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Bézout's theorem for non-commutative projective spaces

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Abstract

We prove a version of Bézout's theorem for non-commutative analogues of the projective spaces $\mathbb{P}^n$. © 2001 Elsevier Science B.V. All rights reserved.

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0. Introduction

Throughout we work over an algebraically closed field k. We establish a version of Bézout's Theorem for non-commutative projective spaces, quantum $\mathbb{P}^n$'s for short. Let Y be a quantum $\mathbb{P}^n$, and $K_0(Y)$ the Grothendieck group of the category of noetherian Y-modules. Then the alternating sum of the dimension of the Ext-groups gives a bilinear form

$$\cdot : K_0(Y) \times K_0(Y) \to \mathbb{Z}$$

for which

$$\mathscr{M} \cdot \mathscr{N} = \deg \mathscr{N} \cdot \deg \mathscr{M}$$

whenever $\mathscr{M}$ and $\mathscr{N}$ are noetherian Y-modules such that $\dim \mathscr{N} + \dim \mathscr{M} = n$. The dimension and degree are defined in terms of intersection with "virtual linear subspaces" of Y (Definition 8.4).

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A quantum $\mathbb{P}^n$ is, roughly speaking, an abelian category which enjoys many of the good properties of $\mathsf{Qcoh}\,\mathbb{P}^n$, the category of quasi-coherent sheaves of $\mathcal{O}_{\mathbb{P}^n}$-modules, where $\mathcal{O}_{\mathbb{P}^n}$ is the sheaf of regular functions on the usual commutative $\mathbb{P}^n$. See Definitions 8.1 and 7.1.

There are several problems that must be faced in developing an intersection theory for non-commutative spaces. There can be rather few points on a non-commutative space, so intersection should not be defined in terms of counting points. Serre's result that the form $\sum_{i=0}^{n}(-1)^i \dim_k H^0(Y, \mathrm{Tor}_i^Y(M,N))$ agrees with the intersection multiplicity for a wide range of schemes Y suggests that one should define intersection in homological terms. However, the non-commutativity means that right modules are not usually left modules, so Tor-groups cannot be used. Therefore, when the sum

$$(\mathcal{M},\mathcal{N}) := \sum_{i=0}^{n}(-1)^i \dim_k \mathrm{Ext}_Y^i(\mathcal{M},\mathcal{N})$$

makes sense for all Y-modules, we define the intersection multiplicity as

$$\mathcal{M}\cdot\mathcal{N} := (-1)^{\dim\mathcal{N}}(\mathcal{M},\mathcal{N}).$$

To check that this gives a good theory for the non-commutative analogues of $\mathbb{P}^n$, we first determine $K_0(Y)$ when Y is a non-commutative projective scheme having a homogeneous coordinate ring of finite global dimension (Theorem 2.3). We define the support filtration on $K_0(Y)$ in Section 4. Section 6 shows that $K_0(Y)$ has certain good properties (which always hold in the commutative case) if Y has a homogeneous coordinate ring which is Auslander–Gorenstein. In Section 5 we define the Euler form on $K_0(Y)$ using the alternating sum of dimensions of Ext groups. In Section 8 we show that if Y is a quantum $\mathbb{P}^n$, as defined in Definition 8.1, then $K_0(Y)$ is isomorphic to $K_0(\mathbb{P}^n)$, and this isomorphism is compatible with the Euler forms. We do not know whether this isomorphism is compatible with the two support filtrations, though it is for a quantum $\mathbb{P}^2$. We use the Euler form to define dimension and degree, and then prove Bézout's Theorem.

In Section 9 we look more closely at a quantum $\mathbb{P}^2$. We define a Picard group for a non-commutative surface Y as a suitable subquotient of $K_0(Y)$, and show that it is isomorphic to $\mathbb{Z}$ for a quantum $\mathbb{P}^2$. [3]

1. Basic definitions

Throughout this paper, k denotes an algebraically closed field, and A is a right noetherian, connected graded k-algebra such that $A_1 \neq 0$.

[3] After writing an earlier version of this paper we learned from Peter Jorgensen that he has developed an intersection theory for non-commutative surfaces using the Euler form on the Grothendieck group. Although his point of view is similar to ours, the results obtained are different. We say more about the comparison between his results and ours in Section 11. The final version of this paper benefitted from conversations with him, and we thank him for sharing his ideas with us.

We write $\mathsf{GrMod}\,A$ for the category of graded right A-modules, and $\mathsf{grmod}\,A$ for the full subcategory of noetherian modules. The degree shift functors $M \mapsto M(n)$ on $\mathsf{GrMod}\,A$ are defined by $M(n)=M$ as an A-module, with the grading $M(n)_i=M_{n+i}$. If M and N are graded modules, we write $\underline{\mathrm{Ext}}^j_A(M,N)$ for the direct sum of all $\mathrm{Ext}^j_A(M,N)_i$.

The full subcategory of $\mathsf{grmod}\,A$ (resp., $\mathsf{GrMod}\,A$) consisting of (direct limits of) finite-dimensional modules is denoted $\mathsf{fdim}\,A$ (resp., $\mathsf{Fdim}\,A$). These are localizing subcategories. We write $\mathsf{Tails}\,A:=\mathsf{GrMod}\,A/\mathsf{Fdim}\,A$ for the quotient category, and write $\pi : \mathsf{GrMod}\,A \to \mathsf{Tails}\,A$ for the quotient functor. $\mathsf{Tails}\,A$ is a Grothendieck category, and following [12,15], we call it the *projective quasi-scheme* associated to A.

For the rest of the paper we will write

$$Y = \mathsf{Mod}\,Y = \mathsf{Tails}\,A.$$

The objects in $\mathsf{Mod}\,Y$ are called Y-modules. The confusing notation $Y = \mathsf{Mod}\,Y$ is a device to make us think that a quasi-scheme *is* its category of quasi-coherent modules. We use Y when we wish to think of it as a geometric object, and use $\mathsf{Mod}\,Y$ when we want to remind ourselves that we have a category. Since A is right noetherian, $\mathsf{GrMod}\,A$, and hence $\mathsf{Mod}\,Y$, is locally noetherian. We write $\mathsf{mod}\,Y$ for the full subcategory of $\mathsf{Mod}\,Y$ consisting of the noetherian Y-modules.

We use roman letters $L, M, N, \ldots,$ for graded A-modules, and script letters $\mathscr{L}, \mathscr{M}, \mathscr{N}, \ldots,$ for the corresponding Y-modules, $\pi L, \pi M, \pi N, \ldots.$ We define

$$\mathcal{O}_Y := \pi A,$$

and call it the structure module of Y. The cohomology groups of a Y-module $\mathscr{M}$ are, by definition,

$$H^q(Y, \mathscr{M}) := \mathrm{Ext}^q_Y(\mathcal{O}_Y, \mathscr{M}).$$

The basic properties of $H^q(Y, -)$ can be found in [7].

The Grothendieck group of $\mathsf{mod}\,Y$ is denoted $K_0(Y)$. If $\mathscr{M} \in \mathsf{mod}\,Y$, we denote its image in $K_0(Y)$ by $[\mathscr{M}]$. The degree shift functor on A-modules induces an auto-equivalence of $\mathsf{mod}\,Y$, and hence an automorphism of $K_0(Y)$. We make $K_0(Y)$ a $\mathbb{Z}[s,s^{-1}]$-module with s acting as the shift functor (-1).

Warning. $K_0(Y)$ does not have a natural ring structure because we cannot tensor together two right A-modules. There are two caveats to this. First, one can have $\mathsf{Tails}\,A \cong \mathsf{Tails}\,A'$ with A' commutative, but A not commutative; in other words, a commutative scheme can have a non-commutative homogeneous coordinate ring. For example, $k[x,y]$ with the relation $yx = qxy$, where $0 \neq q \in k$, is a homogeneous coordinate ring of $\mathbb{P}^1$. Second, it may happen that $K_0(Y)$ is isomorphic as a $\mathbb{Z}[s,s^{-1}]$-module to a quotient of $\mathbb{Z}[s,s^{-1}]$, so it inherits a ring structure compatible with the module structure. This second caveat applies when Y is a quantum $\mathbb{P}^n$ below.

2. Hilbert series and $K_0(Y)$

In this section A is also assumed to have finite global homological dimension.

