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A quotient stack related to the Weyl algebra

S. Paul Smith [1]

Department of Mathematics, Box 354350, University of Washington, Seattle, WA 98195, USA

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ABSTRACT

Let A denote the ring of differential operators on the affine line with its two usual generators t and $\frac{d}{dt}$ given degrees $+1$ and -1 respectively. Let $\mathcal{X}$ be the stack having coarse moduli space the affine line $\operatorname{Spec} k[z]$ and isotropy groups $\mathbb{Z}/2$ at each integer point. Then the category of graded A-modules is equivalent to the category of quasi-coherent sheaves on $\mathcal{X}$.

1. Introduction

1.1. Let k be a field of characteristic zero. All vector spaces and algebras in this paper are taken over k.

The first Weyl algebra is the ring $A = k\langle x, y\rangle/(xy - yx - 1)$. We impose a $\mathbb{Z}$-grading on it by setting $\deg x = 1$ and $\deg y = -1$. There is an isomorphism between A and the ring of differential operators with polynomial coefficients on the affine line $\operatorname{Spec} k[t]$ that is given by sending x to “multiplication by t” and y to $-d/dt$.

E-mail address: smith@math.washington.edu.

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Our main result is that the category $\mathsf{Gr}A$ of $\mathbb{Z}$-graded A-modules is equivalent to the category of quasi-coherent sheaves on a quotient stack $\mathcal{X}$ whose coarse moduli space is the affine line $\operatorname{Spec} k[z]$, and whose stacky structure consists of stacky points $B\mathbb{Z}_2$ supported at each integer point $n \in \mathbb{Z} \subset \mathbb{A}^1_k$. We write

$$\mathsf{Gr}A \equiv \mathsf{Qcoh}\mathcal{X} \tag{1-1}$$

to denote this equivalence.

1.2. We now describe $\mathcal{X}$.

Let $\mathbb{Z}_{\text{fin}}$ be the group of finite subsets of $\mathbb{Z}$ with group operation given by "exclusive or". Let G be the affine group scheme whose coordinate ring is the group algebra $k\mathbb{Z}_{\text{fin}}$ with its usual Hopf algebra structure. Since $\mathbb{Z}_{\text{fin}}$ is 2-torsion and is generated by the singleton sets $\{n\}$ a k-valued point $g \in G$ corresponds to a function $\mathbb{Z} \to \{\pm 1\}$, $n \mapsto g(\{n\})$. We write g_n for $g(\{n\})$.

We define an action of G on the ring

$$C := k[z]\big[\sqrt{z-n} \mid n \in \mathbb{Z}\big]$$

by $g.\sqrt{z-n} := g_n\sqrt{z-n}$, and the stack-theoretic quotient

$$\mathcal{X} := \left[\frac{\operatorname{Spec} C}{G}\right].$$

Its coarse moduli space is the affine line $\operatorname{Spec} k[z]$. Max Lieblich tells me that $\mathcal{X}$ is an algebraic stack whose diagonal is locally of finite type but not quasi-compact (even though it is unramified).

1.3. The action of G on C corresponds to the $\mathbb{Z}_{\text{fin}}$-grading on C given by

$$\deg \sqrt{z-n} = \{n\}.$$

A standard result for quotient stacks says that $\mathsf{Qcoh}\mathcal{X}$ is equivalent to the category of G-equivariant sheaves on $\operatorname{Spec} C$ or, equivalently, that there is an equivalence

$$\mathsf{Qcoh}\mathcal{X} \equiv \mathsf{Gr}(C, \mathbb{Z}_{\text{fin}}),$$

where $\mathsf{Gr}(C, \mathbb{Z}_{\text{fin}})$ is the category of $\mathbb{Z}_{\text{fin}}$ graded C-modules. Under this equivalence locally free coherent $\mathcal{O}_{\mathcal{X}}$-modules correspond to finitely generated projective graded C-modules; for example, $\mathcal{O}_{\mathcal{X}}$ corresponds to C.

We note that C is isomorphic to the polynomial ring $k[x_n \mid n \in \mathbb{Z}]$ modulo the relations $x_n^2 + n = x_m^2 + m$ for all m and n, with the grading given by $\deg x_n := \{n\}$, and the isomorphism given by $z \leftrightarrow x_0^2$, and $\sqrt{z-n} \leftrightarrow x_n$. We therefore think of k-valued points of $\operatorname{Spec} C$ as elements in $k^{\mathbb{Z}}$.

The map $(a_i)_{i \in \mathbb{Z}} \mapsto (a_1 \cdots a_{2g+1}, a_0^2)$ is a surjective morphism from $\operatorname{Spec} C$ to a hyperelliptic curve of genus g. If k is not of characteristic two the fibers of this morphism are uncountable. When $k = \mathbb{C}$ with its usual topology and $\mathbb{C}^{\mathbb{Z}}$ is given the product topology, and $\operatorname{Max} C$ is viewed as a subspace of $\mathbb{C}^{\mathbb{Z}}$ with the subspace topology, the fibers are Cantor sets.

1.4. At first sight, the equivalence (1-1) is surprising. The Weyl algebra is an infinite dimensional k-algebra having no two-sided ideals other than zero and the ring itself so has no non-zero finite dimensional modules. As such it is "very non-commutative". In stark contrast, C is not only commutative but is even a graded PID meaning it is a domain and every graded ideal is principal. Graded right ideals in A need not be principal. Moreover, C is a directed union of Dedekind domains. Although C is not noetherian, it is noetherian from the graded perspective, meaning that $\mathsf{Gr}(C, \mathbb{Z}_{\text{fin}})$ is a locally noetherian category. In particular, it has a set of noetherian generators.

1.5. The relation between A, C, and $\mathcal{X}$, can be viewed in the following way. Let A be the k-linear abelian category $\mathsf{Gr}A$ but forget for the moment that it is $\mathsf{Gr}A$ and consider it just as an abelian category. The endomorphism ring of the identity functor id_{A} is a polynomial ring in one variable. In the equivalences of A with $\mathsf{Gr}A$, $\mathsf{Gr}C$, and $\mathsf{Qcoh}\mathcal{X}$, $\mathrm{End}(\mathrm{id}_{\mathsf{A}})$ identifies with $A_0 = k[t\frac{d}{dt}]$, $C_\varnothing = k[z] = k[x_0^2]$, and $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, respectively. The Picard group, $\operatorname{Pic}\mathsf{A}$, of A is defined as the group of auto-equivalences modulo isomorphism. (It *does* contain the usual Picard group of $\mathcal{X}$ where one identifies an invertible $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{L}$ with the auto-equivalence $\mathcal{L} \otimes -$.) Thinking of elements of $\operatorname{Pic}\mathsf{A}$ as being like invertible sheaves, or line bundles, one is led to associate to each subgroup Γ of $\operatorname{Pic}\mathsf{A}$ a "homogeneous coordinate ring"

$$\bigoplus_{F \in \Gamma} \mathrm{Hom}(\mathrm{id}_{\mathsf{A}}, F)$$

where $\mathrm{Hom}(\mathrm{id}_{\mathsf{A}}, F)$ consists of natural transformations from id_{A} to F. This point of view lies at the heart of the work of Artin and Zhang's conception of non-commutative algebraic geometry [1]. There are particular subgroups of $\operatorname{Pic}\mathsf{A}$ isomorphic to $\mathbb{Z}_{\mathrm{fin}}$ and $\mathbb{Z}$, and the "homogeneous coordinate rings" associated to these subgroups are isomorphic to C and A, respectively. The subgroup isomorphic to $\mathbb{Z}_{\mathrm{fin}}$ also identifies with $\operatorname{Pic}\mathcal{X}$. In some sense, x_n, or rather the result of its action on id_{A}, is an endo-functor of A that is a square root of the endo-functor of $\mathsf{Gr}A$ that is given by a left action of the operator $t\frac{d}{dt} - n$ on graded right A-modules (see Sections 9.4 and 3.6).

The subgroup of $\operatorname{Pic}(\mathsf{Gr}A)$ isomorphic to $\mathbb{Z}_{\mathrm{fin}}$ was found by Sue Sierra [5].

1.6. The stimulus for this paper was Sierra's work on the graded Weyl algebra [5] and especially her "picture"

$$\cdots \;\text{———}\; :\text{———}\; :\text{———}\; :\text{———}\; :\text{———}\; :\text{———}\; \cdots \tag{1-2}$$

of the simple graded A-modules which reminded this author of a stack on the affine line with stacky structure $B\mathbb{Z}_2$ at each integer point. Each point in Sierra's picture represents a simple graded A-module: if $\lambda \in k - \mathbb{Z}$ there is a single simple graded A-module up to isomorphism, namely $A/(xy - \lambda)A$; if $n \in \mathbb{Z}$ there are two simple modules,

$$X(n) := \left(\frac{A}{xA}\right)(n) \quad \text{and} \quad Y(n) := \left(\frac{A}{yA}\right)(n-1).$$

(The isomorphism between A and $\mathcal{D}(\mathbb{A}^1)$ may be chosen so that $Y(1)$ corresponds to the natural module $k[t]$ and $X(1)$ corresponds to the module $k[t, t^{-1}]/k[t]$.) There is a non-split extension of $X(n)$ by $Y(n)$ and a non-split extension of $Y(n)$ by $X(n)$ for each n.

The underlying line in (1-2) should be thought of as $\operatorname{Spec} k[xy]$ and the two points at $n \in \mathbb{Z}$ represent, in some sense, the two formal square roots of $-xy - n$ (because $-xy$ corresponds to $t\frac{d}{dt}$).

We call the two points at $n \in \mathbb{Z} \subset k$ "fractional points". There are two reasons for this. First, if $n \in \mathbb{Z}$ and $\lambda \in k - \mathbb{Z}$ there is an equality $[X(n)] + [Y(n)] = [A/(xy - \lambda)A]$ in the Grothendieck group of $\mathsf{Gr}A$, and under the equivalence with $\mathsf{Qcoh}\mathcal{X}$, $[A/(xy - \lambda)A]$ identifies with the skyscraper sheaf $\mathcal{O}_\lambda$ at the point $\lambda \in \operatorname{Spec} k[z]$. Second, there is some consistency with the notion of a "fractional brane" or "brane fractionation", where a brane represented by a point in the Azumaya locus "fractionates" when it moves to the non-Azumaya locus.

Sierra's picture can also be viewed as a depiction of the stack $\mathcal{X}$. The line given by collapsing the "fractional points" is the coarse moduli space $\operatorname{Spec} k[z]$ of $\mathcal{X}$ and the two points at n correspond to the skyscraper sheaf $\mathcal{O}_n = k[z]/(z - n)$ endowed with the trivial and sign representations of the isotropy group at n, and $\lambda \in k - \mathbb{Z}$ corresponds to $\mathcal{O}_\lambda = k[z]/(z - \lambda)$. If χ_{sgn} and χ_{triv} denote the sign and trivial representations of the appropriate isotropy groups, then under the equivalence of categories $\mathsf{Gr}A \equiv \mathsf{Qcoh}\mathcal{X}$ there are correspondences

$$X(n) \leftrightsquigarrow \begin{cases} \mathcal{O}_n \otimes \chi_{triv} & \text{if } n \leqslant 0, \\ \mathcal{O}_n \otimes \chi_{sgn} & \text{if } n \geqslant 1, \end{cases}$$

$$Y(n) \leftrightsquigarrow \begin{cases} \mathcal{O}_n \otimes \chi_{sgn} & \text{if } n \leqslant 0, \\ \mathcal{O}_n \otimes \chi_{triv} & \text{if } n \geqslant 1. \end{cases}$$

Under the direct image functor for the morphism from $\mathcal{X}$ to its coarse moduli space "half" the $X(n)$s and "half" the $Y(n)$s are sent to zero.

Under the equivalence with $\mathsf{Gr}C$, $X(n)$ corresponds to $C/(x_n)$ when $n \leqslant 0$ and to $Cx_n/(x_n^2)$ when $n \geqslant 1$; similarly, $Y(n)$ corresponds to $C/(x_n)$ when $n \geqslant 1$ and to $Cx_n/(x_n^2)$ when $n \leqslant 0$.

1.7. Further evidence of a possible relation between $\mathsf{Gr}A$ and $\mathsf{Qcoh}\mathcal{X}$ is the behavior of the Ext-groups between simple modules. Sierra showed that the only non-trivial extensions between non-isomorphic simple graded A-modules are the following: For all $i, j \in \mathbb{Z}$

$$\operatorname{ext}^1_A\big(X(i), Y(j)\big) \cong \operatorname{ext}^1_A\big(Y(j), X(i)\big) \cong \begin{cases} k & \text{if } i = j, \text{ and} \\ 0 & \text{if } i \neq j \end{cases} \tag{1-3}$$

where ext^1_A denotes Ext^1_A in $\mathsf{Gr}A$ [5, Lem. 4.3]. This is "the same" as the behavior of the ext-groups between the simple objects in $\mathsf{Qcoh}\mathcal{X}$.

1.8. Using the equivalence between $\mathsf{Gr}A$ and $\mathsf{Qcoh}\mathcal{X}$, the direct image functor for the morphism from $\mathcal{X}$ to its coarse moduli space transfers to a functor $\mathsf{Gr}A \to \mathsf{Mod}k[z]$. That functor sends a graded A-module to its degree zero component. For example, A viewed as a graded right A-module is sent to $k[xy]$ which identifies with $k[z]$. We therefore write z for the element xy of A and think of it both as an element of A and as the coordinate function on the affine line that is the coarse moduli space for $\mathcal{X}$, i.e., $k[z] = C^G = C_\varnothing$. We note that $k[z]$ is equal to $k[x_n^2]$ for all $n \in \mathbb{Z}$.

1.9. Having obtained the equivalence between $\mathsf{Gr}A$ and $\mathsf{Qcoh}\mathcal{X}$ or, equivalently, with $\mathsf{Gr}(C, \mathbb{Z}_{\text{fin}})$, one can obtain alternative proofs of many of Sierra's results by transferring results from $\mathsf{Gr}(C, \mathbb{Z}_{\text{fin}})$ to $\mathsf{Gr}A$. This is a good thing because the fact that C is a graded PID makes the study of its graded modules quite straightforward.

To illustrate this point we compute the Grothendieck and Picard groups of $\mathsf{Gr}(C, \mathbb{Z}_{\text{fin}})$ directly using C rather than using the equivalence of categories and quoting Sierra's result that computes those invariants for $\mathsf{Gr}A$.

1.10. Section 2 is preparatory, setting up notation, and recalling some well-known facts. Section 3 concerns the Picard group $\operatorname{Pic}(R, \Gamma)$ of the category $\mathsf{Gr}(R, \Gamma)$ of graded modules over a ring R graded by an abelian group Γ. The results there may be of independent interest. The notion of an "almost-automorphism" of (R, Γ) is introduced and we show that every almost-automorphism determines an auto-equivalence of $\mathsf{Gr}(R, \Gamma)$. Proposition 3.5 shows that a pair of auto-equivalences F and G such that $F(R(i)) \cong G(R(i))$ for all $i \in \Gamma$ are naturally isomorphic if the endomorphism ring of every homogeneous component R_i is isomorphic to R_0. There is a group homomorphism $\operatorname{Pic}(R, \Gamma) \to \operatorname{Aut}\operatorname{End}(\mathrm{id}_{\mathsf{Gr}(R,\Gamma)})$. In Proposition 3.9 a criterion, which is satisfied by $(A, \mathbb{Z})$ and $(C, \mathbb{Z}_{\text{fin}})$, is given that implies there is a ring isomorphism $R_0 \to \operatorname{End}(\mathrm{id}_{\mathsf{Gr}(R,\Gamma)})$. In particular, in this situation every graded right R-module can be given the structure of an R_0-R-bimodule.

Section 4 concerns the structure of C as an ungraded ring and also examines the "variety" $X \subset \mathbb{C}^{\mathbb{Z}}$ of which C is the coordinate ring. By definition, X is the zero locus of the equations $x_n^2 + n = x_m^2 + m$, $m, n \in \mathbb{Z}$. The topological structure of X is examined when $\mathbb{C}^{\mathbb{Z}}$ is given various topologies and X is given the subspace topology. For example, when $\mathbb{C}^{\mathbb{Z}}$ is given the product topology the fibers of each projection $x_n : X \to \mathbb{C}$ are Cantor sets. With the box topology X becomes discrete. With an appropriate embedding in $\ell^\infty(\mathbb{Z})$, X has uncountably many connected components, all homeomorphic to one another and permuted by the action of the group $\{\pm 1\}^{\mathbb{Z}}$ acting by coordinate-wise multiplication.

Each component can be given the structure of a Riemann surface with respect to which the coordinate functions x_n are holomorphic.

Section 5 establishes the properties of C as a graded ring and culminates in a proof that the categories $\mathsf{Gr}(A,\mathbb{Z})$ and $\mathsf{Gr}(C,\mathbb{Z}_{\mathrm{fin}})$ are equivalent. However, that equivalence is *not* used in Sections 6—8. Thus, the results in those sections are independent of Sierra's work, and provide alternative proofs of several of her results.

Section 6 classifies the simple graded C-modules and focuses on those that are supported at the stacky points on $\mathcal{X}$. We call those *special*. They correspond to the graded A-modules labelled $X(n)$ and $Y(n)$ above. As in Sierra's analysis, they play a central role in this paper. For example, Corollary 6.9 shows that an auto-equivalence of $\mathsf{Gr}(C,\mathbb{Z}_{\mathrm{fin}})$ is determined up to isomorphism by its action on the isomorphism classes of the special simple modules.

The special simples may be characterized as those simple graded modules S such that $\mathrm{ext}^1(S,S')$ is non-zero for some simple module S' that is not isomorphic to S. They may also be characterized as those simple graded modules S such that $\hom(P,S)=0$ for some non-zero projective graded module (Proposition 6.3). Sierra exploits the first characterization in her paper whereas we choose to exploit the second characterization in this paper so as to provide a different perspective.

Section 7 computes the Grothendieck group of the category of finitely generated graded C-modules or, equivalently, that of the G-equivariant locally free sheaves on $\operatorname{Spec} C$. We compute that Grothendieck group as an explicit quotient ring of the group algebra $\mathbb{Z}\mathbb{Z}_{\mathrm{fin}}$. In passing, we prove that the isomorphism classes of finitely generated projective graded C-modules or, equivalently, locally free G-equivariant sheaves on $\operatorname{Spec} C$, are in natural bijection with the finite multi-sets of integers.

Section 8 shows that the translation and reflection symmetries of Sierra's picture (1-2) can be implemented at the functorial level by the functor τ_* induced by the automorphism $\tau: z\mapsto z+1$, or $x_n\mapsto x_{n-1}$, of C, and by the functor φ_* induced by the almost-automorphism $\varphi: x_n\mapsto x_{-n}$, $x_n^2\mapsto -x_{-n}^2$, $z\mapsto -z$, respectively. The reflection symmetry cannot be induced at the functorial level by an automorphism of C unless k contains $\sqrt{-1}$. The main result in Section 8 is the computation of $\operatorname{Pic}(C,\mathbb{Z}_{\mathrm{fin}})$. We show it fits into a sequence

$$1\to\mathbb{Z}_{\mathrm{fin}}\to\operatorname{Pic}(C,\mathbb{Z}_{\mathrm{fin}})\to\operatorname{Iso}(\mathbb{Z})\to 1$$

where $\operatorname{Iso}(\mathbb{Z})$ is the isometry group of $\mathbb{Z}$, the infinite dihedral group, abstractly.

Section 9 makes a direct comparison between $\mathsf{Gr}(A,\mathbb{Z})$ and $\mathsf{Gr}(C,\mathbb{Z}_{\mathrm{fin}})$. Proposition 9.2 shows how the special simples over each ring correspond under the equivalence of categories—the correspondence is not what the notation might lead one to expect. Theorem 9.6 shows that the auto-equivalences ι_J, $J\in\mathbb{Z}_{\mathrm{fin}}$, found by Sierra correspond to the Serre twists (J) on $\mathsf{Gr}(C,\mathbb{Z}_{\mathrm{fin}})$. Proposition 9.5 shows that the Serre twist, (1), on $\mathsf{Gr}(A,\mathbb{Z})$ corresponds to the auto-equivalence $(\{1\})\circ\tau_*$ of $\mathsf{Gr}(C,\mathbb{Z}_{\mathrm{fin}})$. Proposition 9.3 shows that the auto-equivalence of $\mathsf{Gr}(A,\mathbb{Z})$ induced by the automorphism $x\mapsto y$ and $y\mapsto -x$ corresponds to the auto-equivalence $\tau_*\varphi_*$ of $\mathsf{Gr}(C,\mathbb{Z}_{\mathrm{fin}})$ induced by the almost-automorphism $\tau\varphi$.

The equivalence between $\mathsf{Gr}(A,\mathbb{Z})$ and $\mathsf{Gr}(C,\mathbb{Z}_{\mathrm{fin}})$ was proved in Section 5 by starting with C and then showing that A was the endomorphism ring of a certain bigraded P-module. In Section 10 we take the opposite approach and show that C can be constructed from $\mathsf{Gr}(A,\mathbb{Z})$ as a sort of twisted homogeneous coordinate ring.

