SOME FINITE DIMENSIONAL ALGEBRAS RELATED TO ELLIPTIC CURVES

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ABSTRACT. The Koszul dual of a Sklyanin algebra A is a finite dimensional graded algebra depending on an elliptic curve, a translation automorphism, and an integer $n \geq 3$; it may be defined as $\mathrm{Ext}_A^*(k,k)$. The representation theory and structure of these algebras is studied by using the functor $\mathrm{Ext}_A^*(-,k)$ to transfer results from the Sklyanin algebras to the finite dimensional algebras. We show that their representation theory is closely related to the elliptic curve and the automorphism.

0. INTRODUCTION

Given an elliptic curve E over an algebraically closed field k, a translation automorphism σ of E, and an integer $n \geq 3$, we define in Section 10 a finite dimensional algebra $B_n(E,\sigma)$ depending on this data. Its Hilbert series is the same as that of the exterior algebra $\Lambda(k^n)$. In particular, $B_n(E,\sigma)$ is local. It is also of wild representation type, and a Frobenius algebra (in fact, symmetric when $n = 3$). The construction is such that E is naturally embedded in $\mathbb{P}(B_1)$, the projective space of 1-dimensional subspaces of the degree one component of $B_n(E,\sigma)$.

These properties of $B = B_n(E,\sigma)$ are proved in an indirect fashion; the starting point is that B is a Koszul algebra, and its properties are consequences of properties of its Koszul dual $A_n(E,\sigma)$. Since this paper is aimed at those whose main interest is finite dimensional algebras, we will treat $B_n(E,\sigma)$ as the primary object. However, $A_n(E,\sigma)$ is the object of primary interest to the author. It is a Sklyanin algebra, and has been the object of intense study over the past 6 or 7 years. A survey of what is known about $A_4(E,\sigma)$ may be found in [24].

Each $A = A_n(E,\sigma)$ is a connected graded algebra whose defining relations are homogeneous of degree two; its Koszul dual is, by definition, $\mathrm{Ext}_A^*(k,k)$ endowed with the Yoneda product. The contravariant functor $\mathrm{Ext}_A^*(-,k)$, sending graded A-modules to graded B-modules, is the vehicle used for transferring properties from A to B.

The basic properties of $A_n(E,\sigma)$ are reviewed in Section 8. The key result, due to Tate and van den Bergh [35], is that $A_n(E,\sigma)$ is a quantum polynomial ring (Definition 8.5); the terminology suggests that $A_n(E,\sigma)$ is a non-commutative deformation of a polynomial ring—like the polynomial ring, it is generated in degree one, has Hilbert series $(1-t)^{-n}$, is right and left noetherian, is a domain, has global homological dimension n, is Auslander-Gorenstein (Definition 4.2), and Cohen-Macaulay (Definition 8.4). The Frobenius property for $B_n(E,\sigma)$ is equivalent to

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the Gorenstein property for $A_n(E,\sigma)$ (Proposition 5.10); the Hilbert series for $B_n(E,\sigma)$ is obtained from that for $A_n(E,\sigma)$ via the functional equation for Koszul algebras (5-1).

Although the representation theory of $A_n(E,\sigma)$ is not well-understood (except for $n = 3$ and 4), there is, for each n, a particularly important class of graded modules which is understood. These are the linear modules (Definition 9.1). They are analogues of linear subspaces of the projective space $\mathbb{P}^{n-1}$. Their properties are discussed in Section 9. The most important point for the present paper is that linear modules have linear resolutions (Definition 1.6), and $\operatorname{Ext}^*_A(-,k)$ is a duality between the categories of graded modules over A and B having linear resolutions (Corollary 6.4). For each effective divisor D on E of degree $d \leq n$, there is an associated linear module $M(D)$; the B-module $L(D) := \operatorname{Ext}^*_A(M(D),k)$ is indecomposable, cyclic, graded, and has Hilbert series $(1+t)^{n-d}$ (hence dimension 2^{n-d}). These are the B-modules about which we have most information. (For the exterior algebra, the analogous modules are $\Lambda(k^n)/I$ where I is the left ideal generated by a d-dimensional subspace of k^n, the degree one component of $\Lambda(k^n)$.) That part of the Auslander-Reiten quiver for $B_3(E,\sigma)$ containing the modules $L(p)$, $p \in E$, is described.

Our approach to the study of $B_n(E,\sigma)$ requires some technical background, which the earlier part of the paper provides. Section 1 gives basic terminology and results on the category of graded modules, with particular attention paid to linear resolutions. Section 2 recalls the Yoneda product, and gives a result comparing two Yoneda Ext-algebras, one for left, and one for right, modules. Section 3 considers the Frobenius property for connected graded algebras, and defines a 'symmetrizing automorphism' ν which measures the failure of a Frobenius algebra to be symmetric—ν is inner if and only if the algebra is symmetric. The Auslander-Reiten translation $(D \circ Tr)M$ is isomorphic to $\nu_*(\Omega^2 M)$, where ν_* is the pull-back functor along ν and Ω^2 is the second syzygy functor. Section 4 considers consequences of a version of the Gorenstein property for non-commutative connected graded algebras. Theorem 4.3 shows that if A is such an algebra (also left noetherian and of finite global dimension), then $\operatorname{Ext}^*_A(k,k)$ is Frobenius; in particular $B_n(E,\sigma)$ is Frobenius. Section 5 gives some background on Koszul algebras—we restrict attention to those which are k in degree zero. In Section 6 we show that for a Koszul algebra $\operatorname{Ext}^*_A(-,k)$ is a duality for modules having a linear resolution. The close relation of this to the Koszul duality results of Beilinson-Ginsburg-Soergel [5] is briefly discussed in Section 7. Sections 8 and 9 discuss the Sklyanin algebras, and in Section 10 we finally get to the finite dimensional algebras $B_n(E,\sigma)$.

Sections 11 and 12 attempt to put some of the results for Sklyanin algebras in context. The general features of $A_n(E,\sigma)$ show up in many other situations. First, there are many quantum polynomial rings. The 3-dimensional ones have been classified by Artin-Schelter [1] and Artin-Tate-van den Bergh [2]: they are classified by geometric data consisting of a scheme and an automorphism of it, the scheme being either $\mathbb{P}^2$ or a cubic divisor in $\mathbb{P}^2$. The most interesting case is that of a smooth cubic with a translation automorphism, which gives the algebras $A_3(E,\sigma)$. An interesting class of quantum polynomial rings are the homogenized enveloping algebras in Section 12. All quantum polynomial rings are Koszul algebras (Theorem 5.11), and their duals are finite dimensional graded algebras which are Frobenius and have the same Hilbert series as an exterior algebra.

Part of the motivation for this work is the way in which the cohomology ring $H^*(G,k) \cong \mathrm{Ext}^*_{kG}(k,k)$ of a finite group is used to study its k-linear representations. For any (finite dimensional) algebra R with a distinguished module, say k itself for simplicity, one may assign to an R-module M the graded module $\mathrm{Ext}^*_R(M,k)$ over the graded algebra $\mathrm{Ext}^*_R(k,k)$ with its Yoneda product; if one knows a lot about $\mathrm{Ext}^*_R(k,k)$ this provides a method for analyzing M.

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The results in Section 6 were intended to appear as part of [29]; I am grateful to J.T. Stafford for agreeing to allow them to appear in the present paper.

1. Graded Algebras

Throughout we work over a fixed field k. Unadorned tensor products will denote tensor products over k.

Let A be a $\mathbb{Z}$-graded k-algebra. We are interested in the category $\mathrm{GrMod}(A)$ of graded A-modules, and its full subcategory $\mathrm{grmod}(A)$ of finitely generated modules. The morphisms in these categories, denoted $\mathrm{Hom}_{\mathrm{Gr}}(N,M)$, are the A-module maps $f : N \to M$ such that $f(N_i) \subset M_i$ for all i. More generally, if $f : N \to M$ satisfies $f(N_i) \subset M_{i+d}$ for all $i \in \mathbb{Z}$, we say that the **degree** of f is d; we write $\mathrm{Hom}_A(N,M)_d$ for the A-module maps of degree d , and define

$$\underline{\mathrm{Hom}}_A(N,M) = \bigoplus_{d\in\mathbb{Z}} \mathrm{Hom}_A(N,M)_d.$$

We write $\underline{\mathrm{Ext}}^i_A(N,M)$ for the derived functors of $\underline{\mathrm{Hom}}_A(N,M)$. Since $\mathrm{GrMod}(A)$ has enough projectives and enough injectives, $\underline{\mathrm{Ext}}^i_A(N,M)$ may be computed from a projective resolution of N (or an injective resolution of M) in $\mathrm{GrMod}(A)$. If $Q_\bullet \to N$ is such a projective resolution, then the grading on each $\underline{\mathrm{Hom}}_A(Q_p,M)$ induces a grading on $\underline{\mathrm{Ext}}^p_A(N,M)$; we denote the degree j component by $\underline{\mathrm{Ext}}^p_A(N,M)_j$, and view $\underline{\mathrm{Ext}}^*_A(N,M) = \oplus_{p\in\mathbb{Z}} \underline{\mathrm{Ext}}^p_A(N,M)$ as a bigraded vector space with $(p,j)^{\text{th}}$ component $\underline{\mathrm{Ext}}^p_A(N,M)_j$; we will refer to j as the **degree** and p as the **position**.

If N is finitely generated, then $\underline{\mathrm{Hom}}_A(N,M) = \mathrm{Hom}_A(N,M)$, the space of all A-module homomorphisms, and similarly $\underline{\mathrm{Ext}}^*_A(N,M) = \mathrm{Ext}^*_A(N,M)$.

A tensor product of graded vector spaces is always given the **tensor product grading** $(U \otimes V)_n = \sum_{i+j=n} U_i \otimes V_j$. The left derived functors of $\otimes_A$ behave as usual, giving Tor groups which are endowed with a graded vector space structure.

A graded vector space V is **bounded below** if $V_i = 0$ for $i \ll 0$.

A graded vector space V is **locally finite** if $\dim_k(V_i) < \infty$ for all i. The **Hilbert series** of a locally finite vector space V is the formal series

$$H_V(t) := \sum (\dim_k V_i)t^i.$$

The shift functor [1] on $\mathrm{GrMod}(A)$ is defined by $M[1] = M$ as an A-module, but with grading $M[1]_i = M_{i+1}$. Thus

$$\underline{\mathrm{Hom}}_A(N,M) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{Gr}}(N,M[n]).$$

If $M \in \mathrm{GrMod}(A)$, M^* denotes $\underline{\mathrm{Hom}}_k(M,k)$; this has a natural structure of a graded *right* A-module.

Definition 1.1. A k-algebra R is **augmented**, if there is a distinguished k-algebra map $\varepsilon : R \to k$, **the augmentation**. There are two distinguished modules over an augmented algebra, namely the left and right **trivial** modules $R/\ker(\varepsilon)$. We will denote these by k and k_R respectively.

A graded algebra A is **connected** if $A_0 = k$ and $A_i = 0$ for $i < 0$; in that case, A is an augmented algebra, with $\ker(\varepsilon) = A_{\geq 1}$.

Typical examples of augmented k-algebras are commutative local rings, Hopf algebras (group algebras and enveloping algebras) where ε is the co-unit, and connected graded algebras. Later, we will consider $\mathrm{Ext}^*_R(k,k)$; for example, if R is the group algebra kG, this is the cohomology ring $H^*(G,k)$.

Definition 1.2. A complex $\cdots \to P_{n+1} \xrightarrow{d} P_n \xrightarrow{d} P_{n-1} \to \cdots$ of modules over an augmented k-algebra is **minimal** if, for each n, $\ker(d) \subset \ker(\varepsilon)P_n$. In particular, there is a notion of a **minimal projective resolution** of a module, any two of which are isomorphic.

For each $n \geq 0$ we define the **syzygy functors** Ω^i by $\Omega^n M = \ker(P_{n-1} \to P_{n-2})$, where $\cdots \to P_1 \to P_0 \to M \to 0$ is a minimal projective resolution of M.

Connected algebras behave somewhat like commutative local rings: for example, there is a version of Nakayama's Lemma, and this has the usual consequences.

Lemma 1.3. *If A is connected, and $M \in \mathrm{GrMod}(A)$ is bounded below, then $A_{\geq 1}M = M$ if and only if $M = 0$.*

Proposition 1.4. *If A is connected and $\underline{\mathrm{Ext}}^{n+1}_A(k,k) = 0$, then $\mathrm{projdim}(M) \leq n$ for all $M \in \mathrm{grmod}(A)$.*

Proposition 1.5. *If A is connected, then all projectives in $\mathrm{GrMod}(A)$ which are bounded below are free.*

A free module in $\mathrm{GrMod}(A)$ can be written as $A \otimes_k V$ where V is a graded vector space and $A \otimes_k V$ is given the tensor product grading.

Definition 1.6. A module $M \in \mathrm{GrMod}(A)$ has a **linear resolution** if it has a projective resolution of the form

$$\cdots \to A \otimes_k V_r \to \cdots \quad \cdots \to A \otimes_k V_1 \to A \otimes_k V_0 \to M \to 0 \tag{1-1}$$

in which each V_i is concentrated in degree i. We write $\mathrm{Lin}(A)$ and $\mathrm{lin}(A)$ for the full subcategories of $\mathrm{GrMod}(A)$ and $\mathrm{grmod}(A)$ consisting of the modules having linear resolutions.

If (1-1) is a minimal resolution of M, then $\underline{\mathrm{Ext}}^p_A(M,k) \cong V_p^*$ and $\mathrm{Tor}^A_p(k_A, M) \cong V_p$, so we have the following result.

Lemma 1.7. *Let $M \in \mathrm{GrMod}(A)$. The following conditions are equivalent:*

1. *M has a linear resolution;*
2. *the minimal projective resolution of M is linear;*
3. *$\mathrm{Tor}^A_p(k_A, M)_j = 0$ if $j \neq p$, for all p;*
4. *$\underline{\mathrm{Ext}}^p_A(M,k) = \underline{\mathrm{Ext}}^p_A(M,k)_{-p}$ for all p.*

If M has a linear resolution, then $H_M(t) = H_A(t).H_{\underline{\mathrm{Ext}}^*_A(M,k)}(-t)$.

Lemma 1.8. *Let $0 \to L[-1] \to M \to N \to 0$ be an exact sequence in $\mathrm{GrMod}(A)$.*

1. *If L and M have linear resolutions, so does N.*

2. *If M and N have linear resolutions, and L is generated in degree zero, then L has a linear resolution.*

Proof. We will apply Lemma 1.7 to the exact sequence

$$\cdots \to \operatorname{Tor}_n(k_A, M) \to \operatorname{Tor}_n(k_A, N) \to \operatorname{Tor}_{n-1}(k_A, L)[-1] \to \operatorname{Tor}_{n-1}(k_A, M) \to \cdots .$$

(1) Since L and M have linear resolutions, $\operatorname{Tor}_n(k_A, M)$ and $\operatorname{Tor}_{n-1}(k_A, L)[-1]$ are concentrated in degree n, whence so is $\operatorname{Tor}_n(k_A, N)$.

(2) By hypothesis, $\operatorname{Tor}_n(k_A, N)$ is concentrated in degree n and $\operatorname{Tor}_{n-1}(k_A, M)$ is concentrated in degree $n-1$, so the only degrees in which $\operatorname{Tor}_{n-1}(k_A, L)[-1]$ might be non-zero are $n-1$ and n. However, $L[-1]$ is generated in degree 1, so the $(n-1)^{\text{th}}$ term in its minimal projective resolution is zero in degree $\leq n-1$. Thus $\operatorname{Tor}_{n-1}(k_A, L)[-1]$ is concentrated in degree n, so L has a linear resolution. □

Remark 1.9. The category $\operatorname{Lin}(A)$ is rarely abelian. On the positive side $\operatorname{Lin}(A)$ has cokernels. To see this, suppose that $0 \to L \to M \to N \to 0$ is exact, and that L and M have linear resolutions. From the long exact sequence for $\operatorname{Tor}^A_*(k_A, -)$, one sees that $\operatorname{Tor}^A_p(k_A, N)_j = 0$ if $j \neq p-1$ or $j \neq p$. However, M (and hence N) is generated by its degree zero component so, if $\cdots \to Q_1 \to Q_0 \to N \to 0$ is a minimal projective resolution, then the degree j component of Q_p is zero whenever $j < p$; thus $\operatorname{Tor}^A_p(k_A, N)_{p-1} = 0$, whence N has a linear resolution by Lemma 1.7.

