Besicovitch's Approach to Kakeya's Conjecture and a Counterexample Related to Fubini's Theorem

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Abstract

In 1919 A. S. Besicovitch's interest in plane integration led him to construct an integrable function defined in the plane that is not integrable as an iterated integral for any pair of mutually orthogonal directions. Later, in 1928, he noticed that this construction could be modified so that it would suffice as a counterexample to S. Kakeya's 1917 conjecture that the smallest area required to rotate a unit line segment continuously in a plane is $\pi/8$. Besicovitch was able to show that this continual rotation is possible within a set of arbitrarily small area. In this paper we reconstruct both Besicovitch's integration example and his counterexample to Kakeya's conjecture via methods developed by K. J. Falconer and Besicovitch himself.

This paper will investigate two questions that were each posed in the first part of the twentieth century. The first, asked by A. S. Besicovitch, is a problem in Riemann integration and reads

I) 'Given a function of two variables, Riemann integrable on a plane domain, does there always exist a pair of mutually perpendicular directions such that repeated simple integration along these two directions exists and gives the value of the integral over the domain?' [3]

The second was inspired by S. Kakeya in 1917. Kakeya was considering the question

II) Given a unit line segment in a plane, what is the figure having smallest area in which the segment is able to rotate 180° while remaining within the figure?

For example a circle having diameter 1 would suffice by simply allowing the line segment to rotate about its fixed center. However, the rotation of a segment is also possible within an equilateral triangle of unit height, which has less area than the circle. Kakeya conjectured that if the figure
were required to be convex, then the figure of smallest area that allowed for the rotation of a unit segment would in fact be an equilateral triangle with unit height. In 1921, J. Pál proved this to be correct [5]. Kakeya noted that if the convexity restriction were not imposed then there exists a figure with an area even smaller than the equilateral triangle that allows for the rotation of a segment, namely the three-cusped hypocycloid with area $\pi/8$ pictured below in Figure 1 [5]. Convexity is however a very strong assumption. Lifting this requirement allows for many alternate figures in which a unit line segment can be completely rotated. Thus motivating the question in II), which became known as the Kakeya Conjecture.

![Hypocycloid Diagram](image)

Figure 1: The hypocycloid with an area of $\pi/8$ allows for the continuous rotation of a unit segment.

In the year 1919 Besicovitch provided a solution to his integration question in I). He was able to show that the answer to I) is a resounding 'No' [2], thus reinforcing that plane integration and iterated integration are inherently different objects. To do this, Besicovitch constructed a set of Jordan measure zero that contained a unit line segment in each direction. He then manipulated this set and cleverly defined a function that is integrable in the plane but is not as an iterated integral for any pair of mutually orthogonal directions. It was using this set that Besicovitch was later able to construct a set that would supply an answer to II), i.e. that there exists such a set with arbitrarily small area! Our objective here is to reproduce the answers to I) and II), which we will do by first constructing a set of arbitrarily small area that contains a unit line segment in each direction, as Besicovitch originally did in 1919. However, the downside of this is that the set Besicovitch constructed was arbitrarily large, multiply-connected and rather difficult to construct. On the other hand, in past years there have been vast improvements on Besicovitch's original construction. For instance, Cunningham constructs a simply-connected Kakeya set with arbitrarily small area that is contained in a ball of diameter 1 [4]! Due to its simplicity however, we will follow a construction by Falconer [5] to demonstrate the solution to II), that there exists such a figure with arbitrarily small area. Upon creating this set, we will then diverge from
Falconer’s construction and follow that of Besicovitch in [2] to obtain an answer to the question in I). It is interesting to note that Besicovitch also constructed a function that is integrable as an iterated integral for every pair of orthogonal directions yet is not integrable in the plane [2]. This further strengthens the notion that plane and iterated integration are not innately the same. In this paper we will examine Besicovitch’s construction of this function as well.

For ease of reference we will adopt the term Kakeya Set to denote a set in a plane in which a unit line segment can be turned through 180° by a continuous movement, where by continuous movement of the segment we mean that the endpoints of the segment follow paths that are images of continuous functions [1]. Furthermore, recall that when dealing with compact sets the notions of Jordan measure and Lebesgue measure are equivalent. Hence, we will unambiguously refer to the Jordan/Lebesgue measure of a compact set $S$ as its measure or its area and denote it by $A(S)$. Lastly, all integrals considered are Riemann integrals.

1 Preliminaries.

The following will be needed to construct solutions to both I) and II); these results are slight modifications of those made by Falconer in [5]. The main idea in this section is to be able to take a triangle, slice it up into a sufficiently large amount of smaller triangles, then to translate these smaller triangles in such a manner that the resulting figure has as small of an area as we please. We will do this to assure that if the original triangle allows for the complete revolution of a unit segment then the resulting figure will as well. This process will take a number of steps. This first lemma will provide us with some useful estimates on these ‘slicing up’ and ‘translating’ operations.

