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The Global Homological Dimension of the Ring of Differential Operators on a Nonsingular Variety over a Field of Positive Characteristic

S. P. SMITH*

Mathematics Institute, University of Warwick, Coventry CV4 7AL, United Kingdom

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1. INTRODUCTION

Let k be a commutative ring, and A a commutative k-algebra. In Section 2 we define $D(A)$ the ring of k-linear differential operators on A. If k is a field of characteristic zero and A is the coordinate ring of a nonsingular affine algebraic variety over k, then it is shown in [3, 5] that the global homological dimension of $D(A)$ (gl. dim $D(A)$), equals the dimension of the variety.

Here we prove that if k is an algebraically closed field of characteristic $p > 0$, and A is the coordinate ring of a nonsingular affine algebraic variety, X, over k then $\text{gl. dim}\, D(A) = \dim X$. In [5] it is shown that the weak global dimension of $D(A)$ (w-dim. $D(A)$), equals dim X.

As $D(A)$ is not noetherian there is no apriori reason for w-dim $D(A)$ and gl. dim $D(A)$ to be equal. It is a relatively straightforward matter to see that $D(A)$ is a union of subalgebras each of which has global dimension equal to dim X, and so a theorem of Berstein [2] gives $\text{gl. dim}\, D(A) \leqslant \dim X + 1$. So the point is to show that in this particular situation Berstein's result can be improved to show $\text{gl. dim}\, D(A) \leqslant \dim X$. That the global dimension is bounded below by $\dim X$ is a consequence of the fact that $w\text{-dim}\, D(A) = \dim X$.

2. DIFFERENTIAL OPERATORS

Let k be a commutative ring, and A a commutative k-algebra. Then $\text{End}_k A$ may be made into an $A \otimes_k A$-module by defining $((a \otimes b)\theta)(c) =$

* Present address: Department of Mathematics, University of Washington, Seattle, WA 98195.

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$a\theta(bc)$ for $\theta \in \mathrm{End}_k A$ and $a, b, c \in A$. We write $[a, \theta]$ for $(a \otimes 1 - 1 \otimes a)\theta$, so $[a, \theta](b) = a\theta(b) - \theta(ab)$.

DEFINITION 2.1. The space of *k-linear differential operators of order* $\leqslant n$ *on* A is defined inductively by $\mathrm{Diff}_k^{-1} A = 0$, and for $n \geqslant 0$ $\mathrm{Diff}_k^n A = \{\theta \in \mathrm{End}_k A \mid [a, \theta] \in \mathrm{Diff}_k^{n-1} A$ for all $a \in A\}$. The *ring of k-linear differential operators on* A is $D(A) = \bigcup_{n=0}^{\infty} \mathrm{Diff}_k^n A$.

Remark 2.2. (1) $\mathrm{Diff}_k^n A$ is an $A \otimes A$-submodule of $\mathrm{End}_k A$.

(2) If $\theta \in \mathrm{End}_k A$, then $\theta \in \mathrm{Diff}_k^n A$, if and only if, for all $a_0, a_1, ..., a_n \in A$ one has $[a_0[a_1 \cdots [a_n, \theta] \cdots]] = 0$.

(3) The reader is referred to [6, 7, 8] for an introduction to differential operators on commutative rings.

DEFINITION 2.3. Denote by $\mu: A \otimes_k A \to A$ the multiplication map $\mu(a \otimes b) = ab$. This is a k-algebra map (also an A-module map for either the right or left A-module structure on $A \otimes A$). Thus $I = \ker \mu$ is an ideal of $A \otimes_k A$.

THEOREM 2.4 (Heynemann–Sweedler [7], Grothendieck [6]). *Let* $\theta \in \mathrm{End}_k A$. *Then* $\theta \in \mathrm{Diff}_k^n A$, *if and only if,* $I^{n+1} \cdot \theta = 0$.

From now on, char $k = p > 0$, and $A = k[t_1, ..., t_n]$ is a finitely generated commutative k-algebra. We also assume k is contained in A.

DEFINITION 2.5. For $r \geqslant 0$, define A_r to be the subalgebra of A generated by k and all elements a^{p^r} with $a \in A$. Clearly $A = A_0 \supset A_1 \supset \cdots$, and for $r > s$, A_s is a finitely generated A_r-module.

LEMMA 2.6. $A_r = k[t_1^{p^r}, ..., t_n^{p^r}]$.