1.11. The results about graded C-modules can be translated into results about $\mathsf{Qcoh}\mathcal{X}$ or, equivalently, about the G-equivariant sheaves on $\operatorname{Spec} C$. For example

(1) the Grothendieck group $K_0(\mathcal{X})$ is a free abelian group of countable rank, and we present it as an explicit quotient of the group algebra $\mathbb{Z}\mathbb{Z}_{\mathrm{fin}}$ in Theorem 7.4;
(2) the invertible $\mathcal{O}_{\mathcal{X}}$-modules are, up to isomorphisms, the twists of $\mathcal{O}_{\mathcal{X}}$ by the characters of G or, equivalently, $\operatorname{Pic}\mathcal{X}\cong\mathbb{Z}_{\mathrm{fin}}$ (Corollary 5.7);
(3) every locally free $\mathcal{O}_{\mathcal{X}}$-module is a direct sum of invertible $\mathcal{O}_{\mathcal{X}}$-modules (Proposition 5.5);
(4) the locally free $\mathcal{O}_{\mathcal{X}}$-modules are, up to isomorphisms, in natural bijection with the finite multisets of integers.

2. Preliminaries

2.1. The Weyl algebra A

The $\mathbb{Z}$-grading on A given by

$$\deg x = 1, \qquad \deg y = -1$$

is sometimes called the Euler grading because A_n consists of those operators/elements a such that $[D, a] = na$ where D is the Euler vector field/derivation

$$D = t\frac{d}{dt}.$$

2.2. The twist functor on GrA

For each $n \in \mathbb{Z}$, we define Serre's twist automorphism $M \mapsto M(n)$ on GrA by declaring that $M(n)$ is equal to M as a right A-module but the grading is now given by

$$M(n)_i = M_{n+i}.$$

This notation differs from Sierra's: her primary twist functor is denoted by $M \mapsto M\langle n\rangle$ where $M\langle n\rangle = M$ and

$$M\langle n\rangle_i = M_{i-n}.$$

Thus $\langle n\rangle = (-n)$.

2.3. The $\mathbb{Z}_{\mathrm{fin}}$-grading on C

Let $\mathbb{Z}_{\mathrm{fin}}$ be the group of finite subsets of the integers with group operation

$$I \oplus J := I \cup J - I \cap J = (I - J) \cup (J - I).$$

The identity is the empty set $\varnothing$. It is easy to see that $\mathbb{Z}_{\mathrm{fin}}$ is the direct sum of the two-element subgroups $\{\varnothing, \{i\}\}$, $i \in \mathbb{Z}$.

Define the commutative ring

$$C := k[x_n \mid n \in \mathbb{Z}] \quad \text{modulo the relations,}$$
$$x_n^2 + n = x_m^2 + m, \quad \text{for all } n, m \in \mathbb{Z}.$$

We make C a $\mathbb{Z}_{\mathrm{fin}}$-graded ring by declaring that

$$\deg x_n = \{n\}.$$

The identity component, $C_\varnothing$, is equal to $k[x_n^2]$ for all $n \in \mathbb{Z}$. For each $I \in \mathbb{Z}_{\mathrm{fin}}$ we define

$$x_I := \prod_{i \in I} x_i$$

with the convention that $x_\varnothing = 1$.[2] The homogeneous components of C are $C_I := C_\varnothing x_I$. We will see below that C is a domain so each C_I is isomorphic to $C_\varnothing$ as a $C_\varnothing$-module.

2.3.1. The ring $\mathbb{Z}_{\mathrm{sub}}$

Let $\mathbb{Z}_{\mathrm{sub}}$ denote the set of all subsets of $\mathbb{Z}$. Then $\mathbb{Z}_{\mathrm{sub}}$ is a commutative ring with identity with respect to the product given by intersection and addition given by $\oplus$. Its identity is $\mathbb{Z}$ and its zero element is $\varnothing$. Thus $\mathbb{Z}_{\mathrm{fin}}$ is a subring of $\mathbb{Z}_{\mathrm{sub}}$ without identity. Since every element of $\mathbb{Z}_{\mathrm{sub}}$ is idempotent, $\mathbb{Z}_{\mathrm{sub}}$ is a Boolean ring.

We note the identity $I - J = I \oplus (I \cap J) = I \cap (\mathbb{Z} \oplus J)$.

2.3.2. Notation for $\mathbb{Z}_{\mathrm{fin}}$

If $J \in \mathbb{Z}_{\mathrm{fin}}$ and $n \in \mathbb{Z}$ we adopt the notation:

- $n + J := \{n + j \mid j \in J\}$;
- $nJ := \{nj \mid j \in J\}$.

2.4. Categories of graded modules

If R is a ring graded by a group Γ we write $\mathsf{Gr}(R, \Gamma)$ for the category of Γ-graded right R-modules with degree-preserving homomorphisms. If $\Gamma = \mathbb{Z}$ we often write $\mathsf{Gr} R$ for $\mathsf{Gr}(R, \mathbb{Z})$.

We will write $\hom_R(M, N)$ for the degree preserving R-module homomorphisms from one Γ-graded R-module M to another N. We denote the right derived functors of $\hom_R$ by ext^i_R, $i \geqslant 0$.

There is a category of graded rings in which the objects are pairs (R, Γ) consisting of a group Γ and a Γ-graded ring R. Morphisms are pairs $(\alpha, \bar\alpha) : (R, \Gamma) \to (R', \Gamma')$ where $\bar\alpha : \Gamma \to \Gamma'$ is a group homomorphism and $\alpha : R \to R'$ is a ring homomorphism such that $\alpha(r_i) \subset R'_{\bar\alpha i}$ for all $i \in \Gamma$.

Associated to $(\alpha, \bar\alpha)$ is a functor $\alpha^* : \mathsf{Gr}(R, \Gamma) \to \mathsf{Gr}(R', \Gamma')$ and its right adjoint $\alpha_* : \mathsf{Gr}(R', \Gamma') \to \mathsf{Gr}(R, \Gamma)$. If $M \in \mathsf{Gr}(R', \Gamma')$, then $(\alpha_* M)_i := M_{\bar\alpha i}$ for all $i \in \Gamma$ and R acts on $\alpha_* M$ via the homomorphism α.

If $m \in M_{\bar\alpha i}$ we will label it as $\alpha_* m$ when we think of it as an element in $\alpha_* M$. Hence $x.\alpha_* m = \alpha_*(\alpha(x)m)$.

2.5. The affine group scheme G

The group algebra of $\mathbb{Z}_{\mathrm{fin}}$ is

$$k\mathbb{Z}_{\mathrm{fin}} = \frac{k[u_i \mid i \in \mathbb{Z}]}{(u_i^2 - 1 \mid i \in \mathbb{Z})}$$

where u_i is an alias for the element $\{i\}$. The group algebra is given its usual Hopf algebra structure and we define the affine group scheme

$$G := \operatorname{Spec} k\mathbb{Z}_{\mathrm{fin}}.$$

The letter G will denote both the group scheme and the group of k-valued points of $\operatorname{Spec} k\mathbb{Z}_{\mathrm{fin}}$.

Let $\{\pm 1\}^{\mathbb{Z}}$ be the group of functions $\mathbb{Z} \to \{\pm 1\}$. There is an isomorphism $G \to \{\pm 1\}^{\mathbb{Z}}$ that sends $g \in G$ to the function $\mathbb{Z} \to \{\pm 1\}$ given by $i \mapsto u_i(g)$. Thus, G is isomorphic to a countable product of copies of $\{\pm 1\}$. When we consider an element $g \in G$ as a function $\mathbb{Z} \to \{\pm 1\}$ that function will always be given by $g(i) = u_i(g)$.

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[2] Whenever things are indexed by elements of $\mathbb{Z}_{\mathrm{fin}}$ we write x_i rather than $x_{\{i\}}$ for the element indexed by $\{i\}$.

2.6. The stack

We define an algebraic action of G on $\operatorname{Spec} C$ by declaring that $g \in G$ acts on x_i by

$$g.x_i = g(i)x_i,$$

and define the quotient stack

$$\mathcal{X} := \left[\frac{\operatorname{Spec} C}{G}\right].$$

The category of quasi-coherent $\mathcal{O}_{\mathcal{X}}$-modules is denoted by $\mathsf{Qcoh}\mathcal{X}$ and is equivalent to the category of G-equivariant C-modules—we will usually think of it in this way.

The invariant subring of C is $C^G = C_\varnothing = k[x_n^2]$ for all n. The coarse moduli space of $\mathcal{X}$ is therefore the affine line $\operatorname{Spec} k[x_0^2]$.

We will denote k-valued points in $\operatorname{Spec} C$ by tuples $(a_i) \in k^{\mathbb{Z}}$, $a_i \in k$, such a point corresponding to the maximal ideal

$$\sum_{i \in \mathbb{Z}} (x_i - a_i).$$

The relations for C imply that at most one a_i is zero. Therefore the points having a non-trivial isotropy group are those for which one a_i is zero. The isotropy group at such a point is isomorphic to $\mathbb{Z}_2$. Such points are those where x_0^2 takes an integer value. Hence all the stacky structure on $\mathcal{X}$ occurs at the integer points $x_0^2 = n$, $n \in \mathbb{Z}$, on the coarse moduli space $\operatorname{Spec} k[x_0^2]$.

2.7. Because $\mathcal{O}(G)$ is the group algebra $k\mathbb{Z}_{\mathrm{fin}}$ a rational representation of G is the same thing as a $\mathbb{Z}_{\mathrm{fin}}$-graded vector space. In particular, the $\mathbb{Z}_{\mathrm{fin}}$-grading on C is that induced by the action of G on $\mathcal{O}(C)$.

It is a standard result that the category of G-equivariant C-modules is equivalent to the category of $\mathbb{Z}_{\mathrm{fin}}$-graded C-modules, so there is an equivalence of categories

$$\mathsf{Qcoh}\mathcal{X} \equiv \mathsf{Gr}(C, \mathbb{Z}_{\mathrm{fin}})$$

where the latter denotes the category of $\mathbb{Z}_{\mathrm{fin}}$-graded C-modules.

Our main result, namely that

$$\mathsf{Gr}A \equiv \mathsf{Qcoh}\mathcal{X},$$

will be proved by showing that $\mathsf{Gr}A \equiv \mathsf{Gr}(C, \mathbb{Z}_{\mathrm{fin}})$.

3. Auto-equivalences of categories of graded modules

The Picard group of a category is its group of auto-equivalences modulo natural isomorphism.

Let (R, Γ) be a graded ring. We write $\operatorname{Pic}(R, \Gamma)$ for the Picard group of $\mathsf{Gr}(R, \Gamma)$. As remarked in Section 2.4, an automorphism $(\alpha, \bar{\alpha})$ of (R, Γ) induces an automorphism α_* of $\mathsf{Gr}(R, \Gamma)$. We write $[\alpha_*]$ for the isomorphism class of α. This passage $\alpha \rightsquigarrow [\alpha_*]$ can be mimicked for certain maps $\alpha : R \to R$ that are not automorphisms.

3.1. Almost-automorphisms of (R, Γ)

Let k be a subring of R_0 that is central in R. Let $k^\times$ denote the group of units in k.

An almost-automorphism of (R, Γ) is a triple $(\alpha, \bar{\alpha}, \lambda)$ consisting of

(1) an automorphism $\bar{\alpha}$ of Γ,
(2) a k-module automorphism $\alpha : R \to R$ of R such that $\alpha(R_i) = R_{\bar{\alpha} i}$ for all i, and
(3) a normalized 2-cocycle $\lambda : \Gamma \times \Gamma \to k^\times$, $(i, j) \mapsto \lambda_{ij}$, i.e., $\lambda_{00} = 1$ and

$$\lambda_{h,i+j}\lambda_{ij} = \lambda_{h+i,j}\lambda_{hi} \tag{3-1}$$

for all $h, i, j \in \Gamma$, such that

$$\alpha(xy) = \lambda_{hi}\alpha(x)\alpha(y)$$

for all $x \in R_h$ and all $y \in R_i$ and all $h, i \in \Gamma$.

3.1.1. Because $\lambda_{00} = 1$, the restriction of α to R_0 is a k-algebra automorphism of R_0.

3.1.2. An automorphism $(\alpha, \bar{\alpha})$ of (R, Γ) is an almost-automorphism with $\lambda_{hi} = 1$ for all h and i.

3.1.3. I am grateful to Margaret Beattie for the following observation.

Suppose λ is a normalized 2-cocycle and $(\alpha, \bar{\alpha})$ a pair satisfying conditions (1) and (2) in Section 3.1. Let $\nu = \lambda \circ (\bar{\alpha}^{-1} \times \bar{\alpha}^{-1})$. Then ν is also a normalized 2-cocycle and there is a standard construction of a new graded k-algebra (R^ν, Γ) which is (R, Γ) as a graded k-module, but endowed with a new multiplication

$$x * y := \nu_{ij} xy$$

for $x \in R_i$ and $y \in R_j$. Beattie observed that $(\alpha, \bar{\alpha}, \lambda)$ is an almost automorphism of R if and only if $(\alpha, \bar{\alpha})$ is an isomorphism of graded k-algebras $(R, \Gamma) \to (R^\nu, \Gamma)$.

Lemma 3.1. *The set of k-linear almost-automorphisms of a graded k-algebra (R, Γ) form a group with respect to the product*

$$(\alpha, \lambda) * (\beta, \nu) := (\alpha\beta, \xi), \quad \textit{where } \xi_{ij} := \lambda_{\bar{\beta} i, \bar{\beta} j}\nu_{ij} \textit{ for all } i \textit{ and } j.$$

Proof. It is straightforward to check that $(\alpha\beta, \xi)$ is an almost-automorphism. The identity automorphism $(\mathrm{id}_R, \mathrm{id}_\Gamma)$ has the property that $(\mathrm{id}_R, 1) * (\alpha, \lambda) = (\alpha, \lambda) * (\mathrm{id}_R, 1) = (\alpha, \lambda)$, so is an identity for the product $*$.

If (α, λ) is an almost-automorphism so is (α^{-1}, ζ) where

$$\zeta_{ij} := \lambda^{-1}_{\bar{\alpha}^{-1} i, \bar{\alpha}^{-1} j}$$

for all i and j. It is easy to see that $(\alpha, \lambda) * (\alpha^{-1}, \zeta)$ and $(\alpha^{-1}, \zeta) * (\alpha, \lambda)$ are equal to $(\mathrm{id}_R, 1)$. The set of almost-automorphisms is therefore a group. □

We write $\mathbb{A}\mathrm{ut}(R, \Gamma)$ for the group of almost automorphisms of (R, Γ).

3.2. For each $(i, j) \in \Gamma \times \Gamma$, let $\bar{R}_{i,j} := R_{i-j}$ and define

$$\bar{R} := \bigoplus_{i,j \in \Gamma} \bar{R}_{i,j}.$$

The components $\bar{R}_{i,j}$ can be viewed as the sets of morphisms $j \to i$ in the category $\mathsf{C}(R, \Gamma)$ whose objects are the elements of Γ, and in which composition of morphisms is given by multiplication in R. Thus $\bar{R}$ is a ring but does not have an identity if Γ is infinite, though it does have "local" units.

Let $(\alpha, \bar{\alpha}, \lambda)$ be an almost-automorphism of (R, Γ). Define a k-linear map $\widetilde{\alpha} : \bar{R} \to \bar{R}$ by

$$\widetilde{\alpha}(x) := \lambda_{i-j,j}\alpha(x)$$

for $x \in \bar{R}_{i,j}$. Then $\widetilde{\alpha}$ is an algebra automorphism of $\bar{R}$.

We define an automorphism $F = F_{\alpha,\bar{\alpha},\lambda} : \mathsf{C}(R, \Gamma) \to \mathsf{C}(R, \Gamma)$ of $\mathsf{C}(R, \Gamma)$ by declaring that $Fi := \bar{\alpha}i$, $i \in \Gamma$, and on a morphism $x \in \bar{R}_{i,j}$ its action is

$$x \mapsto Fx := \lambda_{i-j,j}\alpha(x).$$

3.3. Automorphisms of $\mathsf{Gr}(R, \Gamma)$ *induced by almost-automorphisms*

Let α be an almost-automorphism of (R, Γ). Let $M \in \mathsf{Gr}(R, \Gamma)$. We define $\alpha_* M$ to be M endowed with the grading

$$(\alpha_* M)_h := M_{\bar{\alpha}h}.$$

We write $\alpha_* m$ for an element $m \in M$ viewed as an element of $\alpha_* M$. It is easy to see that $\alpha_* M$ becomes a graded R-module when the action of $x \in R_i$ on an element $\alpha_* m \in (\alpha_* M)_h$ is defined to be

$$(\alpha_* m).x := \alpha_*\big(m\alpha(x)\big)\lambda_{hi}.$$

The only point to be checked is the associative law, $((\alpha_* m).x).y = (\alpha_* m).(xy)$, which follows from the identity (3-1).

Lemma 3.2. *Let α be an almost-automorphism of (R, Γ). If $f : M \to N$ is a homomorphism of graded R-modules, then the map $\alpha_* f$ defined by*

$$(\alpha_* f)(\alpha_* m) := \alpha_*(fm)$$

is a homomorphism $\alpha_* M \to \alpha_* N$. With these definitions, α_* becomes an automorphism of $\mathsf{Gr}(R, \Gamma)$.

3.3.1. There is another way to view the auto-equivalence α_* associated to an almost automorphism by using Beattie's observation in Section 3.1.3. First, there is an identification $\mathsf{Gr}(R, \Gamma) = \mathsf{Gr}(R^\nu, \Gamma)$ because every left R-module M may be viewed as a left R^ν-module with respect to the action

$$x * m := \nu_{ij} x m$$

for $x \in R_i$ and $m \in M_j$. To avoid confusion, we will write M^ν for M viewed as an R^ν-module and write m^ν to denote the element m in M viewed as an element in M^ν. We now label the homomorphism $(\alpha, \bar{\alpha})$ of graded rings as $(\beta, \bar{\beta}) : (R, \Gamma) \to (R^\nu, \Gamma)$. As remarked at the end of Section 2.4, there is an equivalence $\beta_* : \mathsf{Gr}(R^\nu, \Gamma) \to \mathsf{Gr}(R, \Gamma)$ given by the following rule: if M is an R^ν-module, then $\beta_* M$ is an R-module with $x \in R_i$ acting on $\beta_* m \in (\beta_* M)_j = M_{\bar{\beta}j}$ by $x.\beta_* m = \beta_*(\beta(x)m)$.

If we now identify the domain of β_* with $\mathsf{Gr}(R^\nu, \Gamma)$ of ν with $\mathsf{Gr}(R, \Gamma)$ then β_* is the auto-equivalence α_*. To see this suppose that $M \in \mathsf{Gr}(R, \Gamma)$ and consider the action of $x \in R_i$ on an element $\beta_*(m^\nu)$ in $\beta_*(M^\nu)_j = (M^\nu)_{\bar{\beta} j} = M_{\bar{\beta} j} = M_{\bar{\alpha} j}$. We have

$$x.\beta_*\big(m^\nu\big) = \beta_*\big(\beta(x) * m^\nu\big) = \beta_*\big(\nu_{\bar{\beta}i,\bar{\beta}j}\big(\beta(x)m\big)^\nu\big) = \lambda_{ij}\beta_*\big(\big(\alpha(x)m\big)^\nu\big).$$

Stripping away the superfluous notation this reads $x.m = \lambda_{ij}\alpha(x)m$ which is, indeed, the action of x on $\alpha_* M$.

3.3.2. Warning

Some care must be taken when identifying $\alpha_* M$ with its underlying set M, even when α is an automorphism. Suppose $c \in R_j$ belongs to the center of R. Then the multiplication map $\rho_{c,M} : M \to M(j)$, $\rho_{c,M}(m) := mc$ is a homomorphism of graded R-modules. By definition, $\alpha_*(\rho_{c,M})$ is the homomorphism $\alpha_* M \to \alpha_*(M(j))$ given by

$$\big(\alpha_*(\rho_{c,M})\big)(\alpha_* m) = \alpha_*\big(\rho_{c,M}(m)\big) = \alpha_*(mc) = \alpha_*(m).\alpha^{-1}c$$

so

$$\alpha_*(\rho_{c,M}) = \rho_{\alpha^{-1}c,\alpha_* M}.$$

Thus $\alpha_*(\rho_{c,M})$ is multiplication by $\alpha^{-1}c$ on $\alpha_* M$ but when/if $\alpha_* M$ is identified with its underlying set M, $\alpha_*(\rho_{c,M})$ is multiplication by c. When (α, λ) is an almost automorphism $\alpha_*(\rho_{c,M})$ acts on $(\alpha_* M)_i$ as multiplication by $\lambda_{ij}\alpha^{-1}c$.

Lemma 3.3. *Let (α, λ) be an almost-automorphism of (R, Γ) and $h \in \Gamma$. The map $\theta : R(h) \to \alpha_*(R(\bar{\alpha}h))$ defined by*

$$\theta(x) := \lambda_{h,i}^{-1}\alpha_*(\alpha x) \quad \textit{if}\, x \in R(h)_i$$

is an isomorphism of graded right R-modules. In particular, α viewed as a map $R \to \alpha_* R$ is an isomorphism of right R-modules.