**Lemma 1.** Let $T_1$ and $T_2$ be adjacent triangles with bases on a line $L$, each having base length $b$ and height $h$. Take $\alpha \in (\frac{1}{2}, 1)$ and consider the new figure $\overline{T_2}$, which is the translation of $T_2$ a distance $2(1 - \alpha)b$ along $L$ in the direction of $T_1$. Denote by $S$ the union of $T_1$ and $\overline{T_2}$. Then:

i) The figure $S = T_1 \cup \overline{T_2}$ consists of a triangle, $T$, having base on line $L$, and two smaller triangles, $t_1$ and $t_2$, which we will call auxiliary triangles. The triangle $T$ is similar to $T_1 \cup T_2$ and is positioned in a similar manner.

ii) $A(T) = \alpha^2 A(T_1 \cup T_2)$.

iii) The difference in area between $T_1 \cup T_2$ and $S$ is given by

$$A(T_1 \cup T_2) - A(S) = A(T_1 \cup T_2)(1 - \alpha)(3\alpha - 1).$$
Proof. Consider Figures 2 and 3; we will prove i--iii each in turn.

Figure 2: $T_2$ is being translated to overlap $T_1$. Note $S = t_1 \cup t_2 \cup T$.

i) Clearly $m\angle 1 = m\angle 1'$ and $m\angle 2 = m\angle 2'$ where these angles are those indicated in Figure 3. Hence $T \approx T_1 \cup T_2$ (the symbol $\approx$ is used here to denote similarity). The latter assertion in i) regarding the positioning of the triangles is obvious.

Figure 3: The triangle in (a) is similar to triangle $T$ in (b).

ii) We see that $T$ has a base of length $2b - 2(1 - \alpha)b = 2\alpha b$; whereas $T_1 \cup T_2$ has base of length $2b$. So the ratio of similitude between $T$ and $T_1 \cup T_2$ is $\alpha$, hence $A(T) = \alpha^2 A(T_1 \cup T_2)$.

iii) Let $l$ be the line parallel to $L$ and passing through the common vertex shared by $t_1$ and $t_2$ as in Figure 4. Consider first the triangle $t_1$. $l$ divides $t_1$ into two triangles, call them $t_{1,1}$ and $t_{1,2}$, where $t_{1,2}$ is between $l$ and $L$. Let $x$ be the length of line segment $l \cap t_1$. We will think of $\alpha$ as a translating factor and allow it to vary. Notice that $x$ depends linearly on $\alpha$, so treating $x$ as a function in $\alpha$ and noting that $x|_{\alpha=1} = 0$ and $x|_{\alpha=\frac{1}{2}} = \frac{b}{2}$, we see that $x = b(1 - \alpha)$. Notice that $t_{1,1} \approx T_1$ and $t_{1,2} \approx T_2$, each having a ratio of similitude $\frac{x}{b} = 1 - \alpha$. Hence $A(t_1) = A(t_{1,1} \cup t_{1,2}) = (1 - \alpha)^2 A(T_1 \cup T_2)$. 

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Figure 4: Lines $L$ and $l$ are parallel. Notice that $t_{1,1} \approx T_1$ and $t_{1,2} \approx T_2$ (see Figure 2)

For $t_2$ let $y$ be the length of $t_2 \cap l$. By the same argument as above we get $y = b(1 - \alpha)$, so $x = y$ and we can conclude that $t_1$ is congruent to $t_2$. Now the total area of $S$ is given by

$$A(S) = A(T \cup t_1 \cup t_2) = A(T) + A(t_1) + A(t_2)$$

$$= \alpha^2 A(T_1 \cup T_2) + 2(1 - \alpha)^2 A(T_1 \cup T_2)$$

and, after a little algebra, we arrive at

$$A(T_1 \cup T_2) - A(S) = A(T_1 \cup T_2)(1 - \alpha)(3\alpha - 1).$$

Q.E.D.

Now we will use the estimates from the previous lemma to perform the aforementioned ‘slicing up’ and ‘translating’ of a triangle to form a new figure with arbitrarily small area.

**Lemma 2.** Consider a triangle, $T$, having base on a line $L$. Partition the base of $T$ into $2^k$ congruent segments and join each of the endpoints of these segments to the vertex opposite the base, forming $2^k$ triangles, $T_1, \ldots, T_{2^k}$. Then for every $\epsilon > 0$ there exists $K$ so that when $k > K$ it is possible to translate these $2^k$ triangles along $L$ to form a new figure, $S$, in such a manner that $A(S) < \epsilon$.

**Remark.** The translation of each triangle $T_i$ as described in Lemma 2 is applied to the triangle and its boundary. That is, the image of each $T_i$ under the translation is a closed figure. Thus the resultant figure, $S$, is compact. Notice that since some boundaries are shared by two triangles we have ‘added in’ $2^k - 1$ more line segments. This however does not effect area since
there are only finitely many of these ‘new’ segments.

