Combinatorial Algorithms in the Character Theory of the Symmetric Group

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1 Introduction

In this paper I will attempt three goals. The first is to provide an overview of representation theory aimed at an undergraduate who has taken at least one course in group theory. My second goal is to use the foundations introduced in the overview to explore the beautiful field of symmetric group representations. Finally, in the last section I will introduce an open problem in this field and present some results from my own research. For those that do not have a background in group theory and yet would like to approach this material, I recommend consulting Chapters 2 and 6 of Artin’s ”Algebra” [1]. This work does not require any prior knowledge of combinatorics, but for those who would like an introduction to the subject I would suggest Richard Brualdi’s ”Introductory Combinatorics” [2].

2 An Overview of Group Representations

2.1 Basic Concepts

Representation theory is a useful tool in the study of finite groups since it allows us to view a group as the set of linear transformations of a vector space. In this way we can convert questions in group theory into questions in linear algebra, which might be easier to solve. Depending on the source, representation theory can be discussed either in terms of matrices or modules. In this paper we will primarily stick to the matrix language, but for several results, we will need to also make use of the module interpretation. Also note that when we discuss vector spaces here, we will be assuming that we are working over \( \mathbb{C} \).

To begin with, we define a matrix representation of a group.

Definition 2.1. A matrix representation of a group \( G \) is a group homomorphism \( \Phi : G \rightarrow GL_d \). Where \( GL_d \) is the general linear group of invertible matrices.

Note that we call \( d \) the degree of the representation and will later denote it as \( \text{deg}\Phi \).

Example 2.2. Take the two element cyclic group \( G = \{ \epsilon, g \} \). Obviously, if we let \( \Phi : G \rightarrow GL_d \) be defined by \( \Phi_1(\epsilon) = \Phi_1(g) = 1 \), then this satisfies the requirements of a homomorphism, since for any combinations of \( g_1, g_2 \) belonging to \( G \), \( \Phi_1(g_1g_2) = 1 \neq 1(1) = \Phi_1(g_1)\Phi_1(g_2) \). Another representation however is found by letting \( \Phi_2 : G \rightarrow GL_d \) be defined by \( \Phi_2(\epsilon) = 1 \) and \( \Phi_2(g) = -1 \). Note that

\[
\begin{align*}
\Phi_2(\epsilon) &= \Phi_2(g) = -1 = (-1)1 = \Phi_2(g)\Phi_2(\epsilon) \\
\Phi_2(\epsilon) &= \Phi_2(g) = -1 = 1(-1) = \Phi_2(\epsilon)\Phi_2(g) \\
\Phi_2(\epsilon) &= \Phi_2(g) = 1 = (-1)(-1) = \Phi_2(g)\Phi_2(g)
\end{align*}
\]
\[ \Phi_2(\epsilon) = \Phi_2(\epsilon) = 1 = 1(1) = \Phi_2(\epsilon)\Phi_2(\epsilon). \]

So \( \Phi_2 \) is certainly a group homomorphism. In this case both \( \Phi_1 \) and \( \Phi_2 \) map into the set of \( 1 \times 1 \) matrices, but as we will see this is not the case in general.

We have succeeded in jumping from an arbitrary group \( G \) to a group of invertible matrices \( \Phi(G) \). Recall now from linear algebra that matrices are linear transformations of a vector space. This suggests that we can also view a group \( G \) as the linear transformations of some vector space. In light of this we define a \( G \)-module as follows.

**Definition 2.3.** Let \( V \) be a vector space. Then \( V \) is a \( G \) \(-\) module if there exists a homomorphism \( \Phi : G \rightarrow GL(V) \), where \( GL(V) \) is the set of all invertible, linear transformations from \( V \) to itself.

**Example 2.4.** One \( G \)-module that exists for every group is the one formed by the group acting on itself. If \( G \) is a finite group, then the elements \( g_i \) in \( G \) act on the vector space

\[ C[G] = \{ a_1 \epsilon + a_2 g_1 + a_3 g_2 + \ldots + a_{n+1} g_n : a_i \in \mathbb{C} \} \]

by

\[ g_i(a_1 \epsilon + a_2 g_1 + \ldots + a_{n+1} g_n) = a_1 g_i \epsilon + a_2 g_i g_2 + \ldots + a_{n+1} g_i g_n \]

where \( g_i g_j \) is the usual multiplication operation of \( G \).

It can be verified that this fulfills the requirements above for a \( G \)-module. We call it the left regular representation. The right representation can be gotten by multiplying from the other side.

In many cases it is important to be able to move readily between the matrix and the \( G \)-module interpretations. Luckily, such a transition is not overly cumbersome. It can be done as follows. Take a group \( G \). If \( V \) is a \( G \)-module, we can take any basis \( \mathcal{B} \) of \( V \). Then we calculate \( \Phi(g) \) to be the matrix of the linear transformation \( g \in G \) in the basis \( \mathcal{B} \). If \( G \) has a matrix representation \( \Phi \) of degree \( d \), let \( V \) be the vector space \( V^d \) consisting of all column vectors of length \( d \). Then we define \( gv \) by \( \Phi(g)v \), where we use the normal matrix multiplication. So it is relatively simple to convert back and forth.