Proof. [3] Let $x \in R(h)_i$. Then $x \in R_{h+i}$, so $\alpha x \in R_{\bar{\alpha}h+\bar{\alpha}i} = R(\bar{\alpha}h)_{\bar{\alpha}i}$. Hence $\alpha_*(\alpha x) \in \alpha_*(R(\bar{\alpha}h))_i$. Thus θ preserves degree.

Because α is bijective θ is too.

To see that θ is a right R-module homomorphism, suppose that $x \in R(h)_i$ and $y \in R_j$. Then

$$\begin{aligned}
\theta(x.y) &= \alpha_*\big(\alpha(xy)\big)\lambda_{h,i+j}^{-1} \\
&= \alpha_*\big(\alpha(x)\alpha(y)\big)\lambda_{i+h,j}\lambda_{h,i+j}^{-1} \\
&= \alpha_*\big(\alpha(x)\alpha(y)\big)\lambda_{hi}^{-1}\lambda_{ij} \\
&= \alpha_*(\alpha x).y\lambda_{hi}^{-1} \\
&= \theta(x).y
\end{aligned}$$

so θ is an R-module homomorphism. □

[3] This proof does not use the fact that $\lambda_{00} = 1$.

Proposition 3.4. *Let (α, λ) be an almost-automorphism of (R, Γ) and $h \in \Gamma$. There is an isomorphism of functors*

$$(h) \circ \alpha_* \cong \alpha_* \circ (\bar{\alpha}h).$$

If α is an automorphism, then $(h) \circ \alpha_* = \alpha_* \circ (\bar{\alpha}h)$.

Proof. First we show there is an isomorphism of graded R-modules

$$(\alpha_* M)(h) \cong \alpha_*\big(M(\bar{\alpha}h)\big)$$

for every $M \in \mathsf{Gr}(R, \Gamma)$.

Let $i \in \Gamma$. The degree i components of $\alpha_*(M(\bar{\alpha}h))$ and $(\alpha_* M)(h)$ are equal to $M_{\bar{\alpha}h+\bar{\alpha}i}$. Let $m \in M_{\bar{\alpha}h+\bar{\alpha}i}$. For the purposes of this proof, we will write $\widetilde{m}$ for m viewed as an element of $(\alpha_* M)(h)$ and $\widehat{m}$ for m viewed as an element of $\alpha_*(M(\bar{\alpha}h))$.

The map

$$\psi : (\alpha_* M)(h) \to \alpha_*\big(M(\bar{\alpha}h)\big), \quad \psi(\widetilde{m}) := \lambda_{h,i}^{-1}\widehat{m} \quad \text{for } m \in (\alpha_* M)(h)_i$$

preserves degree and is bijective because the λ_{hi}s are units. Furthermore, if $y \in R_j$, then

$$\begin{aligned}
\psi(\widetilde{m}.y) &= \psi\big(\widetilde{m\alpha(y)}\big)\lambda_{i+h,j} \\
&= \widehat{m\alpha(y)}\lambda_{h,i+j}^{-1}\lambda_{i+h,j} \\
&= \widehat{m\alpha(y)}\lambda_{ij}\lambda_{h,i}^{-1} \\
&= \widehat{m}.y\lambda_{hi}^{-1} \\
&= \psi(\widetilde{m}).y
\end{aligned}$$

so ψ is an R-module homomorphism, and hence an isomorphism of graded R-modules.

Let $f : M \to N$ be a homomorphism between graded R-modules. We write $\widetilde{f}$ and $\widehat{f}$ respectively for the homomorphisms obtained by applying the functors $(h) \circ \alpha_*$ and $\alpha_* \circ (\bar{\alpha}h)$ to f. Of course, $\widetilde{f}$ and $\widehat{f}$ are just the map f on the underlying set M. It is easy to see that

$$\begin{tikzcd}
(\alpha_* M)(h) \arrow[r, "\widetilde{f}"] \arrow[d, "\psi_M"'] & (\alpha_* N)(h) \arrow[d, "\psi_N"] \\
\alpha_*(M(\bar{\alpha}h)) \arrow[r, "\widehat{f}"'] & \alpha_*(N(\bar{\alpha}h))
\end{tikzcd} \tag{3-2}$$

commutes. Hence the ψ_Ms collectively give a natural isomorphism $(h) \circ \alpha_* \to \alpha_* \circ (\bar{\alpha}h)$.

When α is an algebra automorphism we can take $\lambda_{hi} = 1$ for all $h, i \in \Gamma$, so the map ψ is the identity map id_M. Hence $(h) \circ \alpha_* = \alpha_* \circ (\bar{\alpha}h)$. $\square$

Lemma 3.3 is a consequence of Proposition 3.4 and the fact that $\alpha : R \to \alpha_* R$ is an isomorphism of graded right R-modules.

3.4. Isomorphisms between auto-equivalences

Let (R, Γ) be a graded ring and M a graded R-module. For each $y \in R_j$ and each $i \in \Gamma$, define

$$f_{i,y} : R(i) \to R(i+j), \quad f_{i,y}(x) := yx.$$

Then $f_{i,y}$ is a homomorphism of graded right R-modules.

Proposition 3.5. *Let F and G be auto-equivalences of $\mathsf{Gr}(R, \Gamma)$ such that*

$$F\big(R(i)\big) \cong G\big(R(i)\big)$$

for all $i \in \Gamma$. If the map $R_0 \to \mathrm{End}_{R_0}(R_j)$ sending $a \in R_0$ to the map $b \mapsto ab$ is an isomorphism for all $j \in \Gamma$, then

$$F \cong G.$$

Proof. If the proposition is true when $G = \mathrm{id}_{\mathsf{Gr}(R,\Gamma)}$, then it holds for all G because the general result is obtained by applying the special case to $F'G$ where F' is a quasi-inverse to F. We will therefore assume that $F(R(i)) \cong R(i)$ for all $i \in \Gamma$ and show that $F \cong \mathrm{id}_{\mathsf{Gr}(R,\Gamma)}$.

By hypothesis, there are isomorphisms $\phi_i : R(i) \to F(R(i))$ for all $i \in \Gamma$.

Fix $i \in \Gamma$. Let $y \in R_j$. Then

$$\phi_{i+j}^{-1} \circ F(f_{i,y}) \circ \phi_i : R(i) \to R(i+j)$$

is a homomorphism of graded right R-modules, so is left multiplication by a unique element $\theta_i(y) \in R_j$. That is,

$$\phi_{i+j}^{-1} \circ F(f_{i,y}) \circ \phi_i = f_{i,\theta_i(y)}.$$

We therefore have a map $\theta_i : R(j) \to R(j)$ for all $j \in \Gamma$. If $a \in R_0$, then $f_{i,ay} = f_{i+j,a} \circ f_{i,y}$, so $f_{i,\theta_i(ay)} = f_{i,a\theta_i(y)}$. Hence θ_i is a right R_0-module homomorphism.

If $w \in R_j$, then $\phi_{i+j} \circ f_{i,w} \circ \phi_i^{-1} : F(R(i)) \to F(R(i+j))$, but $F : \hom(R(i), R(i+j)) \to \hom(F(R(i)), F(R(i+j)))$ is bijective because F is an auto-equivalence so

$$\phi_{i+j} \circ f_{i,w} \circ \phi_i^{-1} = F(f_{i,y})$$

for some $y \in R_j$. Hence $f_{i,w} = f_{i,\theta_i(y)}$. This proves that θ_i is bijective and hence an isomorphism of right R_0-modules.

In particular, $\theta_0 : R(j) \to R(j)$ is an isomorphism of right R_0-modules for every j, so there are units $u_j \in R_0$ such that $\theta_0(y) = u_j y$ for all $y \in R_j$.

Let $z \in R_k$. Then $f_{i+j,zy} = f_{k,z} \circ f_{i,y}$ so

$$\theta_i(zy) = \theta_{i+j}(z)\theta_i(y). \tag{3-3}$$

In particular, taking $i = 0$ we see that

$$u_{j+k} zy = \theta_j(z) u_j y$$

for all $y \in R_j$ and all $z \in R_k$.

For each $j \in \Gamma$ define $\tau_j : R(j) \to F(R(j))$ to be $\tau_j := u_j\phi_j$. Let $z \in R_k$. Then the diagram

$$\begin{tikzcd} R(j) \arrow[r, "\tau_j"] \arrow[d, "f_{j,z}"'] & F(R(j)) \arrow[d, "F(f_{j,z})"] \\ R(j+k) \arrow[r, "\tau_{j+k}"'] & F(R(j+k)) \end{tikzcd}$$

commutes because if $y \in R(j)$, then

$$\begin{aligned} \big(F(f_{j,z}) \circ \tau_j\big)(y) &= \big(F(f_{j,z}) \circ u_j\phi_j\big)(y) \\ &= \big(F(f_{j,z}) \circ \phi_j\big)(u_j y) \\ &= (\phi_{j+k} \circ f_{j,\theta_j(z)})(u_j y) \\ &= \phi_{j+k}\big(\theta_j(z)u_j y\big) \\ &= \phi_{j+k}(u_{j+k}zy) \\ &= \tau_{j+k}(zy) \\ &= (\tau_{j+k} \circ f_{j,z})(y). \end{aligned}$$

Thus, the τ_js taken together provide a natural isomorphism from the restriction of F to the full subcategory of $\mathsf{Gr}(R,\Gamma)$ consisting of the $R(j)$s to the identity functor on that subcategory.

Because the $R(j)$s generate $\mathsf{Gr}(R,\Gamma)$ the isomorphism τ extends to an isomorphism $\tau : F \to \mathrm{id}_{\mathsf{Gr}(R,\Gamma)}$. (One defines $\tau_M : M \to FM$ for a general M by writing M as the cokernel of a map $P \to Q$ where P and Q are direct sums of various $R(j)$s.) $\square$

The hypotheses of Proposition 3.5 hold if R_0 is an integrally closed commutative noetherian domain and each R_i is isomorphic to a non-zero ideal of R_0.

3.4.1. Proposition 3.5 applies to A and C

Both A_0 and $C_\varnothing$ are isomorphic to a polynomial ring $k[z]$ and all the homogeneous components of A and C are rank one free $k[z]$-modules so the hypotheses of Proposition 3.5 are satisfied by A and C.

Sierra uses the conclusion of Proposition 3.5 for the Weyl algebra although her proof that the conclusion of Proposition 3.5 holds for the Weyl algebra is rather different from our proof of Proposition 3.5.

3.5. The Picard group Pic(R,Γ)

There are several well-understood connections between $\mathrm{Pic}(R,\Gamma)$ and other invariants of (R,Γ). For example, there is a group homomorphism $\Gamma \to \mathrm{Pic}(R,\Gamma)$ that sends $i \in \Gamma$ to the twist functor $M \mapsto M(i)$. The kernel of this map is the subgroup of Γ consisting of those i for which R_i contains a unit. The homogeneous units in $(A,\mathbb{Z})$ and $(C,\mathbb{Z}_{\mathrm{fin}})$ belong to the identity component of the ring so the map $\Gamma \to \mathrm{Pic}(R,\Gamma)$ is injective in those two cases.

The assignment $(\alpha,\bar{\alpha},\lambda) \rightsquigarrow [\alpha_*]$ gives a map $\mathbb{A}\mathrm{ut}(R,\Gamma) \to \mathrm{Pic}(R,\Gamma)$ and by Proposition 3.4 the image of this map is contained in the normalizer of the image of Γ in $\mathrm{Pic}(R,\Gamma)$.

3.5.1. Notation

If $\tau : F \to G$ is a natural transformation we write τ_M for the associated map $FM \to GM$.

Proposition 3.6. *Let* A *be an additive category. There is group homomorphism*

$$\Phi : \operatorname{Pic}(\mathsf{A}) \to \operatorname{Aut}\bigl(\operatorname{End}(\mathrm{id}_{\mathsf{A}})\bigr)$$

defined as follows: If F is an auto-equivalence of A, *G a left adjoint to F, and hence a quasi-inverse to F, and $\eta : \mathrm{id}_{\mathsf{A}} \to FG$ the unit,*

$$\Phi\bigl([F]\bigr) : \operatorname{End}(\mathrm{id}_{\mathsf{A}}) \to \operatorname{End}(\mathrm{id}_{\mathsf{A}}), \quad \Phi\bigl([F]\bigr)(\tau)_M := \eta_M^{-1} \circ F(\tau_{GM}) \circ \eta_M.$$

Proof. It is easy to check that $\Phi([F])$ is an automorphism of the ring $\operatorname{End}(\mathrm{id}_{\mathsf{A}})$ and we omit the details. It is easy to check that $\Phi([F][F']) = \Phi([F])\Phi([F'])$ *provided* Φ is well defined. To check Φ is well defined it suffices to show that $\Phi([F])$ is the identity if $F \cong \mathrm{id}_{\mathsf{A}}$.

Let $\theta : F \to \mathrm{id}_{\mathsf{A}}$ be an isomorphism. Let $\tau \in \operatorname{End}(\mathrm{id}_{\mathsf{A}})$. The large rectangle in the diagram

$$\begin{tikzcd} M \arrow[r, "\eta_M"] \arrow[d, "\tau_M"'] & FGM \arrow[r, "\theta_{GM}"] \arrow[d, dashed] & GM \arrow[d, "\tau_{GM}"] \\ M \arrow[r, "\eta_M"'] & FGM \arrow[r, "\theta_{GM}"'] & GM \end{tikzcd}$$

commutes because τ is a natural transformation. If the dashed arrow is $F(\tau_{GM})$ then the right-hand square commutes because θ is a natural transformation $F \to \mathrm{id}_{\mathsf{A}}$. If the dashed arrow is $FG(\tau_M)$ then the left-hand square commutes because η is a natural transformation $\mathrm{id}_{\mathsf{A}} \to FG$. Because the horizontal arrows in the diagram are isomorphisms, it follows that

$$F(\tau_{GM}) = FG(\tau_M).$$

In particular,

$$\Phi\bigl([F]\bigr)(\tau)_M = \eta_M^{-1} \circ F(\tau_{GM}) \circ \eta_M = \eta_M^{-1} \circ FG(\tau_M) \circ \eta_M = \tau_M.$$

This holds for all M so $\Phi([F])(\tau) = \tau$. But τ was arbitrary, so $\Phi([F])$ is the identity. Hence Φ is a well-defined homomorphism. □

One should check that the definition of $\Phi([F])$ does not depend on the choice of G.

Suppose R_0 is central in R. For each $a \in R_0$ and $M \in \mathsf{Gr}(R,\Gamma)$, let $\mu_{a,M} : M \to M$ be $\mu_{a,M}(m) := ma$ for $m \in M$. Let $\mu_a : \mathrm{id}_{\mathsf{Gr}(R,\Gamma)} \to id_{\mathsf{Gr}(R,\Gamma)}$ be the natural transformation $(\mu_a)_M := \mu_{a,M}$. Then the map

$$\mu : R_0 \to \operatorname{End}(\mathrm{id}_{\mathsf{Gr}(R,\Gamma)}), \quad a \mapsto \mu_a \tag{3-4}$$

is a ring isomorphism.

Proposition 3.7. *Let (R, Γ) be a graded ring and suppose that R_0 belongs to the center of R. Let Φ be the group homomorphism in Proposition* 3.6*, and let μ be the isomorphism in* (3-4)*. Then the map*

$$\alpha : \mathrm{Pic}(R, \Gamma) \to \mathrm{Aut}(R_0), \quad [F] \mapsto \alpha_F := \mu^{-1} \circ \Phi\big([F]\big) \circ \mu$$

is a group homomorphism and, if $M \in \mathsf{Gr}(R, \Gamma)$, then $Ma = 0$ if and only if $(FM).\alpha_F(a) = 0$.

Proof. Let F be an auto-equivalence of $\mathsf{Gr}(R, \Gamma)$, G a left adjoint to F, and $\eta : \mathrm{id}_{\mathsf{Gr}(R,\Gamma)} \to FG$ the unit. By definition, $\mu \circ \alpha_F = \Phi([F]) \circ \mu$, so

$$\begin{aligned}(FM).\alpha_F(a) &= \mathrm{Image}\big(\mu\big(\alpha_F(a)\big)_{FM}\big)\\ &= \mathrm{Image}\big(\Phi\big([F]\big)(\mu_a)_{FM}\big)\\ &= \mathrm{Image}\big(\eta_{FM}^{-1} \circ F(\mu_{a,GFM}) \circ \eta_{FM}\big).\end{aligned}$$

Hence $(FM).\alpha_F(a) = 0$ if and only if $\mu_{a,GFM} = 0$, if and only if $(GFM).a = 0$, if and only if $Ma = 0$. □

More succinctly, $\alpha_F(a)$ is the unique $b \in R_0$ such that $\Phi([F])(\mu_a) = \mu_b$.

The next section examines a situation where there is a homomorphism $R_0 \to \mathrm{End}(\mathrm{id}_{\mathsf{Gr}(R,\Gamma)})$ without the hypothesis that R_0 is central in R. The result applies to $(A, \mathbb{Z})$ and is used implicitly in the proof of [5, Thm. 5.5].

3.6. R_0-R-bimodules

Sierra exploits to advantage the fact that every graded right A-module can be made into an A_0-A-bimodule. Proposition 3.9, the hypotheses of which are satisfied by A and C, gives a criterion on a graded ring (R, Γ) that implies every graded right R-module can be made into an R_0-R-bimodule. First we need a lemma.

Lemma 3.8. *Suppose that*

(1) *R_0 is commutative*;
(2) *R_i is a torsion-free left R_0-module for all $i \in \Gamma$*;
(3) *$R_0 x = x R_0$ for all $x \in R_i$ and all $i \in \Gamma$*;
(4) *$R_0 x \cap R_0 y \neq 0$ for all $x, y \in R_i - \{0\}$ and all $i \in \Gamma$.*

Then there is a homomorphism $\Gamma \to \mathrm{Aut}(R_0)$, $i \mapsto \theta_i$, such that $xa = \theta_i(a)x$ for all $a \in R_0$, all $x \in R_i$, and all $i \in \Gamma$.

Proof. Fix $i \in \Gamma$. Let $a \in R_0$.

Claim. *There is a unique $a' \in R$ such that $xa = a'x$ for all $x \in R_i$.*

Proof. Let $x, y \in R_i - \{0\}$. By hypothesis (2), there are elements $a', a'' \in R_0$ such that $xa = a'x$ and $ya = a''y$. By hypothesis (4), there are $b, c \in R_0$ such that $bx = cy \neq 0$. Therefore

$$a'bx = ba'x = bxa = cya = ca''y = a''cy = a''bx,$$

and $(a' - a'')bx = 0$. But $bx \neq 0$ so hypothesis (2) implies that $a' = a''$. □

It follows that there is a well-defined map $\theta_i : R_0 \to R_0$ such that $xa = \theta_i(a)x$ for all $a \in R_0$ and all $x \in R_i$. Let $b \in R_0$. It is clear that $\theta_i(a + b) = \theta_i(a) + \theta_i(b)$. Also, $\theta_i(ab)x = xab = \theta_i(a)xb = \theta_i(a)\theta_i(b)x$

so, by hypothesis (2), $\theta_i(ab) = \theta_i(a)\theta_i(b)$. Hence θ_i is an endomorphism of R_0. By hypothesis (3), θ_i is surjective. By hypothesis (2), θ_i is injective. Hence $\theta_i \in \mathrm{Aut}(R_0)$.

Let $j \in \Gamma$. Let $a \in R_0$, $x \in R_i$, and $y \in R_j$. Then

$$\theta_{i+j}(a)xy = xya = x\theta_j(a)y = \theta_i\theta_j(a)xy$$

so, by hypothesis (2), $\theta_{i+j} = \theta_i\theta_j$. Hence $i \mapsto \theta_i$ is a group homomorphism, as claimed. $\square$

Proposition 3.9. *Suppose that (R, Γ) satisfies the hypotheses in Lemma 3.8 and let $\theta_i \in \mathrm{Aut}(R_0)$, $i \in \Gamma$, be defined as in Lemma 3.8. Then*

(1) *every $M \in \mathsf{Gr}(R, \Gamma)$ is an R_0-R-bimodule with respect to the action*

$$a.m := m\theta_{-i}(a) \quad \text{for } m \in M_i \text{ and } a \in R_0;$$

(2) *the map*

$$\mu : R_0 \to \mathrm{End}(\mathrm{id}_{\mathsf{Gr}(R,\Gamma)}), \quad \mu(a)_M : M \to M, \quad \mu(a)_M(m) := a.m,$$

is an isomorphism of rings.

Proof. (1) If $a, b \in R_0$, then

$$(ba).m = (ab).m = m\theta_{-i}(ab) = m\theta_{-i}(a)\theta_{-j}(b) = (a.m)\theta_{-i}(b) = b.(a.m)$$

so M is a left R_0-module. If $y \in R_j$, then

$$a.(my) = my\theta_{-i-j}(a) = my\theta_{-j}\theta_{-i}(a) = m\theta_{-i}(a)y = (a.m)y.$$

Hence M is an R_0-R-bimodule.

(2) Because left multiplication by $a \in R_0$ is an R-module endomorphism of M, $\mu(a)$ is a natural transformation. It is easy to check that μ is a homomorphism. It is obviously injective.