Proof. Fix $\epsilon > 0$. We would like to employ Lemma 1, however in doing so we must specify a value for $\alpha \in (\frac{1}{2}, 1)$. To attain the desired result the value $\alpha$ must be chosen so that

$$\frac{1 + (1 - \frac{\epsilon}{A(T)})^{1/2}}{3 - (1 - \frac{\epsilon}{A(T)})^{1/2}} < \alpha < 1. \quad (1)$$

In order to justify that it is even possible to pick an $\alpha$ satisfying (1) we must show that the quantity $\frac{1 + (1 - \frac{\epsilon}{A(T)})^{1/2}}{3 - (1 - \frac{\epsilon}{A(T)})^{1/2}}$ in (1) is indeed less than 1. By assumption

$$0 < \frac{\epsilon}{A(T)},$$

$$1 > 1 - \frac{\epsilon}{A(T)} > 0.$$  

Taking the square root of both sides and multiplying by 2 gives

$$2 > 2 \left(1 - \frac{\epsilon}{A(T)}\right)^{1/2},$$

hence

$$3 - \left(1 - \frac{\epsilon}{A(T)}\right)^{1/2} > 1 + \left(1 - \frac{\epsilon}{A(T)}\right)^{1/2}.$$  

Dividing both sides by $3 - \left(1 - \frac{\epsilon}{A(T)}\right)^{1/2}$ yields the desired result.

Our other preliminary is to specify a value for $K$. Although at the moment the motivation may seem ambiguous, let

$$K > \log_2 \left[1 - (1 - \frac{\epsilon}{A(T)})^{1/2}\right], \quad (2)$$

and choose $k > K$. With this value of $k$, construct the $2^k$ triangles in the manner indicated in the hypotheses. By the following repeated application of the previous lemma we will construct the desired set $S$.

Step 1. Consider two consecutive triangles $T_{2i-1}$ and $T_{2i}$. Translate $T_{2i}$ along $L_i$ in the direction of $T_{2i-1}$, a distance of $2(1 - \alpha)b$, where $b$ is the length of the base (the side lying on $L$) of each $T_j$. This translation forms a new figure, which we will call $S^1_i$. By Lemma 1 the figure $S^1_i$ is the union of two auxiliary triangles and a triangle $T^1_i$ that is similar to $T_{2i-1} \cup T_{2i}$ and positioned in a similar manner; furthermore $A(T^1_i) = \alpha^2 A(T_{2i} \cup T_{2i-1})$. The reduction of area in replacing $T_{2i-1} \cup T_{2i}$ with the new figure $S^1_i$ is $A(T_{2i} \cup T_{2i-1})(1 - \alpha)(3\alpha - 1)$. Repeating for each
$1 \leq i \leq 2^{k-1}$ yields a collection, $\{S^1_i\}$, of new figures. Figures 6 and 5 illustrate the collection $\{S^1_i\}$ for $k = 3$.

![Figure 5: One particular $S^1_i, T^1_i$.](image)

![Figure 6: Translating the $T_i$ to form the $S^1_i$ when $k = 3$.](image)

**Step 2.** We will now perform a similar operation with consecutive $S^1_i$. Let $1 \leq i \leq 2^{k-2}$ and translate $S^1_{2i}$ along $L$ in the direction of $S^1_{2i-1}$. Call this new figure $S^2_i$. See Figure 7. Restricting our attention to the action of $T^1_{2i+1}$ relative to $T^1_{2i}$ in this translation we notice that one side of $T^1_{2i-1}$ is parallel and congruent to the opposite side of $T^1_{2i}$. Hence, Lemma 1 allows us to perform this translation so that $S^2_i$ contains some triangle $T^2_i$ where $A(T^2_i) = \alpha^2 \left( A(T^1_{2i-1}) + A(T^1_{2i}) \right)$ and the reduction of area achieved by replacing $T^1_{2i-1} \cup T^1_{2i}$ by $S^2_i$ is at least

$$(1 - \alpha) (3\alpha - 1) \left( A(T^1_{2i-1}) + A(T^1_{2i}) \right)$$

$$= (1 - \alpha)(3\alpha - 1)\alpha^2 A(T^1_{2i-3} \cup T^1_{2i-2} \cup T^1_{2i-1} \cup T^1_{2i}).$$

**Step 3.** Inductively, let $j \leq k$ and suppose we have $j$ collections of figures $\{S^m_i\}_{i=1}^{2^{k-m}}$ for each $m \leq j$, satisfying the following conditions:

i) each $S^m_i$ lies on $L$,

ii) each $S^m_i$ contains some triangle $T^m_i$ that also lies on $L$,
iii) for a fixed $m$ the $T_i^m$ are disjoint,

iv) for a fixed $m$ consecutive $T_i^m$ have one pair of congruent, parallel sides,

v) $A(T_{2i-1}^{m-1} \cup T_{2i}^{m-1}) - A(S_i^m) \geq (1 - \alpha)(3\alpha - 1) \left( A(T_{2i-1}^{m-1}) + A(T_{2i}^{m-1}) \right)$,

vi) $A(T_i^m) = \alpha^2 A(T_{2i-1}^{m-1} \cup T_{2i}^{m-1})$.

With $1 \leq i \leq 2^{k-j}$ translate $S_{2i}^j$ along $L$ to overlap $S_{2i-1}^j$ obtaining a new figure $S_{2i}^{j+1}$. Lemma 1 tells us that this can be done so that each of the six conditions above are satisfied for $m = j + 1$, thus completing the inductive step.

