EXPLORATIONS IN COHOMOLOGY

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1. INTRODUCTION

The story of this thesis begins in the summer of 2005, when I was frantically writing my application for a Rhodes scholarship. Part of the application was a study proposal, in which I was to detail my intended course of study at Oxford. Since the backbone of the Oxford educational system is the close relationship between the advisor and student, I was forced to make a pitch for studying with someone in the math department. This put me in awkward position. I had just finished second-year calculus and an introductory course in modern algebra, and was in no place to understand the stated research goals of any of the Oxford faculty. Yet I had to choose, so I did so by the only method I could devise. I printed out the list of Fields Medalists and the list of the Oxford faculty and hoped for a match. It was my luck that there was one: Daniel Quillen. In my proposal, I had only to state that he was a Fields Medalist and needed no further justification for wanting to study with him.

That summer, I became curious as to what kind of work Quillen had done to win the Fields Medal. Some preliminary investigations uncovered such phrases as “mod p cohomology ring”. It was then that I wrote my algebra professor, John Palmieri, and asked whether it would be feasible to write a senior thesis on Quillen’s two seminal papers from 1971, “The Spectrum of an Equivariant Cohomology Ring” parts I and II [4]. In retrospect, I am absolutely stunned that his response was ‘sure’. He was quick to add that it would probably take me two solid quarters of study to even understand the statement of the so-called Stratification Theorem and that I might not even get to the proof.

As it turns out, Quillen retired this year, so I will be unable to study with him at Oxford. A conference was held in Oxford on his 65th birthday, May 22, 2006. Perhaps I will one day have the pleasure of meeting him.

Now to the content of the paper. I will start by computing the cohomology of a cyclic group, then discuss how to compute the cohomology of the Grassmannian. These sections are intended as illustrations of two flavors of cohomology – cohomology of a group and cohomology of a space. The rest of the paper examines the motivation for Quillen’s theorems and gives some examples of the theorem in action.

2. THE RING STRUCTURE OF $H^*(G, k)$ WHEN $G$ IS CYCLIC

This section comes from a presentation I gave with Ariana Dundon in Winter 2005 in Julia Pevtsova’s cohomology class. We followed the presentation in [3].

Let $G = \langle g | g^p = 1 \rangle$ be the cyclic group of order $p$, $p$ prime, and look at the familiar periodic resolution

$$
\ldots \rightarrow kG \xrightarrow{T=1+g+\ldots+g^{p-1}} kG \xrightarrow{(1-g)} kG \xrightarrow{c} k \rightarrow 0
$$

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where $T$ denotes multiplication by $T$, $(1 - g)$ denotes multiplication by $(1 - g)$, $\varepsilon$ is the augmentation map, and $k$ is a field of characteristic $p$. We wish to describe the map $\Delta^*: H^*(G \times G, k \otimes k) \to H^*(G, k \otimes k)$ explicitly. To do so, letting $X \to k$ denote the familiar periodic resolution above, we look at the $G \times G$-projective resolution $X \otimes X \to k$. What we need is a map of complexes $D: X \to X \otimes X$ satisfying $D(gx) = \Delta(g)D(x)$. We define this map $D$ by $D = \sum D_{r,s}$, and

$$D_{r,s}: X_{r+s} \to X_r \otimes X_s$$

defined by $D_{r,s}(x_{r+s}) = x_r \otimes x_s$ if $r$ is even, $D_{r,s}(x_{r+s}) = x_r \otimes g x_s$ if $r$ is odd and $s$ is even, and $D_{r,s}(x_{r+s}) = \sum_{0 \leq i, j < p} g^i x_r \otimes g^j x_s$ if $r$ and $s$ are both odd. Here, $x_{r+s}$ is a generator for the free rank 1 module $kG = X_{r+s}$.

Some things to check: Why does this map work? (If it does, it's guaranteed to be the right map up to chain homotopy.) Why does this map commute with the differentials in $X$ and $X \otimes X$?

Armed with our diagonal map, we may now compute products for our cyclic group $G$ using the cup product. Let $a$ and $b$ be elements that represent cohomology classes in degrees $r$ and $s$ respectively. We can compute the cup product using the following map, with $a \in \text{Hom}_G(X_r, k)$ and $b \in \text{Hom}_G(X_s, k)$:

$$\text{Hom}_G(X_r, k) \times \text{Hom}_G(X_s, k) \to \text{Hom}_{G \times G}(X_r \otimes X_s, k \otimes k) \xrightarrow{D^*} \text{Hom}_G(X_{r+s}, k \otimes k)$$

Then the cup product of these classes is $a \otimes b$ if $r$ or $s$ is even and $\sum_{0 \leq i, j < p} g^i a \otimes g^j b$ if $r$ and $s$ are odd (here the tensor product is taken over $k$). Notice that if $s$ is even, then $g b = b$ since the cohomology group is $kG$ an even degree. Now since $k$ is a trivial $G$-module, we have that the cup product is $ab$ if $r$ or $s$ is even and $0$ if $r$ and $s$ are odd. If $p = 2$, we get

$$H^*(G, k) = k[\eta, \xi | \deg \eta = 1, \deg \xi = 2]$$

Otherwise, we get

$$H^*(G, k) = k[\eta, \xi | \deg \eta = 1, \deg \xi = 2/(\eta^2)]$$

Now we can use the Kunneth formula to compute $H^*(\mathbb{Z}/p \times \mathbb{Z}/p, k)$. Let $M$ be a $kG$-module and $N$ a $kH$-module. Let $G$ and $H$ equal $\mathbb{Z}/p$ and assume a resolution $X$, as previously. Then $X \otimes X \to k$ is a $k(G \times H)$-projective resolution. Then the map of complexes

$$\text{Hom}_G(X, M) \otimes \text{Hom}_H(X, N) \to \text{Hom}_{G \times H}(X \otimes X, M \otimes N)$$

defined by $f \otimes g \mapsto f \times g$ where $(f \times g)(x \otimes y) = f(x) \otimes g(y)$ is an isomorphism. Thus

$$H^*(G \times H, M \otimes N) = H^*(\text{Hom}_G(X, M) \otimes \text{Hom}_H(X, N))$$