To see that μ is surjective, let $\tau : \mathrm{id}_{\mathsf{Gr}(R,\Gamma)} \to \mathrm{id}_{\mathsf{Gr}(R,\Gamma)}$ be a natural transformation. Define $a := \tau_R(1)$. Let $r \in R_i$, and let $\lambda_r : R(-i) \to R$ be the map $\lambda_r(x) = rx$. The diagram

$$\begin{tikzcd} R(-i) \arrow[r, "\lambda_r"] \arrow[d, "\tau_{R(-i)}"'] & R \arrow[d, "\tau_R"] \\ R(-i) \arrow[r, "\lambda_r"'] & R \end{tikzcd}$$

commutes so

$$r\tau_{R(-i)}(1) = \tau_R\lambda_r(1) = \tau_R(r) = \tau_R(1.r) = ar = r\theta_{-i}(a).$$

Hence $\tau_{R(-i)}(1) = \theta_{-i}(a)$. Now consider an arbitrary graded right R-module M and an element $m \in M_i$. Let $\lambda_m : R(-i) \to M$ be the map $f(x) = mx$. The diagram

$$\begin{tikzcd} R(-i) \arrow[r, "\lambda_m"] \arrow[d, "\tau_{R(-i)}"'] & M \arrow[d, "\tau_M"] \\ R(-i) \arrow[r, "\lambda_m"'] & M \end{tikzcd}$$

commutes so

$$a.m = m\theta_{-i}(a) = \lambda_m\big(\tau_{R(-i)}(1)\big) = \tau_M\lambda_m(1) = \tau_M(m)$$

for all $m \in M$. It follows that $\tau_M = \mu(a)_M$ and that $\tau = \mu(a)$. Hence μ is surjective. $\square$

3.7. There is one further way in which an auto-equivalence of $\mathsf{Gr}(R,\Gamma)$ can induce an automorphism of R_0. The map

$$\lambda_j : R_0 \to \hom\big(R(j), R(j)\big), \quad \lambda_j(a)(x) := ax$$

is an isomorphism of rings. The following result is therefore clear.

Lemma 3.10. *Let F be an auto-equivalence of $\mathsf{Gr}(R,\Gamma)$. Let $f : \hom(R,R) \to \hom(FR,FR)$ be the isomorphism $g \mapsto Fg$. If $FR \cong R(j)$, then $\lambda_j^{-1} \circ f \circ \lambda_0$ is an automorphism of R_0.*

4. *C* as an ungraded ring

In this section, as in others, we assume that k is of characteristic zero.

The results in this section are not required for the proof of the main result in the paper but C is an interesting example of a class of commutative rings not commonly encountered so we establish its basic properties here.

4.1. If $I \subset \mathbb{Z} - \{0\}$, we write R_I for the subring of C generated by $\{x_0\} \cup \{x_n \mid n \in I\}$.

Proposition 4.1. *The ring C is an ascending union of Dedekind domains, and is flat over each of those Dedekind domains.*

Proof. It is clear that C is the ascending union of the subrings R_I where the union is taken over any ascending and exhaustive chain of finite subsets $I \subset \mathbb{Z} - \{0\}$. Such a subring is isomorphic to the ring

$$S_I := \frac{k[t, X_n \mid n \in I]}{\mathfrak{a}}$$

where $\mathfrak{a}$ is the ideal generated by the elements

$$g_n := X_n^2 - t^2 + n, \quad n \in I.$$

(The element t corresponds to x_0.)

Let I be a finite subset of $\mathbb{Z} - \{0\}$. Let $k(t)$ be the rational function field over k and let F be a splitting field for the polynomial

$$f(X) = \prod_{n \in I}\big(X^2 - t^2 + n\big).$$

Constructing F as a tower of quadratic extensions, it is easy to see that the integral closure of $k(t)$ in F is isomorphic to S_I, and hence to R_I. Therefore R_I is a Dedekind domain.

If $I \subset J \subset \mathbb{Z} - \{0\}$ are finite subsets, then R_I is contained in R_J and R_J is a finitely generated torsion-free, and hence projective, R_I-module. Hence, for every I, C is a directed union of finitely generated projective R_I-modules, and is therefore a flat R_I-module. $\square$

4.1.1. There are other ways to prove Proposition 4.1. For example, one can prove directly, using the Jacobian criterion, that the rings S_I in its proof are regular of Krull dimension one.

4.1.2. Suppose k is algebraically closed and fix elements $\sqrt{-n}$ in k. Let $k[[z]]$ be the ring of formal power series. There is a homomorphism $\varphi : C \to k[[z]]$ given by

$$\varphi(x_0) := z, \qquad \varphi(x_n) := \sqrt{-n}\left(1 - \frac{z^2}{n}\right)^{1/2}, \quad n \neq 0,$$

where $(1 - z^2/n)^{1/2}$ denotes the Taylor series expansion for $\sqrt{1 - z^2/n}$ centered at $z = 0$. The restriction of φ to the Dedekind domains S_I appearing in the proof of Proposition 4.1 is injective, so φ is injective on C.

Proposition 4.2. *The ring C has the following properties*:

(1) *It is an integrally closed non-noetherian domain.*
(2) *Its transcendence degree is one.*
(3) *Suppose $k = \mathbb{C}$. If $\mathfrak{m}$ is a maximal ideal in C, then* $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = 1$.
(4) *Every finitely generated ideal in C is projective and generated by $\leqslant 2$ elements.*
(5) *Let d be a positive integer. The ring homomorphism $\gamma : C \to C$ defined by $\gamma(x_n) := x_{nd}/\sqrt{d}$ is an isomorphism from C onto its subalgebra $k[x_{nd} \mid n \in \mathbb{Z}]$.*

Proof. (1) Of course, C is a domain because it is an ascending union of domains. It is integrally closed because it is a directed union of integrally closed rings.

To show C is not noetherian it suffices to show that $C \otimes_k \bar{k}$ is not noetherian so we may, and will, assume k is algebraically closed.

For each integer N, let $\mathfrak{a}_N$ be the ideal generated by the elements $x_d + \sqrt{-d}$ for $d \leqslant N$. Then $\mathfrak{a}_N \subset \mathfrak{a}_{N+1}$ but $\mathfrak{a}_N \neq \mathfrak{a}_{N+1}$.

(2) Since the field $\operatorname{Fract} C$ is the union of finite extensions of $k(x_0)$, it is clear that x_0 is a transcendence basis for $\operatorname{Fract} C$.

(3) Because C has countable dimension whereas the rational function field $\mathbb{C}(t)$ has uncountable dimension, $\mathfrak{m}$ is generated by $\{x_n - z_n \mid n \in \mathbb{Z}\}$ for suitable elements $z_n \in \mathbb{C}$. We write z for the point $(z_n)_{n \in \mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$ and think of it as a closed point of $\operatorname{Spec} C$.

The same argument as for the polynomial ring in a finite number of variables shows that $\mathfrak{m} = \mathfrak{m}^2 + \sum_n k(x_n - z_n)$.

Fix an integer r. Because $z \in \operatorname{Spec} C$, $z_n^2 + n = z_r^2 + r$ for all n, whence

$$z_n^2 - z_r^2 = r - n = x_n^2 - x_r^2.$$

Therefore $\mathfrak{m}^2 + k(x_r - z_r)$ contains

$$\frac{1}{2}\big((x_r - z_r)^2 - (x_n - z_n)^2\big) + z_r(x_r - z_r) = z_n(x_n - z_n).$$

If $z_n \neq 0$, then $x_n - z_n \in \mathfrak{m}$. In particular, if all z_n are non-zero, then $\mathfrak{m}^2 + k(x_r - z_r) = \mathfrak{m}$.

On the other hand, suppose $z_r = 0$. Then $z_n \neq 0$ for all $n \neq r$ so the argument just given shows that $x_n - z_n \in \mathfrak{m}^2 + kx_r$. Of course, $x_r \in \mathfrak{m}^2 + kx_r$ too, and therefore $\mathfrak{m}^2 + kx_r = \mathfrak{m}$.

(4) A finitely generated ideal in C is generated by an ideal in R_I for some finite subset $I \subset \mathbb{Z} - \{0\}$. However, every ideal in R_I is projective. If $\mathfrak{a}$ is an ideal in $R = R_I$, then the kernel of the multiplication map $C \otimes_R \mathfrak{a} \to C\mathfrak{a}$ is isomorphic to $\operatorname{Tor}_1^R(C, R/\mathfrak{a})$ which is zero because C is flat over R. Thus $C\mathfrak{a} \cong C \otimes_R \mathfrak{a}$ and hence $\mathfrak{a}$ is a projective C-module.

Every ideal in a Dedekind domain can be generated by $\leqslant 2$ elements.

(5) This is straightforward. $\square$

4.2. The results in this section will not be used elsewhere in this paper.

Let $k = \mathbb{C}$, and give $\mathbb{C}$ its usual topology.

Let's write $\mathbb{C}^{\mathbb{Z}}$ for the linear dual of the vector space $\bigoplus_{n \in \mathbb{Z}} \mathbb{C}x_n$ and view the x_ns as coordinate functions on $\mathbb{C}^{\mathbb{Z}}$. The $\mathbb{C}$-valued points of $\operatorname{Spec} C$ are the points in

$$X := \left\{ z = (z_n)_{n \in \mathbb{Z}} \;\middle|\; z_n^2 = z_0^2 - n \text{ for all } n \in \mathbb{Z} \right\} \subset \mathbb{C}^{\mathbb{Z}}.$$

We now consider the question of whether X can be given the structure of a Riemann surface. In order to preserve the usual connection between complex algebraic curves and Riemann surfaces, we are particularly interested in whether X can be made into a Riemann surface in such a way that the coordinate functions x_n are holomorphic. We will show this cannot be done when $\mathbb{C}^{\mathbb{Z}}$ is given the product or the box topologies. On the other hand, if $\mathbb{C}^{\mathbb{Z}}$ is identified with $\ell^\infty(\mathbb{Z})$ in a suitable way, then X has uncountably many connected components, all homeomorphic to one another, and each component can be given the structure of a Riemann surface in such a way that each x_n is a holomorphic function.

I thank Robin Graham for telling me the following result and allowing me to include it here.

Proposition 4.3. *If $\mathbb{C}^{\mathbb{Z}}$ is given the box topology, then X is discrete.*

Proof. For $t \in \mathbb{C}$ and $r \in \mathbb{R}_{>0}$, let $D(t, r)$ denote the open disk of radius r centered at t. Fix a point $z = (z_n)$ in X. If $z_n = 0$, let $r_n := \frac{1}{2|n|+1}$. If $z_n \neq 0$, let

$$r_n := \min\left\{ |z_n|, \frac{1}{2|n|+1} \right\}.$$

Since $|z_n| \to \infty$ as $|n| \to \infty$, $r_n = |z_n|$ for only finitely many n. Since

$$U := \prod_{n \in \mathbb{Z}} D(z_n, r_n)$$

is an open neighborhood of z in the box topology, to show X is discrete it suffices to show that $U \cap X = \{z\}$.

Suppose that $y = (y_n) \in X - \{z\}$. There are two cases.

(1) Suppose $y_0^2 = z_0^2$. Then $y_n = -z_n \neq 0$ for some n. For that n, $|y_n - z_n| = 2|z_n|$ so $y_n \notin D(z_n, r_n)$. Hence $y \notin U$.

(2) Suppose $y_0^2 \neq z_0^2$. If $a, b \in \mathbb{C}$ are fixed and $w \in \mathbb{C}$, then

$$\left| \frac{\sqrt{a+w} + \sqrt{b+w}}{w} \right| \to 0$$

regardless of which branches of the square root function are chosen, and regardless of whether the branches chosen for $\sqrt{a+w}$ and $\sqrt{b+w}$ are the same or not. Therefore, if $|n| \gg 0$,

$$
\begin{aligned}
|y_n - z_n| &= \left|\sqrt{y_0^2 - n} - \sqrt{z_0^2 - n}\,\right| \\
&= \left|\frac{y_0^2 - z_0^2}{\sqrt{y_0^2 - n} + \sqrt{z_0^2 - n}}\right| \\
&> \frac{1}{|n|} \\
&> \frac{1}{2|n| + 1}.
\end{aligned}
$$

Hence $y \notin U$, and X is discrete, as claimed. □

I thank Lee Stout for telling me the following result and allowing me to include it here.

Proposition 4.4. *Let $\mathbb{C}$ have its usual topology, $\mathbb{C}^{\mathbb{Z}}$ the product topology, and X the subspace topology.*

(1) *Every fiber of every x_k is homeomorphic to a Cantor set.*
(2) *X cannot be made into a complex manifold in such a way that any of the coordinate functions is holomorphic.*

Proof. Fix a point $y = (y_n) \in X$ and an integer k. Let $x_k : X \to \mathbb{C}$ be the function taking the k-th coordinate.

(1) Let F be the fiber of x_k over the point y_k. Then

$$F = \prod_{n<k} \{-y_n, y_n\} \times \{y_k\} \times \prod_{n>k} \{-y_n, y_n\}.$$

However, at most one y_n is zero so F is homeomorphic to a countable product of copies of the discrete space $\{\pm\}$ endowed with the product topology. Therefore F is a Cantor set.

(2) Let U be any open neighborhood of y. By shrinking U we can assume there is a positive integer $N > |k|$ and $\varepsilon > 0$ such that

$$U = X \cap \bigl\{(z_n) \bigm| |z_n - y_n| < \varepsilon \text{ if } -N \leqslant n \leqslant N\bigr\}.$$

Then U contains a point z such that $z_k \neq y_k$. For each integer n in $[-N, N]$, choose the branch of the square root function such that $y_n = \sqrt{x_k^2 + k - n}$ and define $s_n(\xi) = \sqrt{\xi^2 + k - n}$ for ξ in a sufficiently small disk centered at y_k.

Hence U contains the set

$$F' := \bigl\{(x_n) \bigm| x_n = y_n \text{ if } -N \leqslant n \leqslant N\bigr\}.$$

This is also a Cantor set, and $z_k - y_k$ vanishes on it. But every point of F' is a limit point in F'. An analytic function on an open set U that vanishes on a subset having a limit point is identically zero in a neighborhood of that limit point (the coefficients in the Taylor series expansion around the limit point are zero). Hence $z_k - y_k$ vanishes on U. But that is absurd because U contains a point (x_n) with $x_k \neq y_k$. □

The space of doubly infinite $\mathbb{C}$-valued sequences $\xi = (\xi_n)$ such that $|\xi_n|$ is bounded is denoted by $\ell^\infty(\mathbb{Z})$. It is a Banach space with respect to the norm

$$\|\xi\| = \sup_n\{\xi_n\}.$$

Let $f : X \to \ell^\infty(\mathbb{Z})$ be the map

$$f(z) = \left(\frac{z_n}{2n+1}\right).$$

Write $Y := f(X)$. If y_n, $n \in \mathbb{Z}$, are the obvious coordinate functions on $\ell^\infty(\mathbb{Z})$, then Y is the locus cut out by the equations

$$(1+2n)^2 y_n^2 = y_0^2 - n. \tag{4-1}$$

The ring $\mathbb{C}[y_n \mid n \in \mathbb{Z}]$ where the y_ns satisfy the relations (4-1) is isomorphic to C.

We write $\{\pm 1\}^{(\mathbb{Z})}$ for the subgroup of $\{\pm 1\}^{\mathbb{Z}}$ consisting of the functions $\mathbb{Z} \to \{\pm 1\}$ that take the value -1 only finitely often. It is a countable direct sum of copies $\{\pm 1\}$.

The next result is due to Robin Graham. I am grateful for his allowing me to include it here.

Proposition 4.5. *Let $\ell^\infty(\mathbb{Z})$ have its usual topology, and give Y the subspace topology. Then*

(1) *Y has uncountably many connected components,*
(2) *all those components are homeomorphic to one another,*
(3) *they are permuted transitively by the action of $\{\pm 1\}^{\mathbb{Z}}$,*
(4) *each component is stable under the action of $\{\pm 1\}^{(\mathbb{Z})}$, and*
(5) *each component can be given the structure of a Riemann surface in such a way that C consists of holomorphic functions.*

5. The $\mathbb{Z}_{\mathrm{fin}}$ graded ring C

In this section we establish the basic properties of C *as a graded ring*. One of the main results is that every graded ideal of C is principal. Because C is also a domain the standard results about modules over a PID carry over to the category of graded C-modules. In particular, every projective graded C-module is a direct sum of twists of C.

We end the section with the proof that $\mathsf{Gr}A \equiv \mathsf{Gr}(C, \mathbb{Z}_{\mathrm{fin}})$. This is done by exhibiting a bigraded A-C-bimodule that is, as a C-module, a projective generator in $\mathsf{Gr}(C, \mathbb{Z}_{\mathrm{fin}})$and has A as its graded endomorphism ring.

Most questions about C reduce to combinatorial questions about $\mathbb{Z}_{\mathrm{fin}}$.

5.1. The homogeneous components of C are

$$C_J = C_\varnothing x_J.$$

Lemma 5.1. *Let I, J, I', $J' \in \mathbb{Z}_{\mathrm{fin}}$. Then*

(1) $x_I x_J = x_{I\cap J}^2 x_{I\oplus J}$;
(2) $(Cx_I)_J = C_\varnothing x_{I-J}^2 x_J$;
(3) *the following conditions are equivalent*:
 (a) $x_I x_J = x_{I'} x_{J'}$;
 (b) *$I \cap J = I' \cap J'$ and $I \oplus J = I' \oplus J'$*;
 (c) *$I \cap J = I' \cap J'$ and $I \cup J = I' \cup J'$.*

Proof. (1) This follows from the identity $I \cup J = (I \cap J) \sqcup (I \oplus J)$.

(2) We have $(Cx_I)_J = C_{I\oplus J} x_I = C_\varnothing x_{I\oplus J} x_I = C_\varnothing x_{I-J}^2 x_J$.

(3) The equivalence of (b) and (c) follows from the identity

$$I \cup J = I \oplus J \oplus (I \cap J).$$

The equivalence of (a) and (b) follows from (1). $\square$

The identity component of the ring obtained by inverting all non-zero homogeneous elements of C is the field of fractions of $C_\varnothing$, the rational function field $k(x_0^2)$.

5.2. Graded ideals in C

Because each x_I is a regular element of C,

$$Cx_I \cong C(I).$$

Here $C(I)$ is the degree-shifted module: C viewed as a graded module with 1 placed in degree $I\,(=-I)$.

Lemma 5.2. *Let $I, J \in \mathbb{Z}_{\mathrm{fin}}$. Then*

(1) $Cx_I + Cx_J = Cx_{I\cap J}$ *and*
(2) $Cx_I \cap Cx_J = Cx_{I\cup J}$.

Proof. (1) Since $I = (I - J) \sqcup (I \cap J)$ and $J = (J - I) \sqcup (I \cap J)$, we have

$$Cx_I + Cx_J = (Cx_{I-J} + Cx_{J-I})x_{I\cap J}.$$

However, $I - J$ and $J - I$ are disjoint so x^2_{I-J} and x^2_{J-I} are relatively prime elements of $C_\varnothing$ whence $Cx_{I-J} + Cx_{J-I} = C$. Hence $Cx_I + Cx_J = Cx_{I\cap J}$.

(2) Since

$$\begin{aligned}(Cx_I \cap Cx_J)_K &= C_\varnothing x^2_{I-K}x_K \cap C_\varnothing x^2_{J-K}x_K\\ &= C_\varnothing x^2_{(I-K)\cup(J-K)}x_K\\ &= C_\varnothing x^2_{(I\cup J)-K}x_K\\ &= (Cx_{I\cup J})_K\end{aligned}$$

for all $K \in \mathbb{Z}_{\mathrm{fin}}$, $Cx_I \cap Cx_J = Cx_{I\cup J}$ as claimed. $\square$

Proposition 5.3. *Every graded ideal of C is generated by a single homogeneous element.*

Proof. Let $\mathfrak{a}$ be a non-zero graded ideal of C. Let d be a non-zero element of $C_\varnothing = k[z]$ of minimal z-degree with the property that $dx_I \in \mathfrak{a}$ for some $I \in \mathbb{Z}_{\mathrm{fin}}$. Let J be of minimal cardinality such that $dx_J \in \mathfrak{a}$.

Let fx_I be an arbitrary element of $\mathfrak{a}$ with $0 \neq f \in C_\varnothing$. Let h be the greatest common divisor of d and f in $C_\varnothing$. Both x_I and x_J divide $x_{I\cup J}$ so $dx_{I\cup J}$ and $fx_{I\cup J}$ belong to $\mathfrak{a}$. Therefore $hx_{I\cup J}$ belongs to $\mathfrak{a}$. But $\deg h \leqslant \deg d$ so the choice of d implies that $\deg h = \deg d$. Therefore d divides f.