Now it is natural to ask how many distinct representations there are for a given group. We found two in the example above, but we can construct another by letting

\[ \Phi(\epsilon) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \Phi(g) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \]

To help answer this question we need the following definition.

**Definition 2.5.** Let \( V \) be a \( G \)-module. Then a subset \( W \) of \( V \) is called a \( sub-module \) if \( W \) is closed under the action of \( G \).
Equivalently, this means that a subset $W$ of $V$ is a sub-module if it is a G-module independent of $V$. It turns out that breaking representations down into submodules is important to understanding the fundamental properties of a group’s representations. This sort of exercise is common in mathematics, we do a similar thing when breaking a group down into subgroups in order to understand the structure of the original group. One might ask to what extent we can break a G-module down into submodules. Is it possible to do this indefinitely? The answer turns out to be no. This leads to the concept of reducibility. We phrase it both in terms of G-modules and matrices.

**Definition 2.6.** [8] A matrix representation $\Phi: G \to \text{GL}_d$ is called reducible if there is a basis for $V$, $B$, such that for every $g \in G$, $\Phi(g)$ has block matrix form.

$$\Phi(g) = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$$

Where $A$ and $C$ are square matrices (possibly of different size), $0$ is a zero matrix, and $B$ need not be a square matrix. A G-module is irreducible if it does not contain any nontrivial submodules.

Pursuing further this strategy of isolating submodules from a vector space $V$, we have the following definition.

**Definition 2.7.** [8] Let $V$ be a vector space, and let $W$ and $U$ be subspaces of $V$. Then $V$ is the direct sum of $U$ and $W$, written as $U \oplus W = V$ if every $v \in V$ can be written uniquely as $u + w = v$ for $u \in U$ and $w \in W$. If $V$ is a G-module and $U$ and $W$ are G-submodules, then we say that $U$ and $W$ are complements.

Equivalently, if $X$ is a matrix, then $X$ is the direct sum of matrices $A$ and $B$, written $A \oplus B = X$, if $X$ has the block diagonal form

$$X = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$  

Note that if $A$ is repeated $m$ times in the direct sum we write

$$A_1 \oplus A_2 \oplus \ldots \oplus A_m \oplus B = mA \oplus B.$$ 

Now we may state the immensely important Maschke’s Theorem.

**Theorem 2.8.** (Maschke’s Theorem) Let $G$ be a finite group, and let $V$ be a nonzero G-module. Then

$$V = W_1 \oplus W_2 \oplus \ldots \oplus W_k$$

where each $W_i$ is an irreducible G-submodule of $V$. In terms of matrices, if $G$ is a finite group and $\Phi$ is its matrix representation with dimension $d > 0$, then
there is a fixed matrix \( T \) such that every \( \Phi(g), \ g \in G \) has the form

\[
T\Phi(g)T^{-1} = \begin{bmatrix}
\Phi_1(g) & 0 & 0 & \ldots & 0 \\
0 & \Phi_2(g) & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \Phi_k(g)
\end{bmatrix}
\]

where each \( \Phi_i \) is an irreducible matrix representation of \( G \).

This implies that the representation of every finite group with non-zero dimension is what we called completely reducible.

We have looked at the maps from groups to linear transformations of vector spaces and maps from groups to groups of invertible matrices, but what about maps between different G-modules? This question motivates the following definition.

**Definition 2.9.** Let \( V \) and \( W \) be \( G \)-modules. A \( G \) - homomorphism \( \beta: V \rightarrow W \) is a linear transformation from \( V \) to \( W \) such that for all \( g \in G \) and \( v \) in \( V \) we have

\[
\beta(gv) = g\beta(v).
\]

In terms of matrices, this says that a matrix representation \( \Phi \) is homomorphic to a matrix representation \( \Theta \) if there is a matrix \( T \) such that for all \( g \in G \) and \( v \) in \( V \),

\[
T\Phi(g)v = \Theta(g)Tv.
\]

If \( \beta \) is bijective, then we say that \( V \) and \( W \) are \( G \)-isomorphic. In this case \( T \) has to be invertible, and hence we can say that \( T\Phi(g)T^{-1} = \Theta(g) \).

### 2.2 Group Characters

In mathematics we often look for properties of objects that remain invariant under some set of actions. It turns out that the irreducible representations of a group have such a property, one that is familiar from linear algebra.

**Definition 2.10.** Let \( g \in G \), and let \( \Phi \) be a matrix representation of \( G \). Then the character \( \chi_\Phi \) is the trace of \( \Phi(g) \), denoted \( tr \Phi(g) \).