But $\operatorname{Lin}(A)$ does not have kernels: if $A = k[x]$, the kernel of the augmentation $\varepsilon : A \to k$ is Ax which is zero in degree zero, so cannot have a linear resolution. Although $Ax[1]$ *has* a linear resolution, the full subcategory of modules which are shifts of modules in $\operatorname{Lin}(A)$ is still not abelian: the kernel of $\binom{\varepsilon}{0} : A \oplus k \to k$ is $Ax \oplus k$, no shift of which is in $\operatorname{Lin}(A)$.

2. The Yoneda Product

In this section, R will initially denote an arbitrary ring, $\operatorname{Mod}(R)$ its category of left R-modules, and $\operatorname{mod}(R)$ its full subcategory of finitely generated modules.

Let X, Y, and Z be left R-modules. The **Yoneda product** is, by definition, a map

$$\operatorname{Ext}^p_R(Y, Z) \otimes \operatorname{Ext}^q_R(X, Y) \to \operatorname{Ext}^{p+q}_R(X, Z), \tag{2-1}$$

which is associative in the obvious sense. Thus $\operatorname{Ext}^*(Y, Y)$ becomes a graded algebra, with degree p-component $\operatorname{Ext}^p_R(Y, Y)$, and $\operatorname{Ext}^*_R(X, Y)$ becomes a graded left $\operatorname{Ext}^*_R(Y, Y)$-module.

The Yoneda product (2-1) may be defined as follows. Let $P_\bullet \to X$ and $Q_\bullet \to Y$ be projective resolutions. Suppose that $[\beta] \in \operatorname{Ext}^i_R(Y, Z)$ and $[\alpha] \in \operatorname{Ext}^j_R(X, Y)$ are represented by $\beta \in \operatorname{Hom}_R(Q_i, Z)$ and $\alpha \in \operatorname{Hom}_R(P_j, Y)$. There is a lift $\alpha_i : P_{i+j} \to Q_i$ of α, so $\beta \circ \alpha_i : P_{i+j} \to Z$ gives a class in $\operatorname{Ext}^{i+j}_R(X, Z)$. By definition $[\beta][\alpha] := [\beta \circ \alpha_i]$. The following picture illustrates what is happening:

$$\begin{array}{ccccccccccc}
P_{i+j} & \longrightarrow & \cdots & \longrightarrow & P_j & \longrightarrow & \cdots & \longrightarrow & X & \longrightarrow & 0 \\
{\scriptstyle\alpha_i}\big\downarrow & & & & {\scriptstyle\alpha_0}\big\downarrow & & & & & & \\
Q_i & \longrightarrow & \cdots & \longrightarrow & Q_0 & \longrightarrow & Y & \longrightarrow & 0 & & \\
{\scriptstyle\beta}\big\downarrow & & & & & & & & & & \\
Z & & & & & & & & & &
\end{array} \tag{2-2}$$

For an augmented k-algebra R, the Yoneda product on $\operatorname{Ext}^*_R(k,k)$ has a direct description using the syzygy functors. First, $\operatorname{Ext}^n_R(k,k) \cong \operatorname{Hom}_R(\Omega^n k, k)$ for each $n \geq 0$. Second, there is a natural map

$$\operatorname{Hom}_R(\Omega^i k, k) \times \operatorname{Hom}_R(\Omega^j k, k) \to \operatorname{Hom}_R(\Omega^{i+j} k, k),$$

namely $(\alpha, \beta) \mapsto \alpha \circ \Omega^i(\beta)$, which puts an algebra structure on

$$\bigoplus_{i \geq 0} \operatorname{Hom}_R(\Omega^i k, k) \cong \operatorname{Ext}^*_R(k,k).$$

It is easy to show that this agrees with the product defined earlier.

The above considerations apply to a connected graded algebra A. If X, Y, and Z are graded modules, then the Yoneda product respects the grading on each Ext-group; that is,

$$\operatorname{Ext}^p_A(Y,Z)_i \otimes \operatorname{Ext}^q_A(X,Y)_j \to \operatorname{Ext}^{p+q}_A(X,Z)_{i+j}. \tag{2-3}$$

Thus $\underline{\operatorname{Ext}}^*_A(Y,Y)$ becomes a bigraded algebra and $\underline{\operatorname{Ext}}^*_A(X,Y)$ becomes a bigraded left $\underline{\operatorname{Ext}}^*_A(Y,Y)$-module.

Lemma 2.1. *If A is a locally finite, connected, graded k-algebra, then there is an isomorphism of graded algebras*

$$\underline{\operatorname{Ext}}^*_A(k_A, k_A) \cong \underline{\operatorname{Ext}}^*_A(k,k)^{\mathrm{op}}.$$

Proof. (Zhang) Let ${}_A\mathcal{C}$ denote the full subcategory of $\operatorname{GrMod}(A)$ consisting of the locally finite modules M such that $M_n = 0$ for $n \ll 0$. Let $\mathcal{D}_A$ denote the full subcategory of $\operatorname{GrMod}(A^{\mathrm{op}})$ consisting of the locally finite modules M such that $M_n = 0$ for $n \gg 0$. The functor $\operatorname{Hom}_{\mathrm{Gr}}(-,k)$ is an equivalence of categories ${}_A\mathcal{C} \to \mathcal{D}_A^{\mathrm{op}}$, sending k to k_A.

The modules in the minimal projective resolution of k belong to ${}_A\mathcal{C}$, so ${}_A\mathcal{C}$ has enough projectives, and $\underline{\operatorname{Ext}}^*_{\mathcal{C}}(k,k) \cong \underline{\operatorname{Ext}}^*_A(k,k)$. Similarly, the modules in the minimal injective resolution of k_A belong to $\mathcal{D}_A$, so $\underline{\operatorname{Ext}}^*_{\mathcal{D}}(k_A,k_A) \cong \underline{\operatorname{Ext}}^*_A(k_A,k_A)$. The equivalence $\operatorname{Hom}_{\mathrm{Gr}}(-,k)$ ensures that $\underline{\operatorname{Ext}}^*_{\mathcal{C}}(k,k) \cong \underline{\operatorname{Ext}}^*_{\mathcal{D}^{\mathrm{op}}}(k_A,k_A)$; the result now follows from the fact that $\underline{\operatorname{Ext}}^*_{\mathcal{D}^{\mathrm{op}}}(k_A,k_A) \cong \underline{\operatorname{Ext}}^*_{\mathcal{D}}(k_A,k_A)^{\mathrm{op}}$. □

3. Frobenius Algebras

Definition 3.1. A finite dimensional k-algebra R is Frobenius if, as left R-modules, $R \cong R^*$, where $R^* = \operatorname{Hom}_k(R,k)$ is given the left module structure induced by the right regular action of R on itself.

A Frobenius algebra is characterized by the fact that there is a non-degenerate bilinear pairing (a Frobenius pairing) $(-,-) : R \times R \to k$ such that $(ab,c) = (a,bc)$ for all $a,b,c \in R$ (which shows that R is Frobenius if and only if R^{op} is) – the isomorphism ${}_RR \xrightarrow{\sim} R^*$ is given by $r \mapsto (-,r)$. For more details see [8, Section 60].

A Frobenius algebra is injective as a module over itself.

Lemma 3.2. *Let R be a finite dimensional, connected graded k-algebra such that $R_n \neq 0$ but $R_{\geq n+1} = 0$. The following are equivalent:*

1. *R is Frobenius;*
2. $\dim_k(R_n) = 1$ *and the map $(-,-) : R \times R \to k$, defined by*

$$(a,b) = \text{the component of } ab \text{ in } R_n,$$

is a Frobenius pairing;

3. *$R \cong R^*[-n]$ as graded left R-modules.*

Proof. First (2) implies (3) because the map $a \mapsto (-, a)$ is a degree zero left R-module map $R \to R^*[-n]$, injective by the non-degeneracy hypothesis, and hence an isomorphism by finite dimensionality. Clearly (3) implies (1), so it remains to prove that (1) implies (2).

Since R is graded, $R_{\geq 1}$ is a nilpotent ideal, whence k is the unique simple R-module by the connectedness hypothesis. Thus R is the only indecomposable projective, so by the Frobenius hypothesis (which implies that R is quasi-Frobenius), the socle of R is 1-dimensional [8, Theorem 58.12]. But R_n is in the socle, so $\dim_k(R_n) = 1$. Since R is injective, it is the injective envelope of its socle, whence R_n is essential in R. Therefore, if $0 \neq a \in R_i$, there exists $b \in R_{n-i}$ such that $0 \neq ab \in R_n$, whence the bilinear map $(-, -)$ is non-degenerate, as claimed. □

If R is as in Lemma 3.2, the connectedness hypothesis implies that there is a *unique* isomorphism $R \mapsto R^*[-n]$ up to scalar multiples, and hence that the non-degenerate pairing in (2) is unique up to scalar multiples.

If R is a Frobenius algebra, then R^* is isomorphic to R both as a left and as a right R-module *but not necessarily as an R-R-bimodule.* However, any bimodule isomorphic to R on both the left and the right is of a special form.

Notation . Let R be a ring and $\nu \in \operatorname{Aut}(R)$ an automorphism of R. We write ${}^{\nu}R^1$ for the R-R-bimodule which is R as an abelian group, endowed with the action

$$a.x.b = a^{\nu}xb \tag{3-1}$$

for $a, b \in R$ and $x \in {}^{\nu}R^1 = R$, where the right hand side of (3-1) is given by the usual multiplication in R. In (3-1), a^{ν} denotes the image of a under ν.

Lemma 3.3. *Let R be a Frobenius algebra with Frobenius pairing $(-, -)$. There exists $\nu \in \operatorname{Aut}(R)$, unique up to an inner automorphism, such that $R^* \cong {}^{\nu}R^1$. Moreover,*

1. *$(a, b) = (b^{\nu}, a)$ for all $a, b \in R$;*
2. *$(a, b) = (a^{\nu}, b^{\nu})$ for all $a, b \in R$;*
3. *if R is connected graded, then ν can be chosen so it preserves degree, and*
4. *if R_n is the socle of R, then $ab = b^{\nu}a$ whenever $a \in R_i$ and $b \in R_{n-i}$.*

Proof. Since ${}^{\nu}R^1 \cong {}^{\mu}R^1$ if and only if $\mu^{-1}\nu$ is an inner automorphism, up to an inner automorphism, ν does not depend on the choice of Frobenius pairing.

Since the pairing is non-degenerate, for a given $b \in R$, there is a unique element $b^{\nu} \in R$ such that $(b^{\nu}, -) = (-, b)$; this defines a k-linear map $\nu : R \to R$. Clearly $1^{\nu} = 1$. If $a, b, c \in R$, then

$$((bc)^{\nu}, a) = (a, bc) = (ab, c) = (c^{\nu}, ab) = (c^{\nu}a, b) = (b^{\nu}, c^{\nu}a) = (b^{\nu}c^{\nu}, a)$$

whence $\nu(bc) = \nu(b)\nu(c)$, showing that ν is an algebra homomorphism.

The map $\Psi : R \to R^*$ defined by $\Psi(a) = (a, -)$ satisfies

$$(a.\Psi(b).c)(x) = \Psi(b)(cxa) = (b, cxa) = (bcx, a) = (a^{\nu}bc, x) = \Psi(a^{\nu}bc)(x),$$

so is a bimodule map ${}^{\nu}R^1 \to R^*$, and hence an isomorphism, as claimed.

Part (2) is straightforward.

If R is connected graded, then (3) and (4) follow easily using the Frobenius pairing described in Lemma 3.2. □

The automorphism ν is called the symmetrizing automorphism. It measures the failure of a Frobenius algebra to be symmetric (R is symmetric if and only if ν is an inner automorphism). For example, if $R = \Lambda(k^n)$ is the exterior algebra, then on R_i, ν is multiplication by $(-1)^{i(n-1)}$; thus ν is the identity when n is odd, and is multiplication by $(-1)^i$ on R_i when n is even.

The automorphism $\nu^{-1} : R \to R$ is a left module isomorphism ${}^{\nu}R^1 \xrightarrow{\sim} R$.

Consider the functor $\nu_* = {}^{\nu}R^1 \otimes_R - : \mathrm{Mod}(R) \to \mathrm{Mod}(R)$. Clearly $\nu_*(R) \cong R$, however, if $\rho : R \to R$ is right multiplication by r, a little care is required to observe that $\nu_*(\rho) : R \to R$ is given by right multiplication by $\nu^{-1}(r)$.

It is not difficult to check that ν_* is naturally isomorphic to the functor induced by $\nu : R \to R$ which sends modules over the second copy of R to modules over the first copy of R by pulling back along ν; thus $\nu_* M$ is M as an abelian group, but the R-action is given by $x.m = x^{\nu}m$ for $x \in R$ and $m \in M$, where $x^{\nu}m$ is computed using the original action of R on M.

Proposition 3.4. *Let R be a Frobenius algebra with symmetrizing automorphism ν. Then the Auslander-Reiten translation $D \circ Tr$ applied to a finite dimensional module M is given by*

$$(D \circ Tr)M \cong \nu_*(\Omega^2 M),$$

where $\Omega^2 M$ denotes the second syzygy of M.

Proof. Let $0 \to \Omega^2 M \to P_1 \to P_0 \to M \to 0$ be the start of a minimal projective resolution of a left R-module M. Write $F := \mathrm{Hom}_k(\mathrm{Hom}_R(-, R), k)$. This is an exact covariant functor since R is injective as a left R-module. Hence we obtain an exact sequence

$$0 \to F(\Omega^2 M) \to FP_1 \to FP_0 \to FM \to 0.$$

By definition $F(\Omega^2 M) \cong (D \circ Tr)M$.

Now, consider F applied to a homomorphism $\rho : R \to R$ which is given by right multiplication by r. By the Frobenius property, $FR \cong R$. Identifying FR with R, then $F(\rho)$ is right multiplication by $\nu^{-1}(r)$, which by the comments prior to the proposition is $\nu_*(\rho)$. It follows that F is naturally equivalent to ν_*, which completes the proof. □

4. Gorenstein Algebras

Definition 4.1. An augmented k-algebra R is Gorenstein if $\mathrm{injdim}(R) = n < \infty$ and, as right R-modules,

$$\mathrm{Ext}^i_R(k, R) \cong \begin{cases} 0 & \text{if } i \neq n, \\ k_R & \text{if } i = n. \end{cases} \tag{4-1}$$

Definition 4.2. A noetherian ring R satisfies the Auslander condition if, for all $M \in \mathrm{mod}(R)$ and all $N \subset \mathrm{Ext}^j_R(M, R)$, $\mathrm{Ext}^i_R(N, R) = 0$ whenever $i < j$. If R is both Gorenstein and Auslander, we say it is Auslander-Gorenstein.

Warning: Our use of the terminology 'Auslander-Gorenstein' is non-standard. Usually, one requires only that R satisfy the Auslander condition and be of finite injective dimension.

A commutative ring R is usually said to be Gorenstein if $\mathrm{injdim}(R_{\mathfrak{m}}) < \infty$ for all maximal ideals $\mathfrak{m}$. For a commutative local ring, the condition (4-1) is equivalent to the requirement that $\operatorname{injdim}(R) < \infty$ [19, Theorem 18.1]. A commutative Gorenstein ring satisfies the Auslander condition but, for a general augmented algebra, the condition $\operatorname{injdim}(R) < \infty$ need not imply the Auslander condition.