Notice that condition ii) provides us with countable additivity, hence by repeated application of condition v) above we have

$$A(S_i^k) \leq \left[ A(T_1^{k-1}) + A(T_2^{k-1}) \right] - (1 - \alpha)(3\alpha - 1) \left[ A(T_1^{k-1}) + A(T_2^{k-1}) \right]$$

$$\leq \left[ A(S_1^{k-1}) + A(S_2^{k-1}) \right] - (1 - \alpha)(3\alpha - 1) \left[ A(T_1^{k-1}) + A(T_2^{k-1}) \right]$$

$$\leq \left[ A(T_1^{k-2}) + \ldots + A(T_4^{k-2}) \right] - (1 - \alpha)(3\alpha - 1) \left[ A(T_1^{k-2}) + \ldots + A(T_4^{k-2}) \right] -$$

$$(1 - \alpha)(3\alpha - 1) \left[ A(T_1^{k-1}) + A(T_2^{k-1}) \right]$$

$$\vdots$$

$$\leq \left[ A(T_1) + \ldots + A(T_{2^k}) \right] - (1 - \alpha)(3\alpha - 1) [A(T_1) + \ldots + A(T_{2^k})] -$$

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\[(1 - \alpha)(3\alpha - 1) \left[ A(T_1^1) + \ldots + A(T_{2(k-1)}^{k-1}) \right] - \ldots - (1 - \alpha)(3\alpha - 1) \left[ A(T_1^{k-1}) + A(T_2^{k-1}) \right]. \]

Now use condition vi) to get

\[= A(T) - (1 - \alpha)(3\alpha - 1)(1 + \alpha^2 + \ldots + \alpha^{2(k-1)})A(T)\]

\[= \left(1 - \frac{(3\alpha - 1)(1 - \alpha^{2k})}{1 + \alpha}\right)A(T).\]

Thus our sought after estimate is

\[A(S_k^1) \leq \left(1 - \frac{(3\alpha - 1)(1 - \alpha^{2k})}{1 + \alpha}\right)A(T)\]

and we will take \(S_k^1\) to be our set \(S\). We will now manipulate equations (1) and (2) to complete the proof. Solving (1) for the quantity \((1 - \frac{\epsilon}{A(T)})^{1/2}\) gives us

\[\left(1 - \frac{\epsilon}{A(T)}\right)^{1/2} < \frac{3\alpha - 1}{1 + \alpha},\]

and by our choice of \(k\) in (2) we get

\[1 - \alpha^{2k} < \left(1 - \frac{\epsilon}{A(T)}\right)^{1/2}.\]

So by putting these together with (3) we arrive at

\[A(S) < \epsilon.\]

\[Q.E.D.\]

Remark. Notice that by fixing the position of the first subtriangle, \(T_1\), and performing the above operations with respect to \(T_1\), each \(T_i\) will have moved no more than \(b\) along \(L\). See Figure 8. We will use this idea in the theorem below.

**Theorem 1.** With the same notation as Lemma 2, given some open set \(V \supset T\) and \(\epsilon > 0\), the construction of \(S\) can be done so that \(S \subset V\) and \(A(S) < \epsilon.\)

\footnote{Since \(\alpha < 1\) we have that \(f(x) = \alpha^{2x}\) is a decreasing function, so the inequality in (2) is reversed when both sides are raised to a power of \(\alpha^2\).}
Figure 8: The red triangles are the images of the gray triangles after translating. Notice that all of the translating is done with respect to $T_1$ and each triangle has been translated no more than $b$.

Proof. Fix $\epsilon > 0$ and let $V$ be an open set containing $T$. Notice that the only part of $V$ that we are interested in is that which is very close to the triangle $T$, so since $T$ is bounded we may as well assume that $V$ is bounded as well. Later on this will allow us the luxury of using compactness. We want to be able translate the points in $T$ to make its area very small, but we cannot translate the points very far since we need to stay within $V$. So we first find an upper bound, $\delta$, exhibiting the property that if any point of $T$ is translated a distance less than $\delta$ in any direction then the point will still be in $V$.

To find $\delta$ we define a function $\phi : T \to \mathbb{R}$ so that

$$\phi(p) = \min \{|p - x| : x \in \partial V\},$$

where $\partial V$ denotes the boundary of $V$. Note that by the extreme value theorem the distance function achieves a minimum on $\partial V$, so $\phi$ is well defined. Furthermore, a calculation shows that $\phi$ is continuous, so by the compactness of $T$ and a second application of the extreme value theorem, $\phi$ achieves a minimum. This minimum will be our $\delta$. Now choose $n > b/\delta$, where $b$ is the length of the base of $T$. Divide up the base of $T$ into $n$ congruent line segments and connect the endpoints of these segments to the opposite vertex of $T$, obtaining $n$ subtriangles, $T_1, \ldots, T_n$, as in Figure 9(a). We will now apply Lemma 2 to these subtriangles (observe that we are not applying Lemma 2 to the larger triangle $T$). The case for $n = 4$ is shown in Figure 9.