We show that if $Y = \mathsf{Tails}\,A$, then $K_0(Y)$ may be described in terms of the Hilbert series of A.

Definition 2.1. The *Hilbert series* of $M \in \mathsf{grmod}\,A$ is the Laurent power series

$$H_M(t) := \sum_{i=-\infty}^{\infty} (\dim_k M_i) t^i.$$

Since A is right noetherian, and $\dim A_i < \infty$ for all i, this makes sense.

Thus $H_{A(-n)}(t) = t^n H_A(t)$.

Since A is connected, finitely generated graded projective A-modules are free. Thus, each $M \in \mathsf{grmod}\,A$ has a minimal projective resolution in which each term is a finite direct sum of shifts of A.

Definition 2.2. If the minimal projective resolution of $M \in \mathsf{grmod}\,A$ is of the form

$$\cdots \to \bigoplus_{j=1}^{r_d} A(-\ell_{dj}) \to \cdots \to \bigoplus_{j=1}^{r_0} A(-\ell_{0j}) \to M \to 0, \tag{2.1}$$

the *characteristic polynomial* of M is

$$q_M(t) := \sum_{i=0}^{\infty} (-1)^i \sum_{j=0}^{r_i} t^{\ell_{ij}}. \tag{2.2}$$

By the hypotheses on A, the resolution in (2.1) is finite, so $q_M(t) \in \mathbb{Z}[t, t^{-1}]$, and

$$q_M(t) = H_M(t).H_A(t)^{-1}.$$

Furthermore, if $\dim_k A = \infty$, then $H_A(t)$ has a pole at $t = 1$, so $1 - t$ divides $q_k(t)$.

Theorem 2.3. *Let A be a connected graded noetherian k-algebra of finite global dimension. Set $Y = \mathsf{Tails}\,A$ and let $q = H_A(T)^{-1}$ denote the inverse of the Hilbert series for A. Then*

$$K_0(Y) \cong \mathbb{Z}[T, T^{-1}]/(q) \cong \mathbb{Z}[T]/(q)$$

and for each $M \in \mathsf{grmod}\,A$, the isomorphism sends $[\pi M]$ to the characteristic polynomial $q_M(T)$. In particular, $[\mathcal{O}_Y(n)]$ is sent to T^{-n}.

Proof. The localization sequence for K-theory gives an exact sequence

$$K_0(\mathsf{fdim}\,A) \xrightarrow{v} K_0(\mathsf{grmod}\,A) \to K_0(\mathsf{mod}\,Y) \to 0. \tag{2.3}$$

Recall that $\mathbb{Z}[s, s^{-1}]$ acts naturally on $K_0(\mathsf{grmod}\,A)$ by $s.[M] = [M(-1)]$. Thus (2.3) is a sequence of $\mathbb{Z}[s, s^{-1}]$-modules.

Since A is right noetherian of finite global dimension, every $M \in \mathsf{grmod}\, A$ has a finite resolution with each term a finite direct sum of shifts of A. Therefore,

$$K_0(\mathsf{grmod}\, A) \cong \mathbb{Z}[T, T^{-1}]$$

with $[A(n)] \leftrightarrow T^{-n}$. This is a $\mathbb{Z}[s, s^{-1}]$-module isomorphism if s acts on the right-hand side as multiplication by T.

If $M \in \mathsf{grmod}\, A$ has a minimal resolution of the form (2.1), then under the isomorphism,

$$[M] \leftrightarrow \sum_{i=0}^{\infty} (-1)^i \sum_{j=0}^{r_i} T^{\ell_{ij}} = q_M(T).$$

Each $M \in \mathsf{fdim}\, A$ has a finite filtration by graded submodules such that the slices are annihilated by $\mathfrak{m} = A_{\geq 1}$. Therefore, by Dévissage,

$$K_0(\mathsf{fdim}\, A) \cong K_0(\mathsf{grmod}\, A/\mathfrak{m}) = K_0(\mathsf{grmod}\, k) \cong \mathbb{Z}[T, T^{-1}].$$

Under the last isomorphism, $[k] \leftrightarrow 1$. Again, these are $\mathbb{Z}[s, s^{-1}]$-module isomorphisms if s acts as multiplication by T. Hence the image of v is generated by $[k]$ as a $\mathbb{Z}[s, s^{-1}]$-module. Therefore,

$$\begin{aligned} K_0(\mathsf{mod}\, Y) &\cong K_0(\mathsf{grmod}\, A)/\mathrm{Im}\, v \\ &\cong \mathbb{Z}[T, T^{-1}]/([k]) \\ &= \mathbb{Z}[T, T^{-1}]/(q_k(T)). \end{aligned}$$

Since $q_k(0)=1$, T is a unit in $\mathbb{Z}[T]/(q_k(T))$, so there is a ring isomorphism $\mathbb{Z}[T]/(q) \cong \mathbb{Z}[T, T^{-1}]/(q)$. $\square$

If $A = k[x_0, \dots, x_n]$ is the commutative polynomial ring with its standard grading then $H_A(T) = (1 - T)^{-n-1}$. Since $\mathsf{Tails}\, A \cong \mathsf{Qcoh}\, \mathbb{P}^n$, the theorem gives $K_0(\mathbb{P}^n) \cong \mathbb{Z}[T]/(1 - T)^{n+1}$. The same result holds for the quantum $\mathbb{P}^n$'s that are defined in Definition 8.1.

3. The rank function $K_0(Y) \to \mathbb{Z}$

In this section A and Y are as in Section 1, with the further requirement that A is a domain. By Goldie's Theorem, A has a division ring of fractions.

Definition 3.1. Define

$$\mathrm{Fract}_{\mathrm{Gr}}\, A := \{ab^{-1} \mid a, b \in A \text{ are homogeneous, and } b \text{ is regular}\}.$$

The *function field*, $k(Y)$, of Y is the degree zero component of $\mathrm{Fract}_{\mathrm{Gr}}\, A$. It is a division algebra. The *generic point* of Y is $\eta := \mathsf{Mod}\, k(Y)$.

By the graded version of Goldie's Theorem [11, Chapter C, Corollary I.1.7 and Chapter A, Theorem I.5.8], $\mathrm{Fract}_{\mathrm{Gr}}\, A \cong k(Y)[z, z^{-1}; \sigma]$, a skew Laurent extension. Since

A is a domain, and $A_1 \neq 0$, z has degree one. Therefore $\mathrm{Fract}_{\mathrm{Gr}} A$ is strongly graded [11, Section I.3].

Since $A_1 \neq 0$ and A is a domain, $Y \neq \emptyset$.

Proposition 3.2. (1) *There is a homomorphism* $K_0(Y) \to K_0(\eta)$ *defined by*

$$[\pi M] \mapsto [(M \bigotimes_A \mathrm{Fract}_{\mathrm{Gr}} A)_0].$$

(2) *There is a homomorphism*

$$\mathrm{rank} : K_0(Y) \to \mathbb{Z}$$

which sends each $[\mathcal{O}_Y(n)]$ *to* 1.

(3) *The kernel of the rank function equals* $(s-1).K_0(Y)$.

Proof. (1) The functor $-\otimes_A \mathrm{Fract}_{\mathrm{Gr}} A$ kills all finite-dimensional modules. Composing it with the exact functor which takes the degree zero part, one obtains an exact functor $\Phi : \mathsf{GrMod}\, A \to \mathsf{Mod}\, k(Y)$. Since Φ kills finite-dimensional modules it induces an exact functor $\mathsf{Mod}\, Y \to \mathsf{Mod}\, \eta$, and hence a homomorphism as claimed.

We also note that $\Phi(A(n)) = (A(n) \otimes_A \mathrm{Fract}_{\mathrm{Gr}} A)_0 = (\mathrm{Fract}_{\mathrm{Gr}} A)_n$, and this is isomorphic to $(\mathrm{Fract}_{\mathrm{Gr}} A)_0 = k(Y)$ because $\mathrm{Fract}_{\mathrm{Gr}} A$ is strongly graded.