Proof. By induction it suffices to prove the result for $r = 1$. Clearly $k[t_1^p, ..., t_n^p] \subset A_1$, so only the reverse inclusion must be established. Let $a \in A$, and write $a = \sum_J \lambda_J t^J$, where $\lambda_J \in k$ and $J = (j_1, ..., j_n)$ is a multi-index and $t^J = t_1^{j_1} \cdots t_n^{j_n}$ (there is not necessarily a unique such expression for a). As char $k = p$, if $u, v \in A$ then $(u + v)^p = u^p + v^p$, so by induction (on the number of nonzero λ_J occurring in the expression for a), $a^p = \sum_J \lambda_J^p (t^J)^p$. But $(t^J)^p = (t_1^p)^{j_1} \cdots (t_n^p)^{j_n} \in k[t_1^p, ..., t_n^p]$. Hence $a^p \in k[t_1^p, ..., t_n^p]$. ∎

This shows that we could actually define A_r to be $k[t_1^{p^r}, ..., t_n^{p^r}]$ and that such a definition is independent of the choice of generators for A.

THEOREM 2.7. $D(A) = \bigcup_{r=0}^{\infty} \mathrm{End}_{A_r} A$.

Proof [5, Lemma 3.3]. Let $\theta \in D(A)$ of order $< p^r$. As $(1 \otimes t_j - t_j \otimes 1) \in I$ for all j, $(1 \otimes t_j - t_j \otimes 1)^{p^r} = 1 \otimes t_j^{p^r} - t_j^{p^r} \otimes 1 \in I^{p^r}$. Hence $0 = (1 \otimes t_j^{p^r} - t_j^{p^r} \otimes 1) \cdot \theta = -[t_j^{p^r}, \theta]$. Thus the action of θ on A commutes with the action of A_r on A given by multiplication, as $A_r = k[t_1^{p^r}, \ldots, t_n^{p^r}]$. That is, $\theta \in \mathrm{End}_{A_r} A$. Hence $D(A) \subseteq \bigcup_{r=0}^{\infty} \mathrm{End}_{A_r} A$.

Conversely, let $\theta \in \mathrm{End}_{A_r} A$. Then certainly $\theta \in \mathrm{End}_k A$. We claim that θ is a differential operator of order $\leqslant s = np^r - 1$. To see this note that I^{s+1} is generated (as an ideal) by all $(1 \otimes t_1 - t_1 \otimes 1)^{j_1} \cdots (1 \otimes t_n - t_n \otimes 1)^{j_n}$ where $j_1 + \cdots + j_n = np^r$. This is because I is generated by $1 \otimes t_1 - t_1 \otimes 1, \ldots, 1 \otimes t_n - t_n \otimes 1$. As $j_1 + \cdots + j_n = np^r$ some $j_i \geqslant p^r$, and thus $(1 \otimes t_i - t_i \otimes 1)^{p^r} = 1 \otimes t_i^{p^r} - t_i^{p^r} \otimes 1$ divides $(1 \otimes t_1 - t_1 \otimes 1)^{j_1} \cdots (1 \otimes t_n - t_n \otimes 1)^{j_n}$. But $0 = -[t_i^{p^r}, \theta] = (1 \otimes t_i - t_i \otimes 1)^{p^r} \cdot \theta$, hence $(1 \otimes t_1 - t_1 \otimes 1)^{j_1} \cdots (1 \otimes t_n - t_n \otimes 1)^{j_n} \cdot \theta = 0$ and $I^{s+1} \cdot \theta = 0$ as required. ∎

Notation. Write $D_r = \mathrm{End}_{A_r} A$, so $D(A) = \bigcup_{r=0}^{\infty} D_r$. Also write $D = D(A)$.

Note that the action of A on itself by multiplication, enables us to consider A as a subalgebra of D. In fact $A = D_0$, and $D_0 \subset D_1 \subset \cdots$. Also notice that A_r is contained in the centre of D_r. For each r, A is a finitely generated A_r module, generated by $t_1^{j_1} \cdots t_n^{j_n}$ with $0 \leqslant j_i < p^r$ for $i = 1, \ldots, n$. Hence D_r is a finitely generated A_r-module. In particular D_r is a finitely generated module over its centre, so satisfies a polynomial identity. However, D will not in general satisfy a polynomial identity (when A is a polynomial ring over k, D does not satisfy a polynomial identity).

The following is used in Section 3.