Consider some $T_i$. By Lemma 2 we can divide $T_i$ into smaller subsubtriangles, so that, upon translation, we obtain a figure $S_i$ where $A(S_i) < \epsilon/n$. Furthermore, by the remark following Lemma 2, this can be done so that each subsubtriangle has moved a distance less than the length of the base of $T_i$. Or equivalently, the elements of $S_i$ have been moved no more than $b/n < \delta$, so the elements of $S_i$ are still contained in $V$. Repeating this for each of the $T_i$, one
Figure 9: Illustrated above is the case for $n = 4$. (b) shows Lemma 2 applied to $T_3$. (c) shows Lemma 2 applied to all of the subtriangles. Notice that Lemma 2 is applied to the $T_i$, not $T$.

obtains a new collection of figures, $\{S_i\}$. Let $S$ be the union of the $S_i$. We thus have

$$A(S) = A(\cup_{i=1}^{n} S_i) \leq \sum_{i=1}^{n} A(S_i) < \epsilon,$$

and furthermore, in obtaining $S$ from $T$ the elements have been translated no more than $\delta$, so $S \subset V$ as desired.

$Q.E.D.$

**Theorem 2.** There exists a bounded set with zero area, containing a unit line segment in every direction from $0^\circ$ to $90^\circ$. Furthermore, given a line $L$ this set can be constructed so that each segment has an endpoint on $L$ and all of the segments lie on the same side of $L$.

**Proof.** Let $S_1$ be an isosceles right triangle with unit height and having its longest base on a line $L$. $S_1$ is measurable so there exists an open cover of $S_1$ with measure as close to $A(S_1)$ as we please. Let $V_1 \supset S_1$ be an open set such that $A(V_1) \leq 2A(S_1)$, where $V_1$ denotes closure. Notice that by taking an intersection, the set $V_1$ is easily contained in a ball of radius 2, see Figure 10. Now applying Theorem 1 to $S_1$ we obtain a new closed figure, $S_2 \subset V_1$, that is the finite union of triangles, each having base on line $L$ and with $A(S_2) \leq 2^{-2}$.

Since $S_2$ is measurable, there exists an open set, $V_2$, containing $S_2$ and with measure as close to $A(S_2)$ as we please. So choose $V_2$ so that $A(V_2) \leq 2A(S_2)$. $V_1$ contains $S_2$ and is open so, by taking an intersection if necessary, this can be done so that $S_2 \subset V_2 \subset V_1$. By applying Theorem 1 again to each of the triangles that make up $S_2$ we obtain a new set, $S_3$, contained in $V_2$, with $A(S_3) \leq 2^{-3}$.

Repeating this process iteratively we obtain two collections of sets, $\{V_i\}$ and $\{S_i\}$, such that each set satisfies the following properties:
Figure 10: Taking an intersection of $V_1$ and a circle of radius 2 that contains $S_1$ ensures that $V_1$ is bounded.

i) each $V_i$ is open,

ii) $S_i \subset V_i$,

iii) $V_i \subset V_{i-1}$,

iv) $A(V_i) \leq 2A(S_i) \leq 2^{-i+1}$,

v) each $S_i$ is the finite union of triangles having unit height and base on $L$; hence each $S_i$ has a unit line segment in each direction between $0^\circ$ and $90^\circ$ that lies on one side of $L$ and has an endpoint on $L$.

Let

$$S = \bigcap_{i=1}^{\infty} V_i.$$  

Notice that $S$ is closed and $S \subset V_1$, the latter of which is bounded so $S$ is compact. Hence the notions of Jordan measure and Lebesgue measure are equivalent. Property iv) above implies that $S$ has zero Lebesgue measure so $A(S) = 0$. It remains to show that $S$ contains a line segment in every direction between $45^\circ$ and $135^\circ$ (measured with respect to $L$) as this statement, along with property v), implies that each line segment lies on one side of $L$ and contains an endpoint on $L$. So let $\theta$ be an angle in the range $[45, 135]$. By property v) for each $i$ there exists some line segment $M_i \subset S_i$ making an angle $\theta$ with $L$. Property ii) then tells us that $M_i \subset V_i$. Let $x_i$ be the $x$-coordinate of the endpoint of $M_i$ that lies on $L$, see Figure 11. It then follows from property iii) that $\{x_i\} \subset V_1$. Compactness of $V_1$ allows us to pass to a convergent subsequence of $\{x_i\}$. So let $x$ be the limit of this subsequence and let $M$ be the line segment corresponding to this value of $x$; that is, the line segment whose endpoint lying on $L$ has $x$-coordinate $x$. Notice that if $i \geq j$, then $M_i \subset V_j$ by property iii). So since the $M_i$ become arbitrarily close to $M$. 

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So each point of $M$ is a limit point for a sequence of points in the $M_i$. Finally, since each $V_i$ is closed we get that $M \subset \overline{V}_i$ for each $i$. Hence $M \subset S$ as desired.

Figure 11: The bolded line segment is $M_i$ and intersects the $x$-axis at $x_0$, forming an angle $\theta$. $S_i$ is illustrated by the gray line segments and $V_i$ is the open set containing $S_i$.

Q.E.D.

2 Rotating Line Segments.

To construct a Kakeya set with arbitrarily small area we will begin with two right isosceles triangles with unit height. When these triangles are situated so that their longest bases are orthogonal a Kakeya set is formed. In Theorem 3 below we will cut each of these triangles into smaller triangles and move them around to make the set have a really small area. However, in moving them around we may have lost the ability to move a line segment continuously between the triangles. The following lemma will allow us a way to compensate for this.