If \( V \) is a \( G \)-module, then the character of \( V \) is the trace of the matrix representation of \( V \).

Now, it is not immediately clear that the character of a \( G \)-module \( V \) is well defined since there is often more than one way to represent \( V \) in terms of matrices. However, recall from above that if two matrix representations \( \Phi \) and \( \Theta \) are isomorphic, then there exists a matrix \( T \) such that for every \( g \in G \),

\[
\Theta(g) = T\Phi(g)T^{-1}.
\]

From linear algebra though, we know that the trace of a matrix is invariant under conjugation, \( tr(\Theta(g)) = tr(T\Phi(g)T^{-1}) = tr(\Theta(g)) \). Therefore giving us the fact that isomorphic representations have the same character.
Group characters have some other nice properties. One of these is the following.

**Theorem 2.11.** Let $\Phi$ be a matrix representation of a group $G$. Let $K$ be a conjugacy class of $G$. Then if $g, h \in K$, $\chi_\Phi(g) = \chi_\Phi(h)$.

**Proof.** If $g, h \in K$, then for some $k \in G$, $kgk^{-1} = h$. So $\Phi(kgk^{-1}) = \Phi(h)$ which implies that $\Phi(k)\Phi(g)\Phi(k^{-1}) = \Phi(h)$ (because $\Phi$ is a homomorphism). Then letting $T = \Phi(k)$, which also means that $T^{-1} = \Phi(k^{-1})$, we get that $T\Phi(g)T^{-1} = \Phi(h)$. We saw before that $\text{tr}(T\Phi(g)T^{-1}) = \text{tr}(\Phi(h))$ which implies that $\chi_\Phi(g) = \chi_\Phi(h)$. \qed

Because all the elements of a given conjugacy class $K$ in $G$ have the same character, if for a given representation one knows the character for at least one element of every conjugacy class, one knows the character for every element in that group. Through this naturally comes the concept of a character table.

**Definition 2.12.** If $G$ is a group, then the **character table** of $G$ is a matrix where for each conjugacy class there is a corresponding column and for each irreducible representation there is a corresponding row. A given cell then tells us the value of a character for a given conjugacy class and a given irreducible representation.

**Example 2.13.** Using our first example, we know that two representations of $G = \{e, g\}$ are

$$\Phi_1(e) = \Phi_1(g) = 1,$$

and

$$\Phi_2(e) = 1, \Phi_2(g) = -1.$$  

Obviously the characters of these representations would be

$$\chi_{\Phi_1}(e) = \chi_{\Phi_1}(g) = 1,$$

and

$$\chi_{\Phi_2}(e) = 1, \chi_{\Phi_2}(g) = -1.$$  

Now $G$ has two conjugacy classes, one containing $e$ and the other containing $g$. It also turns out that $\Phi_1$ and $\Phi_2$ are the only irreducible representations. Therefore the character table of $G$ is

$$\begin{bmatrix}
1 & 1 \\
1 & -1
\end{bmatrix}.$$  

We can manipulate these characters in such a way as to determine whether they are irreducible. If we take the standard inner product $((c_1, c_2, \ldots, c_n), (d_1, d_2, \ldots, d_n)) = c_1d_1 + c_2d_2 + \cdots + c_n d_n$ of the two rows in the example above we get

$$((1, 1), (1, -1)) = 1 \cdot 1 + (-1) \cdot 1 = 0$$

$$((1, 1), (1, 1)) = 1 \cdot 1 + 1 \cdot 1 = 2$$
\[(1, -1), (1, -1)\] = 1 \cdot 1 + (-1) \cdot (-1) = 2.

Let us define the inner products between characters as

\[
\langle \chi_\Phi, \chi_\Theta \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_\Phi(g) \overline{\chi_\Theta(g)}.
\]

This definition is equivalent to

\[
\langle \chi_\Phi, \chi_\Theta \rangle = \frac{1}{|G|} \sum_{K} |K| \chi_\Phi(K) \overline{\chi_\Theta(g)} K
\]

because for any two \(g_1, g_2 \in G\), where \(g_1\) and \(g_2\) are in the same conjugacy class then \(\chi_\Theta(g_1) = \chi_\Theta(g_2)\) and the same follows for \(\chi_\Phi\). This brings us to the following useful theorem.

**Theorem 2.14.** Let \(\chi_\Phi\) and \(\chi_\Theta\) be irreducible representations. Then

\[
\langle \chi_\Phi, \chi_\Theta \rangle = \delta_{\Phi, \Theta}
\]

where \(\delta_{\Phi, \Theta}\) is the Kronecker delta.

From this result we can prove a host of interesting facts about the characters of irreducible representations.