PROPOSITION 2.8. *Let M be a D-module with a chain of k-vector subspaces $M_0 \subseteq M_1 \cdots$ such that*

(a) $M = \bigcup_{r=0}^{\infty} M_r$,

(b) *each M_r is a D_r-module, and for large r,* length ${}_{D_r} M_r \leqslant l$. *Then M is of finite length as a D-module, and* length ${}_D M \leqslant l$.

Proof. Suppose $l = 1$. We must show M is a simple D-module. Pick $0 \neq m \in M$, $m' \in M$. For sufficiently large r, $m, m' \in M_r$ and M_r is a simple D_r-module, so there exists $d \in D_r$ with $dm = m'$. Hence M is simple.

Suppose now $l \geqslant 2$, and the proposition is true for integers less than l. If M is a simple D-module we are finished. So suppose M is not simple and pick a proper D-submodule, $N \neq 0$. Put $N_r = M_r \cap N$, notice that $N = \bigcup_{r=0}^{\infty} N_r$, and that each N_r is a D_r-module. We show that for large r, length ${}_{D_r} N_r \leqslant l - 1$. To see this pick $m \in M \backslash N$. There exists s such that $m \in M_r$ for all $r \geqslant s$. But $m \notin N_r$. Hence if $r \geqslant s$, $M_r \supsetneq N_r$, so for large r, length ${}_{D_r} N_r \leqslant l - 1$. Applying the induction hypothesis length ${}_D N \leqslant l - 1$.

We have shown that any proper submodule of M is of finite length $\leqslant l - 1$. Hence length ${}_D M \leqslant l$. ∎

3. Global Dimension

Henceforth, k is an algebraically closed field with char $k=p>0$, X is a nonsingular affine algebraic variety over k, and $A=\mathcal{O}(X)$ is the coordinate ring of X.

Our immediate goal is statements (3) and (4) of Proposition 3.2. We begin with the following Lemma, parts (i) and (ii) of which are to be found in [5].

Lemma 3.1. *Let $R=\mathcal{O}(X)$ be the coordinate ring of a nonsingular affine variety X over an algebraically closed field k of characteristic $p>0$. Let $q=p^s$. Define S to be the image of the map $x\to x^q$ on R. Then:*

(i) *$S\cong R$ as rings,*

(ii) *R is a finitely generated projective S-module,*

(iii) *$\mathrm{Hom}_S(R, S)$ is a finitely generated projective R-module of rank 1.*

Proof. The statements (i) and (ii) appear in [5] as Lemma 3.1 and Proposition 2.2. Because $\mathrm{Hom}_S(R, S)$ is a finitely generated projective S-module, it is also a finitely generated projective R-module by [5, Proposition 2.2]. It follows that the canonical S-algebra homomorphism $R\to \mathrm{End}_R(\mathrm{Hom}_S(R, S))$ is an isomorphism. Hence by [11, Proposition 7.5], $\mathrm{Hom}_S(R, S)$ is of rank 1 as an R-module. ∎

Proposition 3.2. (1) *D_r is Morita equivalent to A_r, the progenerator being the D_r-A_r bimodule A.*

(2) *For $s\geqslant r$, D_s is a finitely generated projective right D_r-module and a generator in* Mod-D_r.

(3) *D is a flat right D_r-module.*

(4) *If M is a simple D_r-module then $D\otimes_{D_r} M$ is a simple D-module.*

(5) *D is a projective right D_r-module, for all $r\geqslant 0$.*

(6) *The above statements are true if "right" is replaced by "left."*

Proof. (1) appears in the proof of [5, Lemma 3.4]. It is a consequence of the definition of D_r and of Lemma 3.1 with $R=A$, $S=A_r$.

In order to prove (2) we recall the following consequence of the Morita Theorems: Let R be a commutative ring, P a progenerator in Mod-R, and set $D=\mathrm{End}_R P$. If M is a D-module, then M is a progenerator in Mod-D if and only if its is a progenerator in Mod-R.

Let $s\geqslant r$. By the previous paragraph, to show D_s is a progenerator in Mod-D_r, it is enough to show it is a progenerator in Mod-A_r. However, (again by the previous paragraph) D_s is a progenerator in Mod-A_s. By [5, Proposition 2.2] this ensures D_s is a projective A_r-module, and a generator (since over a commutative ring any faithful projective module is a generator).

This establishes (2) and (3) is a consequence since $D = \varinjlim_{s \geqslant r} D_s$, is a direct limit of flat D_r-modules.