**Lemma 3.** Let $L_1$ and $L_2$ be parallel lines. For any $\epsilon > 0$ there exists a compact set $E$ with area less than $\epsilon$ in which a unit line segment can be moved continuously from $L_1$ to $L_2$.

**Proof.** Let $\omega$ be the distance between $L_1$ and $L_2$ and let $x_1$ be any point on $L_1$. Refer to Figure 12 below. Take $x_2$ to be a point on $L_2$ a distance $D$ away from the orthogonal projection of $x_1$ onto $L_2$, where

$$D > \frac{\omega}{\tan \pi \epsilon}.$$
Figure 12: Take \( E \) to be the union of \( M, S_1 \) and \( S_2 \). A line segment can move continuously from \( L_1 \) to \( L_2 \) while remaining within \( E \).

Denote by \( M \) the line segment connecting \( x_1 \) with \( x_2 \). The continuous movement we are looking for will take the unit line segment from line \( L_1 \) and rotate it to \( M \) about the point \( x_1 \). Then slide the segment along \( M \) to \( x_2 \) and rotate around \( x_2 \) until it lies on \( L_2 \). To allow for this we must include the two congruent sectors, \( S_1 \) and \( S_2 \), of radius 1 that lie between \( M \) and \( L_1 \) and \( L_2 \), respectively. The angle between \( M \) and each \( L_i \) is \( \tan^{-1} \frac{x}{D} < \epsilon \), so the area of each sector is less than \( \epsilon/2 \). By taking \( E \) to be the union of \( M \) with \( S_1 \) and \( S_2 \) we attain our desired set.

\[ Q.E.D. \]

**Theorem 3.** Given \( \epsilon > 0 \) there, exists a Kakeya set with Jordan measure less than \( \epsilon \).

*Proof.* Take an isosceles right triangle having unit height and longest base on a line \( L_1 \). Lemma 2 enables us to cut up this triangle into \( n \) smaller triangles, \( T_1, \ldots, T_n \), of unit height each having base on \( L_1 \), and then to translate them so that the area of the resulting figure is less than \( \epsilon/6 \). Let \( \tilde{T}_i \) denote the translated image of \( T_i \) for each \( i \). So in each \( \tilde{T}_i \) a unit line segment can be rotated from one side to the other in a continuous fashion. Notice that \( \tilde{T}_i \) and \( \tilde{T}_{i+1} \) have two sides that are parallel. By an application of Lemma 3 we can move the segment from \( \tilde{T}_i \) to \( \tilde{T}_{i+1} \) in an area that is less than \( \epsilon/6n \). So as the segment moves from each \( \tilde{T}_i \) to the next, and hence through 90°, it sweeps out a set with an area that is less than \( \epsilon/3 \). Call this resulting set \( S_1 \).

Now take \( L_2 \) to be a line perpendicular to \( L_1 \) and construct a copy of \( S_1 \), but this time with respect to \( L_2 \). Call this new set \( S_2 \). Notice that these two sets contain one pair of parallel line segments, so by Lemma 3 a unit line segment can be moved continuously from one set to the other while being contained is a set \( E \) having an area that is less than \( \epsilon/3 \). Thus taking the union of \( S_1, S_2 \) and \( E \) we obtain a Kakeya set with area less than \( \epsilon \).

\[ Q.E.D. \]
3 Integration.

Even before learning of Kakeya’s conjecture, Besicovitch was studying Riemann integration in the plane and the conditions under which plane integration is equivalent to iterated integration, such as in the equality observed in Fubini’s theorem (see the Appendix). In [2] he demonstrated the limitations of this equivalence by constructing functions and sets where 1) iterated integrals exist but the corresponding plane integrals do not and 2) where plane integrals exist but the iterated integrals do not. We will look into both of these examples below and will follow Besicovitch’s original construction.

**Theorem 4.** There exists a function $f : \mathbb{R}^2 \to \mathbb{R}$ and a set $S \subset \mathbb{R}^2$ such that the Riemann integral

$$\int \int_S f \, dA$$

exists, but the expression

$$\int_\alpha^\beta \int_{\eta_1}^{\eta_2} f \, d\xi \, d\eta$$

does not exist as a Riemann integral for any pair of orthogonal directions $\eta, \xi$, where $S$ lies between curves $\eta_1$ and $\eta_2$, which range from $\alpha$ to $\beta$ along the $\xi$-direction.
Figure 14: The set $S$ and an arbitrary pair of orthogonal axes, $\eta$ and $\xi$.

*Proof.* Let $B$ be a set as in Theorem 2, taking $L$ to be the $x$-axis where the line segments are positioned in the first quadrant. Let $r_1, r_2, \ldots$ be an enumeration of the rationals in the range $(0, 1)$ and consider the set $\{L_i\}$ of lines where each $L_i$ is the line $y = 2^{-i}$. Denote by $B_i$ the sections of line segments of $B$ that lie between $L_i$ and $L_{i+1}$. For each $i$, translate $B_i$ a distance $r_i$ in the positive $x$-direction. Thus we have a new set which we call $S_1$. See Figure 15.

