* g} \pi^* N \longrightarrow 0 \tag{3.1}$$

for some exact sequence $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ in $\mathsf{gr}(kQ)$. But (3.1) is also obtained by applying π^* to the restriction $0 \to L_{\geq n} \to M_{\geq n} \to N_{\geq n} \to 0$ which splits for $n \gg 0$. Hence (3.1) splits. $\square$

3.2

By Proposition 3.2(5), $\mathcal{O}$ is a projective object in $\mathsf{qgr}(kQ)$.

Lemma 3.3. *$\mathcal{O}$ is a projective object in* $\mathsf{QGr}(kQ)$.

Proof. As noted in the proof of Proposition 3.2(4), an epimorphism in $\mathsf{QGr}(kQ)$ is necessarily of the form $\pi^*g : \pi^*M \to \pi^*N$ for some surjective homomorphism $g : M \to N$ in $\mathsf{Gr}(kQ)$. Let $\eta : \mathcal{O} \to \pi^*N$ be a morphism in $\mathsf{QGr}(kQ)$. Then

$$\eta \in \varinjlim \operatorname{Hom}_{\mathsf{Gr}(kQ)}(kQ_{\geq n}, N/N')$$

where the direct limit is taken over all $n \in \mathbb{N}$ and all $N' \subset N$ such that N/N' is the sum of its finite dimensional submodules, so $\eta = \pi^*h$ for some n, some N', and some $h : kQ_{\geq n} \to N/N'$. Since $kQ_{\geq n}$ is a projective object in $\mathsf{Gr}(kQ)$, h factors through N and for the same reason h factors through g. Hence there is a morphism $\gamma : \mathcal{O} \to \pi^*M$ such that $\eta = (\pi^*g) \circ \gamma$. $\square$

Lemma 3.4. $\operatorname{Hom}_{\mathsf{QGr}(kQ)}(\mathcal{O}, -)$ *commutes with all direct sums in* $\mathsf{QGr}(kQ)$.

Proof. Let

$$\mathcal{M} = \bigoplus_{\lambda \in \Lambda} \mathcal{M}_\lambda$$

be a direct sum in $\mathsf{QGr}(kQ)$. Let M_λ, $\lambda \in \Lambda$, be graded kQ-modules such that $\pi^*M_\lambda = \mathcal{M}_\lambda$. Because π^* has a right adjoint it commutes with direct sums. Hence $\mathcal{M} = \pi^*M$ where $M = \oplus_{\lambda \in \Lambda} M_\lambda$. Because $kQ_{\geq n}$ is a finitely generated module we obtain the second equality in the computation

$$\begin{aligned}
\operatorname{Hom}_{\mathsf{QGr}(kQ)}(\mathcal{O}, \mathcal{M}) &= \varinjlim_n \operatorname{Hom}_{\mathsf{Gr}(kQ)}(kQ_{\geq n}, \oplus_{\lambda \in \Lambda} M_\lambda) \\
&= \varinjlim_n \bigoplus_{\lambda \in \Lambda} \operatorname{Hom}_{\mathsf{Gr}(kQ)}(kQ_{\geq n}, M_\lambda) \\
&= \bigoplus_{\lambda \in \Lambda} \varinjlim_n \operatorname{Hom}_{\mathsf{Gr}(kQ)}(kQ_{\geq n}, M_\lambda) \\
&= \bigoplus_{\lambda \in \Lambda} \operatorname{Hom}_{\mathsf{QGr}(kQ)}(\mathcal{O}, \mathcal{M}_\lambda).
\end{aligned}$$

This proves the lemma. $\square$

3.2.1. Notation

We write $\mathcal{P}_i = \pi^*P_i$ for the images of the indecomposable projectives in $\mathsf{QGr}(kQ)$.

3.2.2

If $\mathcal{S}$ is a set of objects in an additive category A we write $\mathsf{add}(\mathcal{S})$ for the smallest full subcategory of A that contains $\mathcal{S}$ and is closed under direct summands and finite direct sums.

Lemma 3.5. *For every positive integer n, $\mathcal{O}(-n) \in \mathsf{add}(\mathcal{O})$.*

Proof. Let

$$I_m := \{v \in I \mid \text{ there is a path of length } m \text{ that ends at } v\}, \quad m \geq 1,$$
$$I_0 := I - I_1,$$
$$I_\infty := \bigcap_{m=1}^{\infty} I_m,$$
$$\mathsf{T}_m := \mathsf{add}\{\mathcal{P}_i \mid i \in I_m\}, \quad 0 \leq m \leq \infty.$$

The vertices in I_0 are the sources. A vertex is in I_∞ if and only if for every $m \geq 1$ there is a path of length m ending at it. If $m \gg 0$, then

$$I_1 \supset I_2 \supset \cdots \supset I_m = I_{m+1} = \cdots = I_\infty$$

and, consequently,

$$\mathsf{T}_1 \supset \mathsf{T}_2 \supset \cdots \supset \mathsf{T}_m = \mathsf{T}_{m+1} = \cdots = \mathsf{T}_\infty.$$

To prove the lemma it suffices to show that $\mathcal{O}(-1) \in \mathsf{add}(\mathcal{O})$ because, if it is, an induction argument would complete the proof: $\mathcal{O}(-1) \in \mathsf{add}(\mathcal{O})$ implies that $\mathcal{O}(-2) \in \mathsf{add}(\mathcal{O}(-1)) \subset \mathsf{add}(\mathcal{O})$, and so on. But $\mathcal{O}(-1) = \bigoplus_{i \in I} \mathcal{P}_i(-1)$ is a direct sum of an object in $\mathsf{T}_0(-1)$ and an object in $\mathsf{T}_1(-1)$ so it suffices to show that $\mathsf{add}(\mathcal{O})$ contains $\mathsf{T}_0(-1)$ and $\mathsf{T}_1(-1)$.

If j is a sink, then $P_j = ke_j$ so $\mathcal{P}_j = 0$.

Suppose $j \in I_m$ and j is not a sink. There is an exact sequence

$$0 \to \bigoplus_{a \in s^{-1}(j)} P_{t(a)}(-1) \xrightarrow{(\cdot a)} P_j \to ke_j \to 0$$

where ke_j denotes the simple module concentrated at vertex j. Therefore

$$\mathcal{P}_j \cong \bigoplus_{a \in s^{-1}(j)} \mathcal{P}_{t(a)}(-1)$$

for every vertex j. If $a \in s^{-1}(j)$, then $t(a) \in I_{m+1}$. Therefore $\mathsf{T}_m \subset \mathsf{T}_{m+1}(-1)$. On the other hand, if $m \geq 1$ and $i \in I_{m+1}$, there is an arrow a such that $t(a) = i$ and $s(a) \in I_m$ so $\mathcal{P}_i(-1)$ is a direct summand of $\mathcal{P}_{s(a)}$. Hence $\mathsf{T}_{m+1}(-1) \subset \mathsf{T}_m$.

The previous paragraph shows that $\mathsf{T}_0 \subset \mathsf{T}_1(-1)$ and $\mathsf{T}_m = \mathsf{T}_{m+1}(-1)$ for all $m \geq 1$. Thus

$$\mathsf{T}_1 = \mathsf{T}_2(-1) = \cdots = \mathsf{T}_m(-m+1) \tag{3.2}$$

for all $m \geq 1$. For $m \gg 0$, $\mathsf{T}_m = \mathsf{T}_{m+1}$ so $\mathsf{T}_m(-1) = \mathsf{T}_{m+1}(-1) = \mathsf{T}_m$. Hence, for $m \gg 0$, $\mathsf{T}_m = \mathsf{T}_m(n)$ for all $n \in \mathbb{Z}$. Therefore (3.2) implies $\mathsf{T}_1 = \mathsf{T}_1(n)$ for all $n \in \mathbb{Z}$.

Since $\mathcal{O} = \oplus_{i \in I} \mathcal{P}_i$, $\mathsf{T}_1 \subset \mathsf{add}(\mathcal{O})$. Therefore $\mathsf{T}_1(-1) = \mathsf{T}_1 \subset \mathsf{add}(\mathcal{O})$ and $\mathsf{T}_0(-1) \subset \mathsf{T}_1(-2) = \mathsf{T}_1(-1) \subset \mathsf{add}(\mathcal{O})$. $\square$

Proposition 3.6. $\mathsf{qgr}(kQ) = \mathsf{add}(\mathcal{O})$.

Proof. Let $\mathcal{M} \in \mathsf{qgr}(kQ)$. There is some M in $\mathsf{gr}(kQ)$ such that $\mathcal{M} \cong \pi^* M$. But $\pi^* M \cong \pi^*(M_{\geq n})$ for all n so, by Proposition 3.2(1), if $n \gg 0$ there are integers m_i such that

$$\mathcal{M} \cong \bigoplus_{i \in I} \mathcal{P}_i(-n)^{\oplus m_i}.$$

Each $\mathcal{P}_i(-n)$ belongs to $\mathsf{add}(\mathcal{O})$ by Lemma 3.5 so $\mathcal{M} \in \mathsf{add}(\mathcal{O})$. $\square$

Theorem 3.7. *Let* $\mathsf{Mod}\, S(Q)$ *be the category of right* $S(Q)$*-modules. The functor* $\mathrm{Hom}_{\mathsf{QGr}(kQ)}(\mathcal{O}, -)$ *provides an equivalence of categories*

$$\mathsf{QGr}(kQ) \equiv \mathsf{Mod}\, S(Q)$$

that sends $\mathcal{O}$ *to* $S(Q)$*. This equivalence restricts to an equivalence between* $\mathsf{qgr}(kQ)$ *and* $\mathsf{mod}\, S(Q)$*, the category of finitely presented* $S(Q)$*-modules.*

Proof. By Proposition 3.6, $\mathcal{O}$ is a generator in $\mathsf{qgr}(kQ)$. Every object in $\mathsf{QGr}(kQ)$ is a direct limit of objects in $\mathsf{qgr}(kQ)$ so $\mathcal{O}$ is also a generator in $\mathsf{QGr}(kQ)$. Since $\mathcal{O}$ is a finitely generated, projective generator in the Grothendieck category $\mathsf{QGr}(kQ)$,

$$\mathrm{Hom}_{\mathsf{QGr}(kQ)}(\mathcal{O}, -) : \mathsf{QGr}(kQ) \to \mathsf{Mod}\big(\mathrm{End}_{\mathsf{QGr}(kQ)}(\mathcal{O})\big)$$

is an equivalence of categories [27, Example X.4.2]. The result now follows from the isomorphism $\mathrm{End}_{\mathsf{QGr}(kQ)}(\mathcal{O}) \cong S(Q)$ in Theorem 2.5. $\square$

4. Sinks and sources can be deleted

4.1

A vertex is a sink if no arrows begin at it and a source if no arrows end at it.