Write $f = dg$ where $g \in k[z]$. Then

$$(dx_J, fx_I) = (dx_{J-I}, fx_{I-J})x_{I\cap J} = (x_{J-I}, gx_{I-J})\,dx_{I\cap J}.$$

However, x_i is a unit modulo x_j if $i \neq j$, so x_{I-J} is a unit modulo x_{J-I}. Hence

$$(dx_J, fx_I) = (x_{J-I}, g)\, dx_{I\cap J}.$$

But

$$(x_j, g) = \begin{cases} (x_j) & \text{if } x_j^2 | g, \\ C & \text{otherwise,} \end{cases}$$

so

$$(x_{J-I}, g) = (x_K) \quad \text{where } K = \left\{ j \in J - I \;\middle|\; x_j^2 \text{ divides } g \right\}.$$

Therefore

$$(dx_J, fx_I) = (dx_{I\cap J} x_K) = (dx_L)$$

where $L = (I \cap J) \cup K \subset J$. By the choice of J, the cardinality of L can be no smaller than that of J so $L = J$ and $(dx_J, fx_I) = (dx_J)$. It follows that $\mathfrak{a} = (dx_J)$. $\square$

Proposition 5.4. *Let $I, J, I', J' \in \mathbb{Z}_{\text{fin}}$.*

(1) *There is an isomorphism of graded C-modules*

$$Cx_I \oplus Cx_J \cong Cx_{I\cup J} \oplus Cx_{I\cap J}. \tag{5-1}$$

(2) *There is a surjective degree zero C-module homomorphism*

$$Cx_I \oplus Cx_J \to Cx_K$$

if and only if $I \cap J \subset K \subset I \cup J$.

(3) *There is an isomorphism of graded C-modules*

$$Cx_I \oplus Cx_J \cong Cx_{I'} \oplus Cx_{J'}$$

if and only if $I \cup J = I' \cup J'$ and $I \cap J = I' \cap J'$.

Proof. (1) By Lemma 5.2, the exact sequence

$$0 \to Cx_I \cap Cx_J \to Cx_I \oplus Cx_J \to Cx_I + Cx_J \to 0$$

can be rewritten as

$$0 \to Cx_{I\cup J} \to Cx_I \oplus Cx_J \to Cx_{I\cap J} \to 0.$$

The right-most term is projective, so the sequence splits giving the claimed isomorphism.

(2) ($\Leftarrow$) This follows from (1) because the hypothesis on K implies there is a set $L \in \mathbb{Z}_{\text{fin}}$ such that $K \cap L = I \cap J$ and $K \cup L = I \cup J$, namely $L = (I \cup J - K) \cup (I \cap J)$.

($\Rightarrow$) Suppose there is a surjective degree zero C-module homomorphism $Cx_I \oplus Cx_J \to Cx_K$. Because $Cx_I \cong C(I)$, and so on, there is a surjective degree zero C-module homomorphism $C(I \oplus K) \oplus$

$C(J \oplus K) \to C$ and hence a surjective degree zero C-module homomorphism $f : Cx_{I\oplus K} \oplus Cx_{J\oplus K} \to C$. Since f is completely determined by $f(x_{I\oplus K}, 0)$ and $f(0, x_{J\oplus K})$ which must belong to $C_{I\oplus K}$ and $C_{J\oplus K}$ respectively, i.e., to $C_\varnothing x_{I\oplus K}$ and $C_\varnothing x_{J\oplus K}$, the image of f is contained in $Cx_{I\oplus K} + Cx_{J\oplus K}$ which is equal to $Cx_{(I\oplus K)\cap(J\oplus K)}$ by Lemma 5.2. Hence $Cx_{(I\oplus K)\cap(J\oplus K)} = C$. Therefore $(I \oplus K) \cap (J \oplus K) = \varnothing$ and this implies that $I \cap J \subset K \subset I \cup J$.

(3) ($\Leftarrow$) This follows from (5-1).

($\Rightarrow$) Suppose that $Cx_I \oplus Cx_J \cong Cx_{I'} \oplus Cx_{J'}$. Because $Cx_{I'}$ and $Cx_{J'}$ are quotients of $Cx_I \oplus Cx_J$, (2) implies that $I \cap J \subset I' \subset I \cup J$ and $I \cap J \subset J' \subset I \cup J$, i.e., $I \cap J \subset I' \cap J'$, and $I' \cup J' \subset I \cup J$. By symmetry, the reverse inclusions also hold.[4] □

5.3. Torsion-free and projective graded C-modules

A graded module over a graded ring is said to be a

(1) free graded module if it has a basis consisting of homogeneous elements;
(2) projective graded module if it is projective as an object in $\mathsf{Gr}(C, \mathbb{Z}_{\mathrm{fin}})$.

Let M be a graded module over a commutative graded ring R. A homogeneous element $m \in M$ is torsion if $xm = 0$ for some homogeneous regular element $x \in R$. A graded module is torsion if every homogeneous element in it is torsion and torsion-free if its only torsion element is 0. The submodule of M generated by the torsion elements is a torsion module and is called the torsion submodule of M. We will denote it by τM for now. The quotient $M/\tau M$ is torsion-free.

Presumably the following result is already in the literature.

Proposition 5.5. *Let (R, Γ) be a graded ring. Suppose R is commutative, that all homogeneous elements of R are regular, and that every graded ideal of R is principal. Then*

(1) *every graded submodule of a finitely generated free graded module is a free graded module*[5];
(2) *every finitely generated graded R-module is a direct sum of a graded torsion module and a free graded module.*

Proof. (1) Let $f_1, \dots, f_n$ be a homogeneous basis for a graded module F. We argue by induction on n to show that every graded submodule of F is a free graded module. The result is true by hypothesis if $n = 1$ so suppose $n \geqslant 2$.

Let E be a graded submodule of F. Let $\alpha : F \to Rf_1$ be the projection with kernel $F' := Rf_2 \oplus \cdots \oplus Rf_n$. If $E \subset F'$ then E has a homogeneous basis by the induction hypothesis, so we may suppose that $E \not\subset F'$. Then $\alpha(E)$ is a non-zero graded submodule of Rf_n, so is equal to Raf_n for some homogeneous $a \in R$. But Raf_n is isomorphic to a twist of R so the map $\alpha|_E : E \to \alpha(E)$ splits and $E \cong \alpha(E) \oplus (E \cap F')$. By the induction hypothesis, $E \cap F'$ has a homogeneous basis. Hence E has a homogeneous basis.

(2) Let M be a finitely generated graded R-module. Since $M/\tau M$ is torsion-free it suffices to show that a torsion-free finitely generated graded R-module is a free graded module. So, we assume M is torsion-free.

Let $\{m_1, \dots, m_n\}$ be a homogeneous set of generators of M and assume they have been ordered so that $\{m_1, \dots, m_s\}$ is a maximal subset of linearly independent elements. Write $L = Rm_1 + \cdots + Rm_s$. If $s = n$ we are done, so suppose otherwise. Hence, for $i > s$, there is a non-zero homogeneous element $x_i \in R$ such that $x_i m_i \in N$. Set $x = x_{s+1} \cdots x_n$. Then $xM \subset N$. By (1), xM is a free graded module. Let $\delta = \deg x$. Since M is torsion-free the map $M(\delta) \to xM$ given by multiplication by x is an isomorphism. Hence $M(\delta)$, and therefore M, is a free graded R-module. □

[4] Although the equalities $I \cap J = I' \cap J'$ and $I \cup J = I' \cup J'$ imply that $I \oplus J = I' \oplus J'$, this latter equality follows directly from the isomorphism $Cx_I \oplus Cx_J \cong Cx_{I'} \oplus Cx_{J'}$ because taking the second exterior power implies that $Cx_I x_J \cong Cx_{I'} x_{J'}$ whence $\deg x_I x_J = \deg x_{I'} x_{J'}$ and $I \oplus J = I' \oplus J'$.

[5] The finitely generated hypothesis can be removed.

Corollary 5.6. *Every finitely generated graded projective C-module is isomorphic to a direct sum of twists of C. In particular, a rank one projective graded C-module is isomorphic to $C(I)$ for a unique $I \in \mathbb{Z}_{\text{fin}}$.*

Proof. The only point to be checked is that $C(I) \cong C(J)$ if and only if $I = J$. However, the map

$$\rho : C_{I \oplus J} \to \hom\big(C(I), C(J)\big), \quad \rho(a)(m) := am,$$

is an isomorphism and $\rho(a)$ is an isomorphism if and only if a is a unit, but the only homogeneous units in C are the elements of k which have degree $\varnothing$. Hence $C(I) \cong C(J)$ if and only if $I \oplus J = \varnothing$, i.e., if and only if $I = J$. □

Corollary 5.7. $\operatorname{Pic} \mathcal{X} \cong \mathbb{Z}_{\text{fin}}$.

Proof. Let $I \in \mathbb{Z}_{\text{fin}}$. Multiplication in C provides an isomorphism $C(I) \otimes_C C(I) \xrightarrow{\sim} C$ so $C(I)$ is an invertible object in $\mathsf{Gr}(C, \mathbb{Z}_{\text{fin}})$. It remains to show that the $C(I)$s are the only invertible objects in $\mathsf{Gr}(C, \mathbb{Z}_{\text{fin}})$. However, if $P \otimes_C Q \cong C$, then P is projective and necessarily of rank one since it embeds in C. Hence P is isomorphic to some $C(I)$ by Corollary 5.6. □

Proposition 5.8. *Let $\mathcal{S} \subset \mathbb{Z}_{\text{fin}}$. Then the set of projectives $\{Cx_I \mid I \in \mathcal{S}\}$ generates $\mathsf{Gr}(C, \mathbb{Z}_{\text{fin}})$ if and only if*

$$\bigcap_{I \in \mathcal{S}} I = \varnothing \quad \textit{and} \quad \bigcup_{I \in \mathcal{S}} I = \mathbb{Z}.$$

Proof. ($\Rightarrow$) By hypothesis there is a surjective map

$$\bigoplus_{I \in \mathcal{F}} Cx_I \to C$$

for some subset $\mathcal{F} \subset \mathcal{S}$. But the image of every non-zero degree preserving homomorphism $Cx_I \to C$ is contained in Cx_I, so

$$\sum_{I \in \mathcal{F}} Cx_I = C.$$

Since C is cyclic we can assume $\mathcal{F}$ is finite. Hence by repeated applications of Lemma 5.2(1), the intersection of the Is belonging to $\mathcal{F}$ is empty.

Fix an integer n. By hypothesis there is a surjective map

$$\bigoplus_{I \in \mathcal{F}} Cx_I \to Cx_n$$

for some subset $\mathcal{F} \in \mathcal{S}$. In other words, there is a surjective map

$$\bigoplus_{I \in \mathcal{F}} C(I) \to C\big(\{n\}\big)$$

and hence a surjective map

$$\bigoplus_{I \in \mathcal{F}} C\big(I \oplus \{n\}\big) \to C.$$

Since $C(I \oplus \{n\}) \cong Cx_{I\oplus\{n\}}$, it follows from the previous paragraph that the intersection of all the $I \oplus \{n\}$, $I \in \mathcal{F}$, is empty. However, if n does not belong to any of the Is that belong to $\mathcal{F}$, $i \in I \oplus \{n\}$ for all n, a contradiction. It follows that n must belong to some I.

($\Leftarrow$) To prove that $\{Cx_I \mid I \in \mathcal{S}\}$ generates $\mathsf{Gr}(C, \mathbb{Z}_{\mathrm{fin}})$ it suffices, by Corollary 5.6, to show there is a surjective map

$$\bigoplus_{I \in \mathcal{S}} Cx_I \to Cx_K$$

for every $K \in \mathbb{Z}_{\mathrm{fin}}$. By hypothesis, there are elements $I_1, \dots, I_m, I_{m+1}, \dots, I_n$ of $\mathcal{S}$ such that $I_1 \cap \cdots \cap I_m \subset K \subset I_{m+1} \cup \cdots \cup I_n$. Write $I = I_1 \cap \cdots \cap I_n$ and $J = I_1 \cup \cdots \cup I_n$. By Proposition 5.4(1), both Cx_I and Cx_J are quotients of

$$\bigoplus_{j=1}^{n} Cx_{I_j}.$$

However, $I \cap J \subset K \subset I \cup J$ so Proposition 5.4(2) says that Cx_K is a quotient of $Cx_I \oplus Cx_J$. Hence Cx_K is generated by the Cx_I, $I \in \mathcal{S}$. $\square$

Corollary 5.9. *The set of projectives Cx_n, $n \in \mathbb{Z}$, generates* $\mathsf{Gr}C$.

Lemma 5.10. *Let I, $J \in \mathbb{Z}_{\mathrm{fin}}$. Then*

$$\hom_C\big(C(I), C(J)\big) = C_\varnothing \theta_{JI}$$

where $\theta_{JI} : C(I) \to C(J)$ is the map $\theta_{JI}(c) = cx_{I\oplus J}$. Furthermore,

$$(\theta_{JI}\theta_{IJ} - \theta_{JK}\theta_{KJ})(c) = \big(x^2_{I\oplus J} - x^2_{J\oplus K}\big)c.$$

Proof. If P and Q are graded C-modules, the $C_\varnothing$-module structure on $\hom(P, Q)$ is given by $(c.f)(p) = f(cp)$ for $c \in C_\varnothing$, $f \in \hom(P, Q)$, and $p \in P$. It is a standard fact that the map $\rho : C_{I\oplus J} \to \hom_C(C(I), C(J))$ given by

$$\rho(a)(c) = ac$$

is an isomorphism of $C_\varnothing$-modules. Since $C_{I\oplus J} = x_{I\oplus J}C_\varnothing$, $\hom_C(C(I), C(J))$ is generated by $\rho(x_{I\oplus J})$ which is exactly the map θ_{JI}. The final identity follows immediately from the definition of the θs. $\square$

5.4. The elements $x_{[n]}$

We introduce the notation:

$$[n] := \begin{cases} \{1, \dots, n\} & \text{if } n \geqslant 1, \\ \varnothing & \text{if } n = 0, \\ \{n+1, \dots, 0\} & \text{if } n \leqslant -1. \end{cases} \tag{5-2}$$

Lemma 5.11. *The following identities hold*:

(1) $[n] = [-n] + n$;
(2) $[n-1] \oplus \{n\} = [n]$;

(3) $[m] \oplus [n] = [n-m] + m$;
(4) $[m+n] = [m] \oplus ([n] + m)$;
(5) $-[n] = [-n] - 1$.

Corollary 5.12. *The set* $\{Cx_{[n]} \mid n \in \mathbb{Z}\}$ *generates* $\mathsf{Gr}C$.

Proof. This follows from the criterion in Proposition 5.8. $\square$

5.5. Functors between categories of graded modules

Let (R, Δ) and (S, Γ) be graded k-algebras. A bigraded R-S-bimodule is a k-vector space P that is an R-S-bimodule and has a vector space decomposition

$$P = \bigoplus_{(\delta,\gamma)\in\Delta\times\Gamma} P_{(\delta,\gamma)}$$

such that

$$R_\alpha . P_{(\delta,\gamma)} . S_\beta \subset P_{(\alpha+\delta,\gamma+\beta)}$$

for all $\alpha, \delta \in \Delta$ and $\beta, \gamma \in \Gamma$.

For each $\delta \in \Delta$, the subspace

$$P_{(\delta,*)} = \bigoplus_{\gamma\in\Gamma} P_{(\delta,\gamma)}$$

is an S-module and we view it as a Γ-graded S-module by declaring that its degree-γ component is $P_{(\delta,\gamma)}$. The left action of an element r in R_α on $P_{(\delta,*)}$ is therefore a degree preserving S-module homomorphism $P_{(\delta,*)} \to P_{(\alpha+\delta,*)}$, and we therefore obtain a linear map

$$R_\alpha \to \hom_S(P_{(\delta,*)}, P_{(\alpha+\delta,*)}). \tag{5-3}$$

Let M be a Γ-graded right S-module. We define

$$H_S(P, M) := \bigoplus_{\delta\in\Delta} \hom_S(P_{(\delta,*)}, M)$$

with Δ-grading given by

$$H_S(P, M)_\delta := \hom_S(P_{(-\delta,*)}, M).$$

Composition of S-module homomorphisms gives maps

$$\hom_S(P_{(-\delta,*)}, M) \times \hom_S(P_{(-\alpha-\delta,*)}, P_{(-\delta,*)}) \to \hom_S(P_{(-\alpha-\delta,*)}, M)$$

and therefore maps

$$H_S(P, M)_\delta \times R_\alpha \to H_S(P, M)_{\alpha+\delta}$$

that give $H_S(P, M)$ the structure of a Δ-graded right R-module.

In summary, we obtain a functor

$$H_S(P,-):\mathsf{Gr}(S,\Gamma)\to\mathsf{Gr}(R,\Delta)$$

between the categories of graded right modules. A result of del Rio tells us when this is an equivalence of categories.

Theorem 5.13. *(See [2, Thm. 4.7], [3].) With the above notation, suppose that*

(1) $P_{(\delta,*)}$ *is a projective* S*-module for all* $\delta\in\Delta$ *and*
(2) $\{P_{(\delta,*)}\mid\delta\in\Delta\}$ *generates* $\mathsf{Gr}(S,\Gamma)$.

Then $H_S(P,-)$ *is an equivalence of categories.*

Theorem 5.14. $\mathsf{Gr}A\equiv\mathsf{Gr}(C,\mathbb{Z}_{\mathrm{fin}})$.

Proof. Let e_nC, $n\in\mathbb{Z}$, be a rank one free C-module with basis vector e_n placed in degree $[n]$. We define a $\mathbb{Z}\times\mathbb{Z}_{\mathrm{fin}}$ graded vector space P by setting

$$P_{(n,I)}:=e_nC_{I\oplus[n]}.$$

Thus $P_{(n,*)}=e_nC$ is isomorphic to $C([n])$ as a graded right C-module. We give P the structure of an A-C-module by declaring that x and y act on e_nC by

$$x\cdot e_n:=e_{n+1}x_{n+1}\quad\text{and}\quad y\cdot e_n:=e_{n-1}x_n.$$

This *does* make P a left A-module because

$$(xy-yx)e_n=e_n\left(x_n^2-x_{n+1}^2\right)=e_n.$$

With this action P is a bigraded A-C-bimodule.

The action of A_ℓ on P provides a map

$$\rho:A_\ell\to\hom_C\left(P_{(m,*)},P_{(\ell+m,*)}\right).$$

Since $P_{(n,*)}\cong C([n])$, $\hom_C(P_{(m,*)},P_{(\ell+m,*)})$ is generated as a $C_\varnothing$-module by the map $\theta_{[m+\ell],[m]}$ in Lemma 5.10.

If $\ell\geqslant 0$, then

$$x^\ell\cdot e_m=e_{m+\ell}x_{m+1}\cdots x_{m+\ell}=e_{\ell+m}x_{[\ell+m]\oplus[m]}$$

and

$$y^\ell\cdot e_m=e_{m-\ell}x_m\cdots x_{m-\ell+1}=e_{m-\ell}x_{[m-\ell]\oplus[m]}.$$

The actions of x^ℓ and y^ℓ on $P_{(m,*)}$ are therefore the same as the actions of $\theta_{[m+\ell],[m]}$ and $\theta_{[m-\ell],[m]}$ respectively.

Since xy acts on $P_{(n,*)}$ as multiplication by x_n^2,

$$\rho\left(x^\ell A_0\right)=\theta_{[m+\ell],[m]}k\left[x_n^2\right]=\theta_{[m+\ell],[m]}C_\varnothing.$$

Similarly,

$$\rho\big(y^\ell A_0\big) = \theta_{[m-\ell],[m]} k\big[x_n^2\big] = \theta_{[m-\ell],[m]} C_\varnothing.$$

Hence ρ is an isomorphism from A_ℓ to $\hom_C(P_{(m,*)}, P_{(m+\ell,*)})$.

Since this is the case for all ℓ and m, and since the $P_{(n,*)}$s provide a set of projective generators for $\mathsf{Gr}(C, \mathbb{Z}_{\mathrm{fin}})$ the theorem follows from del Rio's result [2, Th. 4.7] (see also [4, Prop. 2.1]). □

Corollary 5.15. $\mathsf{Gr}A \equiv \mathsf{Qcoh}\mathcal{X}$.

6. Simple and projective graded C-modules

As for a Dedekind domain, the classification of all graded C-modules follows easily once one has determined the simple and projective ones. By a simple graded C-module we mean a non-zero graded C-module whose only graded submodules are itself and the zero module. We define a class of simple graded modules that we call *special*. These are the simple $\mathcal{O}_\mathcal{X}$-modules that are supported at the stacky points of $\mathcal{X}$. The importance of these modules is apparent from Proposition 6.8 and Corollary 6.9.

Under the equivalence $\mathsf{Qcoh}\mathcal{X} \equiv \mathsf{Gr}(C, \mathbb{Z}_{\mathrm{fin}})$ the projective graded C-modules correspond to the locally free $\mathcal{O}_\mathcal{X}$-modules.

6.1. The simple graded modules

Proposition 6.1. *The maximal graded ideals of C are the ideals (p) as p ranges over the irreducible elements in $C_\varnothing - \{x_n^2 \mid n \in \mathbb{Z}\}$ and the ideals (x_n) for $n \in \mathbb{Z}$.*

Proof. The ideals in the statement of the proposition are certainly graded ideals.

To show a graded ideal $\mathfrak{a}$ is maximal among graded ideals it suffices to show that every homogeneous element of $C - \mathfrak{a}$ is a unit in $C/\mathfrak{a}$. Every homogeneous element of C is of the form fx_I for some $f \in C_\varnothing$ and some $I \in \mathbb{Z}_{\mathrm{fin}}$.