Nevertheless, the situation for non-commutative rings is similar to the commutative case. For example, Stafford and Zhang [33] have shown for a connected, graded, noetherian algebra which is finite over its center, finite injective dimension implies the Auslander condition (and also the Cohen-Macaulay property as defined in (8.4)). Furthermore, if R is left noetherian, and $M \in \operatorname{mod}(R)$, then $E \otimes_R M \cong \operatorname{Hom}_R(\operatorname{Hom}_R(M, R), E)$ whenever E is an injective right R-module, whence $\operatorname{Tor}_p^R(E, M) \cong \operatorname{Hom}_R(\operatorname{Ext}_R^p(M, R), E)$; the argument in [10, page 40] then shows that if $0 \to R_R \to E^\bullet$ is a minimal injective resolution, then R satisfies the Auslander condition if and only if $\operatorname{flat.dim}(E^i) \leq i$ for all i.

There is a nice connection between Gorenstein and Frobenius algebras which is a consequence of the following modification of a result of J. Zhang [37]; I am grateful for his allowing me to use his unpublished ideas in its proof.

Theorem 4.3. *Let R be a left noetherian, augmented k-algebra. Suppose that R is Gorenstein and* $\operatorname{gldim}(R) = n$. *Then*

1. $\operatorname{Ext}_R^n(k, k) \cong k$;
2. *for each $0 \leq p \leq n$, there is an isomorphism of functors*

$$\eta_p : \operatorname{Ext}_R^p(-, k) \to \operatorname{Ext}_R^{n-p}(k, -)^* \tag{4-2}$$

on $\operatorname{mod}(R)$;

3. *if $M \in \operatorname{mod}(R)$, then the natural equivalence in* (4-2) *is implemented by the Yoneda product*

$$\operatorname{Ext}_R^p(M, k) \times \operatorname{Ext}_R^{n-p}(k, M) \to \operatorname{Ext}_R^n(k, k).$$

That is, $\eta_p([\alpha])([\beta]) = [\alpha][\beta]$ whenever $[\alpha] \in \operatorname{Ext}_R^p(M, k)$ and $[\beta] \in \operatorname{Ext}_R^{n-p}(k, M)$.

Proof. (1) First, $\operatorname{Ext}_R^n(k, k) \neq 0$ by Proposition 1.4 and the hypothesis that $\operatorname{gldim}(R) = n$. Let

$$0 \to P_n \to \cdots \qquad \cdots \to P_0 \to k \to 0 \tag{4-3}$$

be a minimal resolution of k. By the Gorenstein hypothesis,

$$0 \to \operatorname{Hom}_R(P_0, R) \to \cdots \qquad \cdots \to \operatorname{Hom}_R(P_n, R) \to 0 \tag{4-4}$$

is exact except at the n^{th} position where the homology is $\operatorname{Ext}_R^n(k, R) \cong k_R$. Hence (4-4) is a projective resolution of k_R, which is minimal since (4-3) is. The minimality of (4-4) ensures that $\operatorname{Hom}_R(P_n, R) \cong R$, whence $P_n \cong R$. Therefore, applying $\operatorname{Hom}_R(-, k)$ to (4-3), $\operatorname{Ext}_R^n(k, k) \cong k$.

(2) and (3) We identify $\operatorname{Ext}_R^n(k, k)$ with k, and fix $0 \leq p \leq n$. For each $M \in \operatorname{mod}(R)$, the Yoneda product gives a map

$$\eta_M : \operatorname{Ext}_R^p(M, k) \to \operatorname{Ext}_R^{n-p}(k, M)^*,$$

namely $\eta_M([\alpha])([\beta]) = [\alpha][\beta]$. We will show that this yields a natural transformation $\eta : \operatorname{Ext}_R^p(-, k) \to \operatorname{Ext}_R^{n-p}(k, -)^*$.

Let $f : M \to N$ be an R-module map, and consider the diagram

$$\begin{array}{ccc} \operatorname{Ext}^p_R(N,k) & \xrightarrow{\eta_N} & \operatorname{Ext}^{n-p}_R(k,N)^* \\ {\scriptstyle \operatorname{Ext}^p_R(f,k)}\downarrow & & \downarrow{\scriptstyle \operatorname{Ext}^{n-p}_R(k,f)^*} \\ \operatorname{Ext}^p_R(M,k) & \xrightarrow[\eta_M]{} & \operatorname{Ext}^{n-p}_R(k,M)^* \end{array} \tag{4-5}$$

Let $[\alpha] \in \operatorname{Ext}^p_R(N,k)$ and $[\beta] \in \operatorname{Ext}^{n-p}_R(k,M)$. By [6, page 114], the induced maps $\operatorname{Ext}^p(f,k) : \operatorname{Ext}^p_R(N,k) \to \operatorname{Ext}^p_R(M,k)$ and $\operatorname{Ext}^{n-p}(k,f) : \operatorname{Ext}^{n-p}_R(k,M) \to \operatorname{Ext}^{n-p}_R(k,N)$ are given by $[\alpha] \mapsto [\alpha][f]$ and $[\beta] \mapsto [f][\beta]$. Hence, going clockwise round (4-5), the image of $[\alpha]$ sends $[\beta]$ to

$$\eta_N([\alpha])([f][\beta]) = [\alpha]([f][\beta]),$$

and going counter-clockwise round (4-5), the image of $[\alpha]$ sends $[\beta]$ to

$$\eta_M([\alpha][f])([\beta]) = ([\alpha][f])[\beta];$$

by the associativity of the Yoneda product, these agree, so the diagram commutes. Thus the various η_M produce a natural transformation η_p, as in (4-2).

To show that each η_p is a natural isomorphism, we first consider the case $p = 0$. It is easy to check that $(\eta_0)_R : \operatorname{Hom}_R(R,k) \to \operatorname{Ext}^n_R(k,R)^*$ is an isomorphism, whence so is $(\eta_0)_M$ for all $M \in \operatorname{mod}(R)$: to see this consider a presentation $F_1 \to F_0 \to M \to 0$ with F_1 and F_0 finitely generated free R-modules. Thus η_0 is a natural equivalence.

The long exact sequence for $\operatorname{Ext}^{n-*}_R(k,-)^*$ shows that $\{\operatorname{Ext}^{n-i}_R(k,-)^* \mid i \geq 0\}$ is a δ-functor. Furthermore, the Gorenstein hypothesis ensures that $\operatorname{Ext}^{n-i}_R(k,-)^*$ is co-effacable for each $i > 0$, so these give a universal δ-functor [11, Chapter III, Theorem 1.3A]. But $\{\operatorname{Ext}^i_R(-,k) \mid i \geq 0\}$ is also a universal δ-functor, so by the uniqueness of the lifting(s) of η_0 (and η_0^{-1}) in the definition of a universal δ-functor [11], each η_p is a natural isomorphism. □

Corollary 4.4. *Let R be a left noetherian, augmented k-algebra, and suppose that R is Gorenstein and that* $\operatorname{gldim}(R) = n$. *Then*

1. $\operatorname{Ext}^*_R(k,k)$ *is Frobenius;*
2. *if ν is a symmetrizing automorphism for* $\operatorname{Ext}^*_R(k,k)$ *preserving degree, and* $M \in \operatorname{Mod}(R)$, *then the map* $\eta_M : \operatorname{Ext}^*_R(M,k) \to \operatorname{Ext}^{n-*}_R(k,M)^*$, *defined by* (4-2), *induces an isomorphism*

$$\nu_* \operatorname{Ext}^*_R(M,k) \xrightarrow{\sim} \operatorname{Ext}^{n-*}_R(k,M)^*$$

of left $\operatorname{Ext}^*_R(k,k)$*-modules.*

Proof. (1) The Yoneda product on $\operatorname{Ext}^*_R(k,k)$ gives a map

$$(-,-) : \operatorname{Ext}^p_R(k,k) \times \operatorname{Ext}^{n-p}_R(k,k) \to \operatorname{Ext}^n_R(k,k),$$

which we can extend to $\operatorname{Ext}^*_R(k,k) \times \operatorname{Ext}^*_R(k,k) \to \operatorname{Ext}^n_R(k,k)$ by insisting that $(a,b) = 0$ whenever a and b are homogeneous elements such that $\deg(a) + \deg(b) \neq n$. By the associativity of multiplication, $(ab,c) = (a,bc)$ for all a,b,c. The non-degeneracy of $(-,-)$ follows from (4-2).

(2) Let $[\alpha] \in \nu_* \operatorname{Ext}^*_R(M,k)$, $[\beta] \in \operatorname{Ext}^*_R(k,M)$, and $[\xi] \in \operatorname{Ext}^*_R(k,k)$ be homogeneous elements such that $\deg([\alpha]) + \deg([\beta]) + \deg([\xi]) = n$. Then

$$\begin{aligned}\eta([\xi].[\alpha])([\beta]) &= \eta([\xi]^\nu[\alpha])([\beta]) \\ &= [\xi]^\nu[\alpha][\beta] \\ &= [\alpha][\beta][\xi] \qquad \text{by Lemma 3.3,} \\ &= ([\xi].\eta([\alpha])([\beta]),\end{aligned}$$

whence η is a left $\operatorname{Ext}^*_R(k,k)$-module homomorphism. □

Specializing the preceeding results to a connected graded algebra, we recover the following result of Zhang.

Theorem 4.5. [37] *Let A be a left noetherian connected graded k-algebra of global dimension n. Suppose further that A is Gorenstein and that $\underline{\operatorname{Ext}}^n_A(k,A) \cong k_A[\ell]$. Then for each $0 \le i \le n$, there is an isomorphism of functors*

$$\eta_i : \underline{\operatorname{Ext}}^i_A(-,k) \to \underline{\operatorname{Ext}}^{n-i}(k[\ell],-)^*.$$

Corollary 4.6. *Let A be a left noetherian connected graded k-algebra of global dimension n. Suppose further that A is Gorenstein and that $\underline{\operatorname{Ext}}^n_A(k,A) \cong k_A[n]$. Then $M \in \operatorname{GrMod}(A)$ has a linear resolution if and only if $\underline{\operatorname{Ext}}^p_A(k,M)_q = 0$ whenever $p+q \neq 0$.*

Proof. By Theorem 4.5, $\underline{\operatorname{Ext}}^i_A(M,k)_j \cong \left(\underline{\operatorname{Ext}}^{n-i}_A(k[n],M)_{-j}\right)^*$, so by Lemma 1.7, M has a linear resolution if and only if $\underline{\operatorname{Ext}}^{n-i}_A(k[n],M)_{-j} = 0$ whenever $i+j \neq 0$. But this equals $\underline{\operatorname{Ext}}^{n-i}_A(k,M)_{-n-j} = 0$, so the result follows. □

5. Koszul Algebras

Definition 5.1. A graded k-algebra is quadratic if $A = T(V)/(R)$ where

- V is a finite dimensional k-vector space, concentrated in degree 1,
- $T(V)$ is the tensor algebra on V, with the induced grading, and
- (R) is the ideal generated by a subspace $R \subset V \otimes V$.

The dual of such a quadratic algebra is $A^! := T(V^*)/(R^\perp)$, where

$$R^\perp = \{\lambda \in V^* \otimes V^* \mid \lambda(r) = 0 \text{ for all } r \in R\}.$$

We identify $(V \otimes V)^*$ with $V^* \otimes V^*$ by defining $(\alpha \otimes \beta)(u \otimes v) = \alpha(u)\beta(v)$ for $\alpha, \beta \in V^*$ and $u, v \in V$ (not all authors adopt the same convention).

From now on all algebras will be quadratic (and therefore connected).

Proposition 5.2. [18] *Let A be a quadratic k-algebra. Then the subalgebra of $\underline{\operatorname{Ext}}^*_A(k,k)$ generated by $\underline{\operatorname{Ext}}^1_A(k,k)$ equals $\bigoplus_{p \ge 0} \underline{\operatorname{Ext}}^p_A(k,k)_{-p}$, and is isomorphic to $A^!$.*

Remark 5.3. If in the definition of $A^!$ we identify $(V \otimes V)^*$ with $V^* \otimes V^*$ by defining $(\alpha \otimes \beta)(u \otimes v) = \alpha(v)\beta(u)$, then $\underline{\operatorname{Ext}}^*_A(k,k)$ is isomorphic to $(A^!)^{\mathrm{op}}$.

Definition 5.4. A quadratic algebra A is Koszul if k has a linear resolution.

Definition 5.5. Let A be a quadratic algebra. The Koszul complex is

$$K(A) := A \otimes (A^!)^*,$$

where the n^{th} term is $K_n(A) = A \otimes (A^!_n)^*$, which is naturally a subspace of $A \otimes V^{\otimes n}$, and the differential

$$\partial_n : K_n(A) \to K_{n-1}(A)$$

is the restriction of the map $A \otimes V^{\otimes n} \to A \otimes V^{\otimes n-1}$ defined by

$$a \otimes v_1 \otimes \cdots \otimes v_n \mapsto av_1 \otimes v_2 \otimes \cdots \otimes v_n.$$

Each K_n is a graded free left A-module with the tensor product grading, where $\deg(A^!_n)^* = n$. Furthermore, each ∂_n is a left A-module homomorphism of degree zero.

Since $K_0(A) = A \otimes k$, there is an augmentation map $\varepsilon : K_0(A) \to k$.

Lemma 5.6. *The augmented Koszul complex really is a complex in* $\mathrm{GrMod}(A)$.

Proof. Since $(A^!_n)^* \subset R \otimes V^{\otimes n-2}$, and the image of R in A is zero, $\partial^2 = 0$, as required. □

Remark 5.7. If $\{x_\lambda\}$ is a basis for A_1 and $\{\xi_\lambda\}$ its dual basis in A_1^*, then the element $e := \sum_\lambda x_\lambda \otimes \xi_\lambda$ is independent of the choice of basis. Since $K(A) = A \otimes (A^!)^*$ is a right $A \otimes A^!$-module, e acts on it from the right, and as such is a left A-module homomorphism. It is easy to show that this action of e is the same as the action of the differential ∂.

Remark 5.8. The dual of the surjection $(V^*)^{\otimes n} \equiv (V^{\otimes n})^* \to A^!_n$ is an inclusion $(A^!_n)^* \to V^{\otimes n}$, the image of which is

$$\bigcap_{i=0}^{n-2} V^{\otimes i} \otimes R \otimes V^{\otimes n-i-2}.$$

Theorem 5.9. *Let A be a quadratic algebra. The following are equivalent:*

1. *A is Koszul;*
2. *A^{op} is Koszul;*
3. *$A^!$ is Koszul;*
4. *$\underline{\mathrm{Ext}}^*_A(k,k) \cong (A^!)$ as graded k-algebras;*
5. *the augmented Koszul complex is a minimal resolution of k;*
6. *$\underline{\mathrm{Ext}}^p_A(k,k)_j = 0$ if $p + j \neq 0$.*

Further, if A is Koszul, then

$$H_A(t) H_{A^!}(-t) = 1. \tag{5-1}$$

Proof. (1) ⇔ (2) By considering $\mathrm{Tor}^A_*(k_A, k)$, it follows from Lemma 1.7 that k_A has a linear resolution if and only if k does.

(1) ⇔ (3) Certainly A has a linear resolution, so this follows from Theorem 6.3(4) below (the proof of which is independent of the fact that $A^!$ is Koszul) since, in the notation there, $A^\dagger = {}_{A^!}k$.

(1) ⇔ (6) This is the equivalence of statements (1) and (4) in Lemma 1.7.

(4) ⇔ (5) The hypothesis ensures that k has a linear minimal resolution. Since there is a morphism from that resolution to the Koszul complex, comparing homology of the two complexes, it follows that that morphism is an isomorphism.

(4) ⇔ (6) This is Proposition 5.2.

(5) $\Rightarrow$ (6) If the Koszul complex is exact, then it is a linear resolution of k.
Equation (5-1) is a restatement of the final remark in Lemma 1.7. ☐

Proposition 5.10. *Let A be a Koszul algebra of finite global dimension. Then A is Gorenstein if and only if $A^!$ is Frobenius.*

Proof. Since $\mathrm{gldim}(A) < \infty$, $A^!$ is finite dimensional (1.4). Let n denote the global dimension of A.