(2) Since $k(Y)$ is a division algebra, $\dim_{k(Y)} : \mathsf{mod}\, \eta \to \mathbb{Z}$ induces an isomorphism $K_0(\eta) \xrightarrow{\sim} \mathbb{Z}$ sending $k(Y)$ to 1. Composing this with the homomorphism $K_0(Y) \to K_0(\eta)$ obtained in (1), one gets the rank function sending each $[\mathcal{O}_Y(n)]$ to 1. Explicitly, if $\mathcal{M} = \pi(M) \in \mathsf{mod}\, Y$, then

$$\mathrm{rank}[\mathcal{M}] = \dim_{k(Y)}(M \bigotimes_A (\mathrm{Fract}_{\mathrm{Gr}} A))_0. \tag{3.1}$$

(3) If s is *any* auto-equivalence of $\mathsf{mod}\, Y$, then s induces an auto-equivalence on $\mathsf{Mod}\, \eta$. Because $k(Y)$ is a division algebra, it must be sent to itself by this auto-equivalence. Therefore $[\mathcal{M}] - [s\mathcal{M}] \in \ker(\mathrm{rank})$ for all $\mathcal{M} \in \mathsf{mod}\, Y$. Hence $(1-s)$ $K_0(Y)$ is in the kernel of rank. Since the rank function is surjective, and since $K_0(Y)/(1-s)K_0(Y) \cong \mathbb{Z}$, $\ker(\mathrm{rank}) = (1-s)K_0(Y)$. □

4. The support filtration on $K_0(Y)$

We continue to assume that A and Y are as in Section 1.

We want a substitute for the usual filtration on $K_0(Y)$ given by codimension of support.

Definition 4.1. The *cohomological dimension* of Y is

$$\mathrm{cd}\, Y = \max\{d \mid H^d(Y, -) \neq 0\}.$$

Definition 4.2. If $0 \neq \mathscr{M} \in \mathsf{mod}\, Y$, we define the *dimension of the support* of $\mathscr{M}$ to be

$$\mathrm{ds}(\mathscr{M}) := \max\{ j \mid H^j(Y, \mathscr{M}(i)) \neq 0 \text{ for some } i\}.$$

We define $\mathrm{ds}(0) = -1$. (We do not define support.)

Lemma 4.3. *Suppose that $d = \mathrm{cd}\, Y$ is finite. If $0 \leq p \leq d+1$, then*

$$F^p K_0(Y) := \{[\mathscr{M}] - [\mathscr{N}] \mid \mathrm{ds}(\mathscr{M}) \leq d - p \text{ and } \mathrm{ds}(\mathscr{N}) \leq d - p\}$$

is a $\mathbb{Z}[s, s^{-1}]$-submodule of $K_0(Y)$.

Proof. Since the shift functor is an auto-equivalence of $\mathsf{mod}\, Y$, $F^p K_0(Y)$ is stable under the action of s and s^{-1}, so we only need show it is a group. Certainly, $0 \in F^p K_0(Y)$. If $[\mathscr{M}] - [\mathscr{N}] \in F^p K_0(Y)$, so is its negative. If $[\mathscr{M}_1] - [\mathscr{N}_1]$ and $[\mathscr{M}_2] - [\mathscr{N}_2]$ are in $F^p K_0(Y)$, so is their sum $[\mathscr{M}_1 \oplus \mathscr{M}_2] - [\mathscr{N}_1 \oplus \mathscr{N}_2]$ because $\mathrm{ds}(\mathscr{L}_1 \oplus \mathscr{L}_2) = \max\{\mathrm{ds}(\mathscr{L}_1), \mathrm{ds}(\mathscr{L}_2)\}$. □

Therefore there is a filtration

$$K_0(Y) = F^0 K_0(Y) \supset F^1 K_0(Y) \supset \cdots \supset F^{d+1} K_0(Y) = 0$$

of $K_0(Y)$, where $d = \mathrm{cd}\, Y$, by $\mathbb{Z}[s, s^{-1}]$-submodules. Although we have not defined the word "support", we should think of $F^p K_0(Y)$ as consisting of the modules having support of codimension $\geq p$.

5. The Euler form on $K_0(Y)$

In this section A and Y are as in Section 1.

Definition 5.1 (Artin and Zhang [7]). Let A be a noetherian, connected k-algebra. We say that A *satisfies condition* χ if $\dim_k \underline{\mathrm{Ext}}^j_A(k, M) < \infty$ for all j, and all $M \in \mathsf{grmod}\, A$.

The condition χ is rather mild. Every graded quotient of the polynomial ring satisfies it, and so do most non-commutative algebras of importance. The condition is essential to get a theory for non-commutative schemes which resembles the commutative theory. See [7] for more information.

Proposition 5.2 (Artin and Zhang [7, Proposition 3.5(2)]). *Let A be a right noetherian, connected, k-algebra satisfying the condition χ. Set $Y = \mathsf{Tails}\, A$. Then $\mathrm{Ext}^q_Y(\mathscr{M}, \mathscr{N})$ is finite-dimensional for all q and all $\mathscr{M}, \mathscr{N} \in \mathsf{mod}\, Y$.*

Definition 5.3. Suppose for every $\mathscr{M} \in \mathsf{mod}\, Y$ that $\mathrm{Ext}^q_Y(\mathscr{M}, -) = 0$ for $q \gg 0$. The *Euler form* is the bilinear form $(-,-) : K_0(Y) \times K_0(Y) \to \mathbb{Z}$, defined by

$$(\mathscr{M}, \mathscr{N}) = \sum_{i=0}^{\infty} (-1)^i \dim_k \mathrm{Ext}^i_Y(\mathscr{M}, \mathscr{N}). \tag{5.1}$$

The long exact sequence for Ext shows that this depends only on the classes of $\mathscr{M}$ and $\mathscr{N}$ in $K_0(Y)$.

The Euler form is invariant under the automorphism of $K_0(Y)$ induced by the shift functor $\mathscr{M} \mapsto \mathscr{M}(1)$; that is, $(\mathscr{M}(1), \mathscr{N}(1)) = (\mathscr{M}, \mathscr{N})$. Hence, if $K_0(Y)$ is given the obvious ring structure via Theorem 2.3, $(fT^i, gT^i) = (f, g)$ for all $f, g \in \mathbb{Z}[T, T^{-1}]$ and $i \in \mathbb{Z}$. It follows that

$$(f(T), g(T)) = (1, f(T^{-1})g(T)). \tag{5.2}$$

The Euler form has proved useful in the representation theory of finite-dimensional algebras (see [8, Section VIII.3]).

For curves C and D on a smooth commutative projective surface over an algebraically closed field, it is easy to show that the intersection number $C \cdot D$ equals $-(\mathcal{O}_C, \mathcal{O}_D)$.

6. Auslander–Gorenstein algebras

Any reasonable definition of dimension of support should have the property that $\mathrm{ds}(\mathscr{M}) = \max\{\mathrm{ds}(\mathscr{L}), \mathrm{ds}(\mathscr{N})\}$ whenever $0 \to \mathscr{L} \to \mathscr{M} \to \mathscr{N} \to 0$ is exact in $\mathsf{mod}\, Y$. This is necessary for the support filtration to behave as in the commutative case. In order for this property to hold we need additional hypotheses on A. Fortunately, these hypotheses are satisfied for the quantum $\mathbb{P}^n$'s discussed in Section 8 below.

Definition 6.1. A connected graded k-algebra A is *Auslander–Gorenstein* if
- it is right and left noetherian, and
- it has finite self-injective dimension on both sides, say d, and
- it satisfies the *Gorenstein condition*, namely

$$\underline{\mathrm{Ext}}^i_A(k, A) \cong \begin{cases} k(e) & \text{if } i = d, \text{and} \\ 0 & \text{otherwise,} \end{cases}$$

  for some integer e, and
- for every noetherian right A-module M, $\mathrm{Ext}^i_A(N, A) = 0$ for all $i < j$ whenever N is a left submodule of $\mathrm{Ext}^j_A(M, A)$.

If A is Auslander–Gorenstein, we define the *grade* of a non-zero module M in $\mathsf{grmod}\, A$ to be

$$j(M) := \min\{j \mid \mathrm{Ext}^j_A(M, A) \neq 0\}.$$

It is well known that if $0 \neq M \in \mathsf{grmod}\, A$, then $j(M) = \mathrm{injdim}\, A$ if and only if $\dim_k M < \infty$.