**Corollary 2.15.** Let \(\Phi\) be a matrix representation of \(G\) with character \(\chi_\Phi\). Then suppose that \(\Phi\) is isomorphic to the direct product \(m_1 \Phi_1 \oplus m_2 \Phi_2 \oplus \cdots \oplus m_k \Phi_k\), where each \(\Phi_i\) is irreducible and are pairwise inequivalent. Then

1. \(\chi_\Phi = m_1 \chi_{\Phi_1} + m_2 \chi_{\Phi_2} + \cdots + m_k \chi_{\Phi_k}\).
2. \(\langle \chi_\Phi, \chi_\Phi \rangle = m_1^2 + m_2^2 + \cdots + m_k^2\).
3. \(\Phi\) is irreducible if and only if \(\langle \chi_\Phi, \chi_\Phi \rangle = 1\).
4. Let \(\Psi\) be another matrix representation of \(G\). Then \(\Psi\) is isomorphic to \(\Phi\) if and only if \(\chi_\Psi(g) = \chi_\Phi(g)\) for all \(g \in G\).

**Proof.** 1. It should be obvious that the trace of the direct sum of \(m_1\) matrices of \(\chi_{\Phi_1}\), \(m_2\) matrices of \(\chi_{\Phi_2}\), ..., \(m_k\) matrices of \(\chi_{\Phi_k}\) is the same as the sum of their traces, each multiplied by \(m_1, m_2, \ldots, m_k\) respectively.

\[
tr \left[ \begin{array}{ccccccc}
\Phi_{1,1}(g) & 0 & 0 & \cdots & 0 & 0 \\
0 & \Phi_{1,2}(g) & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & 0 \\
0 & 0 & \cdots & \Phi_{2,1}(g) & \ddots & 0 \\
0 & 0 & 0 & 0 & \cdots & \Phi_{k,m_k}(g)
\end{array} \right]
\]
\[= tr \Phi_{1,1} + tr \Phi_{1,2} + \cdots + tr \Phi_{1,m_1} + tr \Phi_{2,1} + \cdots + tr \Phi_{k,m_k}\]
\[= m_1 tr \Phi_1 + m_2 tr \Phi_2 + \cdots + m_k tr \Phi_k = m_1 \chi_{\Phi_1} + m_2 \chi_{\Phi_2} + \cdots + m_k \chi_{\Phi_k}.
\]

2. By definition we have that
\[
\langle \chi_{\Phi}, \chi_{\Phi} \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_{\Phi}(g) \bar{\chi}_{\Phi}(g)
\]
\[= \frac{1}{|G|} \sum_{g \in G} (m_1 \chi_{\Phi_1}(g) + m_2 \chi_{\Phi_2}(g) + \cdots + m_k \chi_{\Phi_k}(g)) (m_1 \bar{\chi}_{\Phi_1}(g) + m_2 \bar{\chi}_{\Phi_2}(g) + \cdots + m_k \bar{\chi}_{\Phi_k}(g))
\]
\[= \frac{1}{|G|} \sum_{g \in G} \sum_{i,j} m_i m_j \chi_{\Phi_i}(g) \bar{\chi}_{\Phi_j}(g) = \sum_{i,j} m_i m_j \langle \chi_{\Phi_i}, \chi_{\Phi_j} \rangle.
\]

For all \(i \neq j\), \(\chi_{\Phi_i}(g) \bar{\chi}_{\Phi_j}(g) = 0\). Hence, the above expression simplifies to
\[
\frac{1}{|G|} \sum_{g \in G} (m_1^2 \chi_{\Phi}(g) + m_2^2 \chi_{\Phi}(g) + \cdots + m_k^2 \chi_{\Phi}(g)).
\]
Since \(\chi_{\Phi_i}(g) \bar{\chi}_{\Phi_i}(g) = 1\), we have
\[
\frac{1}{|G|} \sum_{g \in G} (m_1^2 \chi_{\Phi}(g) + m_2^2 \chi_{\Phi}(g) + \cdots + m_k^2 \chi_{\Phi}(g)) = m_1^2 + m_2^2 + \cdots + m_k^2.
\]

3. From the previous theorem, we already know that if \(\Phi\) is irreducible then \(\langle \chi_{\Phi}, \chi_{\Phi} \rangle = 1\). Now we show that if \(\langle \chi_{\Phi}, \chi_{\Phi} \rangle = 1\), then \(\Phi\) must be irreducible. So suppose that \(\langle \chi_{\Phi}, \chi_{\Phi} \rangle = 1\). Then
\[
\frac{1}{|G|} \sum_{g \in G} \chi_{\Phi}(g) \bar{\chi}_{\Phi}(g) = 1.
\]
From 2 above we know that this implies that
\[
\langle \chi_{\Phi}, \chi_{\Phi} \rangle = m_1^2 + m_2^2 + \cdots + m_k^2 = 1.
\]
Since \(m_i \in \mathbb{Z}\) and \(m_i \geq 0\), it must be the case that one \(m_i = 1\) and all the rest equal zero. So \(\Phi = \Phi_i\). By assumption \(\Phi_i\) is irreducible, so \(\Phi\) is irreducible.