Theorem 4.1. *Suppose quivers* Q *and* Q' *become the same after repeatedly removing sources and sinks and attached arrows. Let* (Q°, I°) *be the quiver without sources or sinks that is obtained from* Q *by this process. Then there is an equivalence of categories*

$$\mathsf{QGr}(kQ) \equiv \mathsf{QGr}(kQ^\circ) \equiv \mathsf{QGr}(kQ').$$

The equivalence of categories is induced by sending a representation $(M_i, M_a; i \in I, a \in Q_1)$ *of* Q *to the representation* $(M_i, M_a; i \in I^\circ, a \in Q_1^\circ)$ *of* Q°*. A quasi-inverse to this is induced by the functor that sends a representation* $(M_i, M_a; i \in I^\circ, a \in Q_1^\circ)$ *of* Q^e *to the representation* $(M_i, M_a; i \in I, a \in Q_1)$ *of* Q *where* $M_i = 0$ *if* $i \notin I^\circ$ *and* $M_a = 0$ *if* $a \notin Q_1^\circ$*.*

4.2

The fact that sinks and sources can be deleted is reminiscent of three other results in the literature.

The category $\mathsf{QGr}(kQ)$ is related to the dynamical system with topological space the bi-infinite paths in Q viewed as a subspace of $Q_1^{\mathbb{Z}}$ and automorphism the edge shift σ defined by $\sigma(f)(n) = f(n+1)$. No arrow that begins at a source and no arrow that ends at a sink appears in any bi-infinite path so, as remarked after Example 2.2.8 in [21], since Q° "contains the only part of Q used for symbolic dynamics, we will usually confine our attention to (quivers such that $Q = Q^\circ$)". See also [21, Proposition 2.2.10].

Second, as remarked on page 18 of [22], "Cuntz–Krieger algebras are the C^*-algebras of finite graphs with no sinks or sources".

Third, in Section 4 of [5] it is shown that the singularity category of an artin algebra is not changed by deleting or adding sources or sinks.

4.3

Theorem 4.1 follows from the next two results.

Proposition 4.2. *Let t be a sink in (Q, I) and Q' the quiver with vertex set $I' := I - \{t\}$ and arrows*

$$Q'_1 := \{\text{arrows in } Q \text{ that do not end at } t\}.$$

Then the functor $i_* : \mathsf{Gr}(kQ') \to \mathsf{Gr}(kQ)$ that sends a representation of Q' to the "same" representation of kQ obtained by putting 0 at vertex t induces an equivalence of categories

$$\mathsf{QGr}(kQ') \equiv \mathsf{QGr}(kQ)$$

that sends $\mathcal{O}'$ to $\mathcal{O}$.

Proof. Since t is a sink, $(kQ)e_t = ke_t$. Hence $e_t(kQ)$ is a two-sided ideal of kQ and, if M is a left kQ module, then e_tM is a submodule of M.

It is clear that i_* is the forgetful functor induced by the homomorphism $kQ \to kQ/(e_t) = kQ'$. The functor i_* has a left adjoint i^* and a right adjoint $i^!$. The functor i^* sends a kQ-module M to M/e_tM. It is easy to see that the counit $i^*i_* \to \mathrm{id}_{\mathsf{Gr}(kQ')}$ is an isomorphism.

Both i_* and i^* are exact.

Since i_* and i^* send direct limits of finite dimensional modules to direct limits of finite dimensional modules they induce functors $\iota_* : \mathsf{QGr}(kQ') \to \mathsf{QGr}(kQ)$ and $\iota^* : \mathsf{QGr}(kQ) \to \mathsf{QGr}(kQ')$. Because $i^*i_* \cong \mathrm{id}_{\mathsf{Gr}(kQ')}$ we have $\iota^*\iota_* \cong \mathrm{id}_{\mathsf{QGr}(kQ')}$.

If $M \in \mathsf{Gr}(kQ)$ there is an exact sequence $0 \to e_tM \to M \to i_*i^*M \to 0$. If a is any arrow, then $ae_t = 0$. Therefore e_tM is a direct sum of 1-dimensional left kQ-modules, hence in $\mathsf{Fdim}(kQ)$. It follows that the unit $\mathrm{id}_{\mathsf{Gr}(kQ)} \to i_*i^*$ induces an isomorphism $\mathrm{id}_{\mathsf{QGr}(kQ)} \cong \iota_*\iota^*$.

Hence ι_* and ι^* are mutually quasi-inverse equivalences.

Since $kQ' = kQ/e_tkQ$, i_* sends kQ' to kQ/e_tkQ. The natural homomorphism $kQ \to kQ/e_tkQ = i_*(kQ')$ becomes an isomorphism in $\mathsf{QGr}(kQ)$ because $e_tkQ \in \mathsf{Fdim}(kQ)$. Hence $\iota_*\mathcal{O}' = \mathcal{O}$. □

Proposition 4.3. *Let s be a source in (Q, I) and Q' be the quiver with vertex set $I' := I - \{s\}$ and arrows*

$$Q'_1 := \{\text{arrows in } Q \text{ that do not begin at } s\}.$$

The functor $i_* : \mathsf{Gr}(kQ') \to \mathsf{Gr}(kQ)$ that sends a representation of Q' to the "same" representation of kQ obtained by putting 0 at vertex s induces an equivalence, ι_*, of categories

$$\mathsf{QGr}(kQ') \equiv \mathsf{QGr}(kQ).$$

Furthermore, $\mathcal{O} \cong \iota_*\mathcal{O}' \oplus \mathcal{P}_s$ where $\mathcal{P}_s = \pi^*(kQe_s)$ and π^* is the quotient functor $\mathsf{Gr}(kQ) \to \mathsf{QGr}(kQ)$.

Proof. Every kQ'-module becomes a kQ-module through the homomorphism $\varphi : kQ \to kQ/(e_s) = kQ'$; this is the exact fully faithful embedding i_*. A right adjoint to i_* is given by the functor $i^!$,

$$i^!M := \mathrm{Hom}_{kQ}(kQ', M) = \{m \in M \mid e_sm = 0\} = (1 - e_s)M.$$

It is clear that the unit $\mathrm{id}_{\mathsf{Gr}(kQ')} \to i^!i_*$ is an isomorphism of functors.

Both i_* and $i^!$ are exact.

Since i_* and $i^!$ send direct limits of finite dimensional modules to direct limits of finite dimensional modules there are unique functors $\iota_* : \mathsf{QGr}(kQ') \to \mathsf{QGr}(kQ)$ and $\iota^! : \mathsf{QGr}(kQ) \to$

$\mathsf{QGr}(kQ')$ such that the diagrams

$$\begin{tikzcd} \mathsf{Gr}(kQ') \arrow[r, "i_*"] \arrow[d] & \mathsf{Gr}(kQ) \arrow[d] \\ \mathsf{QGr}(kQ') \arrow[r, "\iota_*"'] & \mathsf{QGr}(kQ) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathsf{Gr}(kQ) \arrow[r, "i^!"] \arrow[d] & \mathsf{Gr}(kQ') \arrow[d] \\ \mathsf{QGr}(kQ) \arrow[r, "\iota^!"'] & \mathsf{QGr}(kQ') \end{tikzcd}$$

commute [13, Section III.1]. (The vertical arrows in the diagrams are the quotient functors.)

Because $\mathrm{id}_{\mathsf{Gr}(kQ')} \cong i^! i_*$, $\iota^! \iota_* \cong \mathrm{id}_{\mathsf{QGr}(kQ')}$.

If $M \in \mathsf{Gr}(kQ)$, there is an exact sequence $0 \to i_* i^! M \to M \to \bar{M} \to 0$ in which $\bar{M}$ is supported only at the vertex s; a module supported only at s is a sum of 1-dimensional kQ-modules so belongs to $\mathsf{Fdim}(kQ)$. It follows that the unit $i_* i^! \to \mathrm{id}_{\mathsf{Gr}(kQ)}$ induces an isomorphism $\mathrm{id}_{\mathsf{QGr}(kQ)} \cong \iota_* \iota^!$.

Hence ι_* and $\iota^!$ are mutually quasi-inverse equivalences.

The isomorphism $\mathcal{O} \cong \iota_* \mathcal{O}' \oplus \mathcal{P}_s$ is proved in Section 4.5. $\square$

4.4

It need not be the case that the equivalence ι_* in Proposition 4.3 sends $\mathcal{O}'$ to $\mathcal{O}$. We will show that $\iota_* \mathcal{O}' \ncong \mathcal{O}$ for the quivers

$$Q = \quad \begin{tikzcd} s \arrow[r, "a"] & v \arrow[loop right, "b"] \end{tikzcd} \quad \text{and} \quad Q' = \quad \begin{tikzcd} v \arrow[loop right, "b"] \end{tikzcd}$$

Since kQ' is a polynomial ring in one variable, $\mathsf{QGr}(kQ') \equiv \mathsf{Mod}\, k$ and this equivalence sends $\mathcal{O}'$, the image of kQ' in $\mathsf{QGr}(kQ')$, to k. Since ι_* is an equivalence it follows that $\iota_* \mathcal{O}'$ is indecomposable. Now kQ is isomorphic as a graded left kQ-module to the direct sum of the projectives P_s and P_v. Right multiplication by the arrow a gives an isomorphism $P_v \to (P_s)_{\geq 1}(1)$ in $\mathsf{Gr}(kQ)$. Let $\mathcal{P}_s = \pi^* P_s$ and $\mathcal{P}_v = \pi^* P_v$. Then

$$\mathcal{O} = \mathcal{P}_v \oplus \mathcal{P}_s \cong \mathcal{P}_v \oplus \mathcal{P}_v(-1).$$

Hence $\mathcal{O} \ncong \iota_* \mathcal{O}'$.