If A is Auslander–Gorenstein there are three consequences for $\mathsf{Tails}\, A$ which are relevant to us. The first is that grade behaves well on short exact sequences: if $0 \to L \to M \to N \to 0$ is exact in $\mathsf{grmod}\, A$, then $j(M) = \min\{j(L), j(N)\}$. The second, as expressed by the next result, due to Yekutieli and Zhang, is that $\mathsf{Tails}\,A$ satisfies Serre duality (6.1). The third is that it implies the condition χ.

Theorem 6.2 (Yekutieli and Zhang [16]). *Let A be Auslander–Gorenstein of injective dimension $d+1$. Define $\omega^\circ := \mathcal{O}_Y(-e)$, where $e \in \mathbb{Z}$ is determined by $\underline{\mathrm{Ext}}^{d+1}(k,A) \cong k(e)$. Then*

$$H^q(Y,\mathcal{M})^* \cong \mathrm{Ext}_Y^{d-q}(\mathcal{M},\omega^\circ) \tag{6.1}$$

for every $\mathcal{M} \in \mathsf{mod}\,Y$.

We call ω° a *dualizing module* for Y.

Theorem 6.3. *Let A be a connected, Auslander–Gorenstein ring of injective dimension $d+1$. Let $Y = \mathsf{Tails}\,A$. Then*
1. *if $M \in \mathsf{grmod}\,A$, and $\pi M \neq 0$, then $j(M) = d - \mathrm{ds}(\pi M)$;*
2. *if $0 \to \mathcal{L} \to \mathcal{M} \to \mathcal{N} \to 0$ is exact in $\mathsf{mod}\,Y$, then*

$$\mathrm{ds}(\mathcal{M}) = \max\{\mathrm{ds}(\mathcal{L}), \mathrm{ds}(\mathcal{N})\}; \tag{6.2}$$

3. *$\{\mathcal{M} \in \mathsf{mod}\,Y \mid \mathrm{ds}(\mathcal{M}) \leq p\}$ is closed under submodules, quotients, and extensions.*

Proof. (1) Suppose M is not finite dimensional. Then $j(M) = j(M_{\geq n})$ for all n. Write $\mathcal{M} = \pi M$. We have

$$\mathrm{Ext}_Y^q(\mathcal{M}(t),\omega^\circ) \cong \varinjlim \mathrm{Ext}_A^q(M(t)_{\geq n}, A(-e))$$

and because χ holds this directed system is eventually constant [16]. So for large n,

$$\begin{aligned} j(M) = j(M_{\geq n}) &= \min\{q \mid \underline{\mathrm{Ext}}_A^q(M_{\geq n}, A) \neq 0\} \\ &= \min\{q \mid \mathrm{Ext}_A^q(M(t)_{\geq n}, A) \neq 0 \text{ for some } t\} \\ &= \min\{q \mid \mathrm{Ext}_Y^q(\mathcal{M}(t),\omega^\circ) \neq 0 \text{ for some } t\} \\ &= \min\{q \mid H^{d-q}(Y,\mathcal{M}(t)) \neq 0 \text{ for some } t\} \\ &= d - \mathrm{ds}(\mathcal{M}). \end{aligned}$$

(2) Because grade behaves well on exact sequences, if $0 \to \mathcal{L} \to \mathcal{M} \to \mathcal{N} \to 0$ is exact in $\mathsf{mod}\,Y$, then $\mathrm{ds}(\mathcal{M}) = \max\{\mathrm{ds}(\mathcal{L}), \mathrm{ds}(\mathcal{N})\}$. Part (3) now follows. □

Levasseur [10, Theorem 4.8] has shown that if A is Auslander–Gorenstein and has finite global dimension then A is a domain. Thus the hypotheses in the next lemma are satisfied by the quantum $\mathbb{P}^n$'s in Section 8.

Lemma 6.4. *Let A be a connected, Auslander–Gorenstein domain of injective dimension $d+1$. Let $Y = \mathsf{Tails}\,A$. Let $\mathcal{M} \in \mathsf{mod}\,Y$. The following are equivalent:*
1. $\mathrm{rank}\,\mathcal{M} = 0$;
2. $\mathrm{ds}(\mathcal{M}) \leq d - 1$;
3. $[\mathcal{M}] \in F^1 K_0(Y)$.

Proof. Write $\mathscr{M} = \pi M$ with $M \in \mathsf{grmod}\, A$. Let $\omega^\circ = \mathscr{O}_Y(-e)$ be the dualizing module for Y.

(1) $\Rightarrow$ (2): We will show that $H^d(Y, \mathscr{M}(t)) = 0$ for all $t \in \mathbb{Z}$; actually, we will show that $\mathrm{Hom}_Y(\mathscr{M}(t), \omega^\circ) = 0$, and then invoke Serre duality for Y. Since rank $\mathscr{M} = 0$, $M \otimes_A (\mathrm{Fract}_{\mathrm{Gr}}\, A) = 0$. Hence each homogeneous $m \in M$ is annihilated by some regular homogenous element $a \in A$. However,

$$\mathrm{Hom}_Y(\mathscr{M}(t), \omega^\circ) = \varinjlim \mathrm{Hom}_A(M(t)_{\geq n}, A(-e))$$

and each term in this direct limit is zero by the previous sentence. Hence, by Serre duality, $H^d(Y, \mathscr{M}(t)) = 0$.

(2) $\Rightarrow$ (3): By definition of $F^1K_0(Y)$.

(3) $\Rightarrow$ (1): Write $[\mathscr{M}] = [\mathscr{M}'] - [\mathscr{N}']$ with $\mathrm{ds}(\mathscr{M}') \leq d-1$ and $\mathrm{ds}(\mathscr{N}') \leq d-1$. Write $\mathscr{M}' = \pi M'$ with $M' \in \mathsf{grmod}\, A$. By Theorem 6.3, $j(M') > 0$. But it is well known that this is equivalent to $M' \otimes_A \mathrm{Fract}_{\mathrm{Gr}} A = 0$. Thus rank $\mathscr{M}' = 0$. Similarly, rank $\mathscr{N}' = 0$. Since $[\mathscr{M} \oplus \mathscr{N}'] = [\mathscr{M}']$, and rank is an additive function on exact sequences, rank $\mathscr{M} = 0$. $\square$

Proposition 6.5. *Let A be a connected, Auslander–Gorenstein domain of injective dimension $d+1$. Let $Y = \mathsf{Tails}\, A$. Then $F^1K_0(Y) = \ker(\mathrm{rank})$.*

Proof. One inclusion is immediate: an element of $F^1K_0(Y)$ must be of the form $[\mathscr{M}] - [\mathscr{N}]$ with $\mathrm{ds}(\mathscr{M}) \leq d-1$ and $\mathrm{ds}(\mathscr{N}) \leq d-1$; by Lemma 6.4, rank $\mathscr{M} =$ rank $\mathscr{N} = 0$, whence $\mathrm{rank}([\mathscr{M}] - [\mathscr{N}]) = 0$.

To prove the reverse inclusion, we will show that if $[\mathscr{M}] - [\mathscr{N}] \in \ker(\mathrm{rank})$, then $[\mathscr{M}] - [\mathscr{N}] \in F^1K_0(Y)$. (We are using the fact that every element in $K_0(Y)$ is of the form $[\mathscr{M}] - [\mathscr{N}]$ for some $\mathscr{M}, \mathscr{N} \in \mathsf{mod}\, Y$.) Since rank $\mathscr{M} =$ rank $\mathscr{N}$, $\mathscr{M}_\eta \cong \mathscr{N}_\eta$, so there is a morphism $\mathscr{M} \to \mathscr{M}_\eta \to \mathscr{N}_\eta$. Also $\mathscr{N} \mapsto \mathscr{N}_\eta$. If $\mathscr{L}$ denotes the sum of the images of $\mathscr{M}$ and $\mathscr{N}$ in $\mathscr{N}_\eta$, then $\mathscr{L} \in \mathsf{mod}\, Y$. Since the morphisms $\mathscr{M} \to \mathscr{L}$ and $\mathscr{N} \to \mathscr{L}$ localize to isomorphisms, the kernel and cokernel of each morphism has rank zero. By Lemma 6.4, the classes of these kernels and cokernels belong to $F^1K_0(Y)$, hence so do their differences. Thus $[\mathscr{L}] - [\mathscr{M}]$ and $[\mathscr{L}] - [\mathscr{N}]$ are in $F^1K_0(Y)$; taking their difference, we get $[\mathscr{M}] - [\mathscr{N}] \in F^1K_0(Y)$, as required. $\square$

7. Regular algebras

Definition 7.1. Let A be a connected graded k-algebra. We call A an n-dimensional *regular algebra* if

- A is right and left noetherian;
- $\mathrm{gldim}\, A = n$;
- A satisfies the Gorenstein condition, meaning there is an integer e such that

$$\underline{\mathrm{Ext}}^d_A(k, A) \cong \begin{cases} k(e) & \text{if } d = n, \\ 0 & \text{otherwise.} \end{cases}$$

Whenever A is a regular algebra the letter e will refer to the number appearing in the last part of this definition. Thus, the left-most non-zero term in the minimal projective resolution for ${}_Ak$ is $A(-e)$. Therefore $e = \deg H_A(T)^{-1}$. For a polynomial ring on n indeterminates with the standard grading, $e = n$.