Let $\mathfrak{a} = (p)$ where p is an irreducible element of $C_\varnothing$ but not one of the x_n^2. Every element of $C_\varnothing - \mathfrak{a}$ is a unit modulo $\mathfrak{a}$. If $i \in \mathbb{Z}$, then $x_i^2 \in C_\varnothing - (p)$ so x_i^2, and therefore x_i, is a unit modulo $\mathfrak{a}$. It follows that every x_I is a unit modulo $\mathfrak{a}$. Hence if $f \in C_\varnothing$ and $fx_I \notin \mathfrak{a}$, then fx_I is a unit modulo $\mathfrak{a}$. This completes the proof that (p) is a maximal graded ideal.

Now let $\mathfrak{a} = (x_n)$. If $i \in \mathbb{Z} - \{n\}$, then x_i^2 is congruent to a non-zero scalar modulo $\mathfrak{a}$ so is a unit. Since $C_\varnothing = k[x_n^2]$, $\mathfrak{a} \cap C_\varnothing$ is a maximal ideal of $C_\varnothing$. Hence every element in $C_\varnothing - \mathfrak{a}$ is a unit modulo $\mathfrak{a}$. Therefore, if $f \in C_\varnothing$ and $fx_I \notin \mathfrak{a}$, then fx_I is a unit modulo $\mathfrak{a}$. This completes the proof that (x_n) is a maximal graded ideal for all $n \in \mathbb{Z}$.

Now let $\mathfrak{a}$ be an arbitrary maximal graded ideal of C. Then $(C/\mathfrak{a})_\varnothing$ is a field, so $\mathfrak{a}$ contains an irreducible element of $C_\varnothing$, say p, and $\mathfrak{a} \supset (p)$. If $p \notin \{x_n^2 \mid n \in \mathbb{Z}\}$, then (p) is maximal so $\mathfrak{a} = (p)$. On the other hand, if $p = x_n^2$, then $x_n \in \mathfrak{a}$ because $C/\mathfrak{a}$ has no homogeneous zero divisors, and therefore $\mathfrak{a} = (x_n)$. □

6.2. The ordinary simple graded C-modules

The simple graded modules of the form $C/(p)$ where p is an irreducible element of $C_\varnothing$ play virtually no role in this paper. However, the following facts are easily verified:

(1) if p and p' are relatively prime irreducibles, then $C/(p) \ncong C/(p')$ and $\mathrm{ext}^1_C(C/(p), C/(p')) = 0$;
(2) for all $J \in \mathbb{Z}_{\mathrm{fin}}$, $(C/(p))(J) \cong C/(p)$;
(3) $\mathrm{ext}^1_C(C/(p), C/(p)) \cong C/(p)$.

The simple modules of the form $C/(p)$ correspond to the non-stacky points of the coarse moduli space for $\mathcal{X}$.

For each $\lambda \in k - \mathbb{Z}$, we define

$$\mathcal{O}_\lambda := \frac{C}{(x_0^2 - \lambda)}.$$

6.3. *The special simple graded C-modules*

A simple graded C-module is special if it is isomorphic to one of the modules

$$X_n := \frac{C}{(x_n)}, \qquad Y_n := \left(\frac{C}{(x_n)}\right)(\{n\}), \quad n \in \mathbb{Z}.$$

The following observations follow immediately from the definition:

(1) There are non-split exact sequences

$$0 \longrightarrow Y_n \longrightarrow \tfrac{C}{(x_n^2)} \longrightarrow X_n \longrightarrow 0$$

and

$$0 \longrightarrow X_n \longrightarrow (\tfrac{C}{(x_n^2)})(\{n\}) \longrightarrow Y_n \longrightarrow 0.$$

(2) As $C_\varnothing$-modules, the homogeneous components of X_n are

$$(X_n)_I \cong \begin{cases} C_\varnothing/(x_n^2) & \text{if } n \notin I, \\ 0 & \text{if } n \in I. \end{cases}$$

(3) As $C_\varnothing$-modules, the homogeneous components of Y_n are

$$(Y_n)_I \cong \begin{cases} C_\varnothing/(x_n^2) & \text{if } n \in I, \\ 0 & \text{if } n \notin I. \end{cases}$$

(4) $Y_n \ncong X_n$ because $(X_n)_\varnothing \cong k$ but $(Y_n)_\varnothing = 0$, and
(5) $Y_n(\{n\}) = X_n$ because $2\{n\} = 0$.

One may define and/or characterize the special simple modules in terms of their properties inside the category $\mathsf{Gr}(C, \mathbb{Z}_{\text{fin}})$. For example, working with $\mathsf{Gr}A$, Sierra characterizes them as the simple graded modules S for which $\text{ext}^1(S, M) \neq 0$ for some simple graded module $M \ncong S$. In order to offer an alternative to Sierra's characterization we will characterize them as those simples S for which $\text{hom}(P, S) = 0$ for some non-zero projective graded module P (Proposition 6.3).

As we shall see, the isomorphism class of a special simple module is determined by the degrees in which it is zero, and a simple graded module is special if and only if some of its homogeneous components are zero.

Proposition 6.2. *Let $I \in \mathbb{Z}_{\text{fin}}$ and let $n \in \mathbb{Z}$.*

(1) $\text{hom}(C, X_n) \cong k$.
(2) $\text{hom}(C, Y_n) = 0$.

(3) $X_n(I) \cong \begin{cases} X_n & \text{if } n \notin I, \\ Y_n & \text{if } n \in I. \end{cases}$
(4) $\hom(Cx_I, X_n) \cong \begin{cases} k & \text{if } n \notin I, \\ 0 & \text{if } n \in I. \end{cases}$
(5) $I = \{n \in \mathbb{Z} \mid \hom(Cx_I, X_n) = 0\}$.
(6) $I = \{n \in \mathbb{Z} \mid \hom(Cx_I, Y_n) \neq 0\}$.

Proof. (1) We have $\hom(C, X_n) \cong (X_n)_\varnothing \cong C_\varnothing / C_\varnothing x_n^2 \cong k$.

(2) We have $\hom(C, Y_n) \cong (Y_n)_\varnothing = (X_n)_{\{n\}} = 0$.

(3) If $m \neq n$, then the image of x_m in $C/(x_n)$ is a unit so multiplication by x_m is an isomorphism $X_n(\{m\}) \cong X_n$. In general, if $I = \{i_1, \dots, i_t\}$, then

$$X_n(I) = X_n\bigl(\{i_1\}\bigr) \cdots \bigl(\{i_t\}\bigr)$$

so the result follows from the previous sentence.

(4) Since $\hom(Cx_I, X_n) \cong \hom(C, X_n(I))$, this follows from (1) and (3).

(5) and (6) follow from (4). □

Proposition 6.3. *Let S be a simple graded module. The following three conditions are equivalent*:

(1) *S is special*;
(2) *$\hom(P, S) = 0$ for some non-zero projective graded module P*;
(3) *$S_J = 0$ for some $J \in \mathbb{Z}_{\text{fin}}$*.

Proof. By Corollary 5.6, every projective graded C-module is a direct sum of various $C(J)$s, so (2) holds if and only if $\hom(C(J), S) = 0$ for some $J \in \mathbb{Z}_{\text{fin}}$. However, $\hom(C(J), S) \cong S_J$ so (2) holds if and only if $S_J = 0$ for some $J \in \mathbb{Z}_{\text{fin}}$. This proves the equivalence of (2) and (3).

Suppose S is not special. Then $S \cong C/pC$ for some irreducible $p \in C_\varnothing$ and $S_J = C_J/pC_J$ for all $J \in \mathbb{Z}_{\text{fin}}$. But C_J is isomorphic to $C_\varnothing$ as a $C_\varnothing$-module so $S_J \neq 0$ for all $J \in \mathbb{Z}_{\text{fin}}$. On the other hand, if S is special, then S_J *is* zero for some J by parts (5) and (6) of Proposition 6.2. This proves the equivalence of (1) and (3). □

The next result corresponds to Sierra's result [5, Thm. 5.5]. Our proof is a little different. For example, we characterize the special simple graded modules S using Proposition 6.3(2), and we also exploit the fact that C is commutative by using the map $\operatorname{Pic}(C, \mathbb{Z}_{\text{fin}}) \to \operatorname{Aut}(C_\varnothing)$, $[F] \mapsto \alpha_F$, defined in Proposition 3.7.

We write $\operatorname{Iso}(\mathbb{Z})$ for the isometry group of $\mathbb{Z}$ with respect to the metric is $d(m,n) = |m-n|$. The isometries are exactly the maps $n \mapsto \varepsilon n + d$ where $\varepsilon = \pm 1$ and $d \in \mathbb{Z}$. As an abstract group, $\operatorname{Iso}(\mathbb{Z})$ is isomorphic to the dihedral group D_∞.

Theorem 6.4. *There is a group homomorphism*

$$\operatorname{Pic}(C, \mathbb{Z}_{\text{fin}}) \to \operatorname{Iso}(\mathbb{Z}), \quad [F] \mapsto (n \mapsto \varepsilon n + d), \tag{6-1}$$

where $\varepsilon \in \{\pm 1\}$ and $d \in \mathbb{Z}$ are determined by the requirement that

$$FX_n \oplus FY_n \cong X_{\varepsilon n + d} \oplus Y_{\varepsilon n + d} \tag{6-2}$$

for all $n \in \mathbb{Z}$ and

$$F\mathcal{O}_\lambda \cong \mathcal{O}_{\varepsilon \lambda + d} \tag{6-3}$$

for all $\lambda \in k - \mathbb{Z}$.

Proof. Let S and S' be special simple graded modules. By parts (5) and (6) of Proposition 6.2, $S \oplus S' \cong X_m \oplus Y_m$ for some $m \in \mathbb{Z}$ if and only if $\dim_k \hom(C(I), S \oplus S') = 1$ for all $I \in \mathbb{Z}_{\text{fin}}$. Since

$$\dim_k \hom\bigl(F\bigl(C(I)\bigr), FX_n \oplus FY_n\bigr) = \dim_k \hom\bigl(C(I), X_n \oplus Y_n\bigr)$$

and since F permutes the isomorphism classes of rank one projective graded modules, it follows that $FX_n \oplus FY_n \cong X_{g(n)} \oplus Y_{g(n)}$ for a unique $g(n) \in \mathbb{Z}$. Since F is an auto-equivalence g is a permutation of $\mathbb{Z}$.

By Proposition 3.7, F determines an automorphism α_F of $C_\varnothing$ having the property that $a \in C_\varnothing$ annihilates a module M if and only if $\alpha_F(a)$ annihilates FM. Since x_n^2 annihilates $X_n \oplus Y_n$, $\alpha_F(x_n^2)$ is a multiple of $x_{g(n)}^2$.

Write $z = x_0^2$. Since $C_\varnothing$ is the polynomial ring $k[z]$, there is $\varepsilon \in k - \{0\}$ and $d \in k$ such that $\alpha_F(z) = \varepsilon z + d$. Therefore $\varepsilon z - n + d$, which is $\alpha_F(z - n)$, is a scalar multiple of $z - g(n)$. Hence $\varepsilon z - n + d = \varepsilon(z - g(n))$ for all $n \in \mathbb{Z}$. Thus $g(n) = \frac{1}{\varepsilon}(n - d)$ for all $n \in \mathbb{Z}$. It follows that $\varepsilon = \pm 1$ and $d \in \mathbb{Z}$. $\square$

Theorem 8.4 shows that the map (6-2) is surjective and that its kernel is the image of $\mathbb{Z}_{\text{fin}}$.

6.4. *Projective graded modules*

By Corollary 5.6, every projective graded C-module is a direct sum of various $C(I)$s. The next two results are immediate consequences of parts (5) and (6) of Proposition 6.2.

Corollary 6.5. *If P is a rank one projective graded C-module, then*

$$\hom(P, X_n) \neq 0 \quad \Leftrightarrow \quad \hom(P, Y_n) = 0.$$

Corollary 6.6. *A rank one projective graded C-module maps surjectively onto infinitely many X_ns but only finitely many Y_ns.*

Remark 6.7. By Corollary 6.6, the X_ns play a different role in $\mathsf{Gr}(C, \mathbb{Z}_{\text{fin}})$ from the Y_ns. The X_ns are the simple G-equivariant C-modules on which the corresponding isotropy groups act trivially, whereas those isotropy groups act on the Y_ns via the sign representation.

Sierra labels the simple graded A-modules $X(n)$ and $Y(n)$, $n \in \mathbb{Z}$, but her labelling is *not* compatible with ours—her $X(n)$ corresponds to X_n if $n \geqslant 0$ and to Y_n if $n < 0$. Her labelling, which is designed to remind the reader that $X(n)$ (resp., $Y(n)$) is isomorphic as an ungraded A-module to A/xA (resp., A/yA), makes the different properties of the X_ns and Y_ns less apparent.

Proposition 6.8. *Let P and Q be finitely generated projective graded C-modules having the same rank. Then $P \cong Q$ if and only if* $\dim \hom_C(P, Y_n) = \dim \hom_C(Q, Y_n)$ *for all n.*

Proof. ($\Leftarrow$) Suppose $\dim \hom_C(P, Y_n) = \dim \hom_C(Q, Y_n)$ for all n. We will argue by induction on $r := \operatorname{rank} P$.

By Corollary 5.6 and parts (5) and (6) of Proposition 6.2, the result is true when $r = 1$, so we assume that $r \geqslant 2$.

Let $\mathfrak{a}$ be the largest graded ideal of C that is the image of a degree-preserving homomorphism $P \to C$. Since $P \cong Q$, $\mathfrak{a}$ is also the largest graded ideal of C that is the image of a degree-preserving homomorphism $Q \to C$. Since $\mathfrak{a}$ is projective, there are graded projectives P' and Q' of the same rank such that $P \cong P' \oplus \mathfrak{a}$ and $Q \cong Q' \oplus \mathfrak{a}$. It is obvious that $\dim \hom_C(P', Y_n) = \dim \hom_C(Q', Y_n)$ for all n so, by the induction hypothesis, $P' \cong Q'$. It follows that $P \cong Q$.

($\Rightarrow$) This is obvious. $\square$

Corollary 6.9. *Let F and G be auto-equivalences of* $\mathsf{Gr}(C,\mathbb{Z}_{\mathrm{fin}})$. *Then $F \cong G$ if and only if $FS \cong GS$ for all special simples S.*

Proof. Suppose $FS \cong GS$ for all special simples S. Then for every finitely generated graded C-module M and every special simple S,

$$\begin{aligned}\dim\hom(FM, FS) &= \dim\hom(M, S)\\ &= \dim\hom(GM, GS)\\ &= \dim\hom(GM, FS).\end{aligned}$$

But F and G permute the special simples by Theorem 6.4, so

$$\dim\hom(FM, X_i) = \dim\hom(GM, X_i)$$

and

$$\dim\hom(FM, Y_i) = \dim\hom(GM, Y_i)$$

for all i. Now take $M = R(j)$, $j \in \Gamma$. Because $F(R(j))$ and $G(R(j))$ are rank one projectives, it follows from Proposition 6.8 that $F(R(j)) \cong G(R(j))$. Proposition 3.5 now implies that $F \cong G$. $\square$

Corollary 6.10. *Let* A *be a category and suppose that* $F, G : \mathsf{Gr}(C,\mathbb{Z}_{\mathrm{fin}}) \to \mathsf{A}$ *are equivalences of categories. Then $F \cong G$ if and only if $FS \cong GS$ for all special simples S.*

Proof. Let G^{-1} be a quasi-inverse to G. Then $F \cong G$ if and only if $G^{-1}F \cong \mathrm{id}_{\mathsf{Gr}C}$. Hence $F \cong G$ if and only if $G^{-1}FS \cong S$ for all special simples S. The result follows. $\square$

7. The Grothendieck group of $\mathcal{X}$

The Grothendieck group of $\mathcal{X}$, denoted $K_0(\mathcal{X})$, is, by definition, the Grothendieck group of the category of locally free coherent $\mathcal{O}_{\mathcal{X}}$-modules. Under the equivalence $\mathsf{Qcoh}\mathcal{X} \equiv \mathsf{Gr}(C,\mathbb{Z}_{\mathrm{fin}})$ locally free coherent $\mathcal{O}_{\mathcal{X}}$-modules correspond to finitely generated projective graded C-modules.

We write $\mathsf{gr}(C,\mathbb{Z}_{\mathrm{fin}})$, or $\mathsf{gr}\,C$, and P respectively for the full subcategories of $\mathsf{Gr}(C,\mathbb{Z}_{\mathrm{fin}})$ consisting of the finitely generated modules and the finitely generated projective graded modules.

Because C is a graded principal ideal domain, every graded C-module M is isomorphic to P/Q where P and Q are projective, even free, graded modules. The natural map $K_0(\mathsf{P}) \to K_0(\mathsf{gr}\,C)$ is therefore an isomorphism.[6] In particular,

$$K_0(\mathcal{X}) \cong K_0(\mathsf{gr}\,C).$$

7.1. Classification of projective graded modules

If I and J are multi-sets, i.e., sets whose elements have multiplicities, their union as a multi-set will be denoted by $I \boxplus J$.

[6] Hence the natural map $K_0(\mathcal{X}) \to K_0(\mathsf{coh}\mathcal{X})$ is also an isomorphism.

Proposition 7.1. *Let $I_1, \dots, I_r, J_1, \dots, J_r \in \mathbb{Z}_{\text{fin}}$. Then*

$$\bigoplus_{n=1}^{r} C(I_n) \cong \bigoplus_{n=1}^{r} C(J_n)$$

if and only if

$$I_1 \boxplus \cdots \boxplus I_r = J_1 \boxplus \cdots \boxplus J_r.$$

Proof. By Proposition 6.2(6), $\dim \hom(C(J), Y_i)$ is 1 if $i \in J$ and 0 otherwise, so the result follows from Proposition 6.8. □

Let $r \in \mathbb{N}$ and write $\mathbb{Z}_{\text{mult}\leqslant r}$ for the set of all finite multi-sets M of integers such that every element of M has multiplicity $\leqslant r$. Define

$$\Phi_r : \mathbb{Z}_{\text{mult}\leqslant r} \to \mathsf{Gr}(C, \mathbb{Z}_{\text{fin}})$$

by declaring that

$$\Phi_r(M) := C(I_1) \oplus \cdots \oplus C(I_d) \oplus C^{r-d}$$

where $I_1, \dots, I_d$ are the unique elements of $\mathbb{Z}_{\text{fin}}$ such that

$$I_1 \supset \cdots \supset I_d \neq \varnothing \quad \text{and} \quad I_1 \boxplus \cdots \boxplus I_d = M.$$

Corollary 7.2. *Fix a non-negative integer r. Then Φ_r gives a bijection between the elements of $\mathbb{Z}_{\text{mult}\leqslant r}$ and the isomorphism classes of finitely generated projective graded C-modules of rank $\leqslant r$. The inverse to Φ_r sends a module isomorphic to $C(J_1) \oplus \cdots \oplus C(J_r)$ to $J_1 \boxplus \cdots \boxplus J_r$.*

If P is a finitely generated projective graded C-module we write $[P]$ for its class in the Grothendieck group $K_0(\operatorname{gr} C)$.

Corollary 7.3. *Let P and Q be finitely generated projective graded C-modules of the same rank. Then $[P] = [Q]$ if and only if $P \cong Q$.*

Proof. If $[P] = [Q]$, there is a finitely generated graded projective M such that $P \oplus M \cong Q \oplus M$. It follows that $\dim \hom_C(P, Y_n) = \dim \hom_C(Q, Y_n)$ for all n so $P \cong Q$. The reverse implication is trivial. □

Because C is commutative, $K_0(\operatorname{gr} C)$ is a commutative ring with product $[P].[Q] = [P \otimes_C Q]$ where the tensor product is the usual tensor product of graded modules. By Proposition 5.4,

$$\bigl[C\bigl(\{m\}\bigr)\bigr].\bigl[C\bigl(\{n\}\bigr)\bigr] = \begin{cases} [C] & \text{if } m = n, \\ [C(\{m\})] + [C(\{n\})] - [C] & \text{if } m \neq n. \end{cases}$$

In due course, we will see that the classes $[\mathbb{C}(\{n\})]$, $n \in \mathbb{Z}$, provide a $\mathbb{Z}$-basis for $K_0(\operatorname{gr} C)$.

7.2. The homomorphism Υ

As in Section 2.5, we write u_I for the element of the integral group ring $\mathbb{Z}\mathbb{Z}_{\text{fin}}$ corresponding to I. There is a surjective ring homomorphism

$$\Upsilon : \mathbb{Z}\mathbb{Z}_{\text{fin}} \to K_0(\operatorname{gr} C), \quad u_I \mapsto \big[C(I)\big].$$

Theorem 7.4. *The kernel of the homomorphism $\Upsilon : \mathbb{Z}\mathbb{Z}_{\text{fin}} \to K_0(\operatorname{gr} C)$ is the ideal generated by the elements*

$$u_I + u_J - u_{I\cap J} - u_{I\cup J}, \quad I, J \in \mathbb{Z}_{\text{fin}}.$$

Equivalently, $\ker(\Upsilon)$ *is generated by* $\{u_m u_n + 1 - u_m - u_n \mid m \neq n\}$.