Apply the Kunneth formula: there is a split exact sequence

$$0 \to H^*(G, M) \otimes H^*(H, N) \to H^*(G \times H, M \otimes N) \to \text{Tor}_1^k(H^*(G, M), H^*(H, N)) \to 0$$

and since $k$ is a field, we get an isomorphism

$$H^*(G, M) \otimes H^*(H, N) \cong H^*(G \times H, M \otimes N)$$

Now we can actually compute any product of $\mathbb{Z}/p$. Say we have a $d$-fold product of cyclic groups of order $p$, call it $G' = G \times G \times G...$ Then

$$H^*(G', k) \cong H^*(G, k)^{\otimes d} = H^*(G, k) \otimes H^*(G, k) \otimes ...$$
We just computed what the ring $H^*(G, k)$ looks like, so $H^*(G', k)$ is a $d$-fold tensor product of such rings:

$$H^*(G', k) = \begin{cases} 
    k[x_1, \ldots, x_d : \deg x_i = 1], & p = 2 \\
    \Lambda[x_1, \ldots, x_d : \deg x_i = 1] \otimes k[y_1, \ldots, y_d : \deg y_i = 2], & p \neq 2
\end{cases}$$

3. An Unexpected Appearance by Symmetric Functions

The following comes from a presentation I gave in Sara Billey's combinatorics class on May 24, 2006. Many thanks to Steve Mitchell for tirelessly entertaining my questions and teaching me the material in the presentation. The goal is to try and understand the connection between the cohomology of the Grassmannian and the elementary symmetric functions. Along the way, we'll also be able to compute the cohomology of the flag variety.

Define $\Lambda^*_R$ to be the set of all homogeneous symmetric functions of degree $n$. There is a ring isomorphism

$$H^*(G(n, k), Z) \cong \Lambda^*_{\mathbb{Z}[x_1, \ldots, x_n]}$$

Let $F_n \mathbb{C}^{n+k}$ denote ordered $n$-tuples of pairwise orthogonal lines $(L_1, \ldots, L_N)$ in $\mathbb{C}^{n+k}$ for $0 \leq k \leq \infty$. This is often called a flag manifold or a flag variety. Note that $F_n \mathbb{C}^{n+k} \subset \mathbb{C}P^{n+k-1} \times \ldots \times \mathbb{C}P^{n+k-1}$ (n times). Let $G_n \mathbb{C}^{n+k}$ denote $n$-planes in $\mathbb{C}^{n+k}$. Look at the fiber bundle

$$F_n \mathbb{C}^n \to F_n \mathbb{C}^{n+k} \xrightarrow{\pi} G_n \mathbb{C}^{n+k}$$

where $\pi : (L_1, \ldots, L_n) \mapsto L_1 \oplus L_2 \oplus \ldots \oplus L_n$.

**Definition 1.** A map $E \xrightarrow{\pi} B$ is a local product with fiber $F$ if for all $b \in B$ there exists a neighborhood $U$ of $b$ and a homeomorphism $\pi^{-1}U \to U \times F$.

**Example 1.** A good example of a local product is the Möbius band, where $E$ is the Möbius band, $F$ is the unit interval $I$, $B = S^1$, and $\pi$ is the map projecting $E$ onto $B$. For every point on the circle, there is a neighborhood $U$ on which $\pi^{-1}U \cong U \times I$, so locally, the Möbius band looks like a product, but $E \not\cong S^1 \times I$ because of the twist.

In a local product, every fiber $\pi^{-1}b$ is homeomorphic to $F$. For $W \in G_n \mathbb{C}^{n+k}$, $\pi^{-1}W = \{(L_1, \ldots, L_n) : \oplus L_i = W\}$.

**Proposition 1.** $\pi$ is a local product with fiber $F_n \mathbb{C}^n$.

We need to introduce local products in order to state the Key Technical Theorem (KTT), from which all of the results will follow. The proof relies on spectral sequence arguments that I will omit.

**Theorem 1 (KTT).** Let $F \xrightarrow{i} E \xrightarrow{\pi} B$ be a local product with fiber $F$. Suppose

1. $B$ is simply connected.
2. $H_k B, H_k F$ are finitely generated free modules over $\mathbb{Z}$ for all $k$
3. $H_k B, H_k F$ are zero for odd $k$

Then

a) $H^*E$ is a free module over $H^*B$ ($\alpha \in H^*B, x \in H^*E, \alpha \circ x = (\pi^* \alpha)x$). In particular, $\pi^*$ is split injective.
where \( I \) is the ideal generated by \( \pi^*(\mathbb{Z} \delta B) \). In particular, \( i^* \) is surjective.

Recall that \( \mathbb{C} P^\infty \) denotes lines in \( \mathbb{C}^\infty \). Let \( j : F_n \mathbb{C}^\infty \hookrightarrow \mathbb{C} P^\infty \times \cdots \times \mathbb{C} P^\infty \).