Right multiplication by b induces an isomorphism $\mathcal{P}_v \xrightarrow{\sim} \mathcal{P}_v(-1)$.

4.5

We now prove the last sentence of Proposition 4.3.

Because s is a source, the two-sided ideal (e_s) is equal to kQe_s. Hence as a graded left kQ-module $i_*(kQ')$ is isomorphic to kQ/kQe_s which is isomorphic to $kQ(1-e_s)$. The claim that $\mathcal{O} \cong \iota_* \mathcal{O}' \oplus \mathcal{P}_s$ now follows from the decomposition $kQ = kQ(1-e_s) \oplus kQe_s$.

4.5.1

Because $\mathcal{O} \cong \iota_* \mathcal{O}' \oplus \mathcal{P}_s$, $\iota^! \mathcal{O} \cong \mathcal{O}' \oplus \iota^! \mathcal{P}_s$. Moreover, $\iota^! \mathcal{P}_s$ is isomorphic to $\oplus_{s(a)=s} \mathcal{P}'_{t(a)}(-1)$ where $\mathcal{P}'_{t(a)}$ is the image in $\mathsf{QGr}(kQ')$ of $(kQ')e_{t(a)}$.

5. Description of $S(Q)$

We will give two different descriptions of $S(Q)$.

In Section 5.1, we describe $S(Q)$ in terms of its Bratteli diagram. See [4,11] for information about Bratteli diagrams.

In Section 5.3, we show that when Q has no sinks or sources $S(Q)$ is isomorphic to the degree zero component of the Leavitt path algebra, $L(Q)$, associated to Q and, because $L(Q)$ is strongly graded,

$$\mathsf{QGr}(kQ) \equiv \mathsf{Gr}_\ell L(Q) \equiv \mathsf{Mod}_\ell L(Q)_0$$

where the subscript ℓ means *left* modules.

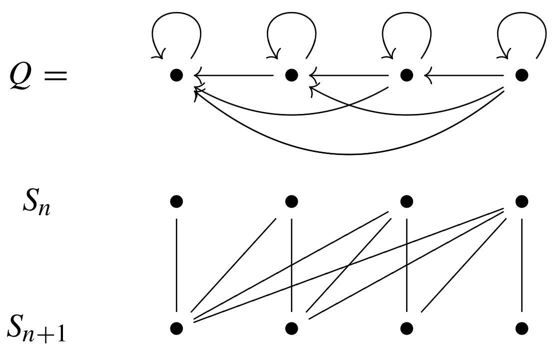
It is well-known that $L(Q)$ is a dense subalgebra of the Cuntz–Krieger algebra $\mathcal{O}_Q$, associated to Q. The philosophy of non-commutative geometry suggests that kQ is a homogeneous coordinate ring for a non-commutative scheme whose underlying non-commutative topological space has $\mathcal{O}_Q$ as its ring of "continuous $\mathbb{C}$-valued functions".

5.1. The Bratteli diagram for $S = S(Q)$

Because kI has $|I|$ isoclasses of simple modules and S_n is the endomorphism ring of a left kI-module, S_n is a product of at most $|I|$ matrix algebras of various sizes; although fewer than $|I|$ matrix algebras may occur in the product it is better to think there are $|I|$ of them with the proviso that some (those corresponding to sources) might be 0×0 matrices.

The nth level of the Bratteli diagram for $S(Q)$ therefore consists of $|I|$ vertices, each denoted by $\bullet$, that we label (n,i), $i \in I$. The vertex labelled (n,i) represents the summand $\mathrm{End}_{kI}(e_i(kQ))$ of S_n; this endomorphism ring is isomorphic to a matrix algebra $M_r(k)$ for some integer r; it is common practice to replace the symbol $\bullet$ at (n,i) by the integer r; we then say that r is the number at vertex (n,i). We do this for some of the examples in Section 6.

We will see that the number of edges from (n,i) to $(n+1,j)$ is the same as the number of edges from i to j in Q. The Bratteli diagram is therefore stationary in the terminology of [12], and the following example, which appears in [19], illustrates how to pass from the quiver to the associated Bratteli diagram.



5.2

Let $C := (c_{ij})_{i,j\in I}$ be the incidence matrix for Q with the convention that

$$c_{ij} = \text{the number of arrows from } j \text{ to } i.$$

The ij-entry in C^n, which we denote by $c_{ij}^{[n]}$, is the number of paths of length n from j to i. The number of paths of length n ending at vertex i is

$$p_{n,i} := \sum_{j\in I} c_{ij}^{[n]}.$$

Proposition 5.1. *The sum of the left kI-submodules of kQ_n isomorphic to E_i is equal to $e_i(kQ_n)$. Its dimension is equal to $p_{n,i}$ and*

$$S_n \cong \bigoplus_{i \in I} \operatorname{End}_{kI}\bigl(e_i(kQ_n)\bigr) \cong \bigoplus_{i \in I} M_{p_{n,i}}(k). \tag{5.1}$$

Referring to the Bratteli diagram for $\varinjlim S_n$, the number at the vertex labelled (n,i) is $p_{n,i}$, and the number of edges from (n,i) to $(n+1,j)$ is c_{ji}.

Composing with the inclusions and projections in (5.1), the components of the map $\theta_n : S_n \to S_{n+1}$ are the maps

$$\operatorname{End}_{kI}\bigl(e_i(kQ_n)\bigr) \to S_n \xrightarrow{\theta_n} S_{n+1} \to \operatorname{End}_{kI}\bigl(e_j(kQ_{n+1})\bigr)$$

that send a matrix in $\operatorname{End}_{kI}\bigl(e_i(kQ_n)\bigr)$ to c_{ji} "block-diagonal" copies of itself in $\operatorname{End}_{kI}\bigl(e_j(kQ_{n+1})\bigr)$.

Proof. The irreducible representation of Q at vertex i is $E_i = ke_i$. Since a path p ends at vertex i if and only if $p = e_i p$, the multiplicity of E_i in a composition series for kQ_n as a left kI-module is

$$\begin{aligned}[kQ_n : E_i] &= \dim_k e_i(kQ_n) \\ &= \text{the number of paths of length } n \text{ ending at } i \\ &= p_{n,i}.\end{aligned}$$

The existence of the left-most isomorphism in (5.1) follows at once from the fact that kI is a semisimple ring; the second isomorphism follows from the analysis in the first part of this paragraph.

Up to isomorphism, $\{E_i^* = \operatorname{Hom}_k(E_i, k) \mid i \in I\}$ is a complete set of simple right kQ modules. It follows that the kI-bimodules

$$E_{ij} := E_i \otimes E_j^*, \quad i, j \in I,$$

form a complete set of isoclasses of simple kI-bimodules. If a is an arrow from j to i there is a kI-bimodule isomorphism $ka \cong E_{ij}$ so the multiplicity of E_{ij} in kQ_1 is the number of arrows from j to i, i.e., $[kQ_1 : E_{ij}] = c_{ij}$. More explicitly, $E_{ij}^{\oplus c_{ij}} \cong e_i(kQ_1)e_j$.

The image in S_{n+1} of a map $f \in S_n$ is the map $kQ_1 \otimes f$. Hence if f is belongs to the component $\operatorname{End}_{kI}\bigl(e_i(kQ_n)\bigr)$ of S_n, the component of $kQ_1 \otimes f$ in $\operatorname{End}_{kI}\bigl(e_j(kQ_{n+1})\bigr)$ is $e_j kQ_1 e_i \otimes f$. But the dimension of $e_j kQ_1 e_i$ is $[kQ_1 : E_{ji}] = c_{ji}$. $\square$

5.3. Leavitt path algebras

Goodearl's survey [17] is an excellent introduction to Leavitt path algebras.

Since $\mathsf{QGr}(kQ)$ is unchanged when Q is replaced by the quiver obtained by repeatedly deleting sources and sinks, the essential case is when Q has no sinks or sources.

For the remainder of Section 5 we assume Q has no sinks or sources. This is equivalent to the hypothesis that $Q = Q^\circ$.

5.3.1

Under the hypothesis that $Q = Q^\circ$,

(1) the Leavitt path algebra of Q, $L(Q)$, is a universal localization of kQ in the sense of [7, Section 7.2] or [23, Ch. 4];

(2) $L(Q)$ is a strongly $\mathbb{Z}$-graded ring and $L(Q)_0 \cong S(Q)^{\mathrm{op}}$ so

$$\mathsf{Gr}_\ell L(Q) \equiv \mathsf{Mod}_\ell L(Q)_0 \equiv \mathsf{Mod}_r S(Q) \equiv \mathsf{QGr}(kQ)$$

where the subscripts ℓ and r denote left and right modules.

(3) $L(Q)$ is a dense subalgebra of the Cuntz–Krieger algebra $\mathcal{O}_Q$.

5.3.2

Statement (1) in 5.3.1 holds for *every* finite Q (i.e., the hypothesis $Q = Q^\circ$ is not needed) [2]. For 5.3.1(3), see [1,22].

The fact that $L(Q)$ is strongly graded when $Q = Q^\circ$ is proved in [18, Theorem 3.11]. We give an alternative proof of this in Proposition 5.5(6). A $\mathbb{Z}$-graded ring R is strongly graded if $R_n R_{-n} = R_0$ for all n. When R is strongly graded $\mathsf{Gr}R$ is equivalent to $\mathsf{Mod}\,R_0$ via the functor $M \rightsquigarrow M_0$ [9, Theorem 2.8]. That explains the left-most equivalence in (2). The second equivalence in 5.3.1(2) follows from the fact that $S(Q)^{\mathrm{op}} \cong L(Q)_0$ which we will prove in Theorem 5.4.