Theorem 7.2. *Let A be an $(n+1)$-dimensional regular algebra, and set $Y = \mathsf{Tails}\,A$. Let $q(t) = H_A(t)^{-1}$. Then*
1. $\operatorname{cd} Y = n$;
2. *Y satisfies Serre duality, with dualizing module $\omega^\circ = \mathscr{A}(-e)$ [16],*
3. *$K_0(Y)$ has basis $\mathscr{A}, \mathscr{A}(-1), \ldots, \mathscr{A}(-e)$.*
4. $(\mathscr{A}, \mathscr{A}(d)) = \begin{cases} \dim A_d & \textit{if } 0 \le d, \\ (-1)^n \dim A_{-d-e} & \textit{if } d \le -e, \\ 0 & \textit{otherwise}. \end{cases}$

Proof. Conditions (1) and (2) were discussed in the previous section.

(3) This is because $\deg q = e$ by the remarks before the theorem.

(4) By [7, Theorem 8.1],

$$H^q(Y, \mathscr{A}(d)) \cong \begin{cases} A_d & \text{if } q = 0 \text{ and } d \ge 0, \\ (A_{-d-e})^* & \text{if } q = n \text{ and } d \le -e, \\ 0 & \text{otherwise}. \end{cases} \tag{7.1}$$

The result follows. $\square$

We now determine the radical of the restriction of the Euler form to $F^1K_0(Y)$.

Lemma 7.3 (Stanley [13, Theorem 1.1, p. 1]). *Let $p, q \in k[t]$ with $q(0) = 1$. Suppose that*

$$\frac{p(t)}{q(t)} = \sum_{i=0}^{\infty} a_i t^i$$

in $k[[t]]$. Let $\alpha_0, \alpha_1, \ldots, \alpha_d \in k$. Then

$$\sum_{i=0}^{d} \alpha_i a_{i+n} = 0$$

for all n if and only if $\alpha_0 + \alpha_1 t + \cdots + \alpha_d t^d$ is in the ideal (q).

Proposition 7.4. *Let A be an $(n+1)$-dimensional regular algebra, and set $Y = \mathsf{Tails}\,A$. Let $q(t) = H_A(t)^{-1}$. Then the left and right radicals of the restriction of $(-,-)$ to $F^1K_0(Y)$ are equal. The radical consists of all scalar multiples of $(1-T)^{-1}q(T)$. In particular, the radical is one dimensional.*

Proof. Recall that $K_0(Y) \cong \mathbb{Z}[T, T^{-1}]/(q)$, and that $F^1K_0(Y)$ is the image of the ideal $(1 - T)$. Write $a_i = \dim A_i$. The Euler form on $K_0(Y)$ may be lifted to a form on

$\mathbb{Z}[T,T^{-1}]$ defined by

$$(T^i,T^j):=\begin{cases} a_{j-i} & \text{if } j-i\geq 0,\\ (-1)^n a_{i-j-e} & \text{if } i-j-e\geq 0,\\ 0 & \text{otherwise.}\end{cases}$$

If J is an ideal in $\mathbb{Z}[T,T^{-1}]$ we write

$$J^{\perp}:=\{g\in\mathbb{Z}[T,T^{-1}]\,|\,(f,g)=0 \text{ for all } f\in J\}$$

and

$$^{\perp}J:=\{g\in\mathbb{Z}[T,T^{-1}]\,|\,(g,f)=0 \text{ for all } f\in J\}.$$

We already know that (q) is in both the right and left radicals of $(-,-)$.

By definition, the right radical of $(-,-)$ restricted to $F^1K_0(Y)$ is $(1-T)^{\perp}$. Modulo (q), $(1-T)$ is spanned by elements of the form $(1-T)^{-1}T^{-j}$ with $j\geq 0$. Therefore, modulo (q), $(1-T)^{\perp}$ consists of all elements $g=\sum_i \lambda_i T^i\in\mathbb{Z}[T]$ such that, for all $j\geq 0$,

$$\begin{aligned}0&=((1-T^{-1})T^{-j},g)=(1,(T^j-T^{j+1})g)\\ &=\sum_i(\lambda_i a_{j+i}-\lambda_i a_{j+i+1})=\sum_i(\lambda_i-\lambda_{i-1})a_{i+j}.\end{aligned}$$

By Lemma 7.3, this can only happen if

$$\sum_i(\lambda_i-\lambda_{i-1})T^i,$$

which equals $(1-T)g$, is in the ideal (q).

A similar argument gives the left radical. Modulo (q), it consists of those $g=\sum_i \delta_i T^{-i}$ such that, for all j,

$$0=(g,(1-T)T^j)=\sum_i(\delta_i a_{i+j}-\delta_i a_{i+j+1})=\sum_i(\delta_i-\delta_{i-1})a_{i+j}.$$

This can only happen if the element

$$\sum_i(\lambda_i-\lambda_{i-1})T^i,$$

which equals $(1-T)g(T^{-1})$, is in the ideal (q). Because A satisfies the Gorenstein condition its Hilbert series satisfies the functional equation $H_A(t^{-1})=(-1)^n t^e H_A(t)$. Therefore $q(T)$ and $q(T^{-1})$ generate the same ideal of $\mathbb{Z}[T,T^{-1}]$. It follows that, if $r(T)=(1-T)^{-1}q(T)$, then $r(T)$ and $r(T^{-1})$ generate the same ideal. We have just shown that g is in the left radical if and only if $g(T^{-1})$ is in the ideal generated by $r(T)$; this is equivalent to the condition that $g(T)$ is in the ideal generated by $r(T^{-1})$. Since this is the same as the ideal generated by $r(T)$, we are done. □

8. Quantum $\mathbb{P}^n$'s

In this section A denotes an $(n+1)$-dimensional quantum polynomial ring, and Y denotes a quantum $\mathbb{P}^n$.

Definition 8.1. An $(n+1)$-dimensional regular k-algebra A is a *quantum polynomial ring* if it is Auslander–Gorenstein, and a domain, and is generated by A_1 as a k-algebra, and has Hilbert series $H_A(t) = (1-t)^{-n-1}$.

A quasi-scheme Y is called a *quantum* $\mathbb{P}^n$ if $\mathsf{Mod}\, Y$ is equivalent to $\mathsf{Tails}\, A$ for some $(n+1)$-dimensional quantum polynomial ring A.

The commutative projective space $\mathbb{P}^n$, more precisely, $\mathsf{Qcoh}\ \mathcal{O}_{\mathbb{P}^n}$, is a quantum $\mathbb{P}^n$. In order to avoid confusion we will write $\mathcal{O}$ for $\mathcal{O}_{\mathbb{P}^n}$, and reserve the letter $\mathscr{A}$ for the image of A in $\mathsf{Mod}\, Y = \mathsf{Tails}\, A$.

Lemma 8.2. *Suppose that Y is a quantum $\mathbb{P}^n$. Then*

1. $K_0(Y) \cong \mathbb{Z}[T]/(T-1)^{n+1}$ *with basis* $\mathscr{A}, \mathscr{A}(-1), \ldots, \mathscr{A}(-n)$.
2. *There is a* $\mathbb{Z}[T, T^{-1}]$*-module isomorphism* $K_0(Y) \to K_0(\mathbb{P}^n)$ *sending* $\mathscr{A}(d)$ *to* $\mathcal{O}(d)$ *for all* d.
3. *This isomorphism is compatible with the Euler forms on* $K_0(Y)$ *and* $K_0(\mathbb{P}^n)$.
4. *The isomorphism in* (2) *commutes with the rank functions.*

Proof. (1) This follows from Theorem 2.3.