Proof. Let $\mathfrak{a}$ be the ideal generated by the elements $u_I + u_J - u_{I\cap J} - u_{I\cup J}$. Then $\mathfrak{a} \subset \ker \Upsilon$ because

$$C(I) \oplus C(J) \cong C(I \cap J) - C(I \cup J). \tag{7-1}$$

In order to shorten the notation we will write $I + J$ rather than $u_I + u_J$ in this proof.

Let $x \in \ker \Upsilon$. By Proposition 7.1 and Corollary 7.3,

$$x = (I_1 + \cdots + I_n) - (J_1 + \cdots + J_n)$$

for some $n \in \mathbb{N}$ and some elements $I_r, J_r \in \mathbb{Z}_{\text{fin}}$ with the property that

$$I_1 \boxplus \cdots \boxplus I_n = J_1 \boxplus \cdots \boxplus J_n.$$

It follows that $I_1 \cap \cdots \cap I_n = J_1 \cap \cdots \cap J_n$ and $I_1 \cup \cdots \cup I_n = J_1 \cup \cdots \cup J_n$.

We will argue by induction on n to show that $x \in \mathfrak{a}$. If $n \leqslant 1$, there is nothing to prove and when $n = 2$ the result follows from (7-1). Suppose that $n \geqslant 3$.

A sequence of elements $K_1, \dots, K_r$ belonging to $\mathbb{Z}_{\text{fin}}$ is said to be decreasing if $K_1 \supset K_2 \supset \cdots \supset K_r$. There is a unique decreasing sequence $K_1, \dots, K_n$ such that

$$I_1 \boxplus \cdots \boxplus I_n = K_1 \boxplus \cdots \boxplus K_n.$$

By Proposition 7.1, $(I_1 + \cdots + I_n) - (K_1 + \cdots + K_n)$ and $(J_1 + \cdots + J_n) - (K_1 + \cdots + K_n)$ belong to $\ker \Upsilon$. If $(I_1 + \cdots + I_n) - (K_1 + \cdots + K_n)$ and $(J_1 + \cdots + J_n) - (K_1 + \cdots + K_n)$ belong to $\mathfrak{a}$ so does x. It therefore suffices to show that x belongs to $\mathfrak{a}$ when $I_1, \dots, I_n$ is decreasing. We assume that is the case.

Let $L := I_1 \cap \cdots \cap I_n$ and define $I'_s := I_s - L$ and $J'_s := J_s - L$ for $1 \leqslant s \leqslant n$, and write

$$x' = \big(I'_1 + \cdots + I'_n\big) - \big(J'_1 + \cdots + J'_n\big).$$

Notice that $x = u_L x'$.

It is clear that $I'_1, \dots, I'_n$ is decreasing, $I'_1 \supset J'_1$, $I'_n = \varnothing$, and

$$I'_1 \boxplus \cdots \boxplus I'_n = J'_1 \boxplus \cdots \boxplus J'_n.$$

Let $K = I'_1 - J'_1$. Then

$$K \boxplus I'_2 \boxplus \cdots \boxplus I'_{n-1} = J'_2 \boxplus \cdots \boxplus J'_n$$

so

$$y := \bigl(K + I'_2 \cdots + I'_{n-1}\bigr) - \bigl(J'_2 + \cdots + J'_n\bigr)$$

belongs to $\ker \Upsilon$. It follows from the induction hypothesis that $y \in \mathfrak{a}$. But $(I'_1 + \varnothing) - (J'_1 + L) \in \mathfrak{a}$ and $x' = y + (I'_1 + \varnothing) - (J'_1 + L)$, so $x' \in \mathfrak{a}$ too. Since $x = x'u_L$, $x \in \mathfrak{a}$. □

Corollary 7.5. *The elements* $[C]$ *and* $[C(\{m\})]$, $m \in \mathbb{Z}$, *provide a* $\mathbb{Z}$-*basis for* $K_0(\operatorname{gr} C)$.

Proof. Let $\mathfrak{a} = \ker(\Upsilon : \mathbb{Z}\mathbb{Z}_{\mathrm{fin}} \to K_0(\operatorname{gr} C))$. Since $u_m u_n \equiv u_m + u_n - 1$ modulo $\mathfrak{a}$, it follows that $\mathbb{Z}\mathbb{Z}_{\mathrm{fin}}/\mathfrak{a}$ is spanned by the images of u_m, $m \in \mathbb{Z}$, and 1.

If there is a relation in $K_0(\operatorname{gr} C)$ of the form

$$\bigl[C\bigl(\{m_1\}\bigr)\bigr] + \cdots + \bigl[C\bigl(\{m_r\}\bigr)\bigr] + d[C] = \bigl[C\bigl(\{n_1\}\bigr)\bigr] + \cdots + \bigl[C\bigl(\{n_s\}\bigr)\bigr] + e[C]$$

for some positive integers d and e, and elements m_i and n_j in $\mathbb{Z}$, then $r + d = s + e$ and the multi-sets $\{\{m_1, \ldots, m_r\}\}$ and $\{\{n_1, \ldots, n_s\}\}$ are equal. Hence $d = e$, and it follows from this that the images of u_m, $m \in \mathbb{Z}$, and 1 in $\mathbb{Z}\mathbb{Z}_{\mathrm{fin}}/\mathfrak{a}$ are linearly independent. □

Corollary 7.6. *If* $\lambda \in k - \mathbb{Z}$, *then* $[\mathcal{O}_\lambda] = 0$ *but* $[X_n] = -[Y_n] \neq 0$ *for every* $n \in \mathbb{Z}$.

Proof. Since $\mathcal{O}_\lambda = C/(x_0^2 - \lambda)$ and $\deg(x_0^2 - \lambda) = \varnothing$, $[\mathcal{O}_\lambda] = 0$. On the other hand, since the $[C(\{n\})]$s form a basis and $X_n = C/Cx_n$, $[X_n] = [C] - [C(\{n\})] \neq 0$. By definition, $Y_n = X_n(\{n\})$, so $[Y_n] = -[X_n]$. □

8. Symmetries and automorphisms of $\mathrm{Gr}(C, \mathbb{Z}_{\mathrm{fin}})$

Consider the diagram

$$\cdots \;\text{———}\; : \text{———}\; : \text{———}\; : \text{———}\; : \text{———}\; : \text{———}\; \cdots$$

in which the underlying line is $\operatorname{Spec} C_\varnothing = \operatorname{Spec} k[z]$ and the two fractional points at the loci $x_n^2 = 0$, $n \in \mathbb{Z}$, represent the special simples X_n and Y_n. There are two obvious symmetries: translation $n \mapsto n + 1$, and reflection about 0. The automorphism $z \mapsto z + 1$ of $k[z]$ extends to an automorphism τ of C, and the automorphism $z \mapsto -z$ of $k[z]$ extends to an almost-automorphism φ of C. (If $\sqrt{-1} \in k$, the automorphism $z \mapsto -z$ of $k[z]$ extends to an automorphism $x_n = \omega : \sqrt{z - n} \mapsto \sqrt{z + n} = \sqrt{-1}x_{-n}$ of C such that $\omega_* = \varphi_*$.)

Theorem 8.1. *There is an automorphism* τ *and an almost-automorphism* φ *such that*

$$\tau_* X_n \cong X_{n+1}, \qquad \tau_* Y_n \cong Y_{n+1}, \qquad \tau_* \mathcal{O}_\lambda \cong \mathcal{O}_{\lambda+1},$$
$$\varphi_* X_n \cong X_{-n}, \qquad \varphi_* Y_n \cong Y_{-n}, \qquad \varphi_* \mathcal{O}_\lambda \cong \mathcal{O}_{-\lambda}$$

for all $n \in \mathbb{Z}$ *and* $\lambda \in k - \mathbb{Z}$.

Proof. Define the automorphism τ by

$$\tau(x_n) = x_{n-1}, \quad n \in \mathbb{Z}.$$

Let τ_* be the associated automorphism of $\mathrm{Gr}(C, \mathbb{Z}_{\mathrm{fin}})$ defined in Section 3.

Then $\tau_* X_n$ is simple, and its degree I component, $(\tau_* X_n)_I$, is equal to $(X_n)_{\bar{\tau} I}$ which is $(X_n)_{I-1}$. By Section 6.3, $(X_n)_{I-1}$ is zero exactly when $n \in I-1$, i.e., when $n+1 \in I$. Hence $\tau_* X_n \cong X_{n+1}$. The proof for $\tau_* Y_n$ is similar. Since $\mathcal{O}_\lambda$ is annihilated by $x_0^2 - \lambda$, $\tau_* \mathcal{O}_\lambda$ is annihilated by $\tau^{-1}(x_0^2 - \lambda) = x_1^2 - \lambda = x_0^2 - 1 - \lambda$. Hence $\tau_* \mathcal{O}_\lambda \cong \mathcal{O}_{\lambda+1}$.

The existence of φ is proved in Proposition 8.2 below. There is an associated automorphism φ_* of $\mathsf{Gr}(C, \mathbb{Z}_{\text{fin}})$. Only two properties of φ are needed for the proof this theorem: $\varphi(C_I) = C_{-I}$ for all $I \in \mathbb{Z}_{\text{fin}}$ and $\varphi(x_0^2) = -x_0^2$. Because $\varphi(C_I) = C_{-I}$ the degree I component of $\varphi_* X_n$, which is, of course, a simple graded C-module, is equal to $(X_n)_{-I}$. By Section 6.3, $(X_n)_{-I}$ is zero exactly when $n \in -I$, i.e., when $-n \in I$. Hence $\varphi_* X_n \cong X_{-n}$. The proof for $\varphi_* Y_n$ is similar. Finally, since $\mathcal{O}_\lambda$ is annihilated by $x_0^2 - \lambda$, $\varphi_* \mathcal{O}_\lambda$ is annihilated by $\varphi^{-1}(x_0^2 - \lambda) = -x_0^2 - \lambda$. $\square$

Proposition 8.2. *Write $z = x_0^2$. There is an almost-automorphism $\varphi : C \to C$ defined by the conditions*:

- *$\varphi : k[z] \to k[z]$ is the k-algebra automorphism $\varphi(z) = -z$ and*
- *$\varphi(a x_J) = \varphi(a) x_{-J}$ for all $a \in k[z]$ and $J \in \mathbb{Z}_{\text{fin}}$.*

Furthermore,

(1) *if $c \in C_I$ and $d \in C_J$, then $\varphi(cd) = (-1)^{|I \cap J|} \varphi(c)\varphi(d)$;*
(2) *$\varphi(x_n^2) = -x_{-n}^2$ for all $n \in \mathbb{Z}$;*
(3) *$\varphi^2 = \mathrm{id}_C$.*

Proof. Let $I, J, K \in \mathbb{Z}_{\text{fin}}$. We write $\lambda_{I,J} := (-1)^{|I \cap J|}$. Since

$$\bigl|K \cap (I \oplus J)\bigr| = |K \cap I| + |K \cap J| \pmod 2$$

it follows that

$$\bigl|K \cap (I \oplus J)\bigr| + |I \cap J| = \bigl|(K \oplus I) \cap J\bigr| + |K \cap I| \pmod 2$$

and hence that

$$\lambda_{K, I \oplus J} \lambda_{I,J} = \lambda_{K \oplus I, J} \lambda_{K,I}.$$

To show that (φ, λ) is an almost-automorphism of C it therefore suffices to prove (1). But first we observe that (2) is true because $x_n^2 = z - n$.

(1) It is enough to check this for $c = x_I$ and $d = x_J$. In that case

$$\begin{aligned}
\varphi(x_I x_J) &= \varphi\bigl(x_{I\cap J}^2 x_{I \oplus J}\bigr) \\
&= (-1)^{|I \cap J|} x_{-(I \cap J)}^2 x_{-(I \oplus J)} \\
&= (-1)^{|I \cap J|} x_{-I} x_{-J} \\
&= (-1)^{|I \cap J|} \varphi(x_I) \varphi(x_J).
\end{aligned}$$

(3) This is clear. $\square$

Proposition 8.3. *If $\sqrt{-1} \in k$, there is an algebra automorphism*

$$\omega : C \to C, \quad \omega(x_n) := \sqrt{-1} x_{-n},$$

such that

$$\omega_* \cong \varphi_*.$$

Proof. It is easy to see that ω does extend to an algebra automorphism. To show that $\omega_* \cong \varphi_*$ it suffices to show that their actions on isomorphism classes of the special simple graded modules are the same. However, $\bar{\omega} = \bar{\varphi}$ because $\omega(C_I) = C_{-I} = \varphi(C_I)$ for all $I \in \mathbb{Z}_{\rm fin}$ so the same argument as was used in Theorem 8.1 for the action of φ_* on the special simples shows that $\omega_* X_n \cong X_{-n}$ and $\omega_* Y_n \cong Y_{-n}$. The result follows. □

Theorem 8.4. *There is an exact sequence*

$$1 \to \mathbb{Z}_{\rm fin} \to \operatorname{Pic}(C, \mathbb{Z}_{\rm fin}) \to \operatorname{Iso}(\mathbb{Z}) \to 1$$

where the map into $\operatorname{Pic}(C, \mathbb{Z}_{\rm fin})$ *sends J to the twist functor (J) and the map out of* $\operatorname{Pic}(C, \mathbb{Z}_{\rm fin})$ *is described in Theorem* 6.4 (*equivalently, it sends F to the automorphism α_F of $C_\varnothing$ defined in Proposition* 3.7).

Proof. Since $\operatorname{Iso}(\mathbb{Z})$ is generated by the maps $n \mapsto n+1$ and $n \mapsto -n$, it follows from Theorem 8.1 that the map $\operatorname{Pic}(C, \mathbb{Z}_{\rm fin}) \to \operatorname{Iso}(\mathbb{Z})$ is surjective.

Suppose $\alpha_F = 1$, i.e., F is an auto-equivalence of $\mathsf{Gr}(C, \mathbb{Z}_{\rm fin})$ such that $FX_n \oplus FY_n \cong X_n \oplus Y_n$ for all $n \in \mathbb{Z}$.

Suppose $FC \cong C(I)$. Since $\hom(C, Y_n) = 0$ for all n, $\hom(C(I), FY_n) = 0$ for all $n \in \mathbb{Z}$. However, $\hom(C(I), Y_n) \neq 0$ if $n \in I$ so, if $n \in I$, then $FY_n \cong X_n$. If $n \notin I$, then $\hom(C(I), X_n) \neq 0$ so $FY_n \cong Y_n$ if $n \notin I$. Hence $FY_n \cong X_n$ if and only if $n \in I$. But $Y_n(I) \cong X_n$ if and only if $n \in I$ so $FY_n \cong Y_n(I)$ for all $n \in \mathbb{Z}$. It follows that $FX_n \cong X_n(I)$ for all $n \in \mathbb{Z}$. Since $FS \cong S(I)$ for all special simples S, $F \cong (I)$. □

There is a $\mathbb{Z}$-linear action of $\operatorname{Pic}(\operatorname{gr} C)$ on $K_0(C, \mathbb{Z}_{\rm fin})$ given by

$$[F] \cdot [M] := [FM].$$

It is simpler to write this as $F.[M] := [FM]$.

Let $\mathfrak{p}$ denote the kernel of the rank function

$$\operatorname{rank} : K_0(\operatorname{gr} C) \to \mathbb{Z}.$$

Since auto-equivalences preserve rank, $\mathfrak{p}$ is stable under the action of $\operatorname{Pic}(C, \mathbb{Z}_{\rm fin})$. Because the rank function is surjective $\mathfrak{p}$ is a prime ideal and

$$K_0(\operatorname{gr} C) = \mathfrak{p} \oplus \mathbb{Z} \cdot [C].$$

Proposition 8.5.

(1) *The elements $[X_n]$, $n \in \mathbb{Z}$, are a basis for $\mathfrak{p}$.*
(2) *The elements $[X_n]$ and $[Y_n]$, $n \in \mathbb{Z}$, form a full set of pairwise distinct representatives of $\mathfrak{p}/2\mathfrak{p}$.*
(3) $\{[X_n], [Y_n] \mid n \in \mathbb{Z}\}$ *and $\mathfrak{p}/2\mathfrak{p}$ are* $\operatorname{Pic}(C, \mathbb{Z}_{\rm fin})$*-torsors.*
(4) $\mathfrak{p}^2 = \mathfrak{p}$.

Proof. (1) If we identify $\mathbb{Z}\mathbb{Z}_{\rm fin}/\ker \Upsilon$ with its image in $K_0(\operatorname{gr} C)$, then

$$1 - u_n = [X_n] = -[Y_n]$$

for all $n \in \mathbb{Z}$. Since $\{1, u_n \mid n \in \mathbb{Z}\}$ is a basis for $\mathbb{Z}\mathbb{Z}_{\text{fin}}/\ker\Upsilon$ and $\operatorname{rank}(X_n) = 0$, the elements $[X_n]$, $n \in \mathbb{Z}$, form a basis for $\mathfrak{p}$.

(2) This follows immediately from (1).

(3) By Theorem 8.1, $\operatorname{Pic}(C, \mathbb{Z}_{\text{fin}})$ acts transitively on the set $\{[X_n], [Y_n] \mid n \in \mathbb{Z}\}$. An auto-equivalence is determined up to isomorphism by its action on the special simples (Corollary 6.10) so the only auto-equivalence that acts trivially on $\{[X_n], [Y_n] \mid n \in \mathbb{Z}\}$ is the identity.

(4) This follows from the fact that $X_i \otimes Y_i \cong Y_i$ and $Y_i \otimes Y_i \cong X_i$. $\square$

9. The correspondence between $\mathsf{Gr}A$ and $\mathsf{Gr}(C, \mathbb{Z}_{\text{fin}})$

In this section we examine the correspondence implemented by the equivalence $\operatorname{Hom}(P, -)$ between various significant features of C and A. The key to doing this is to match up the special simple C-modules with the corresponding simple A-modules.

9.1. *The special simple graded A-modules*

Following Sierra we define the graded simple A-modules

$$X := \frac{A}{xA}, \qquad Y := \left(\frac{A}{yA}\right)(-1).$$

We call the $X(n)$s and $Y(n)$s, $n \in \mathbb{Z}$, the special simple A-modules.

Recall that $A_0 = k[xy]$.

Proposition 9.1. *(See [5, Lemma 4.1].) The simple graded A-modules are*

(1) *the modules $X(n)$ and $Y(n)$, $n \in \mathbb{Z}$, and*
(2) *the modules $A/\mathfrak{m}A$ where $\mathfrak{m}$ is a maximal ideal of A_0 but not one of the ideals $(xy - n)A_0$ for any $n \in \mathbb{Z}$.*

We note that

$$X(n)_m \neq 0 \quad \Leftrightarrow \quad m \leqslant -n$$

and

$$Y(n)_m \neq 0 \quad \Leftrightarrow \quad m \geqslant -n + 1$$

whereas the non-special simple graded A-modules are non-zero in all degrees. It follows that the special simple A-modules can be recognized by the degrees in which they are non-zero.

Proposition 9.2. *Let $H(P, -) : \mathsf{Gr}(C, \mathbb{Z}_{\text{fin}}) \to \mathsf{Gr}A$ be the equivalence in the proof of Theorem* 5.14. *Then*

$$H(P, X_n) \cong \begin{cases} Y(n) & \text{if } n > 0, \\ X(n) & \text{if } n \leqslant 0, \end{cases}$$

and

$$H(P, Y_n) \cong \begin{cases} X(n) & \text{if } n > 0, \\ Y(n) & \text{if } n \leqslant 0. \end{cases}$$

Proof. Since X_n is simple, so is $H(P, X_n)$. Now

$$H(P, X_n)_m = \hom(P_{(-m,*)}, X_n) \cong \hom\bigl(C\bigl([-m]\bigr), X_n\bigr) \cong (X_n)_{[-m]}.$$

Therefore

$$\hom(P, X_n)_m \neq 0 \quad \Leftrightarrow \quad n \notin [-m] \quad \Leftrightarrow \quad \begin{cases} m \geqslant -n+1 & \text{if } n > 0, \\ m \leqslant -n & \text{if } n \leqslant 0. \end{cases}$$

Hence $H(P, X_n)$ is as described. The argument for $H(P, Y_n)$ is similar. □

Let σ be the automorphism of A defined by $\sigma(x) = y$ and $\sigma(y) = -x$. For every $n \in \mathbb{Z}$, $\sigma_*(X(n))$ is isomorphic to A/yA as an ungraded A-module and $\sigma_*(Y(n))$ is isomorphic to A/xA as an ungraded A-module. But $\bar{\sigma}(m) = -m$ for all $m \in \mathbb{Z}$, so

$$\sigma_*\bigl(X(n)\bigr) \cong Y(-n+1) \quad \text{and} \quad \sigma_*\bigl(Y(n)\bigr) \cong X(-n+1) \tag{9-1}$$

for all $n \in \mathbb{Z}$.

Proposition 9.3. *Let P be the bigraded A-C-bimodule in the proof of Theorem* 5.14. *Then*

$$\sigma_* \circ H(P, -) \cong H(P, -) \circ \tau_* \varphi_*.$$

Proof. By Corollary 6.10, it suffices to show that $\sigma_* H(P, S) \cong H(P, \tau_*\varphi_* S)$ for every special simple S.