**Proposition 2.** \( j^* : H^*((\mathbb{C} P^\infty)^n) \to H^*(F_n \mathbb{C}^\infty) \) is an isomorphism.

**Proof.** Follows by induction from the KTT. \( \square \)

We know that \( H^* \mathbb{C} P^\infty = \mathbb{Z}[y], |y| = 2 \), so by the Kunneth formula, \( H^*((\mathbb{C} P^\infty)^n) \cong \mathbb{Z}[y_1, \ldots, y_n] \), with \( |y_i| = 2 \). This gives us an expression for \( H^*(F_n \mathbb{C}^\infty) \) by Proposition 2. Now we're ready to state the first main result.

**Theorem 2.** \( \pi^* : H^*G_n \mathbb{C}^\infty \to H^*F_n \mathbb{C}^\infty \) is an isomorphism onto \( (H^*F_n \mathbb{C}^\infty)^S_n = \mathbb{Z}[\sigma_1, \ldots, \sigma_n] \), where the \( \sigma_i \) denote the elementary symmetric functions.

To see that \( \text{Im} \pi^* \subset (H^*F_n \mathbb{C}^\infty)^S_n \), the main observation is that \( \pi \) factors through \( F_n \mathbb{C}^\infty / S_n \), since two elements of \( F_n \mathbb{C}^\infty \) that differ by a permutation in \( S_n \) are sent to the same element of \( G_n \mathbb{C}^\infty \), namely the \( n \)-plane that they span. Consider the following diagrams:

\[
F_n \mathbb{C}^\infty \xrightarrow{\omega \in S_n} F_n \mathbb{C}^\infty \quad \Rightarrow \quad H^*F_n \mathbb{C}^\infty \xrightarrow{\omega \in S_n} H^*F_n \mathbb{C}^\infty
\]

From the diagram on the right, we can see that for \( \alpha \in H^*G_n \mathbb{C}^\infty \), \( \omega(\pi^*(\alpha)) = \pi^*(\alpha) \), and so the image of \( \pi^* \) must be invariant under the action of \( S_n \), as claimed.

We proceed to prove Theorem 1 in two steps:

1) Show that the ranks of \( H^*(G_n \mathbb{C}^\infty) \) and \( \mathbb{Z}[\sigma_1, \ldots, \sigma_n] \) (as graded free abelian groups) agree in each dimension.

2) Cite the KTT that the map \( \pi^* \) is a split injection.

Together, steps (1) and (2) will prove that \( \pi^* \) is an isomorphism. Why? A split injection guarantees that there is an isomorphism from \( \mathbb{Z}[\sigma_1, \ldots, \sigma_n] \) onto a direct summand of \( H^*(G_n \mathbb{C}^\infty) \), but if their ranks agree in each dimension, there can be no other summand but \( H^*(G_n \mathbb{C}^\infty) \) itself. The question remains: how can we obtain information about the ranks of the rings in each dimension?

As it turns out, there is a useful tool for doing exactly this called a Poincaré polynomial.

**Definition 2.** Suppose we have a graded free abelian group \( A = A_0 \oplus A_1 \oplus A_2 \oplus \cdots \) where each \( A_i \) has finite rank. Then the Poincaré series is given by

\[
f_A(t) = \sum_{n=0}^{\infty} (\text{rank}A_n)t^n \in \mathbb{Z}^+[|t|]
\]

The Poincaré series has the useful properties that \( f_{A \oplus B}(t) = f_A(t) + f_B(t) \) and \( f_{A \oplus B}(t) = f_A(t) \cdot f_B(t) \). For notational purposes, denote

\[
[n](t) = \frac{1-t^n}{1-t} = 1 + t + \cdots + t^{n-1}
\]
and let
\[ [X](t) = f_{\mathcal{H}}(X(t)) \]

**Corollary 1.** Under the conditions of the Key Technical Theorem (KTT), \([E] = [F'][B]\).

**Proof.** From part (b) of the KTT, we have an isomorphism \(H^*E \otimes_{H^*B} \mathbb{Z} \to H^*E\). By part (a), \(H^*E\) is free over \(H^*B\). Taken together, this means that there is a map \(H^*B \otimes H^*E \to H^*E\) which is an isomorphism. So by the multiplicative property of Poincaré series the result follows. \(\square\)

Our intent is to apply this corollary to the fibration \(F_nC^n \to F_nC^\infty \to G_nC^\infty\), which satisfies the conditions of the KTT. If we can deduce the Poincaré polynomials of \(F_nC^n\) and \(F_nC^\infty\) respectively, then we can solve for the Poincaré polynomial of \(G_nC^\infty\) and compare it to that of \(Z[\sigma_1, \ldots, \sigma_n]\).

**Lemma 1.** \([F_nC^n](t) = [n!](t^2)\)

**Proof.** Induct on \(n\). For \(n = 1\) the result is trivial, since \(F_1C^1\) is a point. For the inductive step we can apply Corollary 1 to the fibration \(F_{n-1}C^{n-1} \to F_nC^n \to CP^{n-1}\) where \(p : (L_1, \ldots, L_n) \mapsto L_n\).

The conclusion \([F_nC^n] = [F_{n-1}C^{n-1}][CP^{n-1}] = [(n-1)!(t^2)] \cdot [n](t^2) = [n!](t^2)\). \(\square\)

**Lemma 2.** \([F_nC^\infty](t) = (\frac{1}{1-t^2})^n\)

**Proof.** \(H^*F_nC^\infty = Z[y_1, \ldots, y_n]\) with \([y_i] = 2\). Since \(f_{Z[y]}(t) = 1 + t^2 + t^4 + \ldots = \frac{1}{1-t^2}\), and \(Z[y_1, \ldots, y_n] = Z[y] \otimes Z[y] \otimes \ldots \otimes Z[y]\) \((n\text{ times})\), the multiplicative property of Poincaré polynomials gives us \([F_nC^\infty](t) = (\frac{1}{1-t^2})^n\). \(\square\)

Look at the fibration \(F_nC^n \xrightarrow{i} F_nC^\infty \xrightarrow{\pi} G_nC^\infty\)

\(G_nC^\infty\) is a CW complex with only even-dimensional cells and only finitely many cells in each dimension, therefore we can apply Corollary 1:

\([F_nC^\infty](t) = [F_nC^n](t)G_nC^\infty](t)\)

We already know that \([F_nC^n] = [n!](t^2)\) and \([F_nC^\infty](t) = (\frac{1}{1-t^2})^n\), so we can solve for \([G_nC^\infty](t)\).