(2) Since $K_0(\mathbb{P}^n)$ has a basis $\mathcal{O}, \mathcal{O}(-1), \ldots, \mathcal{O}(-n)$, there is an obvious isomorphism sending $\mathscr{A}(d)$ to $\mathcal{O}(d)$ for $-n \le d \le 0$.

(3) By (7.1), the cohomology groups on Y behave as do those on $\mathbb{P}^n$. Since A has the same Hilbert series as the polynomial ring, $\dim_k H^q(Y, \mathscr{A}(d)) = \dim_k H^q(\mathbb{P}^n, \mathcal{O}(d))$ for all q and d. Therefore $(\mathscr{A}(i), \mathscr{A}(j)) = (\mathcal{O}(i), \mathcal{O}(j))$ for all i and j.

(4) This is clear because $\operatorname{rank} \mathscr{A}(i) = \operatorname{rank} \mathcal{O}(i) = 1$ for all i. $\square$

It follows from Propositions 3.2 and 6.5 that for a quantum $\mathbb{P}^n$, $F^1 K_0(Y) = (1-T)$. We do not know if $F^d K_0(Y) = (1-T)^d$ for all d.

Definition 8.3. Let Y be a quantum $\mathbb{P}^n$. The element $H := [\mathcal{O}_Y] - [\mathcal{O}_Y(-1)]$ in $K_0(Y)$ will be called a *virtual hyperplane*. We call the elements

$$H^d := \sum_{i=0}^{d} (-1)^i \binom{d}{i} [\mathcal{O}_Y(-i)]$$

virtual linear subspaces of Y of codimension d.

The elements $1, H, \ldots, H^n$ are a basis for $K_0(Y)$. It is not known whether every H^d arises as the class of a Y-module. This is an important question.

Definition 8.4. If $\mathscr{M}$ and $\mathscr{N}$ are noetherian Y-modules, we define

$$\mathscr{M} \cdot \mathscr{N} = (-1)^{\dim \mathscr{N}} (\mathscr{M}, \mathscr{N}),$$

where $\dim \mathscr{M}$, the *dimension* of $\mathscr{M}$, is the largest integer d such that $(\mathscr{M}, H^d) \neq 0$. We extend this pairing to a bilinear form on $K_0(Y)$ in the obvious way. Thus $\dim \mathscr{M}$ is the largest integer d for which $\mathscr{M} \cdot H^d \neq 0$.

The *degree* of $\mathscr{M}$ is

$$\deg \mathscr{M} := \mathscr{M} \cdot H^{\dim \mathscr{M}}.$$

For $\mathbb{P}^n$ these agree with the usual definitions.

The next lemma shows that H^p has dimension $n-p$ and degree one, thus justifying our thinking of H^p as a linear subspace of codimension p. This is compatible with the definitions of linear modules in [5].

Lemma 8.5. *Let Y be a quantum $\mathbb{P}^n$. Then*

$$((1-T)^p,(1-T)^q) = \begin{cases} 0 & \text{if } p+q>n, \\ (-1)^p & \text{if } p+q=n. \end{cases}$$

Proof. Making use of (5.2) we have

$$((1-T)^p,(1-T)^q) = (1,(-1)^p T^{-p}(1-T)^{p+q}). \tag{8.1}$$

If $p+q>n$, then $(1-T)^{p+q}=0$ in $K_0(Y)$, so $((1-T)^p,(1-T)^q)=0$. Now suppose that $p+q=n$. By (8.1), it suffices to show that $(T^p,(1-T)^n)=1$ for all $p \le n$. By Proposition 7.4, $(1-T)^n$ is in the radical of the restriction of the form $(-,-)$ to $F^1K_0(Y)$, which is the ideal generated by $1-T$. Therefore,

$$\begin{aligned}(T^p,(1-T)^n) &= (T^n,(1-T)^n) \\ &= \sum_{i=0}^{n}(-1)^i \binom{n}{i} (T^n,T^i) \\ &= \sum_{i=0}^{n}(-1)^i \binom{n}{i} (1,T^{i-n}).\end{aligned}$$

By Theorem 7.2, $(1,T^{i-n})$ is non-zero only when $i=n$, and then $(1,1)=1$. This proves the lemma. □

It follows from the lemma that $\dim H^p = n-p$, and that $\deg H^p = 1$. If $[\mathscr{M}]$ is expressed as a linear combination of the basis elements H^i, then

$$[\mathscr{M}] = (\deg \mathscr{M})H^{n-\dim \mathscr{M}} + \sum_{i>n-\dim \mathscr{M}} c_i H^i \tag{8.2}$$

for some integers c_i.

Theorem 8.6 (Bézout's Theorem)**.** *Let Y be a quantum $\mathbb{P}^n$. If $\mathscr{M}$ and $\mathscr{N}$ are noetherian Y-modules, then*

$$\mathscr{M} \cdot \mathscr{N} = \begin{cases} 0 & \text{if } \dim \mathscr{M} + \dim \mathscr{N} < n, \\ \deg \mathscr{M} . \deg \mathscr{N} & \text{if } \dim \mathscr{M} + \dim \mathscr{N} = n. \end{cases}$$

Proof. If we write $\mathscr{M}$ and $\mathscr{N}$ in the form (8.2), then the result follows immediately from Lemma 8.5. □

9. Surfaces and quantum $\mathbb{P}^2$s

Non-commutative analogues of $\mathbb{P}^2$ are well understood, and provide a testing ground for the problem of developing an intersection theory for curves on non-commutative surfaces.

Naively an intersection theory on a non-commutative surface Y should consist of a set that represents "curves modulo points", together with a $\mathbb{Z}$-valued pairing on that set. The subquotient $F^1K_0(Y)/F^2K_0(Y)$ is a natural candidate for this set if we think of $F^2K_0(Y)$ as classifying the "points" in Y and $F^1K_0(Y)$ as classifying the "curves" modulo some appropriate relations.

The Picard group of a smooth projective surface Y over an algebraically closed field is isomorphic to $F^1K_0(Y)/F^2K_0(Y)$, and the intersection pairing is induced by the negative of the Euler form. The Euler form on $K_0(Y)$ passes to this subquotient because $F^2K_0(Y)$ is contained in the radical of the restriction of the Euler form to $F^1K_0(Y)$. However, there are several non-commutative surfaces for which $F^2K_0(Y)$ is not in the radical of the restriction of the Euler form to $F^1K_0(Y)$. Example 9.2 below gives one such. Another is provided by a smooth quadric Q in the quantum $\mathbb{P}^3$ that has homogeneous coordinate ring the homogenization of the enveloping algebra $U(\mathfrak{sl}_2)$. The surface Q has a homogeneous coordinate ring that is the homogenization of $U(\mathfrak{sl}_2)/I$ where I may be any minimal primitive ideal that annihilates a finite-dimensional simple module.

We write Erad for the radical of the restriction of the Euler form $(-,-)$ to $F^1K_0(Y)$.

Definition 9.1. The *Picard group* of a non-commutative projective surface Y is

$$\operatorname{Pic} Y := F^1K_0(Y)/\mathrm{Erad} \cap F^2K_0(Y).$$

The *intersection pairing*

$$\cdot : \operatorname{Pic} Y \times \operatorname{Pic} Y \to \mathbb{Z},$$

is defined by

$$C \cdot D = -(C,D). \tag{9.1}$$

Example 9.2. The Euler form is not symmetric on the Picard group of the non-commutative surfaces in [14]. These surfaces have the form $Y = \mathsf{Tails}\,A$ where A is a three-dimensional regular algebra with Hilbert series $H_A(t) = (1-t)^{-2}(1-t^n)^{-1}$. Every integer $n \geq 1$ can occur, but here we are interested in those with $n \geq 3$. The minimal resolution of k looks like

$$\begin{aligned} 0 &\to A(-n-2) \to A(-n-1) \oplus A(-n-1) \oplus A(-2) \\ &\to A(-1) \oplus A(-1) \oplus A(-n) \to A \to k \to 0, \end{aligned}$$

so $e = n+2$. By Theorem 2.3 and Propositions 3.2 and 7.4,

$$K_0(Y) \cong \mathbb{Z}[T]/(1-T)^2(1-T^n)$$

and

$$\mathrm{Pic}(Y) \cong (1-T)/(1-T)(1-T^n) \cong \mathbb{Z}^n.$$

Stephenson has shown that there are two different kinds of points on Y. These arise from two different kinds of graded A-modules that are critical of Gelfand–Kirillov dimension one; the standard ones have Hilbert series $(1-t)^{-1}$, and others have Hilbert series $t^i(1-t^n)^{-1}$. By the proof of Theorem 2.3, the class in $K_0(Y)$ of a module $\pi(M)$ is $q_M(T)=H_M(T)H_A(T)^{-1}$. Therefore the standard points have class $(1-T)(1-T^n)$, which is in the radical, and the others have class $T^i(1-T)^2$, which is not in the radical. Therefore the standard points become zero in the Picard group, but the others do not.