5.4

Let E_i be the 1-dimensional left kQ-module supported at vertex i and concentrated in degree zero. Because Q has no sinks E_i is not projective and its minimal projective resolution is

$$0 \longrightarrow \bigoplus_{a \in s^{-1}(i)} P_{t(a)} \xrightarrow{f_i} P_i \longrightarrow E_i \longrightarrow 0 \tag{5.2}$$

where $P_j = (kQ)e_j$ and the direct sum is over all arrows starting at i. Elements in the direct sum will be written as row vectors $(x_a, x_b, \ldots)$ with $x_a \in P_{t(a)}$, $x_b \in P_{t(b)}$, and so on. The map f_i is right multiplication by the column vector $(a, b, \ldots)^{\mathsf{T}}$ where $a, b, \ldots$ are the arrows starting at i, i.e.,

$$f_i(x_a, x_b, \ldots) = (x_a, x_b, \ldots)\begin{pmatrix} a \\ b \\ \vdots \end{pmatrix} = x_a a + x_b b + \cdots \in (kQ)e_i. \tag{5.3}$$

5.4.1. Definition of $L(Q)$ as a universal localization

We refer the reader to [7, Section 7.2] and [23, Ch. 4] for details about universal localization.

Let $\Sigma = \{f_i \mid i \in I\}$ and let

$$L(Q) := \Sigma^{-1}(kQ)$$

be the universal localization of kQ at Σ. Since Q will not change in this section we will often write L for $L(Q)$.

If $x \in kQ$ we continue to write x for its image in L under the universal Σ-inverting map $kQ \to L$. Let

$$P' := \bigoplus_{a \in s^{-1}(i)} Le_{t(a)}.$$

The defining property of L is that the map $kQ \to L$ is universal subject to the condition that applying $L \otimes_{kQ}$-to (5.2) produces an isomorphism

$$\mathrm{id}_L \otimes f_i : P' \xrightarrow{\cdot \begin{pmatrix} a \\ b \\ \vdots \end{pmatrix}} Le_i$$

for all i. Thus $L \otimes_{kQ} E_i = 0$ for all $i \in I$.

Every L-module homomorphism $Le_i \to Le_j$ is right multiplication by an element of L so [23, Theorem 4.1] tells us L is generated by kQ and elements $a^*, b^*, \ldots$ such that the inverse of $\mathrm{id}_L \otimes f_i$ is right multiplication by the row vector $(a^*, b^*, \ldots)$ where $a^* = e_i a^* e_{t(a)}$, etc. In particular, the defining relations for L are given by

$$(a^*, b^*, \ldots) \begin{pmatrix} a \\ b \\ \vdots \end{pmatrix} = \mathrm{id}_{Le_i} \quad \text{and} \quad \begin{pmatrix} a \\ b \\ \vdots \end{pmatrix} (a^*, b^*, \ldots) = \mathrm{id}_{P'}.$$

Since id_{Le_i} is right multiplication by e_i and $\mathrm{id}_{P'}$ is right multiplication by

$$\begin{pmatrix} e_{t(a)} & 0 & 0 & \cdots & 0 \\ 0 & e_{t(b)} & 0 & \cdots & 0 \\ \vdots & & & & \vdots \end{pmatrix},$$

$L = kQ\langle a^* \mid a \in Q_1 \rangle$ modulo the relations

$$\begin{array}{ll} e_{s(a)} a^* e_{t(a)} = a^* & \text{for all arrows } a \in Q_1, \\ aa^* = e_{t(a)} & \text{for all arrows } a \in Q_1, \\ ab^* = 0 & \text{if } a \text{ and } b \text{ are different arrows}, \\ e_i = \sum_{a \in s^{-1}(i)} a^* a & \text{for all } i \in I. \end{array}$$

5.4.2

Our $L(Q)$ is not defined in the same way as the algebra $L_k(Q)$ defined in [17, Section 1]. Because our notational convention for composition of paths is the reverse of that in [17, Section 1.1] the relations for $L(Q)$ just above are the opposite of those for $L_k(Q)$ in [17, Section 1.4]. (If we had defined $L(Q)$ by inverting homomorphisms between *right* instead of left modules we would have obtained the relations in [17, Section 1.4] but then our convention for composition of paths would have created the problems discussed at the end of [17, Section 1.8].) As a consequence, our $L(Q)$ is anti-isomorphic to $L_k(Q)$. *However*, as explained at the end of [17, Section 1.7], $L_k(Q)$ is anti-isomorphic to itself via a map that sends each arrow a to a^* and fixes e_i for each vertex i. Thus, our $L(Q)$ *is* isomorphic to the algebra $L_k(Q)$ defined in [17].

Proposition 5.2. *The algebra $L(Q) = \Sigma^{-1}(kQ)$ is isomorphic and anti-isomorphic to the Leavitt path algebra $L_k(Q)$ defined in* [17, *Section 1.4*].

5.4.3

Our convention for composition of paths is that used by (most of) the finite dimensional algebra community and by Raeburn [22, Remark 1.1.3]. However, it is not the convention adopted in [1] (see [1, Definition 2.9]).

5.5

If $a, b, \dots, c, d \in Q_1$ and $p = dc \dots ba$ we define $p^* := a^*b^* \dots c^*d^*$. If p and q are paths of the same length, then

$$pq^* = \delta_{p,q} e_{t(q)} = \delta_{p,q} e_{t(p)}. \tag{5.4}$$

If p is a path in Q of length n we write $|p| = n$. We give $L(Q)$ a $\mathbb{Z}$-grading by declaring that

$$\deg a = 1 \quad \text{and} \quad \deg a^* = -1 \text{ for all } a \in Q_1.$$

For completeness we include the following well-known fact.

Lemma 5.3. *The degree-n component of $L(Q)$ is*

$$L_n = \operatorname{span}\{p^*q \mid p \text{ and } q \text{ are paths such that } |q| - |p| = n\}.$$

Proof. Certainly $L(Q)$ is spanned by words in the letters a and a^*, $a \in Q_1$. Let w be a non-zero word and ab^* a subword of w with $a, b \in Q_1$. Since $w \neq 0$, $ab^* = aa^* = e_{t(a)}$; but $e_{t(a)}$ can be absorbed into the letters on either side of aa^* so, repeating this if necessary, $w = p^*q$ for some paths p and q. The degree of p^*q is $|q| - |p|$ so the result follows. □

Theorem 5.4. *The algebras $L(Q)_0$ and $S(Q)$ are anti-isomorphic,*

$$L(Q)_0 \cong S(Q)^{\mathrm{op}}.$$

Proof. By definition, $S(Q)$ is the ascending union of its subalgebras $S_n = \operatorname{End}_{kI}(kQ_n)$.

We will sometimes write L for $L(Q)$.

It is clear that L_0 is the ascending union of its subspaces

$$L_{0,n} := \operatorname{span}\{p^*q \mid p, q \in Q_n\}$$

and each $L_{0,n}$ is a subalgebra of L because $(p^*q)(x^*y) = \delta_{xq} p^*y$. It is also clear that $L_{0,n} \subset L_{0,n+1}$ because

$$p^*q = \sum_{a \in s^{-1}(t(q))} p^*a^*aq$$

(this uses the fact that Q has no sinks).

By [6, Proposition 4.1], the linear map $kQ \to L(Q)$, $p \mapsto p$, is injective for any quiver Q. As a consequence, there is a well-defined linear map

$$\Phi_n : L_{0,n} \to \operatorname{End}_{kI}(kQ_n), \qquad \Phi_n(p^*q)(r) := rp^*q \quad (=\delta_{rp} e_{t(p)} q),$$

for $r \in Q_n$. Since

$$\Phi_n(p^*q)\Phi_n(x^*y)(r) = rx^*yp^*q = \Phi_n(x^*yp^*q)(r)$$

Φ_n is an algebra anti-homomorphism. Since $\Phi_n(p^*q)(p) = q$ and $\Phi_n(p^*q)(r) = 0$ if $r \neq p$, Φ_n is injective.

We will now show that $L_{0,n}$ and $\operatorname{End}_{kI}(kQ_n)$ have the same dimension. This will complete the proof that Φ_n is an algebra isomorphism. Since

$$\{\text{non-zero } p^*q \mid p, q \in Q_n\} = \bigsqcup_{i \in I} \{p^*q \mid p, q \in e_i Q_n\} \tag{5.5}$$

it follows that

$$\dim_k L_{0,n} = \sum_{i \in I} |e_i Q_n|^2.$$

On the other hand,

$$kQ_n = \bigoplus_{i \in I} e_i(kQ_n) = \bigoplus_{i \in I} ke_i Q_n$$

and $ke_i Q_n$ is isomorphic as a left kI-module to a direct sum of $|e_i Q_n|$ copies of the simple kI-module ke_i. Hence

$$\operatorname{End}_{kI}(kQ_n) = \bigoplus_{i \in I} \operatorname{End}_k(ke_i Q_n) \cong \bigoplus_{i \in I} M_{|e_i Q_n|}(k)$$

is the direct sum of $|I|$ matrix algebras of sizes $|e_i Q_n|$, $i \in I$. This completes the proof that $L_{0,n}$ and $\operatorname{End}_{kI}(kQ_n)$ have the same dimension. Hence Φ_n is an isomorphism.

Rather than counting dimensions one can give a more honest proof by observing that the elements in $\{p^*q \mid p, q \in e_i Q_n\}$ are a set of matrix units for $\operatorname{End}_k(ke_i Q_n)$ with respect to the basis $e_i Q_n$.

To complete the proof of the theorem we will show the Φ_ns induce an isomorphism between the direct limits by showing that the diagram

$$\begin{tikzcd}
L_{0,n} \arrow[r] \arrow[d, "\Phi_n"'] & L_{0,n+1} \arrow[d, "\Phi_{n+1}"] \\
S_n \arrow[r, "f \mapsto \mathrm{id} \otimes f"'] & S_{n+1}
\end{tikzcd}$$

commutes. To this end, let $p^*q \in L_{0,n}$ where $p, q \in Q_n$, and let $r \in Q_n$ and $a \in Q_1$ be such that $ar \neq 0$. Thus $ar \in Q_{n+1}$ and $a \otimes r \in kQ_1 \otimes_{kI} kQ_n$. Going clockwise around the diagram, $\Phi_{n+1}(p^*q)(ar) = arp^*q$. Going anti-clockwise around the diagram, $(\mathrm{id}_{kQ_1} \otimes \Phi_n)(p^*q)(a \otimes r) = a \otimes \Phi_n(p^*q)(r) = a \otimes rp^*q = arp^*q$. The diagram commutes. $\square$

5.6

Most of the next result, but not part (6), is covered by [3, Section 2.3]. It makes use of a construction in [3] that we now recall.