4. We proved above that if \(\Phi\) and \(\Theta\) are isomorphic then for all \(g \in G\), \(\Phi(g) = \Theta(g)\). Now we want to prove that converse. Suppose that \(\Phi(g) = \Theta(g)\) for all \(g \in G\). We assume \(\Theta \cong q_1 \Phi_1 \oplus q_2 \Phi_2 \oplus \ldots \oplus q_p \Phi_p\). Note that this assumption is valid since if \(\Theta\) does not contain an irreducible that is in \(\Phi\), say \(\Phi_i\), then we simply let \(q_i\) be equal to zero. By the same principle, if \(\Phi_i\) is an irreducible that is not in \(\Phi\), then we can let \(m_i = 0\) in the expansion of \(\Phi\).
Now note that for any \(1 \leq i \leq p\), \(\langle \chi_{\Phi}, \chi_{\Phi i} \rangle = m_i\), since

\[
\langle \chi_{\Phi}, \chi_{\Phi i} \rangle = \frac{1}{|G|} \sum_{g \in G} (m_1 \chi_{\Phi 1}(g) + m_2 \chi_{\Phi 2}(g) + \ldots + m_k \chi_{\Phi k}(g)) \chi_{\Phi i}(g)
\]

\[
= \sum_j m_j m_i \langle \chi_{\Phi j}, \chi_{\Phi i} \rangle = m_i.
\]

Then since \(\Phi(g) = \Theta(g)\), \(q_i = \langle \chi_{\Theta}, \chi_{\Theta i} \rangle = \langle \chi_{\Phi}, \chi_{\Phi i} \rangle = m_i\). Hence for all \(i\), \(q_i = m_i\), and thus \(\Phi\) is isomorphic to \(\Theta\).

\[\square\]

Recall the representation in Example 2 that we called the left regular representation. We formed this representation by letting the group act upon itself. It had a basis consisting of all the elements of the group. Let us number the elements of the group \(G = \{g_1, g_2, \ldots, g_k\}\). It turns out that the matrix version of the left regular representation is formed by letting \(\Phi(g_p)\) be a \(|G| \times |G|\) matrix where cell \((i, j)\) is equal to one if \(g_i\) sends \(g_j\) to \(g_j\). Otherwise cell \((i, j)\) has a zero in it. For example, the matrix corresponding to the left regular representation \(X\) for \(\epsilon, (1, 2) \in S_3\) ordered \(S_3 = \{\epsilon, (12), (13), (23), (123), (132)\}\) is

\[
X(\epsilon) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}, \quad X((12)) = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}.
\]

Notice then that \(\Phi(\epsilon)\) is always the identity matrix with degree \(|G|\) since \(\epsilon\) sends every element \(g_i\) to \(g_i\). Also note that for any non-identity element \(g_i\) in \(G\), there is no \((i, i)\) cell with a 1 in it. If this were the case, for some \(g_i, g_jg_i = g_i\) which multiplying both right sides by \(g_j^{-1}\) gives us \(g_i = \epsilon\), which is a contradiction. Therefore the character of the identity in the group algebra is \(tr(\Phi(\epsilon)) = |G|\), while the character for a non-identity element is \(tr(\Phi(g_i)) = 0\).

We will now use these facts about the group algebra to prove the following theorem.

**Theorem 2.16.** Let \(G\) be a finite group, and let \(\mathbb{C}[G] = \bigoplus m_i V_i\), where the \(V_i\) form a complete list of inequivalent irreducible \(G\)-modules. Then the following is true.

1. \(m_i = \dim(V_i)\).
2. \(\sum_i (\dim(V_i))^2 = |G|\).

**Proof.** First notice that by Maschke’s theorem only a finite number of \(m_i\) will be non-zero.
1. Now from our results above

\[ m_i = \langle V, V_i \rangle = \frac{1}{|G|} \sum_{g \in G} V(g) \overline{V_i(g)}. \]

\( V(g) \) is nonzero only if \( g = \epsilon \). So

\[
\frac{1}{|G|} \sum_{g \in G} V(g) \overline{V_i(g)} = \frac{1}{|G|} V(\epsilon) \overline{V_i(\epsilon)} = \frac{1}{|G|} V(\epsilon) \overline{V_i(\epsilon)}
\]

\[ = \frac{1}{|G|} \times |G| \times \overline{V_i(\epsilon)} = \overline{V_i(\epsilon)} = tr(I_\delta) = \text{dim}(V_i). \]

2. Since \( \text{dim}(V_i) = m_i \),

\[ \sum_i (\text{dim}(V_i))^2 = \sum_i (m_i)^2. \]

And from above,

\[ \sum_i (m_i)^2 = \langle V, V \rangle = \frac{1}{|G|} \sum_{g \in G} V(g) V(g). \]

Since the only element of \( G \) with a nonzero character is \( \epsilon \), we have that

\[ \frac{1}{|G|} V(\epsilon) V(\epsilon) = \frac{1}{|G|} \times |G| \times |G| = |G|. \]

\[ \square \]

This leads us to the following theorem.