Since $H_A(t^{-1})=(1-t^{-1})^{-2}(1-t^{-n})^{-1}=-t^{n+2}H_A(t)$, $(1,T^i)=0$ if $0<i<n+2$. To see that the form is not symmetric, suppose that $n \geq 3$. Then $H_A(t)=1+2t+3t^2+\cdots$. Hence

$$(1-T,(1-T)^2)=1 \quad \text{but} \quad ((1-T)^2,1-T)=0.$$

Notice that $((1-T)^2,(1-T)^2)=1$. The non-standard points behave in $\mathrm{Pic}(Y)$ as if they are curves.

The situation is better for a quantum $\mathbb{P}^2$.

Definition 9.3. Let $Y = \mathsf{Tails}\,A$ be a quantum $\mathbb{P}^2$. A *point* on Y is an isomorphism class of a simple module in $\mathsf{Tails}\,A$. An *irreducible curve* on Y is an isomorphism class in $\mathsf{Tails}\,A$ of a module $\mathscr{M}$ that is critical of Krull dimension one.

Krull dimension was used in the last definition, but the support filtration on $K_0(Y)$ was defined in terms of a homological dimension (Definition 4.2). By the results in [5] these are compatible. Their result is this.

Let $Y=\mathsf{Tails}\,A$ be a quantum $\mathbb{P}^2$, where A is a three-dimensional quantum polynomial ring. If $0 \neq M \in \mathsf{grmod}\,A$, then $j(M) + \mathrm{GKdim}\,M = 3$, and $\mathrm{Kdim}\,M = \mathrm{GKdim}\,M$. Furthermore $\mathrm{Kdim}\,\pi M=\mathrm{Kdim}\,M-1$. Therefore if $\mathscr{M} \in \mathrm{tails}\,A$, then $\mathrm{ds}(\mathscr{M})=\mathrm{Kdim}\,\mathscr{M}$. Hence $F^2K_0(Y)$ is generated by points and $F^1K_0(Y)$ is generated by curves and points. It would be good to know if there is a similar result for all quantum $\mathbb{P}^n$'s.

Thus every simple module over a quantum $\mathbb{P}^2$ has dimension zero. By [5] the points of degree one in a quantum $\mathbb{P}^2$ lie on a commutative curve. By [6], some quantum $\mathbb{P}^2$'s have points of higher degree, and we call these *fat points*.

Proposition 9.4. *Let Y be a quantum $\mathbb{P}^2$. Then*
1. $F^2K_0(Y)=(1-T)^2$;
2. *the degree function defines an isomorphism* $\mathrm{Pic}\,Y \cong \mathbb{Z}$ *sending* $1-T \mapsto 1$;
3. *if $C,D \in \mathrm{Pic}\,Y$ have degrees m and n, then $C \cdot D = mn$.*

Proof. (1) By the previous discussion, a typical element of $F^2K_0(Y)$ is of the form $[\mathscr{M}]-[\mathscr{N}]$ with $\mathrm{Kdim}\,\mathscr{M}=\mathrm{Kdim}\,\mathscr{N}=0$. Since $\mathscr{M}$ and $\mathscr{N}$ have finite length, both $[\mathscr{M}]$ and $[\mathscr{N}]$ are sums of classes of points.

Let p be a point of degree d. By [5,6], $p=\pi(M)$ where M is a graded A-module having a presentation of the form

$$0 \to L'(-d) \to L \to M \to 0,$$

where L and L' are line modules for A. Since $H_L(t)=(1-t)^{-2}$, it follows that in $K_0(Y)$, $[p]=(1-T)-T^d(1-T)=d(1-T)^2$. It follows from this that $F^2K_0(Y)$ is contained in $(1-T)^2$, and it follows from the existence of points of degree one that $(1-T)^2$ is in $F^2K_0(Y)$. Hence $F^2K_0(Y)=(1-T)^2$.

(2) By Propositions 3.2 and 6.5, $F^1K_0(Y)=(1-T)$. By Proposition 7.4, the radical of $(-,-)$ on $F^1K_0(Y)$ is $(1-T)^2$. But this equals $F^2K_0(Y)$, so $\operatorname{Pic} Y \cong (1-T)/(1-T)^2$. It is clear that the degree function sends $(1-T)$ onto $\mathbb{Z}$ with kernel $(1-T)^2$.

(3) This follows from Bezout's Theorem. □

The reason that Ext-groups give the intersection numbers of curves on surfaces is as follows. Let C and D be curves on a smooth commutative irreducible projective surface Y over an algebraically closed field k, and suppose they have no common component. Then $\mathcal{O}_C \otimes \mathcal{O}_D$ is the structure sheaf of the scheme theoretic intersection $C \cap D$. Thus the total intersection multiplicity of C and D is $\dim_k H^0(Y, \mathcal{O}_C \otimes \mathcal{O}_D)$. However, $\mathcal{O}_C \otimes \mathcal{O}_D \cong \mathcal{O}_C \otimes \mathcal{O}_D(C) \cong \mathcal{E}xt^1(\mathcal{O}_C, \mathcal{O}_D)$. It therefore follows from the Grothendieck spectral sequence $H^p(Y, \mathcal{E}xt^q_Y(\mathcal{M},\mathcal{N})) \Rightarrow \operatorname{Ext}^{p+q}_Y(\mathcal{M},\mathcal{N})$ that $H^0(Y, \mathcal{O}_C \otimes \mathcal{O}_D) \cong \operatorname{Ext}^1_Y(\mathcal{O}_C, \mathcal{O}_D)$.

Definition 9.5. We say that two curves $\mathcal{M}$ and $\mathcal{N}$ have *no common component* if $\operatorname{Hom}_Y(\mathcal{M},\mathcal{N})=0$.

Theorem 9.6. *Let Y be a quantum $\mathbb{P}^2$. Suppose that m and n are positive integers. Let*

$$0 \to \mathcal{O}_Y(-m) \to \mathcal{O}_Y \to \mathcal{M} \to 0 \tag{9.2}$$

and

$$0 \to \mathcal{O}_Y(-n) \to \mathcal{O}_Y \to \mathcal{N} \to 0 \tag{9.3}$$

be exact sequences defining "curves" $\mathcal{M}$ and $\mathcal{N}$. If $\mathcal{M}$ and $\mathcal{N}$ have no common component, then

$$\dim_k \operatorname{Ext}^1_Y(\mathcal{M},\mathcal{N}) = mn.$$

Proof. Apply $\operatorname{Hom}_Y(-,\mathcal{O}_Y)$ to (9.2). By (7.1), $\operatorname{Ext}^2_Y(\mathcal{O}_Y,\mathcal{O}_Y)=0$, so by the long exact sequence for Ext, $\operatorname{Ext}^2_Y(\mathcal{M},\mathcal{O}_Y)=0$. Applying $\operatorname{Hom}_Y(\mathcal{M},-)$ to (9.3), the long exact sequence for Ext ends with $\cdots \to \operatorname{Ext}^2_Y(\mathcal{M},\mathcal{O}_Y) \to \operatorname{Ext}^2_Y(\mathcal{M},\mathcal{N}) \to 0$, so $\operatorname{Ext}^2_Y(\mathcal{M},\mathcal{N})=0$. By hypothesis, $\operatorname{Hom}_Y(\mathcal{M},\mathcal{N})=0$, so

$$\dim_k \operatorname{Ext}^1_Y(\mathcal{M},\mathcal{N}) = -([\mathcal{M}],[\mathcal{N}]).$$

By Theorem 8.6, this equals mn. □

It is still not clear what is the right notion of a curve in a non-commutative scheme. Although Y-modules having a presentation of the form (9.2) are reasonable analogues of structure sheaves of curves, there are other modules that might also warrant such an interpretation [1–3].