Figure 15: Illustrated above is the translations of the $B_i$. $S_1$ is the union of the chopped up segments.

Our claim now is that $S_1$ has zero area (Jordan measure). The difficulty comes in that $S_1$ is the result of an infinite number of translations, and Jordan measure does not handle infinities very well. However, these infinite translations all occur very close to the $x$-axis, so we can negate
this problem by placing a sufficiently wide rectangle on the \( x \)-axis, which will cover all of the infinities. We will do this as follows: choose any positive \( \epsilon \) and recall that \( B \) is bounded, so \( S_1 \) must also be bounded since it was constructed from \( B \) by translating distances no greater than 1. Let \( \psi \) be the length of an interval that contains the projection of \( B \) onto the \( x \)-axis. This will be the length of the base of our rectangle. We will take \( \epsilon/2\psi \) to be its height. There is no harm in insisting that this rectangle be closed, so we will require it here as it will benefit us later. Position the base of this rectangle on the \( x \)-axis so that it covers the lower portion of \( S_1 \) as in Figure 16; this is possible because we have chosen the width to be great enough. The remaining uncovered portion of \( S_1 \) consists of a finite number of translates of \( B \), and hence is Jordan measurable, so there exists a finite closed cover of this portion of \( S_1 \) with area less than \( \epsilon/2 \) and we have the freedom to insist that this cover consist solely of rectangles. Thus we have a finite cover of \( S_1 \) that consists of closed rectangles with a total area that is less than \( \epsilon \). As \( \epsilon \) was arbitrary it follows that the Jordan measure is 0.

![Figure 16: The rectangle in red has area \( \epsilon/2 \).](image)

Now let \( S_2 \) be the image of \( S_1 \) after a 90° rotation. Observe that \( A(S_1 \cup S_2) \leq A(S_1) + A(S_2) = 0 \). We are now ready to define the function \( f \) as follows: (i) if \( P \in S_1 \) has a rational \( y \)-coordinate then \( f(P) = 1 \), (ii) if \( P \in S_2 \) has a rational \( x \)-coordinate then \( f(P) = 1 \), (iii) \( f(P) = 0 \) for all other \( P \). This ensures that \( f \) is not integrable as a function of a single variable on any of the line segments of \( S_1 \) or \( S_2 \). Let \( S \) be a ball that contains \( S_1 \cup S_2 \) and its finite cover of closed rectangles that was constructed above. The complement of the cover of \( S_1 \cup S_2 \) is open in \( S \) and since \( f \) is constantly zero off of this cover it follows that \( f \) is continuous there. Thus \( f \) is
discontinuous at most on the cover of rectangles, which can be made to have as small of an area as we please, so the integral

\[ \int \int_S f \, dA \]

exists and equals zero.

Let \( \eta \) and \( \xi \) be an arbitrary pair of orthogonal axes, where the \( \xi \)-axis is chosen to be the one that forms the larger acute angle with the \( x \)-axis. If they both meet the \( x \)-axis at the same angle then just choose one of the axes to be \( \xi \). Call this angle formed \( \phi \). Consider the unit line segment, \( M \), of \( B \) that makes an angle \( \phi \) with the \( x \)-axis, and let \( x_0 \) be the \( x \)-coordinate of the intersection of this line segment with the \( x \)-axis, such a set is guaranteed to exist by the definition of \( B \). Notice that \( S_1 \) contains an enumerable amount of non-collinear line segments that are parallel to \( M \). Let \( L(\phi) \) denote the set of parallel lines that each contain one of these line segments, see Figure 17. Notice that \( L(\phi) \) can be described by \( \{(x, y) : y = \tan(\phi)(x - x_0 - r), \forall r \in \mathbb{Q} \cap (0, 1)\} \). By the density of the rationals it follows that the intersection of \( L(\phi) \) with the \( x \)-axis is dense in the interval \( (x_0, x_0 + 1) \). If we now consider the intersection of \( L(\phi) \) with the \( \eta \)-axis we obtain a set that is dense in an interval of length \( \sin \phi \geq \sqrt{2}/2 \) since \( \phi \geq \frac{\pi}{4} \).  

Figure 17: The dark line segment, \( M \), in (a) is the segment of \( B \) that is parallel to the \( \xi \)-axis. The gray lines in (b) are the lines of \( L(\phi) \) that each contain a line segment from \( S_1 \), which are the chopped up translates of \( M \).

Recall that \( f \) is not integrable over any of the line segments of \( S_1 \) or \( S_2 \), hence the integral

\[ \int_{\eta_1}^{\eta_2} f \, d\xi \]
does not make sense over any of the lines in $L(\phi)$ when considered as a Riemann integral. So when viewed as a function in $\eta$ the expression $\int_{\eta_1}^{\eta_2} f(\xi, \eta) \, d\xi$ is not defined on a dense set of length $\sin \phi$. Hence, in considering the expression

$$\int_{\alpha}^{\beta} \int_{\eta_1}^{\eta_2} f \, d\xi \, d\eta$$

we conclude that it does not exist as a Riemann integral. By the same argument it follows that

$$\int_{\delta}^{\gamma} \int_{\xi_1}^{\xi_2} f \, d\eta \, d\xi$$

does not exist, thus completing the proof.