Let M be a simple D_r-module and let $s \geqslant r$. Consider $A_s \subseteq A_r \subseteq A$. Clearly any simple A-module is a simple A_r-module (since k is algebraically closed). A simple A_r-module is of the form A_r/I for I a maximal ideal. But A is integral over A_r, so there is a maximal ideal J of A such that $I = J \cap A_r$. Hence $A_r/I \to A/J$ is an isomorphism of A_r-modules. Thus any simple A_r-module is also a simple A-module (although not uniquely). Now, if N is a simple A-module then $D_r \otimes_A N \cong (A \otimes_{A_r} \mathrm{Hom}_{A_r}(A, A_r)) \otimes_A N \cong A \otimes_{A_r} N$ (where the final isomorphism uses the fact that $\mathrm{Hom}_{A_r}(A, A_r)$ is a rank 1 projective A-module, and faithful). By the Morita Theorems, $D_r \otimes_A N$ is a simple D_r-module and every simple D_r-module is of the form $D_r \otimes_A N$ for a suitable simple A-module N.

Finally, with M as above, then $M \cong D_r \otimes_A N$ for some simple A-module N. Hence $D_s \otimes_{D_r} M \cong D_s \otimes_{D_r} (D_r \otimes_A N) \cong D_s \otimes_A N$ which is simple. Now we apply Proposition 2.8 to conclude that $D \otimes_{D_r} M$ is a simple D-module; for $s \geqslant r$ write M_s for the D_s-submodule of $D \otimes_{D_r} M$ generated by M, then M_s is certainly of length $\leqslant 1$, being a homomorphic image of $D_s \otimes_{D_r} M$. This completes the proof of (4).

To see that (3) can be improved to show D is projective as a right D_r-module recall [1, Proposition 3]. As $D = \varinjlim_{s \geqslant r} D_s$, it is enough to show that each D_{s+1}/D_s is a projective right D_r-module for $s \geqslant r$. As D_s is a projective right D_r-module, it is enough to show that D_{s+1}/D_s is a projective right D_s-module, or equivalently, that $D_{s+1} = D_s \oplus W$ for some right D_s-module W. Observe that in the proof of (4) we have shown that if M is a simple D_s-module then $D_{s+1} \otimes_{D_s} M$ is a simple D_{s+1}-module, in particular nonzero. In the language of [12] this says that D_{s+1} is a faithfully projective right D_s-module. The conclusion of the Theorem in [12] gives the existence of the required W, proving (5).

Finally to see that the above statements are true if "right" is replaced by "left" is routine. For example, if $s \geqslant r$, to show that D_s is a projective left D_r-module, it is sufficient (by Morita equivalence) to show that $\mathrm{Hom}_{A_r}(A, A_r) \otimes_{D_r} D_s$ is projective as a (left) A_r-module. But this is isomorphic to $A_r \otimes_{A_s} \mathrm{Hom}_{A_s}(A, A_s)$ and this is projective as an A_r-module since $\mathrm{Hom}_{A_s}(A, A_s)$ is a projective A_s-module. We leave the rest of the proof of (6) to the enthusiastic reader! ∎

LEMMA 3.4. *Let J be a left ideal of D_r. There exists a left ideal J' of D_r, containing J such that*

(i) $pd_{D_r}(D_r/J') < \dim X$

(ii) *J'/J is of finite length.*

Proof. D_r is noetherian so define J' to be the largest left ideal containing J such that J'/J is of finite length. Then D_r/J' contains no artinian submodules. Hence by [4, Corollary 5] $pd_{D_r}(D_r/J') \leqslant \text{gl. dim } D_r - 1 = n - 1$.

Alternatively, one can use the fact that D_r is Morita equivalent to A_r and use the fact that the lemma is true for A_r noting that the statements are Morita invariant. ∎

LEMMA 3.5. *Let I be a left ideal of D. Put $I_r = I \cap D_r$. Then for each r there is a left ideal I'_r of D_r containing I_r such that, for all r,*

(i) *I'_r/I_r is of finite length.*

(ii) $pd_{D_r}(D_r/I'_r) < \dim X$.

(iii) $I'_{r-1} \subset I'_r$.