10. Non-commutative versions of $\mathbb{P}^1 \times \mathbb{P}^1$

Among the three-dimensional regular algebras classified in [4] are several families of connected graded k-algebras A for which $\mathsf{Tails}\,A$ should be considered a non-commutative version of $\mathbb{P}^1 \times \mathbb{P}^1$. In this section we consider $K_0(Y)$ for these quasi-schemes.

The algebras just mentioned have the following properties:

- A is generated by A_1 as a k-algebra;
- A is right and left noetherian;
- A is a domain;
- $\operatorname{gldim} A = 3$;
- $\operatorname{Ext}^d_A(k,A) \cong \begin{cases} k(4) & \text{if } d=3, \\ 0 & \text{otherwise.} \end{cases}$
- the minimal resolution of the trivial module is $0 \to A(-4) \to A(-3)^2 \to A(-1)^2 \to A \to k \to 0$;
- $H_A(t)=(1-t)^{-2}(1-t^2)^{-1}$, which is the same as the Hilbert series of a commutative polynomial ring having two generators in degree one and one generator in degree two;
- the Hilbert series of the 2-Veronese subalgebra $A^{(2)} = k \oplus A_2 \oplus A_4 \oplus \cdots$ is the same as the Hilbert series of the commutative ring $k[x_0,x_1,x_2,x_3]/(x_0x_1 - x_2x_3)$.

The categories $\mathsf{Tails}\,A$ and $\mathsf{Tails}\,A^{(2)}$ are equivalent [7], so the last property justifies our thinking of $\mathsf{Tails}\,A$ as a quadric surface in a quantum $\mathbb{P}^3$. Since A has global dimension three, Ext^3 vanishes on $\mathsf{Tails}\,A$, so we should think of $\mathsf{Tails}\,A$ as analogous to a smooth quadric. This is why we think of $\mathsf{Tails}\,A$ as a non-commutative analogue of $\mathbb{P}^1 \times \mathbb{P}^1$.

The simplest example of such an A is the generic Clifford algebra $A = k[x,y]$ with defining relations

$$x^2y = yx^2, \qquad xy^2 = y^2x.$$

The second Veronese of this is the commutative ring $k[x^2, xy, yx, y^2]$ with single defining relation

$$x^2.y^2 = xy.yx.$$

This is the homogeneous coordinate ring of $\mathbb{P}^1 \times \mathbb{P}^1$ embedded in $\mathbb{P}^3$ as a quadric hypersurface. Thus $\mathsf{Tails}\,A \cong \mathsf{Tails}\,A^{(2)} \cong \mathsf{Qcoh}(\mathbb{P}^1 \times \mathbb{P}^1)$. Thus $\mathbb{P}^1 \times \mathbb{P}^1$ can be given a non-commutative homogeneous coordinate ring, and this non-commutative homogeneous coordinate ring is better than any of the commutative homogeneous coordinate rings because it has finite global dimension.

Because A has finite global dimension, it follows from Theorem 2.3 that

$$K_0(Y) \cong \mathbb{Z}[T,T^{-1}]/(T^4 - 2T^3 + 2T - 1).$$

The isomorphism is as $\mathbb{Z}[T,T^{-1}]$-modules. When A is the generic Clifford algebra we get

$$K_0(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathbb{Z}[T,T^{-1}]/(T^4 - 2T^3 + 2T - 1).$$

This is not the way one usually presents $K_0(\mathbb{P}^1 \times \mathbb{P}^1)$; rather, one has

$$K_0(\mathbb{P}^1 \times \mathbb{P}^1) \cong K_0(\mathbb{P}^1) \otimes K_0(\mathbb{P}^1) \cong \mathbb{Z}[x,y]/((x-1)^2,(y-1)^2).$$

Theorem 2.3 does not apply to the commutative ring $k[x_0,x_1,x_2,x_3]/(x_0x_1 - x_2x_3)$ because it has infinite global dimension. Thus, by giving $\mathbb{P}^1 \times \mathbb{P}^1$ a non-commutative coordinate ring, we can compute its K_0 from the Hilbert series as we have just done.

We continue to write $Y = \mathsf{Tails}\,A$ for one of these quantum $\mathbb{P}^1 \times \mathbb{P}^1$'s. It is easy to show that $F^1K_0(Y) = (1-T)$. Point modules over A have a projective resolution of the form

$$0 \to A(-3) \to A(-1) \oplus A(-2) \to A \to M \to 0,$$

so $F^2K_0(Y)$ contains $T^3 - T^2 - T + 1 = (1-T)^2(1+T)$. A little additional work gives equality, so that

$$\operatorname{Pic} Y = F^1K_0(Y)/F^2K_0(Y) \cong (1-T)/(1-T)^2(1+T) \cong \mathbb{Z} \times \mathbb{Z}.$$

Because A is Auslander–Gorenstein, Y satisfies Serre duality, with $\omega^\circ \cong \mathcal{O}_Y(-4)$. The cohomology groups are

$$H^q(Y, \mathcal{O}_Y(d)) \cong \begin{cases} A_d & \text{if } q = 0, \\ 0 & \text{if } q = 1, \\ (A_{-d-4})^* & \text{if } q = 2. \end{cases} \tag{10.1}$$

From this, we can compute the intersection pairing on $\operatorname{Pic} Y$.

The next result may be proved by the methods used in Section 9.

Proposition 10.1. *Let Y be a quantum $\mathbb{P}^1 \times \mathbb{P}^1$. Then* $\operatorname{Pic} Y \cong \mathbb{Z} \times \mathbb{Z}$. *With respect to the ordered basis $\{1-T, T-T^2\}$ for* $\operatorname{Pic} Y$, *the intersection pairing is given by*

$$(a,b) \cdot (c,d) = ad + bc.$$

11. Comparison with Jorgensen's intersection theory

In [9], Jorgensen develops an intersection theory for non-commutative quasi-schemes. In particular, his theory applies to surfaces. He begins with a quasi-scheme Y, and defines an ascending filtration on $K_0(Y)$ using Krull dimension. Our descending filtration using co-dimension of support, which is defined homologically, will often coincide with his filtration. For all known non-commutative projective surfaces the two filtrations agree. The Euler form plays a significant role in [9], and is used there to define an intersection number. Whereas we consider intersecting two arbitrary curves on the surface to obtain an intersection number, Jorgensen requires that one of the curves be an *effective divisor*. Not every curve on a non-commutative surface is an effective divisor. The reader can find the definition of an effective divisor in [9], but without giving that definition we can illustrate their special nature by remarking that in the affine case $Y = \mathsf{Mod}\,R$, effective divisors correspond to two-sided ideals I in R which are invertible as R–R-bimodule, whereas curves correspond to certain one-sided R-modules. Thus effective divisors are in some sense two-sided curves. In particular, a non-commutative surface will often have few effective divisors, though it will have many curves. Indeed it is an unexplained surprise that all known examples of non-commutative projective surfaces have at least one effective divisor, and that divisor is isomorphic to a commutative curve.

Associated to an effective divisor D is a Y-module $\mathcal{O}_D$ which plays the role of the structure sheaf of the hypersurface corresponding to D. Also associated to D is a map $c(D): K_0(Y) \to K_0(Y)$. The map $c(D)$ is the "first Chern class" of D, and its effect on the class $[\mathcal{O}_C]$ of a curve should be thought of as intersecting the curve with the hypersurface corresponding to D. Jorgensen defines

$$\langle D, [\mathcal{O}_C]\rangle := ([\mathcal{O}_Y], c(D)([\mathcal{O}_C])).$$

It is an easy exercise using the definitions of $\mathcal{O}_D$ and $c(D)$ to see that

$$\langle D, [\mathcal{O}_C]\rangle = -([\mathcal{O}_D], [\mathcal{O}_C]) = C \cdot D.$$

Thus the intersection form defined in [9] agrees with that defined in this paper.

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