The groups $\underline{\mathrm{Ext}}^i_A(k, A)$ are the homology groups of the complex obtained by applying $\underline{\mathrm{Hom}}_A(-, A)$ to the Koszul complex for A. That is, they are the homology groups of the complex

$$0 \to A \xrightarrow{d} A^!_1 \otimes A \xrightarrow{d} \cdots \xrightarrow{d} A^!_n \otimes A \to 0 \tag{5-2}$$

of right A-modules, where the differential d is left multiplication by $\sum_\lambda \xi_\lambda \otimes x_\lambda$. Therefore A is Gorenstein if and only if (5-2) is exact except at the final position where its homology is $k_A[n]$. But A^{op} is Koszul, so k_A has a minimal resolution given by the Koszul complex

$$0 \to (A^!_n)^* \otimes A \xrightarrow{\delta} \cdots \xrightarrow{\delta} (A^!_1)^* \otimes A \xrightarrow{\delta} A \to k_A \to 0 \tag{5-3}$$

where δ is left multiplication by $\sum_\lambda \xi_\lambda \otimes x_\lambda$. Thus A is Gorenstein if and only if (5-2) and (5-3) are isomorphic as complexes of right A-modules.

Thus A is Gorenstein if and only if there is an isomorphism $\Phi : A^! \to (A^!)^*[-n]$ of graded vector spaces such that $\Phi \otimes \mathbb{1}_A$ is an isomorphism of complexes; that is, such that $\delta \circ (\Phi \otimes \mathbb{1}) = (\Phi \otimes \mathbb{1}) \circ d$. Given the above descriptions of d and δ, if $\alpha \otimes a \in A^!_i \otimes A$, then

$$\big(\delta \circ (\Phi \otimes \mathbb{1})\big)(\alpha \otimes a) = \sum_\lambda \xi_\lambda \Phi(\alpha) \otimes x_\lambda a$$

and

$$\big((\Phi \otimes \mathbb{1}) \otimes d\big)(\alpha \otimes a) = \sum_\lambda \Phi(\xi_\lambda \alpha) \otimes x_\lambda a.$$

Hence A is Gorenstein if and only if there exists an isomorphism $\Phi : A^! \to (A^!)^*[-n]$ of graded vector spaces such that $\xi_\lambda \Phi(\alpha) = \Phi(\xi_\lambda \alpha)$ for all $\alpha \in A^!$ and all λ. But this is precisely the requirement that Φ be a left $A^!$-module isomorphism so, by Lemma 3.2, the existence of such a Φ is equivalent to the condition that $A^!$ is Frobenius. ☐

Theorem 5.11. (J. Zhang) *Let A be a graded k-algebra. Suppose that A is noetherian, Gorenstein,* $\mathrm{gldim}(A) = n$*, and $H_A(t) = (1-t)^{-n}$. Then A is Koszul.*

Proof. Let $0 \to P_n \to \cdots \to P_0 \to k \to 0$ be a minimal projective resolution of k. Write $P_i = A \otimes_k V_i$ and $P_i^\vee = \underline{\mathrm{Hom}}_A(P_i, A) \cong V_i^* \otimes A$. Then

$$H_A(t)^{-1} = \sum_{i=0}^{n} (-1)^i H_{V_i}(t). \tag{5-4}$$

Let a_i (resp. b_i) denote the least (resp. largest) degree of a component of V_i. Since the resolution is minimal $0 = b_0 = a_0 < a_1 < \cdots < a_n$. In particular, $a_n \geq n$.

Since A is Gorenstein, $0 \to P_0^\vee \to \cdots \to P_n^\vee \to k_A[\ell] \to 0$ is a projective resolution of $k_A[\ell]$ for some integer ℓ. The minimality of $P_\bullet$ ensures that $P_\bullet^\vee$ is also minimal, so $P_n^\vee \cong A[\ell]$, whence $\ell = a_n = b_n \geq n$, and $-b_n < \cdots < -b_1 < -b_0 = 0$.

By hypothesis, the highest degree term occuring in $H_A(t)^{-1}$ is t^n. Since $P_n \cong A[-\ell]$ contributes a term $(-1)^n t^\ell$ to the sum (5-4), which cannot be cancelled out by any other terms in the sum because $\ell = b_n > b_i$ for all $i \neq n$, it follows that $\ell \leq n$. Therefore $\ell = n$, and $a_i = i = b_i$ for all i. In other words, $P_\bullet$ is a linear resolution of k. □

We finish this section with two results which are useful for recognizing Koszul algebras.

Theorem 5.12. [28] *Let A be a Koszul algebra, and suppose that $z \in A_2$ is a central regular element. Then*

1. *$B := A/(z)$ is Koszul, and*
2. *$A^! \cong B^!/(\omega)$ where $\omega \in B^!_2$ is central and regular.*

Theorem 5.13. [15] *Let A be a Koszul algebra, and suppose that $z \in A_1$ is a normal regular element such that $B := A/(z)$ is Koszul. Let R and R' denote the quadratic relations for A and B respectively. Then A is Koszul if and only if the natural map $(A_1 \otimes R) \cap (R \otimes A_1) \to (B_1 \otimes R') \cap (R' \otimes B_1)$ is surjective.*

6. A duality for modules having a linear resolution

We will show that, for a Koszul algebra A, the functor $\underline{\mathrm{Ext}}^*_A(-,k)$ is a duality between $\mathrm{Lin}(A)$ and $\mathrm{Lin}(A^!)$.

Notation . Let R be an augmented k-algebra. We denote the functor $\mathrm{Ext}^*_R(-,k) : \mathrm{Mod}(R) \to \mathrm{Mod}(\mathrm{Ext}^*_R(k,k))$ by

$$M \mapsto M^\dagger := \mathrm{Ext}^*_R(M,k).$$

For example, $R^\dagger \cong k$ and $k^\dagger \cong \mathrm{Ext}^*_R(k,k)$. For connected graded algebras, $M \mapsto M^\dagger$ sends graded modules to graded modules.

Lemma 6.1. *Let A be a connected graded k-algebra. Suppose that L, M and N are graded A-modules having linear resolutions, and that $0 \to L[-1] \to M \to N \to 0$ is an exact sequence. Then there is an exact sequence of left $\underline{\mathrm{Ext}}^*_A(k,k)$-modules*

$$0 \to L^\dagger[-1] \to N^\dagger \to M^\dagger \to 0.$$

Proof. Applying $\underline{\mathrm{Hom}}_A(-,k)$ to the initial exact sequence gives an exact sequence

$$\begin{aligned} \cdots \to \underline{\mathrm{Ext}}^{p-1}_A(M,k)_{-p} \to \underline{\mathrm{Ext}}^{p-1}_A(L[-1],k)_{-p} \to \underline{\mathrm{Ext}}^{p}_A(N,k)_{-p} \to \\ \to \underline{\mathrm{Ext}}^{p}_A(M,k)_{-p} \to \underline{\mathrm{Ext}}^{p}_A(L[-1],k)_{-p} \to \cdots . \end{aligned} \tag{6-1}$$

However, since M and L have linear resolutions, the first and last terms in (6-1) are zero, whence we have an exact sequence

$$0 \to L^\dagger_{-p-1} \to N^\dagger_{-p} \to M^\dagger_{-p} \to 0.$$

Summing over all p gives the result. □

For quadratic algebras the left action of $A^!$ on $M^\dagger$ has a simple description in terms of the differential on the minimal resolution of M.

Proposition 6.2. *Let A be a quadratic algebra and $(A \otimes V_\bullet, d)$ a minimal resolution of $M \in \operatorname{GrMod}(A)$. There is a commutative diagram*

$$\begin{array}{ccc} A^!_1 \otimes V_i^* & \xrightarrow{d^*} & V_{i+1}^* \\ \big\downarrow & & \big\downarrow \\ \underline{\operatorname{Ext}}^1_A(k,k) \otimes \underline{\operatorname{Ext}}^i_A(M,k) & \xrightarrow{\rho} & \underline{\operatorname{Ext}}^{i+1}_A(M,k) \end{array} \tag{6-2}$$

where ρ denotes the Yoneda product and the vertical maps are induced by the isomorphism $V_i^* \to \underline{\operatorname{Ext}}^i_A(M,k)$, $\alpha \mapsto [\varepsilon \otimes \alpha]$.

Proof. Let $\xi \in A^!_1$ and $\alpha \in V_i^*$. The image of $\xi \otimes \alpha$ going in the counter-clockwise direction is $\rho(\xi \otimes [\alpha])$, namely $[(\varepsilon \otimes \xi) \circ \alpha_1]$ where α_1 is the lift of α in the following commutative diagram:

$$\begin{array}{ccccc} V_{i+1} & \xrightarrow{d} & A_1 \otimes V_i & & \\ {\scriptstyle \alpha_1}\big\downarrow & & \big\downarrow{\scriptstyle \alpha_0 = 1 \otimes \alpha} & & \\ A_0 \otimes A_1 & \xrightarrow[\partial]{} & A_1 \otimes A_0 & \xrightarrow[\varepsilon \otimes 1]{} & k \\ {\scriptstyle \varepsilon \otimes \xi}\big\downarrow & & & & \\ k & & & & \end{array}$$

(We have ommitted those parts of the resolutions which are irrelevant to the present proof.) Since $\partial(1 \otimes a) = a \otimes 1$ for $1 \otimes a \in A \otimes A_1$, $\varepsilon \otimes \xi = (\xi \otimes \varepsilon) \circ \partial$. Therefore

$$\begin{aligned} \rho(\xi \otimes [\alpha]) &= [(\varepsilon \otimes \xi) \circ \alpha_1] \\ &= [(\xi \otimes \varepsilon) \circ \partial \circ \alpha_1] \\ &= [(\xi \otimes \varepsilon) \circ (1 \otimes \alpha) \circ d] \\ &= [(\xi \otimes \alpha) \circ d] \\ &= [d^*(\xi \otimes \alpha)]. \end{aligned}$$

However, $[d^*(\xi \otimes \alpha)]$ is the image of $\xi \otimes \alpha$ under the clockwise composition of maps, thus proving the commutativity of the diagram. □

Theorem 6.3. *Let A be a Koszul algebra, and let $M \in \operatorname{Lin}(A)$.*

1. *The minimal projective resolution of M is $A \otimes_k (M^\dagger_\bullet)^*$ with differential given by right multiplication by e (see Remark 5.7);*
2. *$H_M(t) = H_A(t) H_{M^\dagger}(-t)$;*
3. *If $A \otimes D \to A \otimes C \to M$ is the start of a minimal resolution of M, then the minimal resolution of $M^\dagger$ begins $A^! \otimes_k D^\perp \to A^! \otimes_k C^* \to M^\dagger$;*
4. *$M^\dagger$ has a linear resolution as a left $A^!$-module, and $M^{\dagger\dagger} \cong M$.*

Proof. (1) If L is a left $A^!$-module with structure map $\rho : A^! \otimes L \to L$, it is a triviality that $\rho(\xi \otimes \ell) = \sum_\lambda x_\lambda(\xi) \xi_\lambda . \ell = e.(\xi \otimes \ell)$. Hence the map ρ in (6-2) is left multiplication by e, and the dual of (6-2) is the commutative diagram

$$\begin{array}{ccc} V_{i+1} & \xrightarrow{d} & A_1 \otimes V_i \\ \big\downarrow & & \big\downarrow \\ (M^\dagger_{i+1})^* & \xrightarrow{e} & A_1 \otimes (M^\dagger_i)^* \end{array} \tag{6-3}$$

where the bottom map denotes *right* multiplication by e, and the vertical maps are isomorphisms. Hence the vertical maps induce an isomorphism of complexes of graded A-modules

$$\begin{array}{ccccccc} \cdots \longrightarrow & A \otimes V_{i+1} & \xrightarrow{d} & A \otimes V_i & \longrightarrow \cdots \\ & \downarrow & & \downarrow & \\ \cdots \longrightarrow & A \otimes (M^{\dagger}_{i+1})^* & \xrightarrow{e} & A \otimes (M^{\dagger}_i)^* & \longrightarrow \cdots \end{array} \tag{6-4}$$

Therefore $A \otimes (M^{\dagger}_{\bullet})^*$, with differential e, is a minimal resolution of M, as claimed.

(2) This is a trivial consequence of (1).

(3) From the minimal resolution of M, one sees that $M_0 = C$ and that there is an exact sequence $0 \to D \to A_1 \otimes C \to M_1 \to 0$. Hence $M_0^* = C^*$ and $M_1^* \cong D^{\perp} \subset (A_1 \otimes C)^* \cong A^!_1 \otimes C^*$. By (1) and (3), the minimal resolution of $M^{\dagger}$ begins $A^! \otimes M_1^* \to A^! \otimes M_0^* \to M^{\dagger} \to 0$, whence the result.

(4) We will show that the minimal resolution of $M^{\dagger}$ is given by the complex $(A^! \otimes M^*_{\bullet}, \partial)$, where ∂ is right multiplication by $\sum_{\lambda} \xi_{\lambda} \otimes x_{\lambda}$; that is, we will prove the exactness of

$$\cdots A^! \otimes (M_2)^* \to A^! \otimes (M_1)^* \to A^! \otimes (M_0)^* \to M^{\dagger} \to 0.$$

By part (3), the right hand end of this is the beginning of a minimal resolution of $M^{\dagger}$.

We consider the two spectral sequences associated to the bicomplex $(C^{\bullet\bullet}, d', d'')$, where $C^{pq} = A^!_p \otimes A^* \otimes M^{\dagger}_q$ and $d' : C^{pq} \to C_{p+1\,q}$ is right multiplication by $\sum(\xi_{\lambda} \otimes x_{\lambda} \otimes 1)$ and $d'' : C^{pq} \to C^{p\,q+1}$ is left multiplication by $\sum(1 \otimes x_{\lambda} \otimes \xi_{\lambda})$.

We begin with the first filtration. The p^{th} column is $A^!_p \otimes (A \otimes (M^{\dagger}_{\bullet})^*)^*$, where $A \otimes (M^{\dagger}_{\bullet})^*$ is the projective resolution of M in part (1), so the columns are exact except in the 0^{th} row. Therefore

$$'E_2^{pq} = H_I^p H_{II}^q (C^{\bullet\bullet}) = \begin{cases} H^p(A_{\bullet} \otimes M^*, \partial) & \text{if } q = 0, \\ 0 & \text{if } q \neq 0 \end{cases}$$

and $H^n(\text{Tot}(C^{\bullet\bullet})) \cong H^n(A^!_{\bullet} \otimes M^*, \partial)$. With respect to the second filtration,

$$''E_2^{pq} = H_{II}^q (H^p(A^!_{\bullet} \otimes A^*) \otimes M^{\dagger}_{\bullet}) = \begin{cases} H^q({}_A k \otimes M^{\dagger}_{\bullet}) & \text{if } p = 0, \\ 0 & \text{if } p \neq 0, \end{cases}$$

because $A^!_{\bullet} \otimes A^*$ with differential right multiplication by $\sum \xi_{\lambda} \otimes x_{\lambda}$ is the graded k-dual of the Koszul complex (which is a projective resolution of k_A). The differential on ${}_A k \otimes M^{\dagger}_{\bullet}$ induced by d'' is left multiplication by $\sum x_{\lambda} \otimes \xi_{\lambda}$; this is zero since ${}_A k$ is killed by x_{λ}. Therefore $H^q({}_A k \otimes M^{\dagger}_{\bullet}) = M^{\dagger}_q$.

Comparing the two spectral sequences $H^n(A^!_{\bullet} \otimes M^*) = H^n(\text{Tot}(C^{\bullet\bullet})) = M^{\dagger}_n$, which is what we needed to prove. □

Corollary 6.4. *If A is a Koszul algebra, then $\underline{\text{Ext}}^*_A(-, k) : \text{Lin}(A) \to \text{Lin}(A^!)$ is a duality.*

7. Koszul Duality

The functor $\underline{\text{Ext}}^*_A(-, k)$ is closely related to the functor in [5] which is used to establish an equivalence between certain derived categories over a Koszul algebra and its dual. The results in [5] are not used in the rest of this paper; this section is intended to place the methods we use in a broader context.