Let R be a ring and $\phi : (\mathbb{Z}_{\geq 1}, +) \to \operatorname{End}_{\mathsf{Ring}}(R)$ a monoid homomorphism to the monoid of ring endomorphisms of R. Let $\mathbb{Z}[t_+]$ and $\mathbb{Z}[t_-]$ be the polynomial rings on the indeterminates t_+ and t_-. We define the ring $R[t_-, t_+; \phi]$ to be the quotient of the free coproduct

$$\mathbb{Z}[t_-] *_{\mathbb{Z}} R *_{\mathbb{Z}} \mathbb{Z}[t_+]$$

modulo the ideal generated by the relations

$$\begin{aligned} &t_- r = \phi(r) t_-, \quad r \in R, \\ &r t_+ = t_+ \phi(r), \quad r \in R, \\ &t_- t_+ = \phi(1) \\ &t_+ t_- = 1. \end{aligned}$$

The image of the map ϕ in the next result is the subalgebra $t_- L t_+ = eLe$ where e is the idempotent $t_- t_+$.

Proposition 5.5. *We continue to assume Q has neither sinks nor sources and L denotes $L(Q)$. For each $i \in I$ pick an arrow a_i ending at i and define*

$$t_+ := \sum_{i \in I} a_i \quad \text{and} \quad t_- := t_+^*.$$

Define a non-unital ring homomorphism $\phi : L \to L$ by $\phi(x) := t_- x t_+$. Then

(1) $t_+ t_- = 1$;
(2) *If $n > 0$, then $L_n = t_+^n L_0$ and $L_{-n} = L_0 t_-^n$;*
(3) *$L(Q)$ is generated by L_0 and t_+ and t_-;*
(4) *in the notation of [3], $L = L_0[t_-, t_+; \phi]$;*
(5) *if n is positive, $L_n = (L_1)^n$ and $L_{-n} = (L_{-1})^n$;*
(6) *$L(Q)$ is strongly graded.*

Proof. (1) This follows from the fact that $a_i a_i^* = e_i$ and $a_i a_j^* = 0$ if $i \neq j$.

(2) Suppose $n > 0$ and let $b \in L_n$ and $c \in L_{-n}$. Then $b = t_+^n t_-^n b$ and $c = c t_+^n t_-^n$, and $t_-^n b, c t_+^n \in L_0$. This proves (2) and (3) is an immediate consequence.

(4) See [3, Section 2 and Lemma 2.4].

(5) This is proved by induction. For example, assuming $n > 0$ and starting with (2) and the induction hypothesis, $L_n = t_+^n L_0$, we have

$$(L_1)^{n+1} = (L_1)^n L_1 = t_+^n L_0 L_1 = t_+^n L_1 = t_+^{n+1} L_0 = L_{n+1}.$$

The proof for L_{-n} is similar.

(6) Suppose $n > 0$. Then $1 = t_+^n t_-^n \in L_n L_{-n}$ so $L_n L_{-n} = L_0$.

Now

$$1 = \sum_{i \in I} e_i = \sum_{i \in I} \sum_{a \in s^{-1}(i)} a^* a = \sum_{a \in Q_1} a^* a$$

so $1 \in L_{-1} L_1$. It now follows from (5) that $L_{-n} L_n = L_0$. $\square$

Dade's Theorem [9, Theorem 2.8] on strongly graded rings gives the next result.

Corollary 5.6. *If Q has neither sinks nor sources, there is an equivalence of categories*

$$\mathsf{Gr} L(Q) \equiv \mathsf{Mod}\, L_0.$$

Because $L(Q)$ is strongly graded, [9, (2.12a)] tells us that each L_n is an invertible L_0-bimodule and the multiplication in L gives L_0-bimodule isomorphisms

$$L_m \otimes_{L_0} L_n \xrightarrow{\sim} L_{m+n}$$

for all m and n. In other words, if we use the multiplication in L to identify L_1^{-1} with L_{-1} and define $L_1^{\otimes(-r)} := (L_{-1})^{\otimes r}$ for all $r > 0$, then

$$L = \bigoplus_{n=-\infty}^{\infty} L_1^{\otimes n}$$

where the tensor product is taken over L_0.

5.7

We have now completed the proof that

$$\mathsf{QGr}(kQ) \equiv \mathsf{Mod}_r S(Q) \equiv \mathsf{Mod}_\ell L(Q)_0 \equiv \mathsf{Gr}_\ell L(Q)$$

when Q has no sinks or sources. It is possible to prove the equivalence $\mathsf{QGr}(kQ) \equiv \mathsf{Gr}_\ell L(Q)$ directly by modifying the arguments in Section 4 of [25] for the free algebra $k\langle x_0, \dots, x_n\rangle$ so they apply to kQ. The required changes are minimal and straightforward so we leave the details of the next three results to the reader.

The next result is proved in [2, Theorem 4.1] but the following proof is more direct.

Proposition 5.7. *The ring L is flat as a right kQ-module.*

Proof. Since L is the ascending union of the finitely generated free right kQ-modules

$$F_n = \sum_{p\in Q_n} p^*(kQ) = \bigoplus_{p\in Q_n} p^*(kQ) \tag{5.6}$$

it is a flat right kQ-module. $\square$

A version of the following result for finitely presented not-necessarily-graded modules is given in [2, Section 6].

Proposition 5.8. *If $M \in \mathsf{Gr}(kQ)$, then $L \otimes_{kQ} M = 0$ if and only if $M \in \mathsf{Fdim}(kQ)$.*

Proof. The argument in [25, Proposition 4.3] works provided one replaces "free module" by "projective module". $\square$

A version of the next result for finitely presented not-necessarily-graded modules is given in [2, Section 6].

Theorem 5.9. *Let $\pi^* : \mathsf{Gr}(kQ) \to \mathsf{QGr}(kQ)$ be the quotient functor and $i^* = L \otimes_{kQ} - : \mathsf{Gr}(kQ) \to \mathsf{Gr} L$. Then*

$$\mathsf{QGr}(kQ) \equiv \mathsf{Gr} L$$

via a functor $\alpha^* : \mathsf{QGr}(kQ) \to \mathsf{Gr} L$ such that $\alpha^*\pi^* = i^*$.

Proof. The argument in [25, Theorem 4.4] works here. $\square$

5.8

The referee pointed out that the equivalence between $\mathsf{QGr}(kQ)$ and $\mathsf{Gr} L(Q)$ can be understood using ideas about perpendicular subcategories of $\mathsf{Gr}(kQ)$, as in [15], and ideas about universal localization that are implicit in [14].

We defined $L(Q)$ as the universal localization $\Sigma^{-1}(kQ)$. Let $\varphi : kQ \to L(Q)$ be the universal Σ-inverting map. As Schofield remarks, [23, p. 56], φ is an epimorphism in the category of rings. It is also an epimorphism in the category of graded rings. The restriction functor $\varphi_* : \mathsf{Gr}(L(Q)) \to \mathsf{Gr}(kQ)$ therefore embeds $\mathsf{Gr}(L(Q))$ as a fully exact subcategory (see [14, p. 280] for the definition) of $\mathsf{Gr}(kQ)$.

Because $L(Q)$ is flat as a right kQ-module (Proposition 5.7) $\varphi : kQ \to L(Q)$ satisfies the slightly stronger property of being a homological epimorphism in the category of graded rings (i.e., the equivalent properties of [15, Theorem 4.4] are satisfied).

6. Examples

6.1

If $\dim_k kQ < \infty$, then $S(Q) = 0$.

6.2

If Q is the cyclic quiver $\begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & \cdots \arrow[r] & n \arrow[lll, bend left] \end{tikzcd}$ then $S(Q) \cong k^n$.

6.3

By [25], if Q has one vertex and r arrows, then kQ is the free algebra $k\langle x_1, \dots, x_r\rangle$, the Bratteli diagram for $S(Q)$ is

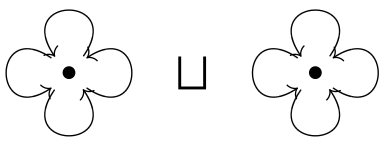
$$1 \overset{\vdots}{\rightrightarrows} r \overset{\vdots}{\rightrightarrows} r^2 \overset{\vdots}{\rightrightarrows} r^3 \overset{\vdots}{\rightrightarrows} r^4 \cdots$$

where there are r arrows between adjacent vertices, and

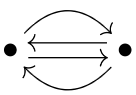
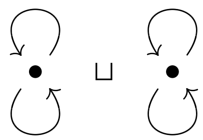
$$S(Q) \cong \varinjlim_n M_r(k)^{\otimes n}.$$

6.4

Different quivers can have a common Veronese quiver. For example,



is the 2-Veronese quiver of both

$Q =$  and $Q' =$ 

It now follows from Theorem 1.8 that $\mathsf{QGr}(kQ) \equiv \mathsf{QGr}(kQ')$ and, by the comments after Theorem 1.8,

$$\begin{aligned} S(Q) &= S(Q^{(2)}) \\ &= S(Q') \\ &\cong \Big(\varinjlim_n M_2(k)^{\otimes n}\Big) \oplus \Big(\varinjlim_n M_2(k)^{\otimes n}\Big). \end{aligned}$$

6.5

It is not obvious that $\mathsf{QGr}(k\langle x,y\rangle/(y^2))$ is equivalent to $\mathsf{QGr}(kQ)$ for some quiver Q.

Proposition 6.1. *Let*

$$Q = \begin{tikzcd} 1 \arrow[loop left] \arrow[r, bend left] & 2 \arrow[l, bend left] \end{tikzcd}$$

The Bratteli diagram for $S(Q)$ is

$$\begin{tikzcd} 1 \arrow[r] \arrow[dr] & 2 \arrow[r] \arrow[dr] & 3 \arrow[r] \arrow[dr] & 5 \arrow[r] \arrow[dr] & 8 \arrow[r] \arrow[dr] & \cdots \\ 1 \arrow[ur] & 1 \arrow[ur] & 2 \arrow[ur] & 3 \arrow[ur] & 5 \arrow[ur] & \cdots \end{tikzcd}$$

and

$$\mathsf{QGr}(kQ) \equiv \mathsf{QGr}\,\frac{k\langle x,y\rangle}{(y^2)}.$$

Proof. We will use the notation E_i and E_{ij} that appears in the proof of Proposition 5.1.