**Theorem 2.17.** \([8]\) Let \( G \) be a finite group, the number of irreducible representations of \( G \) is equal to the number of conjugacy classes of \( G \).

From this result we can see the character table will be a square matrix with dimensions equal to the number of conjugacy classes of the group.

Finally, before moving on to examine in-depth the representations of the symmetric group we should discuss the concepts of restriction and induction, as they will figure prominently into the proof of an important combinatorial algorithm.
The idea of restriction arises naturally when we ask how we might move between representations of a finite group \( G \) and one of its subgroups \( H \). The simpler of these two directions, restriction, comes easily.

**Definition 2.18.** Let \( G \) be a finite group and \( H \) be one of its subgroups. Also, let \( \Phi \) be a matrix representation of \( G \). Then the restriction of \( \Phi \) to \( H \), is defined to be

\[
\Phi \upharpoonright_H (h) = \Phi(h)
\]

for all \( h \in H \). We denote the character of the restriction by \( \chi \Phi \upharpoonright_H (h) \).

It is easy to check that this is indeed a representation of \( H \). Since \( H \) is a subgroup, \( H \) contains the identity \( e \), and because \( \Phi \) is a representation of \( G \),

\[
\Phi(e) = I_d,
\]

where \( d \) is the dimension of the representation. Hence

\[
\Phi \upharpoonright_H (e) = \Phi(e) = I_d.
\]

Next take two elements \( h_1, h_2 \) in \( H \). Then \( h_1 \) and \( h_2 \) are in \( G \), and

\[
\Phi(h_1 h_2) = \Phi(h_1) \Phi(h_2).
\]

And this implies that

\[
\Phi \upharpoonright_H (h_1 h_2) = \Phi(h_1 h_2) = \Phi(h_1) \Phi(h_2) = \Phi \upharpoonright_H (h_1) \Phi \upharpoonright_H (h_2).
\]

So the restriction of \( \Phi \) to \( H \) is indeed a representation.

Next we consider the induced representation from \( H \) to \( G \). This turns out to be a bit more complicated than the restriction. We accomplish it by making use of the familiar concept of the coset.

**Definition 2.19.** Let \( H \) be a subgroup of the finite group \( G \). Also, let \( t_1, \ldots, t_q \) be a transversal for the left cosets of \( H \). Then if \( \Phi \) is a representation of \( H \), then the corresponding induced representation \( \Phi \uparrow^G_H \) is defined to be

\[
\Phi \uparrow^G_H (g) = (\Phi(t_i^{-1} g t_j)) = \\
\begin{bmatrix}
\Phi(t_1^{-1} g t_1) & \Phi(t_1^{-1} g t_2) & \cdots & \Phi(t_1^{-1} g t_q) \\
\Phi(t_2^{-1} g t_1) & \Phi(t_2^{-1} g t_2) & \cdots & \Phi(t_2^{-1} g t_q) \\
\vdots & \vdots & \ddots & \vdots \\
\Phi(t_q^{-1} g t_1) & \Phi(t_q^{-1} g t_2) & \cdots & \Phi(t_q^{-1} g t_q)
\end{bmatrix}
\]

Where \( \Phi(g) \) is the zero matrix if \( g \) not in \( H \).

We now will prove that this is a representation.

**Lemma 2.20.** The induced representation \( \Phi \uparrow^G_H \) is a representation of \( G \).
Proof. First we will show that

$$\Phi \uparrow^G_H (\epsilon) = I_d$$

where $d$ is equal to $d_H \times |G/H|$, and $d_H$ is the dimension of $\Phi$ over $H$. First note that for all $t_i, t_j$ in the transversal,

$$t_i^{-1} e t_j = t_i^{-1} t_j$$

and if $i = j$,

$$t_i^{-1} e t_j = \epsilon.$$

Since

$$\Phi(e) = I_{d_H},$$

then all the block matrices along the diagonal are identity matrices. If $i \neq j$, then we can show that $t_i^{-1} t_j$ does not belong to $H$. If it did, then for some $h \in H$,

$$t_i^{-1} t_j = h$$

which would imply that $t_j = t_i h$. Then $t_i$ and $t_j$ belong to the same coset, which is impossible since $t_i$ and $t_j$ are each representatives from the transversal. Hence $t_i^{-1} t_j$ does not belong to $H$ if $i \neq j$. So $\Phi(t_i^{-1} t_j)$ is the zero matrix. This means that $\Phi \uparrow^G_H (\epsilon) = I_d$.