Letting \(t^2 = z\), we have

\[ [G_nC^\infty](z) = \frac{1}{(1-z)^n} \cdot \frac{(1-z)^n}{(1-z)(1-z^n)(1-z^n-1)\ldots(1-z)} = f_{Z[\sigma_1, \ldots, \sigma_n]}(z) \]

To see the last equality, note that the \((1-z)\) term in the denominator corresponds to the generator \(\sigma_1 = y_1 + y_2 + \ldots + y_n\) and the \((1-z^n)\) term to the generator \(\sigma_n = y_1\ldots y_n\). Part (a) of the KTT tells us that the map \(H^*G_nC^\infty \to Z[\sigma_1, \ldots, \sigma_n]\) is a split injection. But since the ranks agree in each dimension, it must be an isomorphism.

Now we can also say something about \(H^*F_nC^n\). We calculated \([F_nC^n]\), but we can get a much better description of the ring structure.

**Theorem 3.**

\[ i^* : H^*F_nC^n \xrightarrow{\cong} H^*F_nC^n \]

\[ Z[y_1, \ldots, y_n]/(\sigma_1, \ldots, \sigma_n) \]
Proof. 

\[ F_n \mathbb{C}^n \xrightarrow{i} F_n \mathbb{C}^\infty \xrightarrow{\pi} G_n \mathbb{C}^\infty \]

has the property that \( \pi \circ i \) is constant. Thus \( i^* j^* = 0 \), which implies \( i^* \sigma_k = 0 \), \( 1 \leq k \leq n \). This proves the existence of the dotted map. The fact that the map is an isomorphism follows from the KTT. We get an isomorphism \( H^* F_n \mathbb{C}^n \cong H^* F_n \mathbb{C}^\infty / I \) where \( I = (\pi^* (H^{>0} G_n \mathbb{C}^\infty)) \). By Theorem 1, \( I = (\sigma_1, ..., \sigma_n) \), which gives us the result. \( \square \)

4. Motivation for Quillen Stratification Theorem

This section arose from notes from a lecture given by Steve Mitchell in the Quillen seminar in Spring 2006.

Let \( G \) be a discrete group and fix a prime \( p \). We're interested in looking at

\[ H^*_{\text{gr}}(G, \mathbb{Z}/p) = \text{Ext}^*_{\mathbb{Z} G}(\mathbb{Z}, \mathbb{Z}/p) \]

There exists a space \( BG = K(G, 1) \) characterized by

- \( \pi_1 BG = G \)
- Its universal cover, denoted \( EG \), is contractible

(Note: this characterization depends on considering a discrete group \( G \).)

**Theorem 4.** \( H^*_{\text{gr}}(G, \mathbb{Z}/p) = H^*(BG, \mathbb{Z}/p) \)

The main case we're interested in is when \( G \) is finite. What can we say about the graded ring \( H^*(BG, \mathbb{Z}/p) = H^* BG \)?

**Proposition 3.** If \( p \) does not divide the order of \( G \), then \( H^* BG = 0 \).

**Theorem 5.** \( H^* BG \) is finitely generated as a \( \mathbb{Z}/p \)-algebra.

Let's look at some examples:

**Example 2.** This paper began with a detailed calculation of the cohomology of \( (\mathbb{Z}/p)^n \), but here is its ring structure again:

\[ H^*(G', k) = \begin{cases} k[x_1, ..., x_d : \deg x_i = 1], & p = 2 \\ \Lambda[x_1, ..., x_d : \deg x_i = 1] \otimes k[y_1, ..., y_d : \deg y_i = 2], & p \neq 2 \end{cases} \]

**Example 3.** \( H^* BQ_8 = \mathbb{Z}/2[x, y, w] / (x^2 + xy + y^2, xy^2 + y^2) \) with \( p = 2 \)

**Example 4.** \( H^* BD_8 = \mathbb{Z}/2[x, y, w] / (xy) \) with \( p = 2 \)

**Definition.** \( H^* BG \) is called periodic if for some \( d > 0 \) there is \( \alpha \in H^d BG \) such that multiplication by \( \alpha \) induces an isomorphism \( H^k BG \xrightarrow{\alpha} H^{k+d} BG \).

As an example, \( H^* BQ_8 \) is periodic with period 4. This comes from \( w \), which has degree 4.

**Theorem 6.** \( H^* BG \) is periodic if and only if every abelian \( p \)-subgroup of \( G \) is cyclic.

Notice:

1) Every abelian \( p \)-subgroup of \( G \) is cyclic if and only if \( \text{rank}_p G = 1 \), where \( \text{rank}_p G \) is the maximum rank \( k \) such that \((\mathbb{Z}/p)^k \subset G\).

2) \( H^* BG \) is periodic if and only if the Krull dimension of \( H^* BG \) is 1.

Atiyah and Swan conjectured that the Krull dimension of \( H^*(BG, \mathbb{Z}/p) = \text{rank}_p G \). In fact, this is part of Quillen's theorem:
Theorem 7 (Quillen, 1971). A) The Atiyah-Swan conjecture holds. B) Minimal primes in $H^*_G(X)$ are in bijective correspondence with conjugacy classes of maximal $p$-tori. C) $x \in H^*BG$ is nilpotent if $x$ restricts to zero on every $p$-torus.