The powers of the incidence matrix for Q are

$$C^n = \begin{pmatrix} f_n & f_{n-1} \\ f_{n-1} & f_{n-2} \end{pmatrix}$$

where $f_{-1} = 0$, $f_0 = f_1 = 1$, and $f_{n+1} = f_n + f_{n-1}$ for $n \geq 1$. As a kI-bimodule

$$kQ_n \cong E_{11}^{f_n} \oplus E_{12}^{f_{n-1}} \oplus E_{21}^{f_{n-1}} \oplus E_{22}^{f_{n-2}}$$

and as a left kI-module

$$kQ_n \cong E_1^{f_{n+1}} \oplus E_2^{f_n}.$$

It follows that $S_n \cong M_{f_n}(k) \oplus M_{f_{n-1}}(k)$ and the Bratteli diagram is as claimed. But this Bratteli diagram also arises in [26] where it is shown that

$$\mathsf{Mod}\, S(Q) \equiv \mathsf{QGr}\,\frac{k\langle x,y\rangle}{(y^2)}.$$

It also follows from the main result in [19], which was written after this paper, that $\mathsf{QGr}(k\langle x,y\rangle/(y^2))$ is equivalent to $\mathsf{QGr}(kQ)$. $\square$

As explained in [26], we can interpret $k\langle x,y\rangle/(y^2)$, and therefore kQ, as a non-commutative homogeneous coordinate ring of the space of Penrose tilings of the plane. This is consistent with Connes' view [8, Section II.3] of the norm closure of $S(Q)$, over $\mathbb{C}$, as a C^*-algebra coordinate ring for the space of Penrose tilings.

For each integer $r \geq 1$ there is a quiver Q such that $\mathsf{QGr}(kQ)$ is equivalent $\mathsf{QGr}\, k\langle x,y\rangle/(y^{r+1})$; see Section 7.2 and [19].

6.6

The previous example can be generalized as follows.

Proposition 6.2. *Let*

$$Q = \begin{tikzcd} \bullet \arrow[loop left, "m"] \arrow[r, bend left] & \bullet \arrow[l, bend left] \end{tikzcd}$$

where there are m arrows from the left-hand vertex to itself. The Bratteli diagram for $S(Q)$ is

$$\begin{tikzcd} 1 \arrow[r, "m"] \arrow[dr] & q_1 \arrow[r, "m"] \arrow[dr] & q_2 \arrow[r, "m"] \arrow[dr] & q_3 \arrow[r, "m"] \arrow[dr] & \cdots \\ 1 \arrow[ur] & q_0 \arrow[ur] & q_1 \arrow[ur] & q_2 \arrow[ur] & \cdots \end{tikzcd}$$

where there are m arrows from q_n to q_{n+1} in the top row, and the numbers q_n are given by $q_0 = q_{-1} = 1$ and $q_{n+1} = mq_n + q_{n-1}$ for $n \geq 0$. Furthermore, the Hilbert series of kQ, viewed as an element of $K_0(kI)[[t]] = (\mathbb{Z} \times \mathbb{Z})[[t]]$, is

$$H_{kQ}(t) = \frac{1}{1 - mt + t^2}(1 + t, 1 + (1 - m)t)$$

with the first component of $H_{kQ}(t)$ giving the multiplicity in kQ_n of the simple kI-module that is supported at the left-most vertex.

6.7

If Q and Q' become the same after repeatedly deleting sources and sinks, then $\mathsf{QGr}(kQ) \equiv \mathsf{QGr}(kQ')$ by Theorem 4.1. Therefore $S(Q)$ and $S(Q')$ are Morita equivalent, but they need not be isomorphic as the next example shows. The quivers Q and Q' are formed by adjoining a sink, respectively, a source, to

$$Q^\circ = \begin{tikzcd} \bullet \arrow[loop right, "x"] \end{tikzcd} \tag{6.1}$$

By Theorem 4.1

$$\mathsf{QGr}(kQ) \equiv \mathsf{QGr}(kQ') \equiv \mathsf{QGr}(kQ^\circ) \equiv \mathsf{Qcoh}(\operatorname{Proj} k[x]) \equiv \mathsf{Mod}\, k.$$

In this example, $Q' = Q^{\mathrm{op}}$.

Proposition 6.3. *Let*

$$Q = \begin{tikzcd} 1 \arrow[loop left] \arrow[r] & 2 \end{tikzcd} \quad \text{and} \quad Q' = \begin{tikzcd} 1 \arrow[loop left] & 2. \arrow[l] \end{tikzcd}$$

The Bratteli diagram for $S(Q)$ is

$$\begin{tikzcd} 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 \arrow[r] \arrow[ur] & 1 \arrow[r] \arrow[ur] & 1 \arrow[r] \arrow[ur] & 1 \arrow[r] \arrow[ur] & \cdots \end{tikzcd}$$

and that for $S(Q')$ is

$$2 \longrightarrow 2 \longrightarrow 2 \longrightarrow 2 \longrightarrow \cdots$$

Furthermore, $S(Q) \cong k$ and $S(Q') \cong M_2(k)$.

Proof. The incidence matrices for Q and Q' are

$$C = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = C^n \quad \text{and} \quad C' = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = (C')^n$$

so

$$kQ_n \cong E_{11} \oplus E_{21} \quad \text{and} \quad kQ'_n \cong E_{11} \oplus E_{12}.$$

The result follows easily from this. $\square$

Here is another way to show $\mathsf{QGr}(kQ) \equiv \mathsf{Mod}\, k$ for the Q in Proposition 6.3. Write x for the loop at vertex 1 and w for the arrow from 1 to 2. The path algebra is

$$kQ = \begin{pmatrix} k[x] & 0 \\ wk[x] & k \end{pmatrix}.$$

The two-sided ideal generated by e_2 is

$$T = \begin{pmatrix} 0 & 0 \\ wk[x] & k \end{pmatrix},$$

which is the ideal T in Lemma 2.1, T is annihilated on the left by

$$\begin{pmatrix} k[x] & 0 \\ wk[x] & 0 \end{pmatrix}$$

so, as a left kQ-module, T is a sum of finite dimensional modules. Therefore

$$\mathsf{QGr}(kQ) \equiv \mathsf{QGr}\left(\frac{kQ}{T}\right) \equiv \mathsf{QGr}(k[x]) \equiv \mathsf{Qcoh}(\operatorname{Proj} k[x]) \equiv \mathsf{Mod}\, k.$$

6.8

Let $\mathsf{K}(\mathcal{H})$ be the C^*-algebra of compact operators on an infinite dimensional separable Hilbert space $\mathcal{H}$. The direct limit in the category of C^*-algebras of the directed system with Bratteli diagram (6.2) is isomorphic to $\mathsf{K}(\mathcal{H}) \oplus \mathbb{C}\mathrm{id}_{\mathcal{H}}$. The algebra $S(Q)$ in Proposition 6.4 is the algebraic analogue of $\mathsf{K}(\mathcal{H}) \oplus \mathbb{C}\mathrm{id}_{\mathcal{H}}$.

Proposition 6.4. *Let*

$$Q = \begin{tikzcd} 1 \arrow[loop left, "x"] & 2 \arrow[l, "w"'] \arrow[loop right, "y"] \end{tikzcd}$$

The Bratteli diagram for $S(Q)$ is

$$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & 4 \arrow[r] & 5 \arrow[r] & \cdots \\ 1 \arrow[r] \arrow[ur] & 1 \arrow[r] \arrow[ur] & 1 \arrow[r] \arrow[ur] & 1 \arrow[r] \arrow[ur] & 1 \arrow[r] \arrow[ur] & \cdots \end{tikzcd} \tag{6.2}$$

and

$$S(Q) \cong M_\infty(k) \oplus k.I$$

where $M_\infty(k)$ is the algebra without unit consisting of $\mathbb{N} \times \mathbb{N}$ matrices with only finitely many non-zero entries and $M_\infty(k) \oplus k.I$ is the algebra of $\mathbb{N} \times \mathbb{N}$ matrices that differ from a scalar multiple of the $\mathbb{N} \times \mathbb{N}$ identity matrix in only finitely many places.

Proof. One can see directly that

$$kQ \cong \begin{pmatrix} k[x] & k[x]\otimes w\otimes k[y] \\ 0 & k[y] \end{pmatrix}.$$

As a kI-bimodule, $kQ_1 = kx \oplus kw \oplus ky \cong E_{11} \oplus E_{12} \oplus E_{22}$. The nth power of the incidence matrix for Q is

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

so, as a kI-bimodule, $kQ_n \cong E_{11} \oplus E_{12}^{\oplus n} \oplus E_{22}$ and as a left kI-module

$$kQ_n \cong E_1^{\oplus(n+1)} \oplus E_2.$$

Therefore $\operatorname{End}_{kI}(kQ_n) \cong M_{n+1}(k) \oplus k$.

In order to give an explicit description of the homomorphisms

$$\theta_n : \operatorname{End}_{kI}(kQ_n) \to \operatorname{End}_{kI}(kQ_{n+1})$$

we take the ordered basis for kQ_n consisting of the $n+2$ elements

$$x^n, x^{n-1}w, x^{n-2}wy, \dots, xwy^{n-2}, wy^{n-1}, y^n.$$

The linear span of x^n is a kI-bimodule isomorphic to E_{11}. The linear span of the next n elements, those with a w in them, is a kI-bimodule isomorphic to $E_{12}^{\oplus n}$. The linear span of y^n is a kI-bimodule isomorphic to E_{22}. We will write an element of $f \in S_n = \operatorname{End}_{kI}(kQ_n)$ as

$$f = (A, \lambda) \in M_{n+1}(k) \oplus k$$

where A represents the restriction of f to $E_1^{\oplus(n+1)}$ with respect to the ordered basis, and $f(y^n) = \lambda y^n$.