Next we show that if $g_1, g_2 \in G$, then

$$\Phi \uparrow^G_H (g_1 g_2) = \Phi \uparrow^G_H (g_1) \Phi \uparrow^G_H (g_2).$$

This is equivalent to showing that

$$\sum_K \Phi(t_i^{-1} g_1 t_k) \Phi(t_k^{-1} g_2 t_j) = \Phi(t_i^{-1} g_1 g_2 t_j).$$

Now suppose that

$$\Phi(t_i^{-1} g_1 g_2 t_j) = 0.$$

Then $t_i^{-1} g_1 g_2 t_j \neq H$. Since

$$(t_i^{-1} g_1 t_k)(t_k^{-1} g_2 t_j) = t_i^{-1} g_1 g_2 t_j$$

this implies that either $t_i^{-1} g_1 t_k$ or $t_k^{-1} g_2 t_j$ is not in $H$, otherwise their product would certainly be in $G$. Then either $\Phi(t_i^{-1} g_1 t_k)$ or $\Phi(t_k^{-1} g_2 t_j)$ is zero for each $k$. So each product

$$\Phi(t_i^{-1} g_1 t_k) \Phi(t_k^{-1} g_2 t_j) = 0.$$

So

$$\sum_K \Phi(t_i^{-1} g_1 t_k) \Phi(t_k^{-1} g_2 t_j) = 0 = \Phi(t_i^{-1} g_1 g_2 t_j).$$

Next suppose that

$$\Phi(t_i^{-1} g_1 g_2 t_j) \neq 0.$$
This would imply that $t_i^{-1}g_1g_2t_j \in H$. Since $t_1, t_2, \ldots, t_q$ is the traversal of $H$ in $G$, then there is a unique index $m$ such that $t_i^{-1}g_1t_m = h \in H$. Then
\[(t_i^{-1}g_1t_m)^{-1}(t_i^{-1}g_1g_2t_j) = (t_m^{-1}g_1^{-1}t_i)(t_i^{-1}g_1g_2t_j) = t_m^{-1}g_2t_j \in H.\]

Since for no other index $i$ is $(t_i^{-1}g_1t_k) \in H$, then
\[\sum_{k} \Phi(t_i^{-1}g_1t_k)\Phi(t_k^{-1}g_2t_j) = 0 + \cdots + \Phi(t_i^{-1}g_1t_m)\Phi(t_m^{-1}g_2t_j) + \cdots + 0 = \Phi(t_i^{-1}g_1g_2t_j).\]
Where the last equivalence comes from the fact that $\Phi$ is a homomorphism. These two facts show that the induction process does in fact create a new representation for $G$ from the representation of $H$. It is simple to show that the induced representation is well defined [8].

We would hope that the induced representation would turn out to be independent of the choice of $t_1, t_2, \ldots, t_k$ elements in the transversal. This turns out to be the case and we thus have the fact that.

**Theorem 2.21.** [8] Let $H$ be a subgroup of $G$ and $\Phi$ be a matrix representation of $H$. Also let $t_1, t_2, \ldots, t_k$ and $s_1, s_2, \ldots, s_k$ be two transversals of $H$ giving rise to the induced matrix representations of $G\Phi_1$ and $\Phi_2$. Then $\Phi_1$ is equivalent to $\Phi_2$.

As for the character of an induced representation, we have the useful identity known as Frobenius Reciprocity.

**Theorem 2.22.** Let $H$ be a subgroup of $G$. Let $\chi_\Phi$ and $\chi_\Omega$ be characters of $H$ and $G$ respectively. Then
\[\langle \chi_\Phi \uparrow^G_H, \chi_\Omega \rangle = \langle \chi_\Phi, \chi_\Omega \uparrow^G_H \rangle\]
where the left side is calculated over $G$ and the right side is calculated over $H$.

**Proof.** First note that
\[\langle \chi_\Phi \uparrow^G_H, \chi_\Omega \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_\Phi \uparrow^G_H (g) \overline{\chi_\Omega(g^{-1})}\]
\[= \frac{1}{|G|} \times \frac{1}{|H|} \sum_{g \in G} \sum_{k \in G} \chi_\Phi(k^{-1}gk) \overline{\chi_\Omega(g^{-1})}.\]
Notice that this last equivalence comes from
\[\chi_\Phi \uparrow^G_H (g) = \frac{1}{|H|} \sum_{k \in G} \chi_\Phi (k^{-1}gk).\]
We get this in turn from the fact that the trace of $\Phi \uparrow^G_H (g)$ is just the sum of the trace of each of the block matrices on the diagonal. Each of these have the form $\Phi(k^{-1}gk)$. The sum is divided by $|H|$ since we are taking all $k$ in $G$. 