\textit{Q.E.D.}

The following theorem does not use any of the machinery developed previously in this paper, however it is included here because it has results that are so similar to that of the previous theorem. As a note, the following theorem will also be proved in the same fashion as Besicovitch did in his 1919 paper.

\textbf{Theorem 5}. There exists a function $f : \mathbb{R}^2 \to \mathbb{R}$ and a set $S \subset \mathbb{R}^2$ so that the iterated integral

$$\int_{\alpha}^{\beta} \int_{\eta_1}^{\eta_2} f \, d\xi \, d\eta$$

exists for every pair of orthogonal directions $\eta, \xi$, but the expression

$$\int \int_S f \, dA$$

does not exist, where $S$ is bounded in the $\xi$ direction by the curves $\eta_1$ and $\eta_2$ and $\eta$ ranges from $\alpha$ to $\beta$.

\textit{Proof}. Consider the square with vertices $A(0,1)$, $B(1,1)$, $C(1,0)$ and $D(0,0)$. Let $r_1, r_2, \ldots$ be an enumeration of the rationals in $(0,1)$. For each $i$ denote by $L_i$ the translation of line segment $DA$ in the positive $x$-direction a distance of $r_i$. Divide each $L_i$ into $i$ equal pieces and designate the points of division by $a_{i,1}, a_{i,2}, \ldots, a_{i,i-1}$.

We are now going to construct the set of points
that satisfy the properties

i) the distance from $b_{i,j}$ to $a_{i,j}$ is less than $\frac{1}{i}$

ii) no three of the $b_{i,j}$ are collinear.

This can be done as follows: choose $b_{2,1}$ and $b_{3,1}$ to be points within $\frac{1}{2}$ and $\frac{1}{3}$ of $a_{2,1}$ and $a_{3,1}$, respectively. Now take $b_{3,2}$ to be a point within $\frac{1}{3}$ of $a_{3,2}$ and not lying on the line determined by $b_{2,1}$ and $b_{3,1}$. Inductively, take $b_{i,j}$ to be a point within $\frac{1}{i}$ of $a_{i,j}$ and not lying on any of the lines determined by the points $b_{2,1}, b_{3,1}, b_{3,2}, \ldots, b_{i,1}, b_{i,2}, \ldots, b_{i,i-1}$.

Figure 18: Shown here is the case where $i = 8$.

Our claim now reads, the set of all $b_{i,j}$ is dense in the square $ABCD$. Consider an arbitrary point $P_0 = (x_0, y_0)$ in the interior of $ABCD$ and let $\epsilon > 0$ be small enough so that the ball of radius $\epsilon$ centered at $P_0$ is contained in $ABCD$. There are an infinite number of rationals in the interval $(x_0 - \frac{\epsilon}{3}, x_0 + \frac{\epsilon}{3})$ so let $r_n$ be such a rational with $n > \frac{3}{\epsilon}$. The $a_{n,i}$ are within $\frac{1}{n} < \frac{\epsilon}{3}$ of each
other, so there exists some $a_{n,k}$ that is within $\frac{2\varepsilon}{3}$ of $P_0$. The triangle inequality tells us that since $b_{n,k}$ is within $\frac{\varepsilon}{3}$ of $a_{n,k}$ we can conclude $b_{n,k}$ is within $\varepsilon$ of $P_0$, so the $b_{i,j}$ are dense in $ABCD$.

Take the function $f$ to be the characteristic function on the set of all $b_{n,i}$ and let $\xi, \eta$ be an arbitrary set of orthogonal directions. For each fixed value of $\xi$ the function $f$ is $1$ at at most two of the $b_{n,i}$, so $f$ is discontinuous only on a set of (Jordan) measure zero and constantly zero elsewhere on the given line. Hence

$$\int_{\eta}^{n_2} f d\xi = 0$$

for every value of $\eta$, and the integral

$$\int_{a}^{b} \int_{\eta}^{n_2} f d\xi d\eta$$

exists and is also identically zero.

On the other hand, the Riemann integral

$$\int \int_{ABCD} f dA$$

does not exist since $f$ is discontinuous on a dense set in $ABCD$.

Q.E.D.

4 Appendix

Fubini’s Theorem provides us with sufficient conditions for situations in which plane integration and iterated integration are equivalent. Below is a statement of this theorem [6].

Fubini’s Theorem. Let $R = \{ (x,y) \in \mathbb{R}^2 : a \leq x \leq b, c \leq y \leq d \}$, and let $f$ be an integrable function on $R$. Suppose that for each $y \in [c,d]$, the function $f_y$ defined by $f_y(x) = f(x,y)$ is integrable on $[a,b]$, and the function $g(y) = \int_{a}^{b} f(x,y)dx$ is integrable on $[c,d]$. Then

$$\int \int_{R} f dA = \int_{c}^{d} \left[ \int_{a}^{b} f(x,y)dx \right] dy.$$
Remark. Having introduced the notion of a Kakeya set and demonstrating such a set with arbitrarily small area, we are naturally inclined to ask the question: Does there exist a Kakeya set with zero Jordan (or even Lebesgue) measure? As far as I have been able to find this question remains unanswered, so I state it here as an open problem.

References