If $n > 0$, then $\sigma_* H(P, X_n) \cong \sigma_*(Y(n)) \cong X(-n+1)$ and $\sigma_* H(P, Y_n) \cong \sigma_*(X(n)) \cong Y(-n+1)$. If $n \leqslant 0$, then $\sigma_* H(P, X_n) \cong \sigma_*(X(n)) \cong Y(-n+1)$ and $\sigma_* H(P, Y_n) \cong \sigma_*(Y(n)) \cong X(-n+1)$.

Now we consider the action of $H(P, -) \circ \tau_*\varphi_*$. By Theorem 8.1, $\tau_*\varphi_* X_n \cong X_{-n+1}$ and $\tau_*\varphi_* Y_n \cong Y_{-n+1}$. If $n \leqslant 0$, then $-n+1 > 0$ so, by Proposition 9.2, $H(P, -) \circ \tau_*\varphi_* X_n \cong Y(-n+1)$ and $H(P, -) \circ \tau_*\varphi_* Y_n \cong X(-n+1)$. If $n > 0$, then $-n+1 \leqslant 0$ so, by Proposition 9.2, $H(P, -) \circ \tau_*\varphi_* X_n \cong X(-n+1)$ and $H(P, -) \circ \tau_*\varphi_* Y_n \cong Y(-n+1)$. Comparing the results in this and the previous paragraphs, it follows that $\sigma_* \circ H(P, -) \cong H(P, -) \circ \tau_*\varphi_*$. □

Lemma 9.4. *Let*

$$\Sigma := \bigl(\{1\}\bigr) \circ \tau_*.$$

Then Σ is an automorphism of $\mathsf{Gr}(C, \mathbb{Z}_{\mathrm{fin}})$ *and it permutes the isomorphism classes of the special simple modules as in the diagram*:

$$\begin{tikzcd}
\cdots \arrow[r] & X_{-1} \arrow[r] & X_0 \arrow[dr] & X_1 \arrow[r] & X_2 \arrow[r] & \cdots \\
\cdots \arrow[r] & Y_{-1} \arrow[r] & Y_0 \arrow[ur] & Y_1 \arrow[r] & Y_2 \arrow[r] & \cdots
\end{tikzcd}$$

Proof. By Theorem 8.1, $\Sigma X_n \cong X_{n+1}(\{1\})$. Hence $\Sigma X_0 \cong Y_1$. By the remarks at the beginning of Section 6.3, if $n \neq 0$, then $\Sigma X_n \cong X_{n+1}$. Similar considerations apply to ΣY_n. □

Proposition 9.5. *Let $H(P,-) : \mathsf{Gr}(C, \mathbb{Z}_{\mathrm{fin}}) \to \mathsf{Gr}A$ be the equivalence in Theorem* 5.14. *Then there is an isomorphism of functors*

$$H(P,-) \circ \Sigma \cong (1) \circ H(P,-).$$

Proof. By Corollary 6.10, it suffices to show that $H(P,S)(1) \cong H(P,\Sigma S)$ for every special simple S. $\square$

9.2. The automorphisms ι_J

The key to much of Sierra's analysis of $\mathsf{Gr}A$ is her discovery of the automorphisms ι_J, $J \in \mathbb{Z}_{\mathrm{fin}}$, of $\mathsf{Gr}A$ that she describes in [5, Prop. 5.9]. These have the following properties:

(1) $\iota_\varnothing = \mathrm{id}_{\mathsf{Gr}A}$;
(2) $\iota_n = (n) \circ \iota_0 \circ (-n)$ for all $n \in \mathbb{Z}$[7];
(3) $\iota_J = \prod_{j \in J} \iota_j$ for all $J \in \mathbb{Z}_{\mathrm{fin}}$;
(4) $\iota_0^2 \cong \mathrm{id}_{\mathsf{Gr}A}$. This is not an equality.

The third condition says that the map $J \to \iota_J$ from $\mathbb{Z}_{\mathrm{fin}}$ to the automorphism group of $\mathsf{Gr}A$ is a homomorphism of monoids, and the fourth condition implies that the map $\mathbb{Z}_{\mathrm{fin}} \to \mathrm{Pic}(\mathsf{Gr}A)$, $J \mapsto \iota_J$, or, more precisely, $J \mapsto$ the image of ι_J in $\mathrm{Pic}(\mathsf{Gr}A)$, is a group homomorphism.

The functor ι_0 is defined first as an automorphism of the subcategory of $\mathsf{Gr}A$ consisting of the projective graded modules and as an automorphism of that subcategory it is a subfunctor of the identity functor. It follows that every ι_J is also a subfunctor of the identity functor on that subcategory.

Sierra shows that $\hom(P, X \oplus Y) \cong k$ for all rank one graded projectives P (cf. Corollary 5.6 and parts (5) and (6) of Proposition 6.2). The functor ι_0 is then defined on a rank one projective by

$$\iota_0 P := \ker f \quad \text{where } f : P \to X \oplus Y \text{ is any non-zero graded homomorphism.}$$

Equivalently, $\iota_0 P$ is the unique graded submodule of P that fits into an exact sequence

$$0 \to \iota_0 P \to P \to X \oplus Y$$

in which the right-most map is the unique (up to scalar multiple) non-zero map $P \to X \oplus Y$. From the exact sequence $0 \to \iota_0(P(-n)) \to P(-n) \to X \oplus Y$, we see that $\iota_n P$ is the unique submodule of P fitting into an exact sequence

$$0 \to \iota_n P \to P \to X(n) \oplus Y(n)$$

where the right-most map is non-zero.

Theorem 9.6. *Let $J \in \mathbb{Z}_{\mathrm{fin}}$. Then*

$$H(P,-) \circ (J) \cong \iota_J \circ H(P,-).$$

Proof. By [5, Prop. 5.9], ι_n interchanges the isomorphism classes of $X(n)$ and $Y(n)$ and fixes the isomorphism classes of all other $X(m)$s and $Y(m)$s. On $\mathsf{Gr}(C, \mathbb{Z}_{\mathrm{fin}})$, the twist $(\{n\})$ interchanges X_n and Y_n and fixes the isomorphism classes of all other special simples. It now follows from Proposition 9.2

[7] Because our (n) is equal to Sierra's $\langle -n \rangle$, our ι_n is her ι_{-n}.

that $\iota_n H(P,S) \cong H(P, S(\{n\}))$ for all special C-modules S. Hence by Corollary 6.10, $\iota_n \circ H(P,-) \cong H(P,-) \circ (\{n\})$. Thus, if $J = \{j_1, \ldots, j_t\}$, then

$$\begin{aligned}\iota_J \circ H(P,-) &\cong \iota_{j_1} \circ \cdots \circ \iota_{j_t} \circ H(P,-) \\ &\cong H(P,-) \circ \big(\{j_1\}\big) \circ \cdots \circ \big(\{j_1\}\big) \\ &\cong H(P,-) \circ (J),\end{aligned}$$

as required. □

9.3. The monoidal structures

The equivalence of categories does not respect the "natural" internal tensor products on $\mathsf{Gr}A$ and $\mathsf{Gr}(C, \mathbb{Z}_{\mathrm{fin}})$. The tensor product on $\mathsf{Gr}(C, \mathbb{Z}_{\mathrm{fin}})$ is the usual graded tensor product over C. The tensor product on $\mathsf{Gr}A$ is that which exists on the category of $\mathcal{D}_Y$-modules for any smooth variety Y, namely $\mathcal{M} \otimes \mathcal{N} = \mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{N}$ with a derivation δ acting on the tensor product as $\delta \otimes 1 + 1 \otimes \delta$. Specializing to A and taking $k[y]$ as the coordinate ring of the line on which A acts as differential operators (and x acts as $-d/dy$), the internal tensor product on $\mathsf{Gr}A$ is $- \otimes_{k[y]} -$ with x acting on the tensor product as $x \otimes 1 + 1 \otimes x$.

Both tensor products are commutative and

$$X(m) \otimes X(n) \cong X(m+n),$$

$$Y(m) \otimes Y(n) \cong Y(m+n-1) \cong X(m-1) \otimes Y(n)$$

whereas

$$X_i \otimes X_j \cong Y_i \otimes Y_j \cong \delta_{ij} X_i, \qquad X_i \otimes Y_j \cong \delta_{ij} Y_j.$$

The identity for the tensor product in $\mathsf{Gr}A$ is the simple module $X = A/xA$, whereas the identity for the tensor product in $\mathsf{Gr}(C, \mathbb{Z}_{\mathrm{fin}})$ is the projective module C.

The tensor product of two finitely generated C-modules is finitely generated but that property does not hold for A-modules.

9.4. Preparations for Section 10

9.4.1. The left $k[z]$ action on graded right A-modules

Throughout we will write

$$z = xy.$$

The degree zero component of A is therefore

$$A_0 = k[z].$$

Sierra [5, Sect. 4, p. 14] makes the fundamental observation that every graded right A-module M can be given the structure of a *left* $k[z]$-module in such a way that M becomes a $k[z]$-A-bimodule. The left action of z on an element $m \in M_j$ is

$$z.m := m(z-j). \tag{9-2}$$

The left $k[z]$-action commutes with the right A action because for all homogeneous $a \in A$,

$$[a, z] = (\deg a)a. \tag{9-3}$$

Homomorphisms $f : M \to N$ in $\mathsf{Gr}A$ are also homomorphisms of left $k[z]$-modules. Hence $\hom(M, N)$ has induced left and right $k[z]$-module structures given by

$$(z.\psi)(m) = z.\psi(m) \quad \text{and} \quad (\psi.z)(m) = \psi(z.m).$$

Since ψ preserves degree these two $k[z]$-module structures are the same.

When $M = A$ the left action of z on A given by (9-2) is the ordinary left multiplicative action, i.e., $z.a = za$. This follows from (9-3). However, the left action of z on $A(n)$ given by (9-2) coincides with left multiplication by $z + n$. For example, if $\bar{1}$ denotes 1 viewed as an element in $A(n)$ then $z.\bar{1} = \bar{1}(z + n)$ because $\bar{1} \in A(n)_{-n}$.

If $z.M = 0$, then $(z - n).M(n) = 0$.

Recall that $\mathcal{O}_\lambda = A/(z - \lambda)A$. It is straightforward to see that

$$(z - n).\mathcal{O}_n = (z - n).\mathcal{O}_{n+1}(-1) = 0$$

and, as a consequence of either of these facts,

$$(z - n).X(n) = (z - n).Y(n) = 0.$$

9.4.2. Isomorphisms between products of the ι_Js

For each $J \in \mathbb{Z}_{\mathrm{fin}}$, define

$$h_J = \prod_{j \in J} (z - j). \tag{9-4}$$

This polynomial belongs to $k[z] = A_0$.

Because the left action of $(z - n)$ annihilates the non-split extensions between $X(n)$ and $Y(n)$ it follows that

$$\iota_n^2 P = (z - n)P.$$

There is therefore a unique isomorphism

$$\eta^n : \iota_n^2 \to \mathrm{id}_{\mathsf{Gr}A}$$

such that $(\eta^n)_P : \iota_n^2 P \to P$ is left multiplication by $(z-n)^{-1}$ for every projective P. To be precise, η^n is first defined as a natural transformation between the restrictions of the functors to the subcategory of projectives, and as such η^n is multiplication by $(z - n)^{-1}$. The natural transformation η^n "extends" uniquely to a natural transformation between the functors defined on all of $\mathsf{Gr}A$ *but*, as a natural transformation on $\mathsf{Gr}A$, η^n is *not* "multiplication by $(z - n)^{-1}$". Similarly, if $I, J \in \mathbb{Z}_{\mathrm{fin}}$, we define the isomorphism

$$\eta^{IJ} : \iota_{I \oplus J} \to \iota_I \iota_J$$

to be left multiplication by the polynomial h_J in (9-4).

We define the automorphism $\sigma_n : \iota_n^2 A \to A$ to be the isomorphism η^n at A, i.e., $\sigma_n = (\eta^n)_A$. Thus σ_n is left multiplication by $(z-n)^{-1}$. We also define

$$\sigma_J := \prod_{j \in J} \sigma_j : \iota_J^2 A \to A \tag{9-5}$$

and $\sigma_\varnothing = \mathrm{id}_A$.

10. *C* is a twisted homogeneous coordinate ring for Gr*A*

In this section C is constructed directly from A as a sort of twisted homogeneous coordinate ring for GrA. We will show that C is isomorphic as a graded ring to the ring B defined in (10-1).

10.1. The $\mathbb{Z}_{\mathrm{fin}}$-graded ring B

We define the $\mathbb{Z}_{\mathrm{fin}}$-graded ring

$$B := \bigoplus_{J \in \mathbb{Z}_{\mathrm{fin}}} B_J = \bigoplus_{J \in \mathbb{Z}_{\mathrm{fin}}} \hom(A, \iota_J A) \tag{10-1}$$

endowed with the following multiplication: if $f \in B_I$ and $g \in B_J$, then

$$f \cdot g := \iota_{I \oplus J}(\sigma_{I \cap J}) \circ \iota_J(f) \circ g, \tag{10-2}$$

where $\sigma_{I \cap J}$ is defined in (9-5).

The identity component of B is

$$B_\varnothing = \hom(A, A) \cong A_0 = k[z].$$

Lemma 10.1. *The product* (10-2) *on B is associative.*

Proof. It suffices to check that the natural transformations η^{IJ} defined in Section 9.4.2 satisfy the conditions mentioned after [5, Prop. 2.2]. This reduces to checking that the analogue of the commutative diagram [5, (2.5)] really does commute, and that reduces to showing that

$$\iota_I\big(\eta^{J,L}\big) \circ \eta^{I, J \oplus L} = \eta^{IJ} \iota_L \circ \eta^{I \oplus J, L}$$

which, in turn, reduces to showing that

$$h_{J \cap L} h_{I \cap (J \oplus L)} = h_{I \cap J} h_{(I \oplus J) \cap L}.$$

This equality follows from the fact that

$$(J \cap L) \cap I \cap (J \oplus L) = (I \cap J) \cap (I \oplus J) \cap L = \varnothing$$

and

$$(J \cap L) \cup \big(I \cap (J \oplus L)\big) = (I \cap J) \cup \big((I \oplus J) \cap L\big).$$

The last two expressions both equal $(I \cap J) \cup (I \cap L) \cup (J \cap L)$. $\square$

Lemma 10.2. *With the above notation,*

(1) $(\iota_J A)_0 = h_J k[z]$ *where* h_J *is as defined in* (9-4).
(2) *The map*

$$\rho_J : (\iota_J A)_0 \to \hom(A, \iota_J A), \quad \rho_J(m)(a) = ma,$$

is an isomorphism of A_0*-modules.*
(3) *As a right* $B_\varnothing$*-module,* B_J *is freely generated by the elements*

$$b_J := \rho_J(h_J).$$

(4) $b_n^2 + n = b_m^2 + m$ *for all* $m, n \in \mathbb{Z}$.
(5) $b_I b_J = b_{I\cap J}^2 b_{I\oplus J}$.
(6) $b_J = \prod_{j\in J} b_j$.
(7) B *is a commutative* k*-algebra generated by* $\{b_n \mid n \in \mathbb{Z}\}$.

Proof. (1) As noted in the proof of [5, Lemma 5.14],

$$\iota_n A = \begin{cases} x^{n+1}A + (xy+n)A & \text{if } n \geqslant 1, \\ xA & \text{if } n = 0, \\ yA & \text{if } n = -1, \\ y^{-n}A + (xy+n)A & \text{if } n \leqslant -2. \end{cases} \tag{10-3}$$

Hence $(\iota_n A)_0 = (z-n)k[z]$ and the result follows from the fact that

$$\iota_J A = \bigcap_{j\in J} \iota_j A.$$

(2) This is trivial. We will use the isomorphism $\rho_\varnothing : A_0 = k[z] \to B_\varnothing$ to identify $k[z]$ with $B_\varnothing$.

(3) The multiplication $B_J \times B_\varnothing \to B_J$ sends (f, g) to $f \circ g$. Since $B_\varnothing = k[z]$ and $B_J = \rho_J(h_J k[z])$ the result follows.

(4) The multiplication $B_n \times B_n \to B_\varnothing$ is given by

$$b.b' = \sigma_n \circ \iota_n(b) \circ b'$$

so

$$(b_n.b_n)(a) = \sigma_n\big(h_n^2 a\big) = (z-n)^{-1} h_n^2 a = h_n a \in B_\varnothing.$$

Hence $b_n^2 = \rho_\varnothing(h_n) = \rho_\varnothing(z-n) = \rho_\varnothing(z) - n$ and

$$b_n^2 + n = \rho_\varnothing(z) = b_m^2 + m$$

for all $m, n \in \mathbb{Z}$.

(5) By definition of the product in B,

$$\begin{aligned} b_I b_J &= \iota_{I\oplus J}(\sigma_{I\cap J}) \circ \iota_J(b_I) \circ b_J \\ &= \iota_{I\oplus J}(\sigma_{I\cap J}) \circ \iota_J\big(\rho_I(h_I)\big) \circ \rho_J(h_J). \end{aligned}$$

But $\rho_J(h_J)$ is "left multiplication by h_J", $\rho_I(h_I)$ is "left multiplication by h_I", $\iota_J(\rho_I(h_I))$ is the restriction of $\rho_I(h_I)$ so is also "left multiplication by h_I", and $\sigma_{I\cap J}$, and hence $\iota_{I\oplus J}(\sigma_{I\cap J})$, is "left multiplication by $h_{I\cap J}^{-1}$". Hence

$$\begin{aligned} b_I b_J &= (\text{left multiplication by } h_{I\cup J}) : A \to \iota_{I\oplus J} A \\ &= \rho_{I\oplus J}(h_{I\cup J}). \end{aligned}$$

Hence $b_{I\cap J} b_{I\oplus J} = \rho_{I\cup J}(h_{I\cup J}) = b_{I\cup J}$ and therefore

$$\begin{aligned} b_{I\cap J}^2 b_{I\oplus J} &= b_{I\cap J} b_{I\cup J} \\ &= \rho_{(I\cap J)\oplus(I\cup J)}(h_{(I\cap J)\cup(I\cup J)}) \\ &= \rho_{I\oplus J}(h_{I\cup J}) \\ &= b_I b_J \end{aligned}$$

as claimed.

(6) This follows from (5) and an induction argument on $|J|$.

(7) It follows from (5) that $b_m b_n = b_n b_m$ for all $m, n \in \mathbb{Z}$. □

Proposition 10.3. *As $\mathbb{Z}_{\text{fin}}$-graded k-algebras, $B \cong C$.*

Proof. By parts (5) and (6) of Lemma 10.2, the function $\psi(x_n) := b_n$ extends to a well-defined homomorphism $\psi : C \to B$ of $\mathbb{Z}_{\text{fin}}$-graded k-algebras. By part (7) of Lemma 10.2, ψ is surjective.

Since $C_\varnothing$ is a polynomial ring in one variable the restriction of ψ to $C_\varnothing$ is an isomorphism $C_\varnothing \to B_\varnothing$. The kernel of ψ is the sum of its homogeneous components. If $C_J \cap \ker\psi$ were non-zero multiplying it by x_J would produce a non-zero element of $C_\varnothing \cap \ker\psi$; but the latter is zero, so $C_J \cap \ker\psi = 0$ and we conclude that ψ is an isomorphism, as claimed. □

Lemma 10.4 *(Sierra). (See [5, Prop. 4.1].[8]) The set of modules $\{\iota_J A \mid J \in \mathbb{Z}_{\text{fin}}\}$ generates $\mathsf{Gr}A$.*

Because the isomorphisms $\eta^{IJ} : \iota_{I\oplus J} \to \iota_I \iota_J$ satisfy the conditions verified in the proof of Lemma 10.1 there is a well-defined functor

$$F : \mathsf{Gr}A \to \mathsf{Gr}(B, \mathbb{Z}_{\text{fin}}),$$
$$FM := \bigoplus_{J\in\mathbb{Z}_{\text{fin}}} \hom(A, \iota_J M).$$

Because $\{\iota_J A \mid J \in \mathbb{Z}_{\text{fin}}\}$ is a set of projective generators for $\mathsf{Gr}A$, it follows from del Rio's result (Theorem 5.13) that F is an equivalence.

10.2. Final remarks

Is there an a priori reason why the Weyl algebra with the given $\mathbb{Z}$-grading might be so intimately related to a ring like C (or, equivalently, a stack like $\mathcal{X}$)? One explanation is this. A $\mathbb{Z}$-grading typically forces graded modules to behave somewhat like ungraded modules over a ring of dimension one less.

[8] This result is also a consequence of the equivalence $\mathsf{Gr}A \equiv \mathsf{Gr}(C, \mathbb{Z}_{\text{fin}})$ and the fact that the $C(J)$s are a set of generators for $\mathsf{Gr}(C, \mathbb{Z}_{\text{fin}})$.

Since the Weyl algebra has Gelfand–Kirillov dimension two, and since rings of Gelfand–Kirillov dimension one behave a lot like curves, GrA might reasonably be expected to exhibit curve-like features. The stacky behavior corresponds to the existence of non-split extensions between non-isomorphic simples.

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