Example 5. $Q_8$ has one conjugacy class of the 2-torus, namely its central element. The nilpotent elements are $x$ and $y$.

Example 6. $H^*D_8$, however, has no nilpotent elements, so we say that "$H^*BD_8$ is detected on $p$-tori".

Example 7. Let $G = GL_n(F_q)$, where $q = l^k$. We consider only the case where $p \neq l$, since what happens in the case that $p = l$ is a very difficult unsolved problem. Also assume that $p$ divides $q - 1$. Then any $p$-torus in $GL_n(F_q)$ is conjugate to a subgroup of the diagonal matrices $D_n$. There is a unique conjugacy class of maximal $p$-tori, namely those diagonal matrices where the diagonal entries are $p$th roots of unity.

Theorem 8 (Quillen stratification, rough form). Let $V_G$ be the variety defined by $H^*BG$. Then $V_G$ has a stratification into affine subvarieties, where the strata are indexed by conjugacy classes of $p$-tori.

There are some generalizations of the Quillen stratification theorem:
1) $G$ need not be finite; it can be any compact Lie group. For example, $H^*BU(n) = \mathbb{Z}/p[c_1, \ldots, c_n]$.
2) $BG$ can be replaced by any $EG \times_G X$, with $X$ a smooth compact $G$-manifold. ($BG$ is the case $X = \text{a point}$).
3) $G$ could be an "arithmetic group", e.g. $GL_n\mathbb{Z}$.

5. Finite Generation of Cohomology

This section arose from Luke Gutzwiller’s lecture in the Quillen seminar.

We turn our attention to proving the following:

Theorem 9. $H^*BG$ is finitely generated as a $\mathbb{Z}/p$-algebra.

There are two main ways to prove this theorem: topologically and algebraically. The topological proof is the one I will present here, following Venkov’s argument. The algebraic proof, due to Evens, involves the wreath product, tensor induction, and generally seems far more involved. For an exposition of Evens’ proof, see Julia Pevtsova’s notes, posted on her website (www.math.washington.edu/~julia).

First, we want $H^*(G, M) = H^*(BG, M)$, where $M$ is a $G$-module. When $G$ is discrete and finite, we can do this explicitly by giving a concrete construction of $BG$ as a simplicial complex: Start with $EG$, the universal cover of $G$, as a simplicial complex with $n$-simplices $EG^n = \{(g_0, \ldots, g_n) \in G^{n+1}\}$. $EG^n$ is contractible, so it has no cohomology. Define $BG = EG/G$. Then $H^*(G, M) = H^*(BG, M)$ as desired.


Proof. Let $C_n$ be the chain complex for $BG$ and $\tilde{C}_n$ the chain complex for $EG$. Denote $(EG)^n/G = S_n$. Then

$$H^*(BG, k) = H^*(Hom(C_n, k))$$

Note that $Hom(C_n, k) = F(S_n, k)$, $k$-valued functions on $S_n$. This gives

$$H^*(Hom(C_n, k)) = H^*(F(S_n, k)) = H^*(F((EG)^n, k))^G = H^*((Hom(\tilde{C}_n, k))^G)$$
But \( \mathcal{C} \) is a free resolution of \( k \) over \( ZG \): because \( EG \) is contractible, it has trivial cohomology, thus its chain complex is in fact a resolution (the sequence is exact). Thus we can write

\[
H^\bullet((\text{Hom}(\mathcal{C}, k))^G) = H^\bullet(\text{Hom}_{ZG}(\mathcal{C}, k)) = H^\bullet_{\text{grp}}(G, k)
\]

which completes the proof.

What about in general?

**Definition 4.** \( P \xrightarrow{\pi} X \) is a principal \( G \)-bundle if \( P \) is a free \( G \)-space, \( P \to X \) is the map \( P \\to P/G \), and \( X \) is covered by open sets \( U \) such that \( \pi^{-1}U \approx \mathbb{R} \), \( U \times G \) is a map that is equivariant under \( G \)-actions.

It is a fact that \( EG \to BG \) is a principal \( G \)-bundle, and it is also universal, meaning given a map from our total space \( P \) to our base space \( X \), there exists a map \( f \) (unique up to homotopy), \( f : X \to BG \), such that the induced map \( f : P \\to EG \) is a bundle map.

When \( G \) is a compact Lie group, \( BG \), its classifying space, is the base space of a universal principal \( G \)-bundle \( EG \to BG \) if and only if \( BG \) is the quotient of a contractible \( EG \) by a free \( G \)-action. For "nice" \( X \), there is a correspondence between isomorphism classes of principal \( G \)-bundles over \( X \) and \([X, BG]\).

**Example 8.** Take the unitary group \( U(n) \), \( EU(n) = V_n^\infty = \{(v_1, ..., v_n) \in (\mathbb{C}^\infty)^n | v_i v_j = \delta_{ij}\} \), which is contractible. \( BU(n) = EU(n)/U(n) = G_n^\infty \) and \( H^\bullet(BU(n), \mathbb{Z}) = \mathbb{Z}[c_1, ..., c_n] \) with \( |c_i| = 2i \). Recall that we computed this earlier; it is the cohomology of the Grassmannian.

When \( G \) is a finite group, \( G \subset S_{|G|} \subset O(n) \subset U(n) \), so we can clearly embed any finite group into \( U(n) \). In fact, we can embed an arbitrary compact Lie group \( G \) into \( U(n) \) (not obvious). In any case, the point is that we can then use \( V_n^\infty \) as \( EG \), whence \( BG = EU(n)/G \) exists. The fact that we can find a contractible space with a free \( G \)-action \( (EG) \) by "stealing" it from \( U(n) \) is the key to the proof of the theorem, which we now prove.