The homomorphism $\theta_n : S_n \to S_{n+1}$ is defined in Section 1.2. In this example,

$$\begin{aligned} kQ_{n+1} &= kQ_1 \otimes_{kI} kQ_n \\ &= \big(E_{11} \otimes_k E_1^{n+1}\big) \oplus \big(E_{12} \otimes_k E_2\big) \oplus \big(E_{22} \otimes_k E_2\big) \\ &= x \otimes \operatorname{span}\{x^n, x^{n-1}w, x^{n-2}wy, \dots, xwy^{n-2}, wy^{n-1}\} \\ &\quad \oplus (kw \otimes ky^n) \oplus (ky \otimes ky^n). \end{aligned}$$

Therefore

$$\theta_n(A, \lambda) = (A + \lambda e_{n+2,n+2}, \lambda) = \left(\begin{pmatrix} A & 0 \\ 0 & \lambda \end{pmatrix}, \lambda\right).$$

Define $\phi_n : S_n \to M_\infty(k) \oplus kI$ by

$$\phi_n(A, \lambda) := A + \lambda I_{n+1}$$

where $I_n = I - (e_{11} + \cdots + e_{nn}) \in M_\infty(k) \oplus kI$. Since $AI_{n+1} = I_{n+1}A = 0$ and $I_{n+1}^2 = 0$, ϕ_n is a homomorphism of k-algebras sending the identity to the identity. It is straightforward to check that $\phi_{n+1}\theta_n = \phi_n$. It follows that all the ϕ_ns factor through a single homomorphism $\phi : S(Q) = \varinjlim S_n \to M_\infty(k) \oplus kI$. We leave the reader to check that ϕ is an isomorphism. $\square$

7. Relation to finite dimensional algebras with radical square zero

7.1. The work of Xiao-Wu Chen [5]

The singularity category of a left coherent ring R, denoted as $\mathsf{D}_{\mathrm{sg}}(R)$, is the quotient of the derived category $\mathsf{D}^b(\mathsf{mod}\,R)$ of bounded complexes of finitely presented left R-modules by its full subcategory of perfect complexes.

The following is a simplified version of the main result in [5].

Theorem 7.1 (*X.-W. Chen [5]*)**.** *Let Λ be a finite dimensional k-algebra and J its Jacobson radical. Suppose $J^2 = 0$. Viewing J as a left Λ-module, define*

$$S(\Lambda) := \varinjlim \operatorname{End}_\Lambda\bigl(J^{\otimes n}\bigr)$$

and the $S(\Lambda)$-bimodule

$$B := \varinjlim \operatorname{Hom}_\Lambda\bigl(J^{\otimes n}, J^{\otimes n-1}\bigr)$$

where the maps in the directed systems are $f \mapsto \mathrm{id}_J \otimes f$. Then

- *B is an invertible $S(\Lambda)$-bimodule with inverse $\varinjlim \operatorname{Hom}_\Lambda(J^{\otimes n}, J^{\otimes n+1})$;*
- *$S(\Lambda)$ is a von Neumann regular ring;*
- *J is a progenerator in $\mathsf{D}_{\mathrm{sg}}(\Lambda)$ with endomorphism ring $S(\Lambda)$;*
- *$\operatorname{Hom}_{\mathsf{D}_{\mathrm{sg}}(\Lambda)}(J, -)$ is an equivalence of triangulated categories*

  $$\bigl(\mathsf{D}_{\mathrm{sg}}(\Lambda), [1]\bigr) \equiv \bigl(\mathsf{proj}\, S(\Lambda), - \otimes_{S(\Lambda)} B\bigr)$$

  where $\mathsf{proj}\, S(\Lambda)$ is the category of finitely generated projective right $S(\Lambda)$-modules, and $- \otimes_{S(\Lambda)} B$ is the translation functor on $\mathsf{proj}\, S(\Lambda)$.

If the field k in Chen's theorem is algebraically closed then Λ is Morita equivalent to $kQ/kQ_{\geq 2}$ for a suitable quiver Q.

Theorem 7.2. *Let k be a field, Q a quiver, and $\Lambda = kQ/kQ_{\geq 2}$. The rings $S(\Lambda)$ and $S(Q)$ are isomorphic and there is an equivalence of triangulated categories*

$$\bigl(\mathsf{D}_{\mathsf{sg}}(\Lambda), [1]\bigr) \equiv \bigl(\mathsf{qgr}(kQ), (-1)\bigr).$$

Proof. The Jacobson radical of Λ is $J = kQ_{\geq 1}/kQ_{\geq 2}$. We identify J with kQ_1. This identification is compatible with the kI-bimodule structures.

Since $J^2 = 0$,

$$\begin{aligned} J \otimes_\Lambda \cdots \otimes_\Lambda J &= J \otimes_{\Lambda/J} \cdots \otimes_{\Lambda/J} J \\ &= (kQ_1) \otimes_{kI} \cdots \otimes_{kI} (kQ_1) \end{aligned}$$

so $\operatorname{End}_\Lambda(J^{\otimes n}) = \operatorname{End}_{kI}(kQ_n)$; i.e., the individual terms in the directed systems defining $S(\Lambda)$ and $S(Q)$ are the same. But the maps in the directed systems are of the form $f \mapsto \mathrm{id} \otimes f$ in both cases so the direct limits $S(\Lambda)$ and $S(Q)$ are isomorphic.

Since $S(\Lambda)$ is von Neumann regular $\mathsf{proj}\, S(\Lambda)$ is equal to $\mathsf{mod}_r\, S(\Lambda)$, the category of finitely presented right $S(\Lambda)$-modules. The translation functor on $\mathsf{proj}\, S(\Lambda)$ is $M \mapsto M \otimes_{S(\Lambda)} B$. Hence Chen's Theorem says that

$$\bigl(\mathsf{D}_{\mathsf{sg}}(\Lambda), [1]\bigr) \equiv \bigl(\mathsf{mod}_r\, S(Q), - \otimes_{S(Q)} B\bigr).$$

We write Σ for the translation functor $- \otimes_{S(Q)} B$.

An auto-equivalence of an abelian category having a generator is determined by its effect on the generator. The generator $\mathcal{O}$ for $\mathsf{qgr}(kQ)$ corresponds to the generator $S(Q)$ under the equivalence $\operatorname{Hom}_{\mathsf{QGr}}(\mathcal{O}, -) : \mathsf{qgr}(kQ) \equiv \mathsf{mod}\, S(Q)$; since $\Sigma(S(Q)) = B$, the auto-equivalence of $\mathsf{qgr}(kQ)$ that corresponds to Σ is the unique auto-equivalence σ such that $\operatorname{Hom}_{\mathsf{QGr}}(\mathcal{O}, \sigma(\mathcal{O})) = B$. The calculation in the next paragraph shows that $\sigma(\mathcal{O}) = \mathcal{O}(-1)$, so the auto-equivalence of $\mathsf{qgr}(kQ)$ that corresponds to Σ is $\mathcal{F} \mapsto \mathcal{F}(-1)$.

The equivalence $\mathsf{qgr}(kQ) \to \mathsf{mod}_r\, S(Q)$ sends $\mathcal{O}(-1)$ to

$$\begin{aligned}
\operatorname{Hom}_{\mathsf{QGr}}(\mathcal{O}, \mathcal{O}(-1)) &= \varinjlim \operatorname{Hom}_{\mathsf{Gr}(kQ)}(kQ_{\geq n}, kQ(-1)) \\
&= \varinjlim \operatorname{Hom}_{\mathsf{Gr}(kQ)}(kQ_{\geq n}, kQ(-1)_{\geq n}) \\
&= \varinjlim \operatorname{Hom}_{kI}(kQ_n, kQ(-1)_n) \\
&= \varinjlim \operatorname{Hom}_{kI}(kQ_n, kQ_{n-1}) \\
&= \varinjlim \operatorname{Hom}_{kI}(J^{\otimes n}, J^{\otimes(n-1)}) \\
&= B.
\end{aligned}$$

This completes the proof that $(\mathsf{mod}\, S(\Lambda), - \otimes_{S(\Lambda)} B) \equiv (\mathsf{qgr}(kQ), (-1))$. $\square$

7.2. An example

Fix an integer $r \geq 1$, let

$$Q = \begin{tikzcd} 0 \arrow[loop left] \arrow[r] & 1 \arrow[l, bend left] \arrow[r] & 2 \arrow[ll, bend left] \arrow[r] & \cdots & \cdots \arrow[r] & r \arrow[lllll, bend left] \end{tikzcd} \tag{7.1}$$

and define

$$\Lambda_r := kQ/kQ_{\geq 2}.$$

The algebra $k\langle x, y\rangle/(y^{r+1})$ in the next result is studied in [26]. When $r = 1$ it behaves as a non-commutative homogeneous coordinate ring for the space of Penrose tilings of the plane. Thus the equivalence of categories in the next result says that the path algebra of the quiver in (7.1) for $r = 1$ is also a homogeneous coordinate ring for the space of Penrose tilings.

Proposition 7.3. *The following categories are equivalent:*

$$\mathsf{D}_{\mathsf{sg}}(\Lambda_r) \equiv \mathsf{qgr}(kQ) \equiv \mathsf{qgr}\left(\frac{k\langle x, y\rangle}{(y^{r+1})}\right).$$

Proof. The incidence matrix for Q is the $(r+1) \times (r+1)$ matrix

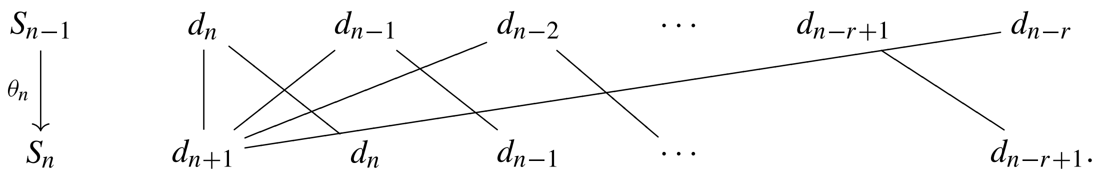
$$C = \begin{pmatrix} 1 & 1 & 1 & \cdots & & 1 \\ 1 & 0 & 0 & \cdots & & 0 \\ 0 & 1 & 0 & \cdots & & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}. \tag{7.2}$$

It follows that $kQ_1 \cong E_0^{r+1} \oplus E_1 \oplus \cdots \oplus E_r$ as a left kI-module and, as a kI-bimodule,

$$kQ_1 \cong E_{10} \oplus E_{21} \oplus \cdots \oplus E_{r\,r-1} \oplus \left(\bigoplus_{i=0}^{r} E_{0i}\right).$$

Thus, the dimension vector for kQ_1 as a left kI-module is $(r+1, 1, \dots, 1)^{\mathsf{T}}$ and the dimension vector for kQ_n as a left kI-module is $C^{n-1}(r+1, 1, \dots, 1)^{\mathsf{T}}$.