Proof. After Lemma 3.4 we need only show we can choose the I'_r such that (iii) is satisfied. Suppose I'_{r-1} has been chosen. Note that $I_r + D_rI'_{r-1}/I_r \cong D_rI'_{r-1}/I_r \cap D_rI'_{r-1}$, which is a homomorphic image of $D_rI'_{r-1}/D_rI_{r-1}$ (since $I_{r-1} \subset I_r$). But $D_rI'_{r-1}/D_rI_{r-1} \cong D_r \otimes_{D_{r-1}} (I'_{r-1}/I_{r-1})$ and hence $I_r + D_rI'_{r-1}/I_r$ is of finite length as a D_r-module. Now apply (3.4) to $J = I_r + DI'_{r-1}$, and put $I'_r = J'$. Since J'/J is of finite length and J/I_r is of finite length, I'_r/I_r is of finite length. ∎

In the next proof we will make frequent use of the fact that if $0 \to X \to Y \to Z \to 0$ is a short exact sequence of modules over a ring R with $pd_R(Y) < n$ and $pd_R(Z) \leqslant n$ then $pd_R(X) < n$. The truth of this can be seen from considering the long exact sequence for Ext; in particular, $\cdots \to \text{Ext}^n(Y, -) \to \text{Ext}^n(X, -) \to \text{Ext}^{n+1}(Z, -) \to \cdots$.

THEOREM 3.6. *If I is a left ideal of D, then $pd_D(I) < \dim X$.*

Proof. Suppose $\dim X = n$. Put $I_r = I \cap D_r$, and choose the I'_r as in the lemma above. Put $T_r = I \cap DI'_r$. First note that T_r/DI_r is a submodule of $DI'_r/DI_r \cong D \otimes_{D_r} (I'_r/I_r)$ and hence of finite length. In particular, T_r is finitely generated as a left ideal, since DI_r is finitely generated. We also have for all r that, $T_{r-1} \subset T_r$ and $I = \bigcup_{r=0}^{\infty} DI_r = \bigcup_{r=0}^{\infty} T_r$.

We will next show that $pd_D(T_r/T_{r-1}) < n$ for all r, and hence by [1, Proposition 3], $pd_D(I) < n$.

As T_r and T_{r-1} are finitely generated, there exists m with $T_r = D(T_r \cap D_m)$ and $T_{r-1} = D(T_{r-1} \cap D_m)$. Hence $T_r/T_{r-1} \cong D \otimes_{D_m} (T_r \cap D_m/T_{r-1} \cap D_m)$.

Because $I/T_r = I/I \cap DI'_r \cong I + DI'_r/DI'_r \subsetneq D/DI'_r$ there is a short exact sequence of D_m-modules $0 \to I/T_r \to D/DI'_r \to Z \to 0$. As $D/DI'_r \cong D \otimes_{D_r} (D_r/I'_r)$ and D is flat as a right D_r-module, applying the functor $D \otimes_{D_r} -$ to a projective resolution of D_r/I'_r (as a D_r-module) gives a projective resolution of D/DI'_r as a D-module. Hence $pd_D(D/DI'_r) < n$ (as $pd_{D_r}(D_r/I'_r) < n$). As D is projective as a left D_m-module, a D-projective resolution for D/DI'_r is also a D_m-projective resolution. Thus $pd_{D_m}(D/I'_r) < n$. But gl. dim $D_m = n$, and hence $pd_{D_m}(Z) \leqslant n$. Now we get $pd_{D_m}(I/T_r) < n$.

Because $T_r \cap D_m/T_{r-1} \cap D_m \cong (T_r \cap D_m) + T_{r-1}/T_{r-1} \subsetneq I/T_{r-1}$ there is a short sequence of D_m-modules

$$0 \to T_r \cap D_m/T_{r-1} \cap D_m \to I/T_{r-1} \to Z \to 0.$$

However, $pd_{D_m}(Z) \leqslant n$ and $pd_{D_m}(I/T_{r-1}) < n$, so $pd_{D_m}(T_r \cap D_m/T_{r-1} \cap D_m) < n$. Applying the exact functor $D \otimes_{D_m} -$ gives $pd_D(T_r/T_{r-1}) < n$. ∎

THEOREM 3.7. gl. dim $D(A) = \dim X$.

Proof. Recall [9, Theorem 9.12] that gl. dim $D = \sup\{pd_D(D/I) \mid I$ is a left ideal of $D\}$. By (3.6), we get gl. dim $D \leqslant \dim X$.

For the reverse inequality, observe that Chase [5] has already shown that w-dim $D(A) \geqslant \dim X$. ∎

Remark. The case of gl. dim $D(k[t]) = 1$ is proved in [10] using a slightly different argument to the above—the proof in [10] is somewhat cleaner than the above, and the comparison of the two proofs might be useful for the reader.

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