Warning: Our definition of $A^!$ in (5.1) is *not* the same as that given in [5, page 5]. Their $A^!$ is the *opposite* of ours. In what follows we will use $A^!$ the same way as we have been, so when we quote results from [5] we will need to replace their use of $A^!$ by $(A^!)^{\text{op}}$. (See Remark 5.3.)

Definition 7.1. Let B be an $\mathbb{N}$-graded k-algebra. If (M, ∂) is a complex of graded B-modules we will write $M = \oplus M^i_j$ where i denotes the position in the complex, and j denotes the degree for the B-module action; thus $\partial : M^i_j \to M^{i+1}_j$.

We define the following categories:

- $C(B)$ = the homotopy category of complexes of graded B-modules, morphisms being homotopy classes of maps of complexes;
- $C^+(B)$ = the full subcategory of $C(B)$ consisting of complexes $M^\bullet_\bullet$ such that $M^i_j = 0$ if $i \gg 0$ or $i + j \ll 0$;
- $C^-(B)$ = the full subcategory of $C(B)$ consisting of complexes $M^\bullet_\bullet$ such that $M^i_j = 0$ if $i \ll 0$ or $i + j \gg 0$;
- $D^+(B)$ and $D^-(B)$ are the quotient categories of $C^+(B)$ and $C^-(B)$ obtained by localizing at the quasi-isomorphisms (they are triangulated categories).

Theorem 7.2. [5] *If A is a Koszul algebra, then there is an equivalence of categories $D^+(A) \to D^-((A^!)^{\text{op}})$.*

The equivalence of categories is induced by the functor by $F : C^+(A) \to C^-((A^!)^{\text{op}})$ defined as follows: if $(M, \partial) \in C^+(A)$, then

$$FM = ((A^!)^{\text{op}} \otimes M, d)$$

where

$$d(a \otimes m) = a \otimes \partial m + (-1)^{i+j} \sum_\lambda \xi_\lambda a \otimes x_\lambda m, \tag{7-1}$$

whenever $a \otimes m \in (A^!)^{\text{op}} \otimes M^i_{i+j}$, and FM is viewed as a complex of left $(A^!)^{\text{op}}$-modules, which in position p is

$$(FM)^p = \bigoplus_{i+j=p} A^! \otimes M^i_j. \tag{7-2}$$

(In (7-1) the product $\xi_\lambda a$ is computed in $A^!$; in [5], it appears as $a\xi_\lambda$, using the product in $(A^!)^{\text{op}}$.)

There is a full embedding of $\text{GrMod}(A)$ in $D^+(A)$ sending a module M to the complex which is M in position zero, and zero elsewhere. When F is applied to a single module M, we obtain the complex

$$\cdots \to A^! \otimes M_p \to A^! \otimes M_{p+1} \to \cdots, \tag{7-3}$$

with differential being left multiplication by $\sum_\lambda \xi_\lambda \otimes x_\lambda$. Sometimes this is quasi-isomorphic to a single $A^!$-module.

Proposition 7.3. *Let A be a Koszul algebra of finite global dimension. Furthermore, suppose that A is Gorenstein with $\underline{\text{Ext}}^n_A(k, A) \cong k[n]$. If $M \in \text{GrMod}(A)$, then FM is*

1. *quasi-isomorphic to the complex, with zero differential, which in the p^{th} position is $\oplus \text{Ext}^j_A(k, M)_{p-j}$;*

2. *quasi-isomorphic to a single $A^!$-module if and only if $M[r] \in \operatorname{Lin}(A)$ for some r.*

Proof. Consider $\underline{\operatorname{Ext}}^*_A(k, M)$. This is computed by taking homology after applying $\underline{\operatorname{Hom}}_A(-, M)$ to $A \otimes (A^!_\bullet)^*$; that is, it is the homology of the complex

$$0 \to M \to A^!_1 \otimes M \to A^!_2 \otimes M \to \cdots, \tag{7-4}$$

with differential left multiplication by $\sum_\lambda \xi_\lambda \otimes x_\lambda$. Since $\deg(A^!_j) = -j$, it follows that $\underline{\operatorname{Ext}}^p(k, M)_q$ is a subquotient of $A^!_p \otimes M_{p+q}$. We can rearrange the complex (7-4) so it is precisely the complex (7-3). Hence, taking homology of $(FM)^\bullet$, which is the same thing as homology of (7-3), we obtain $\oplus_j \underline{\operatorname{Ext}}^j_A(k, M)_{p-j}$ in position p.

Therefore, FM is quasi-isomorphic to a single $A^!$-module if and only if, for some p,

$$\underline{\operatorname{Ext}}^*_A(k, M[p]) = \bigoplus_j \underline{\operatorname{Ext}}^j_A(k, M[p])_{-j}.$$

By Corollary 4.6, this is equivalent to the requirement that $M[p]$ have a linear resolution. □

8. Sklyanin Algebras

From now on we will assume that k is algebraically closed.

Definition 8.1. Let $n \geq 3$, E an elliptic curve over k, and $\sigma : E \to E$ the translation automorphism by a fixed point $\zeta \in \operatorname{Pic}(E) \cong E$. We also introduce the following data and notation:

- $\mathcal{L}$ is a line bundle on E of degree n;
- $V = \mathrm{H}^0(E, \mathcal{L})$, whence $V \otimes V = \mathrm{H}^0(E \times E, \mathcal{L} \boxtimes \mathcal{L})$;
- $\Delta_\sigma := \{(p, \sigma^{n-2}(p)) \mid p \in E\}$, a divisor on $E \times E$;
- $M =$ the fixed points of the involution $(p, q) \mapsto (\sigma^2(q), \sigma^{-2}(p))$;
- a divisor D on $E \times E$ is **allowable** if D is stable under the involution and M occurs in D with even multiplicity;
- $R := \{f \in V \otimes V \mid \operatorname{div}(f) = \Delta_\sigma + D \text{ with } D \text{ allowable}\}$.

The **n-dimensional Sklyanin algebra** is

$$A_n(E, \sigma) := T(V)/(R).$$

By the Riemann-Roch Theorem $\dim(V) = n$ and, since $\mathcal{L}$ is very ample, there is a natural embedding $E \to \mathbb{P}(V^*)$. We will always view E as a subvariety of this particular copy of $\mathbb{P}^{n-1}$; as such it is a degree n curve, meaning that every hyperplane meets E at n points counted with multiplicity.

The dependence of $A_n(E, \sigma)$ on the choice of the line bundle $\mathcal{L}$ is illusory because any two degree n line bundles are pullbacks of one another along an automorphism of E. Hence for each n there is a 2-dimensional family of Sklyanin algebras, one dimension coming from the choice of E, the other from the choice of ζ.

The 3- and 4-dimensional Sklyanin algebras can be defined more simply, either by generators and relations, or by simplifying the geometric method of Definition 8.1. Until one is familiar with the Sklyanin algebras it is a good idea to concentrate on the 3-dimensional case.

Example 8.2. [2] Suppose that $n = 3$. Then there are scalars a, b, c, depending on E and σ, such that $A_3(E, \sigma) \cong k[x, y, z]$ with defining relations

$$\begin{cases} ax^2 + byz + czy = 0, \\ ay^2 + bzx + cxz = 0, \\ az^2 + bxy + cyx = 0. \end{cases} \tag{8-1}$$

The algebra depends on the point $(a, b, c) \in \mathbb{P}^2$, but not all points of $\mathbb{P}^2$ arise from a choice of E and σ. We should think of the algebras corresponding to the missing points as degenerations of Sklyanin algebras. (The polynomial ring occurs when $(a, b, c) = (0, 1, -1)$; this corresponds to the case when σ is the identity.)

In this case, $E \subset \mathbb{P}^2$ is a smooth cubic curve, and the space of relations R consists of all $f \in V \otimes V$ which vanish on the shifted diagonal $\{(p, \sigma(p)) \mid p \in E\}$ in $E \times E$; here we are thinking of $V = A_1$ as linear forms on $\mathbb{P}^2 = \mathbb{P}(V^*)$, and of $V \otimes V$ as bilinear forms on $\mathbb{P}^2 \times \mathbb{P}^2$, which contains a copy of $E \times E$. In other words $R = \mathrm{H}^0(E \times E, (\mathcal{L} \boxtimes \mathcal{L})(-\Delta_\sigma))$.

Example 8.3. [27] Suppose that $n = 4$. Then there are scalars $(\alpha_1, \alpha_2, \alpha_3) \in k^3$ such that $A_4(E, \sigma) = k[x_0, x_1, x_2, x_3]$ with relations

$$\begin{array}{ll} x_0x_1 - x_1x_0 = \alpha_1(x_2x_3 + x_3x_2) & x_0x_1 + x_1x_0 = x_2x_3 - x_3x_2 \\ x_0x_2 - x_2x_0 = \alpha_2(x_3x_1 + x_1x_3) & x_0x_2 + x_2x_0 = x_3x_1 - x_1x_3 \\ x_0x_3 - x_3x_0 = \alpha_3(x_1x_2 + x_2x_1) & x_0x_3 + x_3x_0 = x_1x_2 - x_2x_1 \end{array} \tag{8-2}$$

The parameter $\underline{\alpha}$ lies on the surface $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_1\alpha_2\alpha_3 = 0$, but not all points of this surface arise from some (E, σ).

In this case, $E \subset \mathbb{P}^3$ is a smooth quartic curve. It is the intersection of two quadrics, so is contained in the pencil they generate. Exactly four members of this pencil are singular, the singular locus of each being a single point. We will label these points $\{e_0, e_1, e_2, e_3\}$. Let $\Gamma \subset \mathbb{P}^3 \times \mathbb{P}^3$ denote the union of $\{(p, \sigma^2(p)) \mid p \in E\}$ and the four points (e_i, e_i). Then $A_4(E, \sigma) \cong T(V)/(R)$ where $R = \{f \in V \otimes V \mid f|_\Gamma = 0\}$.

The Sklyanin algebras are remarkably well-behaved and share most good properties of the polynomial ring, except that they are highly non-commutative. Theorem 8.6 makes this precise.

Definition 8.4. Let A be a graded k-algebra of finite injective dimension. The grade of a non-zero A-module M is

$$j(M) := \min\{j \mid \mathrm{Ext}^j(M, A) \neq 0\}.$$

We say that A is Cohen-Macaulay if $\mathrm{GKdim}(M) + j(M) = \mathrm{GKdim}(A)$ for all $0 \neq M \in \mathrm{grmod}(A)$.

(Here $\mathrm{GKdim}(-)$ denotes Gelfand-Kirillov dimension, which is a non-commutative analogue of Krull dimension—if $H_M(t) = f(t)(1-t)^{-r}$, where $r \geq 0$, $f(t) \in \mathbb{Z}[t, t^{-1}]$, and $f(1) \neq 0$, then $\mathrm{GKdim}(M) = r$.)

Definition 8.5. A graded k-algebra, A say, is a quantum polynomial ring if it satisfies the following conditions:

- A is left and right noetherian;
- $H_A(t) = (1-t)^{-n}$ for some n;
- $\mathrm{gldim}(A) = n$ (we call this the dimension of A);
- A is Auslander-Gorenstein and Cohen-Macaulay.

A quantum polynomial ring is a domain [13, Theorem 4.8], a maximal order in its division ring of fractions [32, Theorem 2.10], a Koszul algebra (Theorem 5.11), and $\mathrm{Ext}^n_A(k, A) \cong k[n]$ (by the proof of Theorem 5.11).

Theorem 8.6. [35] *Let $A = A_n(E, \sigma)$. Then*

1. *A is a quantum polynomial ring of dimension n;*
2. *A is a finite module over its center if σ is of finite order (the converse is known to be true when $n = 3$ or 4).*

Remark 8.7. There are plenty of other quantum polynomial rings. Some popular ones are the coordinate rings of quantum affine space and coordinate rings of quantum matrices. The homogenized enveloping algebras in Section 12 are also quantum polynomial rings.

It follows from Theorem 8.6 and (5-1) that the dual of $A_n(E, \sigma)$ has Hilbert series $(1 + t)^n$. Before turning to these finite dimensional algebras which are the subject of this note, we need some information about modules over $A_n(E, \sigma)$.

9. Linear modules over Sklyanin algebras

Definition 9.1. Let A be a graded k-algebra. A module $M \in \mathrm{grmod}(A)$ is d-linear if

- it is cyclic, and
- $H_M(t) = (1 - t)^{-d}$.

A point module (resp. line module) is a linear module for which $d = 1$ (resp. $d = 2$).

The second condition in Definition 9.1 says that M has the same Hilbert series as the homogeneous coordinate ring of a linear subspace of projective space. There is a correspondence between linear modules and linear subspaces of $\mathbb{P}(A_1^*)$: a linear module, M say, determines the subspace $\mathcal{V}(\mathrm{Ann}_{A_1}(M_0)) \subset \mathbb{P}(A_1^*)$, the zero locus of $\{a \in A_1 \mid aM_0 = 0\}$. For example, a point module determines a point, and a line module determines a line, et cetera. It is also clear that if $M \to N$ is a surjection from a line module to a point module, then the corresponding point lies on the corresponding line. (The obvious generalization to other linear modules is true.)

For the polynomial ring $S(V)$ this correspondence sets up a bijection between isomorphism classes of linear modules and linear subspaces of $\mathbb{P}(V^*)$, the subspace $\mathcal{V}(x_1, \dots, x_d)$ corresponding to $S(V)/(x_1, \dots, x_d)$, where $x_1, \dots, x_d \in V$. Here, degree zero maps between linear modules correspond to inclusions of subspaces.

For the Sklyanin algebras linear submodules are of great importance; they play a role analogous to Verma modules (see [30] for example). A key result is to classify them. It turns out that the correspondence above sets up a bijection between isomorphism classes of linear modules and certain linear subspaces of $\mathbb{P}(A_1^*)$. Before describing which subspaces these are we mention the $n = 3$ case. If σ is not of order 3, then the point modules for $A_3(E, \sigma)$ are in bijection with the points of E and the line modules are in bijection with the lines in $\mathbb{P}^2$; the algebra A itself is a linear module which corresponds to the whole projective plane. (If σ is of order 3, then the point modules are in bijection with the points in $\mathbb{P}^2$.) We usually write $M(p)$ for the point module corresponding to the point $p \in E$ and $M(\ell)$ for the line module corresponding to the line ℓ. There is a non-zero map $M(\ell) \to M(p)$ if and only if $p \in \ell \cap E$.

Definition 9.2. A linear subspace $L \subset \mathbb{P}(V^*)$ is a secant to E if $\deg(E \cap L) = \dim(L) + 1$, where $E \cap L$ is the scheme-theoretic intersection of E and L. In this case, we say that L is spanned scheme theoretically by $E \cap L$.

Theorem 9.3. [35] [34] *If L is a secant to E, then $A/AL^\perp$ is a linear module, where $L^\perp \subset A_1$ denotes of the linear forms vanishing on L. Moreover,* $\mathrm{GKdim}(A/AL) = \dim(L) + 1$.

Notation . If $D \in \mathrm{Div}(E)$ is an effective divisor of degree $\leq n$, we write $M(D)$ for the corresponding linear module; that is, $M(D) \cong A/AL^\perp$ where L is the linear subspace of $\mathbb{P}(V^*)$ whose scheme theoretic intersection with E is D. Notice that $\mathrm{GKdim}(M(D)) = \deg(D)$.

Theorem 9.4. [35] [34] *Let D be an effective divisor of degree $\leq n-1$ on E, and let $z \in E$. Then there is an exact sequence of linear modules*

$$0 \to M(D^{\sigma^2} + (z^{\sigma^{2-n}}))[-1] \to M(D + (z)) \to M(D) \to 0.$$

Moreover, $\mathrm{Hom}_{Gr}(M(D + (z)), M(D)) \cong k$.