Define $d_0 = r+1, d_1 = d_2 = \cdots = d_r = 1$, and $d_{n+1} = d_n + \cdots + d_{n-r}$ for $n \geq r$. The dimension vector for kQ_{n+1} as a left kI-module is therefore $(d_{n+1}, d_n, \dots, d_{n-r+1})$ and the Bratteli diagram for $S(Q)$, written from top to bottom, is repeated copies of



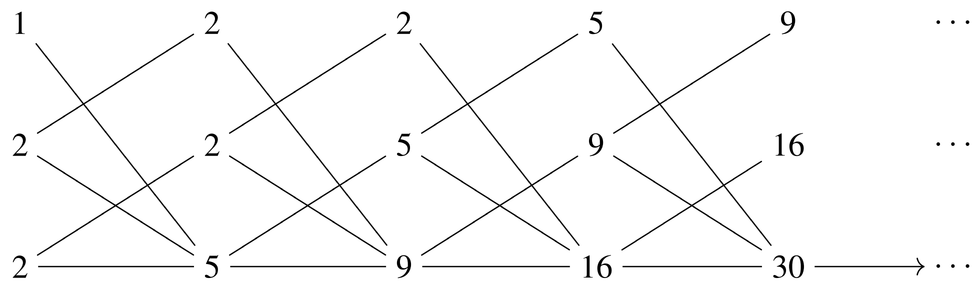
In [26] it was show that

$$\mathsf{QGr}\,\frac{k\langle x, y\rangle}{(y^{r+1})} \equiv \mathsf{Mod}\,R_r \tag{7.3}$$

where R_r is the ultramatricial algebra associated to a Bratteli diagram that has the same underlying (unlabelled) graph as that for $S(Q)$. By Proposition 7.4, R_r and $S(Q)$ are Morita equivalent. $\square$

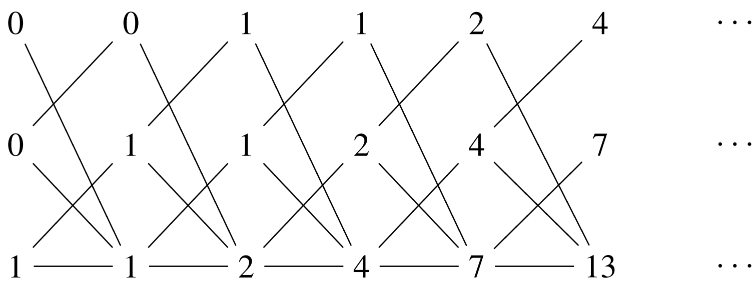
We illustrate the remark in the last paragraph of the previous proof.

The Bratteli diagram for $S(\Lambda_2)$, written from left to right, is



$$S_1(\Lambda_2) \longrightarrow S_2(\Lambda_2) \longrightarrow S_3(\Lambda_2) \longrightarrow S_4(\Lambda_2) \longrightarrow S_5(\Lambda_2) \longrightarrow$$

The Bratteli diagram for the ring R_2 in (7.3), written from left to right, is



7.3

The proof of the next result was shown to me by Ken Goodearl and I thank him for allowing me to include it. Although the result is implicit in [1,10,28], Goodearl's proof is simple and direct.

Proposition 7.4. *Suppose A and B are ultramatricial k-algebras formed from Bratteli diagrams on the same underlying (i.e., unlabelled) directed graph. Then A is Morita equivalent to B.*

Proof (*Goodearl*)**.** The directed graph determines a directed system of free abelian groups whose direct limit as an ordered group is isomorphic to $K_0(A)$ and to $K_0(B)$. Since $K_0(A)$ and $K_0(B)$ are isomorphic as ordered groups Elliott's results show that A and B are Morita equivalent.

To see this directly, choose an ordered group isomorphism $f : K_0(A) \to K_0(B)$, and let P be a finitely generated projective right A-module such that $[P] = f^{-1}([B])$. Since $[B]$ is an order-unit in $K_0(B)$, $[P]$ is an order-unit in $K_0(A)$. This implies that P is a generator in $\mathsf{Mod}\,A$. The category equivalence given by $- \otimes_C P$, where $C = \operatorname{End}_A P$, takes the category of finitely generated projective right C-modules to the category of finitely generated projective right A-modules with C mapping to P. The composition

$$K_0(C) \xrightarrow{\ -\otimes_C P\ } K_0(A) \xrightarrow{\ f\ } K_0(B)$$

is an isomorphism of ordered abelian groups that sends $[C]$ to $[B]$, i.e., it is an isomorphism of ordered abelian groups with order unit. Elliott's classification theorem therefore implies $C \cong B$. But C and A are Morita equivalent via P. $\square$

Acknowledgments

Part of this paper was written during a visit to Bielefeld University in April 2011. I thank Henning Krause for the invitation and the university for providing excellent working conditions.

Conversations with Gene Abrams, Pere Ara, Ken Goodearl, Henning Krause, Mark Tomforde, and Michel Van den Bergh, shed light on the material in this paper and I thank them all for sharing their ideas and knowledge.

I am especially grateful to Xiao-Wu Chen, Roozbeh Hazrat, and the referee, for reading earlier versions of this paper and pointing out typos and obscurities. Their comments and suggestions have improved the final version of this paper. I thank Chen for sending me an early version of his paper [5] and Hazrat for bringing his paper [18] to my attention (see the remark after Theorem 1.3).

References

[1] G. Abrams, M. Tomforde, Isomorphism and Morita equivalence of graph algebras, Trans. Amer. Math. Soc. 363 (2011) 3733–3767.

[2] P. Ara, M. Brustenga, Module theory over Leavitt path algebras and K-theory, J. Pure Appl. Algebra 214 (2010) 1131–1151.

[3] P. Ara, M.A. González-Barroso, K.R. Goodearl, E. Pardo, Fractional skew monoid rings, J. Algebra 278 (1) (2004) 104–126.

[4] O. Bratteli, Inductive limits of finite dimensional C^*-algebras, Trans. Amer. Math. Soc. 171 (1979) 195–234.

[5] X.-W. Chen, The singularity category of an algebra with radical square zero, Doc. Math. 16 (2011) 921–936.

[6] X.-W. Chen, Irreducible representations of Leavitt path algebras, arXiv: 1108.3726v1.

[7] P.M. Cohn, Free Rings and their Relations, second ed., Academic Press, London, 1985.

[8] A. Connes, Noncommutative Geometry, Academic Press, San Diego, 1994.
[9] E.C. Dade, Group graded rings and modules, Math. Z. 174 (1980) 241–262.
[10] D. Drinen, Viewing AF-algebras as graph algebras, Proc. Amer. Math. Soc. 128 (2000) 1991–2000.
[11] E.G. Effros, Dimensions and C^*-Algebras, in: CBMS Regional Conf. Ser. in Math., vol. 46, Amer. Math. Soc., Providence, RI, 1981.
[12] G.A. Elliott, On the classification of inductive limits of sequences of semisimple finite-dimensional algebras, J. Algebra 38 (1976) 29–44.
[13] P. Gabriel, Des Catégories Abéliennes, Bull. Soc. Math. France 90 (1962) 323–448.
[14] P. Gabriel, J.-A. de la Peña, Quotients of representation-finite algebras, Comm. Algebra 15 (1987) 279–307.
[15] W. Geigle, H. Lenzing, Perpendicular categories with applications to representations and sheaves, J. Algebra 144 (1991) 273–343.
[16] K.R. Goodearl, Von Neumann Regular Rings, in: Monographs and Studies in Mathematics, vol. 4, Pitman, London, San Francisco, Melbourne, 1979.
[17] K.R. Goodearl, Leavitt path algebras and direct limits, in: Rings, Modules and Representations, in: Contemp. Math., vol. 480, Amer. Math. Soc., Providence, RI, 2009, pp. 165–187.
[18] R. Hazrat, The graded structure of Leavitt path algebras, arXiv: 1005.1900.
[19] J.H. Hong, W. Szymański, Quantum spheres and projective spaces as graph algebras, Comm. Math. Phys. 232 (2002) 157–188.
[20] H. Krause, The Spectrum of a Module Category, in: Memoirs of the Amer. Math. Soc., No. 707, vol. 149, 2001.
[21] D. Lind, B. Marcus, An Introduction to Symbolic Dynamics and Coding, Camb. Univ. Press, Cambridge, 1995.
[22] I. Raeburn, Graph Algebras, CBMS Regional Conference Series in Mathematics, vol. 103, Published for the Conference Board of the Mathematical Sciences, Washington, DC, by the Amer. Math. Soc., Providence, RI, 2005.
[23] A.H. Schofield, Representations of Rings Over Skew Fields, in: Lond. Math. Soc., Lecture Notes Series, vol. 92, Cambridge University Press, Cambridge, UK, 1985.
[24] S.P. Smith, Shift equivalence and a category equivalence involving graded modules over path algebras of quivers, arXiv:1108.4994.
[25] S.P. Smith, The non-commutative scheme having a free algebra as a homogeneous coordinate ring, arXiv:1104.3822.
[26] S.P. Smith, The space of Penrose tilings and the non-commutative curve with homogeneous coordinate ring $k\langle x, y\rangle/(y^2)$, J. Noncommut. Geom. (in press). arXiv:1104.3811.
[27] B. Stenström, Rings of Quotients, in: Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen, vol. 217, Springer Verlag, Berlin, 1975.
[28] J. Tyler, Every AF-algebra is Morita equivalent to a graph algebra, Bull. Aust. Math. Soc. 69 (2004) 237–240.
[29] A.B. Verevkin, On a non-commutative analogue of the category of coherent sheaves on a projective scheme, Amer. Math. Soc. Transl. Ser. 2 151 (1992).