**Proof of Theorem 9.**

\[
U(n) \to EU(n) \to BU(n)
\]

is a fibration, so

\[
U(n)/G \to EU(n)/G = BG \to BU(n)
\]

is also a fibration. Look at the Serre spectral sequence for its cohomology:

\[
E_2 = H^\bullet(BU(n)) \otimes H^\bullet(U(n)/G)
\]

as \( H^\bullet BU(n) \)-modules. \( E_2 \) is finitely generated as a module over the Noetherian ring \( H^\bullet BU(n) \), implying that all \( E_r \) pages are finitely generated. For \( r \) large enough, all differentials are zero, so \( E_\infty = E_r \) is finitely generated. This gives that \( H^\bullet BG \) is finitely generated over \( H^\bullet BU(n) \) (the associated graded object is finitely generated). Thus we can conclude that \( H^\bullet BG \) is finitely generated as an algebra. \( \square \)

**Corollary 2.** Suppose \( G \) is finite and \( p \) divides the order of \( G \). Then \( H^n(BG, \mathbb{Z}/p) \neq 0 \) for infinitely many \( n \).

**Proof.** If \( G \) is finite and \( p \) divides the order of \( G \), then \( G \) has a subgroup of order \( p \) and \( \mathbb{Z}/p \subset G \). Thus \( H^\bullet(B\mathbb{Z}/p, \mathbb{Z}/p) \) is finitely generated over \( H^\bullet(BG, \mathbb{Z}/p) \). \( \square \)

The corollary implies that we always need to take an infinite-dimensional complex to construct, say, a CW complex, and that the cohomology of finite groups will always be infinite-dimensional.

**Example 9.** \( B(\mathbb{Z}/m) = S^\infty/\mathbb{Z}/m \)
6. Approximating $H^*(G)$

This section arose from Anton Dochtermann’s lecture in the Quillen seminar.

We wish to use equivariant cohomology of $X$, a smooth compact $G$-manifold, to make statements about $H^*_G(X)$. Let $\mathcal{F}$ be a set of subgroups of $G$ closed under conjugation, and let $\mathcal{A}$ be the set of $p$-tori in $G$. We define a category $\mathcal{C}_{tr}$: its objects are pairs $(H, a)$ where $H \subseteq F$ and $a$ is a path component of $X^H$, and its morphisms take a pair of pairs $((H, a), (K, b))$ to the set $\{ x \in G : xHx^{-1} \subseteq K, xa \supseteq b \}$.

Note that when $X$ is a point, the objects of $\mathcal{C}_{tr}$ are elements of $\mathcal{F}$ and its morphisms are inner monomorphisms.

Next let $\tau$ be a functor from $\mathcal{C}_{tr}$ to graded $\mathbb{Z}/p$ algebras given by $\tau((H, a)) = H^*(BH), \tau(c_y) = c^*_y,$ where $c_y$ denotes conjugation by an element $y$ and $c^*_y$ is the induced map on cohomology. We define $\mathcal{F}_c^*(X)$ to be the limit of $\tau$ over the category $\mathcal{C}_{tr}$. Then an element of $\mathcal{F}_c^*(X)$ is a function $f$ that assigns to each element $(H, a)$ an element of $H^*(BH)$ with compatibility: that is, $f((H, a)) = c^*_y f((K, b))$.

When $X$ is a point and $G$ has a unique conjugacy class of maximal $p$-tori, we get a particularly simple expression for $\mathcal{A}_c^*(X)$. If $A$ is the representative of the unique conjugacy class of maximal $p$-tori, then every element of $\mathcal{A}_c^*(X)$ is determined by its image at $A$ and we get $\tau(A) = H^*(BA)$, implying

$$\mathcal{A}_c^*(X) = (H^*(BA))^{W_G(A)}$$

where $W_G(A)$ is the Weyl group of $A$.

**Theorem 10.** For any smooth compact $G$-manifold $X$, the map

$$\rho_x : H^*_G(X) \rightarrow \mathcal{A}_c^*(X)$$

is an isogeny. That is, $\rho_x$ is an $F$-isomorphism with the property that if $\alpha \in \ker \rho_x$, then $\alpha$ is nilpotent, and if $\beta \in \mathcal{A}_c^*(X)$, then $\beta^k \in \text{im}(\rho_x)$.

7. Some Examples

Many thanks to Steve Mitchell and Julia Pevtsova for helping me prepare the following two examples, presented in the Quillen seminar.

**Example 10.** Let $G = A_4$ and $X$ be a point. What is $H^*BG$? We have a short exact sequence

$$\mathbb{Z}/2 \rightarrow A_4 \rightarrow \mathbb{Z}/3$$

From cohomology, we have the following theorem:

**Theorem 11.** Suppose $H$ is a normal subgroup of $G$, and $p$ does not divide the order of $G/H$.

If we are computing cohomology with $\mathbb{Z}/p$ coefficients, then the map $H^*G \rightarrow H^*H$ is an isomorphism onto $(H^*H)^{G/H}$.

Applied to $A_4$, the theorem tells us (since $2$ does not divide $3$) that $H^*(A_4) \cong (H^*(\mathbb{Z}/2)^2)^{\mathbb{Z}/3}$.