Example 9.5. Consider a point module $M(z)$, with $z \in E$. Since $M(z)_0 \cong k$, there is a surjective map $M(z) \to k$. By Theorem 9.4, the kernel of this is isomorphic to $M(z^{\sigma^{2-n}})[-1]$. Hence $M(z)_{\geq r}[-r] \cong M(z^{\sigma^{(2-n)r}})$ for all $r \geq 0$.

Theorem 9.4 also shows that if $n = 3$ and $y, z \in E$, then there is an exact sequence $0 \to M((y^{\sigma^2}) + (z^{\sigma^{-1}}))[-1] \to M((y) + (z)) \to M(z) \to 0$.

Warning. The modules $M(D)$ are not all the linear modules except when $n = 3$. For $n \geq 4$, we call the linear modules not of the form $M(D)$ exceptional. For $n = 4$, there are four exceptional linear modules, namely the point modules corresponding to the 4 points e_i in Example 8.3. With respect to the coordinates $x_0, \dots, x_3$, these points are $(1,0,0,0), \dots, (0,0,0,1)$, and the corresponding point modules are $A/Ax_1 + Ax_2 + Ax_3, \dots, A/Ax_0 + Ax_1 + Ax_2$. For $n \geq 5$ the exceptional linear modules are classified in [34], but we do not need that result here.

Theorem 9.4 is useful for proving homological properties of linear modules by induction. The next definition isolates the property which permits such induction arguments.

Definition 9.6. Let A be a quantum polynomial ring of dimension n. We say that A has the catenarity property for linear modules if, whenever N and N' are linear modules such that

- N is d-linear, with $0 < d < n$, and
- there is a non-injective, surjective map $f : N' \to N$ (for example, take $N' = A$),

then there exist linear modules $L \neq 0$ and M, and a short exact sequence $0 \to L[-1] \to M \xrightarrow{g} N \to 0$ and a factorization $f = gh$ for some $h \in \mathrm{Hom}_{\mathrm{Gr}}(N', M)$. A Hilbert series calculation shows that L and M are both $(d+1)$-linear.

Definition 9.7. Let A be a graded k-algebra. A module $M \in \mathrm{grmod}(A)$ is Cohen-Macaulay if $\underline{\mathrm{Ext}}^p_A(M, A) = 0$ whenever $p \neq j(M)$. If M is Cohen-Macaulay, we define the graded *right* A-module

$$M^\vee := \underline{\mathrm{Ext}}^{j(M)}_A(M, A)[-j(M)].$$

For a fixed integer $n \geq 0$, the functor $M \mapsto M^\vee$ is a duality between the left and right Cohen-Macaulay modules of projective dimension n [14, Proposition 1.10].

Lemma 9.8. [14, Lemma 1.12] *Let* $0 \to L[-1] \to M \to N \to 0$ *be an exact sequence in* $\operatorname{grmod}(A)$. *Suppose that* $j(N) = j(M) + 1$. *Then*

1. *if* L *and* M *are Cohen-Macaulay, so is* N*;*
2. *there is an exact sequence* $0 \to M^\vee[-1] \to L^\vee \to N^\vee \to 0$.

Proof. Write $p = j(M)$. The long exact sequence for $\operatorname{Ext}^*_A(-, A)$ gives a short exact sequence

$$0 \to \underline{\operatorname{Ext}}^p_A(M, A) \to \underline{\operatorname{Ext}}^p_A(L[-1], A) \to \underline{\operatorname{Ext}}^{p+1}_A(N, A) \to 0,$$

so the result follows by shifting degrees by $-p-1$. □

Proposition 9.9. *Let* A *be a quantum polynomial ring of dimension* n*, and suppose that* A *has the catenarity property for linear modules. If* N *is a* d*-linear module, then*

1. N *is a Cohen-Macaulay module with* $j(N) = n - d$*;*
2. N *has a linear resolution;*
3. $N^\vee$ *is a linear right* A*-module.*

Proof. We proceed by downward induction on d. If $d = n$, then $N = A$, and conditions (1)-(3) hold in this case. Now let $d < n$, and suppose the result is true for linear modules of GK-dimension $> d$.

By the catenarity hypothesis, there is an exact sequence $0 \to L[-1] \to M \to N \to 0$, with L and M both $(d+1)$-linear. By the induction hypothesis, L and M satisfy (1)-(3). Since A is Cohen-Macaulay, $j(N) = n - \operatorname{GKdim}(N) = n - d$. Hence by Lemma 9.8, N is Cohen-Macaulay. By Lemma 1.8, N has a linear resolution since L and M do.

It remains to prove (3). It follows from Theorem 6.3 that the minimal resolution of N is of the form

$$0 \to A[-n+d] \to A[-n+d+1]^{n-d} \to \cdots\cdots \to A[-1]^{n-d} \to A \to N \to 0.$$

Applying $\underline{\operatorname{Hom}}_A(-, A)$ to this complex gives a complex

$$0 \leftarrow A[n-d] \leftarrow A[n-d-1]^{n-d} \leftarrow \cdots\cdots \leftarrow A[1]^{n-d} \leftarrow A \leftarrow 0 \tag{9-1}$$

which has homology only at the $A[n-d]$ position, since N is Cohen-Macaulay of grade $n-d$; in other words, (9-1) is a projective resolution of $\underline{\operatorname{Ext}}^{n-d}_A(N, A)$. Therefore $\underline{\operatorname{Ext}}^{n-d}_A(N, A)$ is a cyclic right A-module, generated in degree $d - n$, with Hilbert series $t^{d-n}(1-t)^{-d}$. Shifting the generator to degree zero, we see that $\underline{\operatorname{Ext}}^{n-d}_A(N, A)[d-n]$ is a linear module as claimed. □

The notion of linear modules also applies to right modules. By [35, Proposition 4.1.1], $A_n(E, \sigma)^{\mathrm{op}} \cong A_n(E, \sigma^{-1})$, so there is a version of Theorem 9.4 for linear right modules too. Hence for each effective $D \in \operatorname{Div}(E)$ of degree $\leq n$, we will write $X(D)$ for the right linear module A/LA where L is the linear subspace of $\mathbb{P}(A_1^*)$ spanned by D. It is not immediately apparent that the right linear module $M(D)^\vee$ is one of those linear modules corresponding to a secant (i.e., it is not exceptional); it is though, and for each effective divisor D of degree $\leq n$ we define $D^\vee$ by the requirement that $M(D)^\vee \cong X(D^\vee)$.

Proposition 9.10. *If* $D \in \operatorname{Div}(E)$ *is effective of degree* $d \leq n$*, then* $D^\vee = D^{\sigma^{2(n-d)}}$.

Proof. We proceed by downward induction on d. Suppose that $d = n$. Then $M(D) \cong A/Ax$ where $0 \neq x \in A_1$ is such that $\mathcal{V}(x)$ meets E scheme-theoretically at D. A simple calculation shows that $\underline{\operatorname{Ext}}^1_A(A/Ax, A) \cong (A/xA)[1]$, so $X(D^\vee) \cong A/xA$, whence $D^\vee = D$. Hence the result holds when $d = n$.

Now suppose that $\deg(D) = d$, and the result holds for divisors of degree $> d$. By Theorem 9.4, there is an exact sequence

$$0 \to M(D^{\sigma^2} + (z)^{\sigma^{2-n}})[-1] \to M(D + (z)) \to M(D) \to 0$$

for some $z \in E$, and hence by Lemma 9.8, an exact sequence

$$0 \to X((D + (z))^\vee)[-1] \to X((D^{\sigma^2} + (z)^{\sigma^{2-n}})^\vee) \to X(D^\vee) \to 0.$$

By the induction hypothesis this is the same as

$$0 \to X(D^{\sigma^{2(n-d-1)}} + (z)^{\sigma^{2(n-d-1)}})[-1] \to X(D^{\sigma^{2(n-d)}} + (z)^{\sigma^{n-2d}}) \to X(D^\vee) \to 0.$$

Hence by Theorem 9.4 applied to right modules, using the fact that $A_n(E,\sigma)^{\mathrm{op}} \cong A_n(E,\sigma^{-1})$, the result follows. □

10. The duals of the Sklyanin algebras

Throughout this section we will write $B_n(E,\sigma) = A_n(E,\sigma)^!$. Actually, we will usually fix E, σ, and n, and just write B and A for the algebras.

Since $H_B(t) = (1+t)^n$, which is the Hilbert series of an exterior algebra, B is sometimes called an 'elliptic deformation of the exterior algebra'. Since B is connected graded, it is local. It follows from Proposition 5.10 that B is a Frobenius algebra, and from Proposition 1.4 that $\operatorname{gldim}(B) = \infty$, since $\operatorname{Ext}^*_B(k,k) \cong A$. Furthermore, the isomorphism $B_n(E,\sigma)^{\mathrm{op}} \cong B_n(E,\sigma^{-1})$ follows from the corresponding result for $A_n(E,\sigma)$.

Example 10.1. It follows from Example 8.2 that the dual of a 3-dimensional Sklyanin algebra can be presented as $B = k[x,y,z]$ with defining relations

$$\begin{array}{ll} cyz - bzy, & bx^2 - ayz, \\ czx - bxz, & by^2 - azx, \\ cxy - byx, & bz^2 - axy \end{array} \tag{10-1}$$

for suitable $a, b, c \in k$. It is easy to see that B has basis $\{1, x, y, z, x^2, y^2, z^2, xyz\}$ and that $x^3 \neq 0$, $xy^2 = xz^2 = y^2x = z^2x = 0$. Hence the symmetrizing automorphism ν sends x to x. By symmetry (that is, using the fact that $\mathbb{Z}_3$ acts as automorphisms of A, and therefore of B, cyclically permuting x, y and z) we also have $y^\nu = y$ and $z^\nu = z$. Thus $\nu = 1$, whence B is a symmetric algebra. In particular, the Auslander-Reiten translation is the same as Ω^2, the second syzygy functor.

The elliptic curve E lies in $\mathbb{P}(B_1)$, so the cone over it lies in B_1. Therefore it makes sense to speak of the left ideal of B generated by a point of E, or more generally of the left ideal generated by some linear subspace (such as a secant line to E) of $\mathbb{P}(B_1)$. We will often write Bp for the left ideal generated by the line in B_1 corresponding to a point $p \in E$. We will further abuse notation by saying that

$pq = 0$ in B for points $p, q \in E$ if $\xi\eta = 0$ whenever $\xi, \eta \in B_1$ are in the preimage of p and q.

The following result was brought to my attention by J.T. Stafford.

Lemma 10.2. *Let $A = T(V)/(R)$ be a quadratic algebra. Let $(\xi, \eta) \in V^* \times V^*$. Then $\xi\eta = 0$ in $A^!$ if and only if $f(\xi, \eta) = 0$ for all $f \in R$.*

Proof. By definition of $A^!$, $\xi\eta = 0$ if and only if $\xi \otimes \eta \in R^\perp$, whence the result. □

In particular, since the defining relations of $A_n(E, \sigma)$ vanish on Δ_σ, in B we have $p.p^{\sigma^{n-2}} = 0$ for all $p \in E$. In fact, if $n \neq 4$, then Δ_σ is the zero locus of R, so these are the *only* products in B_1 which are zero. If $n = 4$, there are also four elements $\xi_i \in B_1$ whose squares are zero, namely the preimages of the four points e_i defined in Example 8.3.

One consequence of these observations is that for $n = 3$ and $n = 4$, B can be defined geometrically as $T(V^*)/(R')$ where $R' \subset V^* \otimes V^*$ consists of those $\xi \otimes \eta$ such that the image of (ξ, η) in $\mathbb{P}(V^*) \times \mathbb{P}(V^*)$ belongs to Δ_σ (or $\Delta_\sigma \cup \{(e_i, e_i) \mid 1 \leq i \leq 4\}$ when $n = 4$). This seems an interesting way to define a finite dimensional algebra. (There are analogous definitions for many other finite dimensional graded algebras; for an arbitrary finite dimensional algebra R it seems possible that considerable information may be contained the subvariety of $\mathbb{P}(R) \times \mathbb{P}(R)$ consisting of those (ξ, η) such that $\xi\eta = 0$.)

The best understood (and most important) A-modules are the linear modules. With that in mind we focus attention on the corresponding B-modules, for which we introduce the following notation.

Notation . If $D \in \text{Div}(E)$ is effective of degree $\leq n$, we will write

$$L(D) := M(D)^\dagger = \underline{\text{Ext}}_A^*(M(D), k).$$

Definition 10.3. The **complexity** of a B-module N, denoted $c(N)$, is the least integer c such that there is a constant κ satisfying

$$\dim \text{Tor}_j^B(k_B, N) \leq \kappa . j^{c-1}$$

for all $j \geq 0$.

Proposition 10.4. *Let $D \in \text{Div}(E)$ be an effective divisor of degree $d \leq n$. Then*

1. *$L(D)$ is an indecomposable, cyclic B-module with Hilbert series $(1+t)^{n-d}$;*
2. *$L(D)$ has a linear resolution, and $c(L(D)) = d$;*
3. *if $z \in E$, then there is an exact sequence*

$$0 \to L(D^{\sigma^2} + (z^{\sigma^{2-n}}))[-1] \to L(D) \to L(D + (z)) \to 0.$$

Proof. (1) By Theorem 6.3, $L(D)$ has a minimal projective resolution, which in position p is $B \otimes M(D)_p$. In particular, $L(D)$ is cyclic. Any cyclic B-module is indecomposable (apply $\text{Hom}_k(-, k)$ and use the fact that B is a uniform module since it has a simple essential submodule). Since $H_{M(D)}(t) = (1-t)^{-d}$, it follows from Theorem 6.3(2) that $H_{L(D)}(t)$ is as claimed.

(2) This follows from Theorem 6.3(1), and the fact that the linear module $M(D)$ has Hilbert series $(1-t)^{-d}$.

(3) This follows from Theorem 9.4 and Lemma 6.1. □

Corollary 10.5. *The algebra B has wild representation type.*

Proof. For each $0 < \ell < n$, the effective divisors on E of degree ℓ are parametrized by the ℓ^{th} symmetric power of E which is a variety of dimension ℓ. Hence there is an ℓ-dimensional family of indecomposable cyclic B-modules of dimension $2^{n-\ell}$, namely the various $L(D)$. Thus, by Drozd's Theorem, B is wild. □

We now consider that piece of the Auslander-Reiten quiver for B which contains the modules $L(p)$, for $p \in E$. If $p, q \in E$, we will write $L(pq) = L((p)+(q))$ for the module corresponding to the secant line spanned by p and q.

Proposition 10.6. *Let $p \in E$.*

1. *$L(p)$ is an indecomposable, cyclic module with Hilbert series $(1+t)^{n-1}$.*
2. *There is an exact sequence $0 \to L(p^{\sigma^{2-n}})[-1] \to B \to L(p) \to 0$.*
3. *$\Omega^r L(p) \cong L(p^{\sigma^{(2-n)r}})[-r]$.*
4. *For each $q \in E$, there is an exact sequence*

$$0 \to L(p^{\sigma^2} q)[-1] \to L(p) \to L(pq^{\sigma^{n-2}}) \to 0.$$

Proof. Parts (1), (2) and (4) are all special cases of Proposition 10.4. Part (3) follows from (2) by induction. □

We now restrict our attention to the case $n = 3$, since that is the only case for which we have explicitly computed the symmetrizing automorphism ν (it is the identity when $n = 3$, by Example 10.1).

Proposition 10.7. *Suppose that $n = 3$, that $\sigma^3 \neq 1$. For each $p \in E$, the almost split sequence ending in $L(p)$ is*

$$0 \to L(p^{\sigma^{-2}})[1] \to K(p) \xrightarrow{\pi} L(p) \to 0, \qquad (10\text{-}2)$$

where

1. *the sequence consists of graded modules and degree zero maps, and*
2. *$K(p)$ is indecomposable with Hilbert series $t^{-1}(1+t)^3$.*

Proof. We observed in Example 10.1 that B is a symmetric algebra, whence $\tau = \Omega^2$. Hence, by Proposition 10.6, the almost split sequence ending in $L(p)$ begins with $L(p^{\sigma^{-2}})$. We now compute $\underline{\text{Ext}}^1_B(L(p), L(p^{\sigma^{-2}}))$.