What about the right hand side of the isogeny? $A_4$ has three 2-tori of rank 1 and its maximal 2-torus, $(\mathbb{Z}/2)^2$. Since there is only one maximal 2-torus and all the other 2-tori sit inside it, we are in the aforementioned case and

$$\mathcal{A}_c^* = (H^*((\mathbb{Z}/2)^2))^{W_G(A)}$$

What is the Weyl group? $A_4$ is generated by $(12)(23)$ and a cycle of length 3, say $(123)$. The subgroup $(\mathbb{Z}/2)^2$ can be written in cycle notation as $(e, (12)(34), (13)(24), (14)(23))$. Since this
subgroup is normal and the cycle (123) normalizes the subgroup, we can take $e$, (123), and $(132) = (123)^2$ to be coset representatives for $A_4/(\mathbb{Z}/2)^2$. To see how $\mathbb{Z}/3$ acts on the subgroup, we multiply:

$$(123)(12)(34)(132) = (13)(24)$$
$$(123)(14)(23)(132) = (12)(34)$$

and observe that the $\mathbb{Z}/3$ action permutes the three nontrivial elements of $(\mathbb{Z}/2)^2$. Now we can compute the Weyl group. The normalizer of $(\mathbb{Z}/2)^2$ is $A_4$ itself since $(\mathbb{Z}/2)^2$ is normal. The centralizer is $(\mathbb{Z}/2)^2$ itself. To see this, recall that any element in $A_4$ can be written as a power of the cycle (123) and an element of $(\mathbb{Z}/2)^2$. If we try to conjugate such an element by an element from $(\mathbb{Z}/2)^2$, we will get one of the results from the formulas above, unless the power of (123) was trivial. Then

$$W_{A_4}((\mathbb{Z}/2)^2) = N_{A_4}((\mathbb{Z}/2)^2)/C_{A_4}((\mathbb{Z}/2)^2) = A_4/(\mathbb{Z}/2)^2 = \mathbb{Z}/3$$

Thus the Weyl group is $\mathbb{Z}/3$ and so the isogeny is in fact an isomorphism.

**Example 11.** Now let’s look at an example where the isogeny is not an isomorphism, and there is a layer of nonmaximal $p$-tori that make a difference in computing $A_4^p$.

Let $G = GL(3, \mathbb{F}_p)$, and assume $p > 2$. Let $E$ be a maximal torus in $G$. We wish to identify candidates for $E$, and we can start by examining possible Jordan forms for 3x3 matrices. Since we’re really looking for conjugacy classes of $E$, if a matrix $A$ has Jordan form $t$, then there exists a $B \in G$ such that $BAB^{-1} = t$. This means that we can consider the Jordan forms themselves to be representatives for the maximal $p$-tori, making calculations much easier. We consider two cases.

**CASE I.** $E$ contains an element of the standard Jordan form

$$
\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}
$$

Then there exists a matrix $A \in G$ such that conjugation by $A$ takes the corresponding element to its standard Jordan form. Therefore, the subgroup $AEA^{-1}$ contains the element $t =
\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}$.

Since we are only interested in the conjugacy class of $E$, we may assume that $E$ itself contains $t$.

Next, we compute the centralizer of $t$ in $G$. We find

$$C_G(t) = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix} \mid a, b, c \in \mathbb{F}_p, a \neq 0 \right\}$$

We check that this is an abelian group. The subgroup of all elements of order $p$ has the form

$$E_1 = \left\{ \begin{pmatrix} 1 & b & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid b, c \in \mathbb{F}_p \right\}$$

The 1’s come from the fact that raising $E_1$ to the $p$th power will cause the diagonal entries $a$ to be raised to the $p$th power. The only element in $\mathbb{F}_p^*$ with $a^p = 1$ is 1. Since any torus containing $t$ must belong to the centralizer of $t$, and must consist of elements of order $p$, we conclude that $E_1$ is a maximal torus. It is isomorphic to $\mathbb{Z}/p \times \mathbb{Z}/p$.

We need to compute $N_G(E_1)$ and $C_G(E_1)$. Since $C_G(E_1) \subseteq C_G(t)$, and since $C_G(t)$ centralizes $E_1$, we get right away that $C_G(E_1) = C_G(t)$. For the Normalizer, it is convenient to first compute for which matrices $A \in G$ we may have $ATA^{-1} \in E_1$. This alone gives the condition that
A must be upper-triangular: \( A = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \) plus an extra condition \( af = d^2 \). Then we check that any upper triangular matrix of this form normalizes \( E_1 \). Thus, we get \( N_G(E_1) = \left\{ \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \mid a, b, c, d, f \in \mathbb{F}_p, af = d^2, afd \neq 0 \right\} \). We can write

\[
\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & d/a & x \\ 0 & 0 & f/a \end{pmatrix} \begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix}
\]

for some \( x \in \mathbb{F}_p \). Thus, we can think of \( W_G(E_1) \) as \( \left\{ \begin{pmatrix} s & x \\ 0 & s^2 \end{pmatrix} \mid s \in \mathbb{F}_p^*, x \in \mathbb{F}_p \right\} \).

**CASE II.** Now we assume that the maximal \( p \)-torus \( E \) does NOT have an element of standard Jordan form \( \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \). In other words, any non-trivial element in \( E \) has standard Jordan form \( \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \). Pick any non-trivial element in \( E \) and find a matrix \( A \) which conjugates that element to the matrix \( \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \). This is possible since this matrix has the same Jordan form as \( \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \). Replacing \( E \) by \( AEA^{-1} \) we assume that \( s = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \) is in \( E \).