The minimal projective resolution for $L(p)$ begins

$$\cdots \to B[-2] \xrightarrow{p^{\sigma^{-1}}} B[-1] \xrightarrow{p} B \to L(p) \to 0, \qquad (10\text{-}3)$$

so for $q \in E$, $\underline{\text{Ext}}^1_B(L(p), L(q))$ is the homology of the middle term in the sequence

$$L(q)[2] \xleftarrow{p^{\sigma^{-1}}} L(q)[1] \xleftarrow{p} L(q). \qquad (10\text{-}4)$$

The main point in the computation of this homology is that, for each $r \in E$, there is an exact sequence $0 \to L(q^{\sigma^2} r)[-1] \to L(q) \to L(q^{\sigma} r^{\sigma}) \to 0$. Hence $L(q)_1$ contains a non-zero element whose annihilator is the left ideal of B generated by the secant line spanned by q^{σ^2} and r. As r varies over E, each line in $\mathbb{P}(B_1)$ passing through q^{σ^2} occurs: there is a $\mathbb{P}^1$ of such lines, and these are in natural bijection with the 1-dimensional subspaces of $L(p)_1$ which they annihilate. In particular, $q^{\sigma^2}.L(q)_1 = 0$, and $\dim_k(r.L(q)_1) = 1$ if $q^{\sigma^2} \neq r \in E$.

The homology of (10-4) can now be computed case-by-case, giving

$$\underline{\mathrm{Ext}}^1_B(L(p),L(q))_{-1} = \begin{cases} 0 & \text{if } p \neq q^{\sigma}, \\ k & \text{if } p = q^{\sigma} \end{cases}$$

$$\underline{\mathrm{Ext}}^1_B(L(p),L(q))_{0} = \begin{cases} 0 & \text{if } p \neq q,\, q^{\sigma^3}, \\ k & \text{if } p = q \text{ or } p = q^{\sigma^3} \end{cases}$$

$$\underline{\mathrm{Ext}}^1_B(L(p),L(q))_{1} = \begin{cases} 0 & \text{if } p \neq q^{\sigma^2}, \\ k & \text{if } p = q^{\sigma^2}. \end{cases}$$

In particular,

$$\underline{\mathrm{Ext}}^1_B(L(p),L(p^{\sigma^{-2}})[1])_i = \begin{cases} 0 & \text{if } i \neq 0, \\ k & \text{if } i = 0. \end{cases}$$

Hence the almost split sequence ending in $L(p)$ can be realized as a sequence in $\mathrm{GrMod}(B)$, as described in the statement of the Proposition, and the middle term has Hilbert series $1,3,3,1$, beginning in degree -1.

It remains to show that K is indecomposable. Suppose that $K = Y \oplus Z$ with $Y \neq 0$. For brevity, write L for the image of $L(p^{\sigma^{-2}})[1]$ in K. First, notice that the socle of L is L_1, and socle of K is $L_1 \oplus K_2$. These are essential submodules of L and K respectively.

We may assume, interchanging the roles of Y and Z if necessary, that $\pi(Y) \not\subset L(p)_{\geq 1}$. But $L(p)_{\geq 1}$ is the unique maximal submodule of $L(p)$ so $\pi(Y) = L(p)$. If $\dim_k(Y) = 4$, then the map $\pi : Y \to L(p)$ is an isomorphism, so (10-2) splits; this is absurd, so we conclude that $\dim_k(Y) \geq 5$. Hence $Y \cap L \neq 0$ and, because L_1 is essential in L, $L_1 \subset Y$.

If $x \in K$, we write x_i for its degree i component. If $z \in Z$, then $z_0 \in L$, otherwise $\pi(Bz) = L(p)$, which contradicts the fact that $\dim_k(Z) \leq 3$. Suppose that $z \in Z$ with $z_{-1} \neq 0$. Then $p.z_{-1} \in L_0$ is non-zero since $pL_{-1} \neq 0$. Since $p.L(p)_0 = 0$, $0 \neq pz \in L_0 + L_1 + K_2$. Since $B_1.(L_1 + K_2) = 0$, and no non-zero element of L_0 is killed by all of B_1, it follows that $B_1.pz = L_1$. This contradicts the fact that $Y \cap Z = 0$, so we conclude that $z_{-1} = 0$ for all $z \in Z$.

By the previous paragraph, $Z \subset L_0 + K_1 + K_2$. Hence Y contains an element, $y = y_0 + y_1 + y_2$ say, with $y_0 \notin L$. Since $B_2.L(p)_0 = L(p)_2$, it follows that $B_2 y = B_2 y_0 \neq 0$. Thus $K_2 \subset Y$, whence Y contains $L_1 + K_2$, which is essential in K. This forces $Z = 0$. □

Thus, when $n = 3$, the $L(p)$ form a family of indecomposable cyclic modules, each of dimension 4, parametrized by the points of E; this family is stable under Auslander-Reiten translation, and this corresponds to translation on E by σ^{-2}. In particular, one sees that by choosing the point ζ in Definition 8.1 to be a point of finite order, one can arrange for the Auslander-Reiten translation to be periodic on the $L(p)$'s of any desired order.

Corollary 10.8. *Suppose that $n = 3$ and that σ is of finite order $s \neq 3$. Then $E/\langle\sigma\rangle$ is an elliptic curve, the points of which parametrize tubes for B of rank s, each such tube having at its mouth the modules $L(p)$ for the p's in a single coset of $\langle\sigma\rangle$.*

Remark 10.9. By Proposition 10.7, when σ is of infinite order, the orbit of $L(p)$ under the Auslander-Reiten translation is infinite but each term in it has the same dimension. I am grateful to Shiping Liu for telling me that Ringel [21, Problem 1, page 13] had asked whether this could happen; Liu himself and Schulz [17] gave an example showing that it can. What is interesting to us is that Liu and Schulz's example, like ours, is the Koszul dual of a quantum polynomial ring. They do not describe it in that way, so we briefly give the details.

Fix $0 \neq \rho \in k$, and consider $k[x,y,z]^!$ where $k[x,y,z]$ has defining relations

$$xy + \rho yx = zx + \rho xz = yz + \rho zy = 0.$$

Thus $k[x,y,z]$ is a quantum polynomial ring (essentially because it is an iterated Ore extension with basis $x^i y^j z^k$). It is an easy exercise to write down relations for $k[x,y,z]^!$, and see that there is a homomorphism $k[x,y,z]^! \to T$, where T is the example in [17]; by comparing dimensions this map is an isomorphism. Either by direct computation, or by Liu and Schulz's construction, T is a symmetric algebra, so the Auslander-Reiten translation functor is $D\,\mathrm{Tr} = \Omega^2$.

If $\rho^3 + 1 \neq 0$, then the point modules for $k[x,y,z]$ are parametrized by the three lines $\mathcal{V}(xyz)$ in $\mathbb{P}^2$. Let $\sigma : \mathcal{V}(xyz) \to \mathcal{V}(xyz)$ be the automorphism defined by by $(0,y,z)^\sigma = (0,y,-\rho z)$, $(x,0,z)^\sigma = (-\rho x, 0, z)$ and $(x,y,0)^\sigma = (x,-\rho y,0)$. For each $p \in \mathcal{V}(xyz)$ let $M(p)$ be the corresponding point module for $k[x,y,z]$; there is a short exact sequence $0 \to M(p^{\sigma^{-1}})[-1] \to M(p) \to k \to 0$, and thus a short exact sequence $0 \to L(p^{\sigma^{-1}})[-1] \to k[x,y,z]^! \to L(p) \to 0$ of $k[x,y,z]^!$-modules, where $L(p) = M(p)^\dagger$. Hence $(D\,\mathrm{Tr})L(p) \cong L(p^{\sigma^{-2}})$, thus showing that the orbit of $L(p)$ is infinite whenever ρ is not a root of unity, and that it consists of modules of fixed dimension. The module which is used in [14] to give the orbit of interest is of the form $L(p)$ for $p \in \mathcal{V}(x)$ (actually, they work with right modules, but there is no essential difference).

Some further comments on a class of algebras containing $k[x,y,z]$ are made in Example 11.1.

Definition 10.10. A B-module N is **periodic** if $\Omega^r N \cong N$ for some r, where $\Omega^r N$ denotes the r^{th} syzygy in a minimal resolution of N.

The minimal resolution of $L(p)$ in (10-3) shows a sharp contrast with what happens for finite group algebras. A finitely generated kG-module of complexity one is a direct sum of periodic modules and projective modules [9], but if σ has infinite order, then $L(p)$ is not periodic.

Proposition 10.11. ($n = 3$) *The almost split sequence beginning with the trivial B-module k has indecomposable (and non-projective) middle term.*

Proof. We use the criterion and notation in [7]. To avoid conflicts with the earlier notation in this paper, we will, in this proof, use denote the graded algebra B by the letter B; thus A_i, B_i, $\mathcal{N}$, N and n now have the meanings in [7]. The proof involves a careful examination of the defining equations for D as given in Example 10.1, and Lemma 10.2.

(The reader may need a copy of [7] to follow the proof.) The sets A_0 and B_0 are empty since no non-zero element of D_1 annihilates all of D_1. Hence A_1 and B_1 are non-empty. Suppose that $A_2 \neq \emptyset$. By the defining properties of the sets A_i and B_i, if $0 \neq p \in A_2$, then $B_1 p = 0$, which forces $p \in E$, $B_1 = \{p^i \sigma^{-1}\}$ and $A_2 = \{p\}$, by Lemma 10.2. Since $\cup B_i$ spans D_1, we therefore have $B_2 \neq \emptyset$. By a similar argument, it follows that B_2 and A_1 are singleton sets. Hence A_3 and B_3 are non-empty. However, by the defining properties of the sets A_i and B_i,

$B_3A_2 = 0$, whence $B_3 = B_1$, contradicting the fact that the B_i's are disjoint. So we conclude that $A_2 = \emptyset$. By symmetry $B_2 = \emptyset$. Thus A_1 and B_1 are both of cardinality 3, whence $\mathcal{N} = \{(1,1)\}$, and $N_e = n_e = 1$. Brenner's Theorem now gives the result. □

The duality $\mathrm{Hom}_B(-,k)$ is related to the duality $M \mapsto M^\vee$ on Cohen-Macaulay A-modules which is defined in Definition 9.7.

Proposition 10.12. *Suppose that A is a connected, noetherian k-algebra of finite self-injective dimension. Let $M \in \mathrm{grmod}(A)$ be a Cohen-Macaulay A-module, and set $j = j(M)$. Then, for all p,*

$$\underline{\mathrm{Ext}}^p_A(M,k)^* \cong \underline{\mathrm{Ext}}^{j-p}_A(M^\vee,k)[-j].$$

Proof. The hypotheses imply that Ischebeck's spectral sequence [12]

$$\underline{\mathrm{Ext}}^p_A(\underline{\mathrm{Ext}}^q_A(M,A),k_A) \Rightarrow \mathrm{Tor}^A_{q-p}(k_A,M)$$

collapses. Therefore

$$\underline{\mathrm{Ext}}^p_A(M^\vee[j],k_A) \cong \mathrm{Tor}^A_{j-p}(k_A,M) \cong \underline{\mathrm{Ext}}^{j-p}_A(M,k)^*,$$

as claimed. □

If we also suppose in Proposition 10.12 that $\mathrm{projdim}(M) < \infty$, then the spectral sequence can be avoided, and the result is a consequence of the fact that applying $\underline{\mathrm{Hom}}_A(-,A)$ to a projective resolution of M gives a projective resolution of $M^\vee$.

Corollary 10.13. *As graded right B-modules, $L(D)^* \cong \mathrm{Ext}^*(M(D)^\vee, k_A)[d-n]$, where $d = \deg(D)$.*

11. Algebras defined by geometric data

Throughout this section V is a finite dimensional vector space and $\mathbb{P}$ denotes the projective space $\mathbb{P}(V^*)$.

The definition of $A_n(E,\sigma)$ given in Definition 8.1 is at first sight rather mysterious. The purpose of this section is to dispel some of the mystery.

The defining relations of the commutative polynomial ring $S(V)$ consist of the skew-symmetric tensors in $V \otimes V$, and (provided that $\mathrm{char}(k) \neq 2$) the skew symmetric tensors vanish on the diagonal copy, say Δ, of $\mathbb{P}$ in $\mathbb{P} \times \mathbb{P}$. Conversely, if $f \in V \otimes V$ vanishes on Δ, then f is skew-symmetric. Thus $S(V) = T(V)/(R)$ where

$$R = \{f \in V \otimes V \mid f|_\Delta = 0\}.$$

One need not take the whole diagonal Δ. Let $X \subset \mathbb{P}$ be a subvariety which is not contained in any quadric. Then no non-zero symmetric tensor in $V \otimes V$ can vanish on $\Delta_X = \{(p,p) \mid p \in X\}$, so the skew-symmetric tensors may also be described as

$$\{f \in V \otimes V \mid f|_{\Delta_X} = 0\}.$$

For example, if X is a cubic curve in $\mathbb{P}^2$, then X may be used in this way to define the 3-dimensional polynomial ring.

The simplest non-commutative algebras which can be defined in this sort of way are the twists of the polynomial ring. Let $\sigma \in \mathrm{GL}(V)$, and continue to write σ for the induced automorphisms of $S(V)$ and $\mathbb{P}$. Define a new graded algebra $S(V)^\sigma$ as follows. As a graded vector space $S(V)^\sigma = S(V)$, but multiplication is defined by $f * g = fg^{\sigma^n}$ if $f \in S(V)_n$. It is easy to show that $S(V) \cong T(V)/I$ where I is the two-sided ideal generated by $\{x^\sigma \otimes y - y^\sigma \otimes x \mid x, y \in V\}$. Hence I_2 vanishes on $\{(p, p^\sigma) \mid p \in \mathbb{P}\}$, which is a sort of 'shifted' diagonal copy of $\mathbb{P}$ in $\mathbb{P} \times \mathbb{P}$. In fact, any element of $V \otimes V$ which vanishes on this shifted diagonal is easily shown to belong to I_2. Thus

$$S(V)^\sigma \cong T(V)/(\{f \in V \otimes V \mid f(p, p^\sigma) = 0 \text{ for all } p \in \mathbb{P}\}).$$

These twists, which are quantum polynomial rings, are rather dull because the categories $\operatorname{GrMod}(S(V)^\sigma)$ and $\operatorname{GrMod}(S(V))$ are equivalent [37].

As mentioned in the introduction, the 3-dimensional quantum polynomial rings have been classified. They may all be obtained as follows. Let X be either $\mathbb{P}^2$ or a degree 3 divisor in $\mathbb{P}^2$ considered as a scheme. Let σ be an automorphism of X, and let Γ be the graph of σ, considered as a subscheme of $\mathbb{P}^2 \times \mathbb{P}^2$. Associate to this data the quadratic algebra $T(V)/(R)$ where $R = \{f \in V \otimes V \mid f|_\Gamma = 0\}$. Then, with a few exceptions, A is a 3-dimensional quantum polynomial ring and all such arise this way. Moreover, A is a Koszul algebra, and its Koszul dual is a finite dimensional graded algebra having the same Hilbert series as $\Lambda(k^3)$. There are graded $A^!$-modules $L(p)$ parametrized by the closed points of X, each being cyclic, indecomposable of dimension 4; the Auslander-Reiten translate of $L(p)$ is $L(p^{\sigma^{-2}})[1]$. If $X = \mathbb{P}^2$, then A is a twist of a polynomial ring, as in the previous paragraph, and $A^!$ is then a twist of $\Lambda(k^3)$.

Example 11.1. Let $X \subset \mathbb{P}^2$ be a triangle, say $\mathcal{V}(xyz)$, where x, y, z is a basis for V. Let $\sigma \in \operatorname{Aut}(V)$ be defined by $(0, y, z)^\sigma = (0, \alpha y, z)$, $(x, 0, z)^\sigma = (x, 0, \beta z)$ and $(x, y, 0)^\sigma = (\gamma x, y, 0)$, and suppose that σ does not extend to all of $\mathbb{P}^2$ (equivalently $\alpha\beta\gamma \neq 1$). Let $\Gamma = \{(p, p^\sigma) \mid p \in X\}$, and define $A = T(V)/\{f \in V \otimes V \mid f|_\Gamma = 0\}$. Then it is a straightforward (but tedious) calculation to show that $A \cong k[x, y, z]$ with defining relations $zy = \alpha yz$, $xz = \beta zx$, $yx = \gamma xy$. Again A is a quantum polynomial ring, but now $\operatorname{GrMod}(A)$ is *not* equivalent to $\operatorname{GrMod}(S(V))$.

The construction of $A_n(E, \sigma)$ is related to another natural construction relating a graded algebra with some geometric data. The ideas which follow are found in [2]. Let $A = T(V)/I$, where V is a finite dimensional vector space, and I a graded ideal; we associate to A a sequence of schemes $\Gamma_n \subset \mathbb{P}^{\times n}$, $n \geq 1$, and then to such a sequence we associate a graded algebra, $\bar{A}$ say; moreover, these constructions have the property that there is a canonical graded algebra homomorphism $A \to \bar{A}$. In exceptionally good situations A is actually isomorphic to $\bar{A}$, in which case A determines, and is determined by the geometric data (Γ_n). These constructions are as follows.

Definition 11.2. Let $A = T(V)/I$ be a graded k-algebra. The degree n homogeneous component I_n, of I, is a subspace of $V^{\otimes n}$, so its elements are linear maps $(V^*)^{\otimes n} \to k$ or, equivalently, n-multilinear maps

$$V^* \times \cdots \times V^* \to k.$$

We define an inverse system of schemes $\Gamma = (\Gamma_n)_{n \geq 1}$ by

$$\Gamma_n := \mathcal{V}(I_n) \ \subset \ \mathbb{P} \times \cdots \times \mathbb{P} = \mathbb{P}^{\times n},$$

and morphisms $\pi_n^m : \Gamma_m \to \Gamma_n$ for $m \geq n$ which are the restrictions of the projections $\mathbb{P}^{\times m} \to \mathbb{P}^{\times n}$ onto the first n copies. (The first part of the next Lemma shows that this really is an inverse system.)

Lemma 11.3. *Let $A = T(V)/I$ be a connected, graded algebra. Then*

1. $\Gamma_{n+1} \subset (\mathbb{P} \times \Gamma_n) \cap (\Gamma_n \times \mathbb{P})$, *with equality if* $I_{n+1} = V \otimes I_n + I_n \otimes V$;
2. *if* A *is a quadratic algebra and* $\Gamma_2 = \mathcal{V}(I_2)$ *is the graph of an automorphism* σ *of a subscheme* $X \subset \mathbb{P}$, *then, for all* $n \geq 2$,
$$\Gamma_n = \bigcap_{i=0}^{n-2} \mathbb{P}^{\times i} \times \Gamma \times \mathbb{P}^{\times n-i-2},$$
the scheme-theoretic intersection. In particular, if X *is a variety, then*
$$\Gamma_n = \{(p, p^\sigma, \dots, p^{\sigma^{n-1}}) \mid p \in X\}.$$

Proof. (1) It is clear that $\mathcal{V}(V \otimes I_n) = \mathbb{P} \times \Gamma_n$ and that $\mathcal{V}(I_n \otimes V) = \Gamma_n \times \mathbb{P}$. Since $V \otimes I_n + I_n \otimes V \subset I_{n+1}$ the result follows.

(2) Since $I_n = \sum_{i=0}^{n-2} V^{\otimes i} \otimes I_2 \otimes V^{\otimes n-i-2}$, it is clear that I_n vanishes on the given points. On the other hand, an induction argument, using (1), shows that Γ_n must belong to this set. Hence there is equality, as claimed. □

Example 11.4. What are the spaces Γ_n for the tensor algebra, symmetric algebra, and exterior algebra on a vector space V? Since there are no relations in degree one, $\Gamma_1 = \mathbb{P}$. For the tensor algebra $T(V)$, $I_n = 0$ for all n, so $\Gamma_n = \mathbb{P}^{\times n}$. For the symmetric algebra, $\Gamma_n = \{(p, \dots, p) \mid p \in \mathbb{P}\}$ for all $n \geq 1$. For the exterior algebra, if $\text{char}(k) \neq 2$, $\Gamma_n = \emptyset$ for all $n \geq 2$.

Before defining the graded algebra associated to the data (Γ_n), we realize $V^{\otimes n}$ as the global sections of a line bundle on $\mathbb{P}^{\times n}$. Let $\text{pr}_i : \mathbb{P}^{\times n} \to \mathbb{P}$ be the projection onto the i^{th} component, and define
$$\begin{aligned} \mathcal{O}(1, \dots, 1) &:= \text{pr}_1^* \mathcal{O}_{\mathbb{P}}(1) \otimes \cdots \otimes \text{pr}_n^* \mathcal{O}_{\mathbb{P}}(1) \\ &\cong \mathcal{O}_{\mathbb{P}}(1) \boxtimes \cdots \boxtimes \mathcal{O}_{\mathbb{P}}(1). \end{aligned}$$
Thus $V^{\otimes n} = \mathrm{H}^0(\mathbb{P}^{\times n}, \mathcal{O}(1, \dots, 1))$.

Definition 11.5. Let $\Gamma = (\Gamma_n)_{n \geq 1}$ be a sequence of subschemes $\Gamma_n \subset \mathbb{P}^{\times n}$ such that, for all $n \geq 1$,
$$\Gamma_{n+1} \subset (\mathbb{P} \times \Gamma_n) \cap (\Gamma_n \times \mathbb{P})$$
scheme-theoretically. Let $j_n : \Gamma_n \to \mathbb{P}^{\times n}$ and $i_{mn} : \Gamma_{m+n} \to \Gamma_m \times \Gamma_n$ be the inclusions. Define
$$\begin{aligned} \mathcal{B}_n &:= j_n^* \mathcal{O}(1, \dots, 1) \quad \text{for } n \geq 0, \\ B_n &:= \mathrm{H}^0(\Gamma_n, \mathcal{B}_n), \\ B(\Gamma) &:= \bigoplus_{n=0}^{\infty} B_n. \end{aligned}$$
If the Γ_n's are clear from the context, we just write B for $B(\Gamma)$. We give B a graded algebra structure: the multiplication map $B_m \times B_n \to B_{m+n}$ is the composition
$$\mathrm{H}^0(\Gamma_m, \mathcal{B}_m) \times \mathrm{H}^0(\Gamma_n, \mathcal{B}_n) \xrightarrow{\sim} \mathrm{H}^0(\Gamma_m \times \Gamma_n, \mathcal{B}_m \boxtimes \mathcal{B}_n) \to \mathrm{H}^0(\Gamma_{m+n}, \mathcal{B}_{m+n}),$$
where the second map is induced by the $\mathcal{O}_{\Gamma_{m+n}}$-module map $i_{mn}^*(\mathcal{B}_m \boxtimes \mathcal{B}_n) \to \mathcal{B}_{m+n}$. The product on B is associative because inverse image is functorial.

Example 11.6. If each $\Gamma_n = \mathbb{P}^{\times n}$, then $\mathcal{B}_n = \mathcal{O}(1, \dots, 1)$ so $B_n = V^{\otimes n}$ and $B = T(V)$.

If each $\Gamma_n = \{(p, \dots, p) \mid p \in \mathbb{P}\}$, then $B = S(V)$.

If $\sigma \in \text{Aut}_k \mathbb{P}$, and $\Gamma_n = \{(p, p^\sigma, \dots, p^{\sigma^{n-1}}) \mid p \in \mathbb{P}\}$, then $B = S(V)^\sigma$.

Proposition 11.7. *Let $\Gamma = (\Gamma_n)_{n\geq 1}$ be the sequence of subschemes of $\mathbb{P}^{\times n}$ determined by an algebra $A = T(V)/I$. Then there is a graded algebra homomorphism $A \to B(\Gamma)$.*

Proof. The natural map $V \to \mathrm{H}^0(\Gamma_1, j_1^*\mathcal{O}_{\mathbb{P}}(1)) = B_1$ induces an algebra homomorphism $T(V) \to B$; in degree n this is $V^{\otimes n} \to \mathrm{H}^0(\Gamma_n, j_n^*\mathcal{O}(1,\dots,1))$. The image of I_n under this map is zero because I_n vanishes on Γ_n, so the map $T(V) \to B$ factors through A. □

12. Homogenized Enveloping Algebras

This section describes a class of finite dimensional Koszul algebras whose duals are very well understood. It would be an interesting project to study these finite dimensional algebras with the goal of understanding how the well-known features of the duals are translated into statements about the finite dimensional algebras, and conversely.

Let $\mathfrak{g}$ be a finite dimensional Lie algebra and $U(\mathfrak{g})$ its enveloping algebra. The homogenized enveloping algebra of $\mathfrak{g}$ is

$$A(\mathfrak{g}) := T(\mathfrak{g} \oplus kz)/(R),$$

where z is a new indeterminate, and R is spanned by

$$\{z\otimes x - x \otimes z \mid x \in \mathfrak{g}\} \;\cup\; \{x\otimes y - y\otimes x - [x,y]\otimes z \mid x,y \in \mathfrak{g}\}.$$

Thus z is a central element in $A(\mathfrak{g})$ and $H(\mathfrak{g})/(z-1) \cong U(\mathfrak{g})$. Also notice that $A(\mathfrak{g})/(z) \cong S(\mathfrak{g})$. In fact, $A(\mathfrak{g})$ is the Rees ring of $U(\mathfrak{g})$ associated to the canonical filtration: it is isomorphic to the subalgebra of the polynomial extension $U(\mathfrak{g})[Z]$ generated by $(k+\mathfrak{g})Z$.

Suppose that $x_1,\dots,x_n$ is a basis for $\mathfrak{g}$. It is a consequence of the Poincaré-Birkhoff-Witt Theorem for $\mathfrak{g}\otimes k(z)$ viewed as a Lie algebra over $k(z)$ that $A(\mathfrak{g})$ has basis $\{x_1^{i_1}\cdots x_n^{i_n}z^j\}$. It follows that the Hilbert series of $A(\mathfrak{g})$ is $(1-t)^{-(n+1)}$ and that z is a central regular element of $A(\mathfrak{g})$.

It is not difficult to show that A is a quantum polynomial ring.

Proposition 12.1. *$A(\mathfrak{g})$ is a Koszul algebra.*

Proof. We will use the criterion in Theorem 5.13. The symmetric algebra $S(\mathfrak{g}) \cong A(\mathfrak{g})/(z)$ is a Koszul algebra. Write $V = \mathfrak{g}\oplus kz$, and R and R' for the quadratic relations in $A(\mathfrak{g})$ and $S(\mathfrak{g})$ respectively. We must show that the map $V^{\otimes 3} \to \mathfrak{g}^{\otimes 3}$ sends $V\otimes R \cap R\otimes V$ onto $\mathfrak{g}\otimes R' \cap R'\otimes \mathfrak{g}$.

Since $S(\mathfrak{g})^! = \Lambda(\mathfrak{g}^*)$, $\dim(\mathfrak{g}\otimes R' \cap R'\otimes\mathfrak{g}) = \binom{n}{3}$.

Let $x_1,\dots,x_n$ be a basis for $\mathfrak{g}$. Then $\mathfrak{g}\otimes R'\cap R'\otimes\mathfrak{g}$ contains

$$\sum_{\text{cyclic}} x_h \otimes (x_i\otimes x_j - x_j \otimes x_i) = \sum_{\text{cyclic}} (x_i\otimes x_j - x_j\otimes x_i)\otimes x_h \tag{12-1}$$

where the sum is over the three cyclic permutations (h,i,j), (i,j,h), and (j,h,i) of the three element subset $\{h,i,j\}$ of $\{1,\dots,n\}$. It is easy to see that these sums are linearly independent, and hence a basis for $\dim(\mathfrak{g}\otimes R'\cap R'\otimes\mathfrak{g})$.

On the other hand, the element in (12-1) is the image of

$$\sum_{\text{cyclic}} x_h \otimes (x_i \otimes x_j - x_j \otimes x_i - [x_i, x_j] \otimes z)$$
$$+ z \sum_{\text{cyclic}} (x_h \otimes [x_i, x_j] - [x_i, x_j] \otimes x_h - [x_h[x_i, x_j]] \otimes z)$$
$$= \sum_{\text{cyclic}} (x_i \otimes x_j - x_j \otimes x_i - [x_i, x_j] \otimes z) \otimes x_h + \sum_{\text{cyclic}} ([x_i, x_j] \otimes z - z \otimes [x_i, x_j]) \otimes x_h$$

which belongs to $V \otimes R \cap R \otimes V$. This completes the proof. □

We define the finite dimensional algebra

$$B(\mathfrak{g}) := A(\mathfrak{g})^!.$$

Since $A(\mathfrak{g})$ is closely related to $U(\mathfrak{g})$ which is well-understood, it is reasonable to expect that $B(\mathfrak{g})$ can be understood in great detail. As a first step towards this one might begin with the modules $M^\dagger$ where M is a linear module for $A(\mathfrak{g})$.

Theorem 12.2. [16] *The linear modules over $A(\mathfrak{g})$ are*

1. *the linear modules over $S(\mathfrak{g}) = A(\mathfrak{g})/(z)$, and*
2. *the induced modules $M(f, \mathfrak{h}) := A(\mathfrak{g}) \otimes_{A(\mathfrak{h})} k[z]_f$, where*
   - *$\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$,*
   - *$f \in \mathfrak{h}^*$ satisfies $f([\mathfrak{h}, \mathfrak{h}]) = 0$,*
   - *$k[z]_f$ is the $A(\mathfrak{h})$-module which is the polynomial ring $k[z]$ with z acting by multiplication and $x \in \mathfrak{h}$ acting as multiplication by $f(x)z$.*

The linear modules in (1) are the obvious ones, whereas those in (2) are the homogenizations of the induced modules $U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} k_f \cong U(\mathfrak{g}) \otimes_{A(\mathfrak{g})} M(f, \mathfrak{h})$. If d is the codimension of $\mathfrak{h}$ in $\mathfrak{g}$, then the Hilbert series of $M(f, \mathfrak{h})$ is $(1-t)^{-d}$. Since induced modules for $U(\mathfrak{g})$ are important it is likely that the modules $M(f, \mathfrak{h})^\dagger$ will be important $B(\mathfrak{g})$-modules.

Just as $U(\mathfrak{g})$ is richest when $\mathfrak{g}$ is semisimple (say over $\mathbb{C}$ for simplicity), so too should $A(\mathfrak{g})$ be most interesting in that case. It would probably be interesting to focus on the finite dimensional simple $\mathfrak{g}$ modules. If V is such a module, then there is a 'lifting' of it to $A(\mathfrak{g})$, say $\tilde{V}$, defined as follows. As a graded vector space $\tilde{V} = V \otimes k[z]$, with $\deg(z) = 1$ and $\deg(V) = 0$. Define $\varphi : A(\mathfrak{g}) \to U(\mathfrak{g}) \otimes k[z]$ by $\varphi(x) = x \otimes z$ for $x \in \mathfrak{g}$ and $\varphi(z) = 1 \otimes z$. Let $U(\mathfrak{g}) \otimes k[z]$ act on $\tilde{V}$ in the obvious way, and then let $A(\mathfrak{g})$ act on $\tilde{V}$ through the homomorphism φ.

Now $\tilde{V}$ will rarely have a linear resolution so under the equivalence of categories in Section 7 will give a complex of $A(\mathfrak{g})^!$-modules. These should be important complexes, but it is rather difficult to see what exactly makes them important.

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