Computing the Centralizer of \( s \) in \( G \) we find that it has the form

\[
C_G(s) = \left\{ \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d, e \in \mathbb{F}_p, a^2 d \neq 0 \right\}
\]

Let \( s' \) be some other element in \( E \). Then \( s' \) commutes with \( s \), and, therefore, has the form as above. We also have \( (s')^p = 1 \). Thus, \( a = d = 1 \), i.e. \( s = \begin{pmatrix} 1 & b & c \\ 0 & 1 & d \\ 0 & 0 & 1 \end{pmatrix} \) And we also have that the standard Jordan form of \( s \) is \( \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \). This implies that \( (s' - 1)^2 = 0 \), and therefore \( bd = 0 \). Hence, \( b = 0 \) or \( d = 0 \). We next observe that two matrices of the form \( \begin{pmatrix} 1 & b & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \)

and \( \begin{pmatrix} 1 & 0 & c' \\ 0 & 1 & d \\ 0 & 0 & 1 \end{pmatrix} \) where BOTH \( b, d \) are non-zero do NOT commute. This allows us to conclude
that there are two different maximal \( p \)-tori inside \( G(s) \):

\[
E_2 = \left\{ \begin{pmatrix} 1 & b & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, b, c \in \mathbb{F}_p \right\}
\]

and

\[
E_3 = \left\{ \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & d \\ 0 & 0 & 1 \end{pmatrix}, c, d \in \mathbb{F}_p \right\}
\]

There are still some things to do:
- one has to check that \( E_2, E_3 \) give two different conjugacy classes, i.e. that they are not conjugate
- find centralizers, normalizers and Weyl groups for \( E_2, E_3 \)
- to get the full structure of the category \( \mathcal{F}_G^\ast \) we need to list the tori of rank 1 (isomorphic to \( \mathbb{Z}/p \)) and how they are embedded into the classes of maximal tori. All three of the maximal tori will have lots of cyclic subgroups but the only one which embed into at least two (and, in fact, into all three) is

\[
\left\{ \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, c \in \mathbb{F}_p \right\}
\]

This subgroup, call it \( E_4 \), is invariant under the action of the Weyl groups (again, this must be checked).

Note: The category \( \mathcal{A}_G^\ast \) will consist of three rings, \( k[x_1, x_2]^W(E_1) \), \( k[y_1, y_2]^W(E_2) \), and \( k[z_1, z_2]^W(E_3) \) mapping down to \( k[u]^W(E_4) \), and from there we can take the inverse limit. Note, however, that we can already ascertain some things just from the diagram of the category: first, the Krull dimension is 2, and second, if we consider the rings as varieties, the identification resulting from taking the inverse limit is a line lying in three planes.

This example demonstrates how the Quillen stratification theorem is highly imperfect as an approximation: here, the Weyl groups were difficult to compute and though we got a general geometric picture of the ring structure of the cohomology as three planes intersecting in a line, we didn’t even get a ring as the approximation in the end.

**Example 12.** For a more successful example, we turn to \( D_8 \). Earlier we mentioned that \( H^*D_8 \) is detected on \( p \)-tori – let’s see how this works \( (p = 2) \). (Thanks to Anton Dochtermann for discussing this example with me.)

\( D_8 \) has five 2-tori of rank 1 and two 2-tori of rank 2. Let \( a \) denote vertical reflection, \( b \) horizontal reflection, \( c \) a rotation by \( \pi \), and \( d \) and \( e \) diagonal reflections. Then we have subgroups \( \langle 1, a, b, c \rangle, \langle 1, c, d, e \rangle, \langle a \rangle, \langle b \rangle, \langle c \rangle, \langle d \rangle, \langle e \rangle \) and \( C \) looks like:

```
  \langle 1, a, b, c \rangle
    \langle a \rangle
      \langle b \rangle
    \langle c \rangle
  \langle 1, c, d, e \rangle
    \langle d \rangle
      \langle e \rangle
```

Taking cohomology, we get:
for $1 \leq i \leq 2$, since $H^*(\mathbb{Z}_2) = \mathbb{Z}_2[x_1]$. The maps on $\mathbb{Z}_2[y_1, y_2]$ are identical, replacing the variables. The map from $\mathbb{Z}_2[x_1, x_2]$ to itself sends $x_1 \mapsto x_2$, and similarly for $\mathbb{Z}_2[y_1, y_2]$. When we take the inverse limit, we look for pairs $(f(x_1, x_2), g(y_1, y_2))$ such that $f$ and $g$ are symmetric functions and the elements of the pair get mapped to the same element of $\mathbb{Z}_2[c]$.

Now we can justify ignoring the rings $\mathbb{Z}_2[a]$, $\mathbb{Z}_2[b]$, $\mathbb{Z}_2[c]$, and $\mathbb{Z}_2[d]$. Take the $\mathbb{Z}_2[a]$ ring and suppose the map sending $x_1 \mapsto a$ and $x_2 \mapsto 0$ is called $h$. Then $h(0, x) = h(x, 0)$ since we have to take into account the isomorphism permuting the variables in $\mathbb{Z}_2[x_1, x_2]$. Notice that this is a weaker restriction than we already have, so when we take the inverse limit we don’t have to worry about the other four 2-tori.

Finally, what do we get? Recall that the set of elementary symmetric polynomials in $n$ variables form a basis for the ring of polynomials in $n$ variables. Thus the inverse limit is $X = \langle (x_1 + x_2, 0), (0, y_1 + y_2), (x_1x_2, y_1y_2) \rangle$. Renaming the generators $(x_1 + x_2, 0) = x$, $(0, y_1 + y_2) = y$, and $(x_1x_2, y_1y_2) = z$, we can rewrite $X$ as $\mathbb{Z}_2[x, y, z]/(xy)$ where $|x| = |y| = 1$ and $|z| = 2$. This ring is in fact $H^* D_8$ (recall Example 4), so we have demonstrated another case in which the theorem gives a perfect approximation.

8. Acknowledgments

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References