In all of $G$ there are $|H|$ transversals of the cosets of $H$. These give identical sums. We want to the sum over just one transversal. Hence the division. Now returning to the previous equivalence, we let $r = k^{-1}gk$. So we have

$$\frac{1}{|G|} \times \frac{1}{|H|} \sum_{g \in G} \sum_{k \in G} \chi(\Phi^{-1}gk)\chi_{\Omega}(g^{-1})$$

$$= \frac{1}{|G|} \times \frac{1}{|H|} \sum_{g \in G} \sum_{k \in G} \chi(\Phi(r))\chi_{\Omega}(k^{-1}r^{-1}k).$$

However, because $\chi_{\Phi}$ is constant over conjugacy classes, we have that

$$\frac{1}{|G|} \times \frac{1}{|H|} \sum_{g \in G} \sum_{k \in G} \chi(\Phi(r))\chi_{\Omega}(k^{-1}r^{-1}k)$$

$$= \frac{1}{|G|} \times \frac{1}{|H|} \sum_{g \in G} \sum_{k \in G} \chi(\Phi(r))\chi_{\Omega}(r^{-1}).$$

Since we have eliminated $k$, the sum is constant over this variable and hence

$$\frac{1}{|G|} \times \frac{1}{|H|} \sum_{g \in G} \sum_{k \in G} \chi(\Phi(r))\chi_{\Omega}(r^{-1})$$

$$= \frac{1}{|H|} \sum_{g \in G} \chi(\Phi(r))\chi_{\Omega}(r^{-1}).$$

and since $\Theta(r) = 0$ for all $r \not\in H$, we have that

$$\frac{1}{|H|} \sum_{g \in G} \chi(\Phi(r))\chi_{\Omega}(r^{-1}) = \frac{1}{|H|} \sum_{r \in H} \chi(\Phi(r))\chi_{\Omega}(r^{-1})$$

$$= \langle \chi_{\Phi}, \chi_{\Omega} \mid \frac{1}{H} \rangle.$$ 

\[\square\]

3 Representations of the Symmetric Group

3.1 Partitions and Tableaux

In the remaining sections we want to focus on the symmetric group. Specifically, we would like to construct all its irreducible representations and look at the character table associated with these representations. To do this however, we will need to introduce a number of important concepts.
We begin by considering a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of $n$. To help visualize $\lambda$ we define the Ferrers diagram of $\lambda$.

**Definition 3.1.** Let $\lambda$ be a partition of $n$. Then the Ferrers diagram of $\lambda$ is constructed by placing $\lambda_1$ cells in a row, and then on the row below that placing $\lambda_2$ cells such that the far left cell in the second row is directly below the far left cell in the first row. We continue in this process until we have placed $\lambda_k$ cells in the $\lambda_k$th row.

**Example 3.2.** Let $\lambda = (5, 4, 2, 1)$. Then the Ferrers diagram of $\lambda$ is

```
+---+---+---+---+---+
|   |   |   |   |   |
+---+---+---+---+---+
|   |   |   |   |   |
+---+---+---+---+---+
|   |   |   | X |   |
+---+---+---+---+---+
|   |   |   |   |   |
```

It should be noted that the above definition of the Ferrer's diagram uses English as opposed to French notation.

We can identify a particular cell in a Ferrers diagram the same way that we identify a cell in an matrix. We number the rows from 1 to $k$ with the first row being the top one, and we number of the columns from 1 to $\lambda_i$ with the first column being the one furthest to the left. Using the example above, the cell with the X on it is denoted $(2,3)$.

```
+---+---+---+---+---+
| X |   |   |   |   |
+---+---+---+---+---+
|   |   |   |   |   |
+---+---+---+---+---+
|   |   |   |   |   |
+---+---+---+---+---+
|   |   |   |   |   |
```

Now for a given partition $\lambda$ of $n$, it is easy to associate it with a subgroup of $S_n$. Simply take the set $S_\lambda$ of elements of $S_n$ defined by

$$S_\lambda = S_{\{1,\ldots,\lambda_1\}} \times S_{\{\lambda_1+1,\ldots,\lambda_1+\lambda_2\}} \times \cdots \times S_{\{n-\lambda_k,\ldots,n\}}.$$

**Definition 3.3.** Let $\lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_i\}$ be a partition of $n$. Then the corresponding Young subgroup of $S_n$ is

$$S_\lambda = S_{\{1,2,\ldots,\lambda_1\}} \times S_{\{\lambda_1+1,\lambda_1+2,\ldots,\lambda_1+\lambda_2\}} \times \cdots \times S_{\{n-\lambda_i+1,n-\lambda_i+2,\ldots,n\}}.$$

**Example 3.4.** The Young subgroup corresponding to the partition from the example above would be

$$S_\lambda = \{\pi \in S_n : \pi = \pi_1 \times \pi_2 \times \pi_3 \times \pi_4, \text{ where} \}
\begin{align*}
\pi_1 & \in S_{\{1,2,3,4,5\}}, \\
\pi_2 & \in S_{\{6,7,8,9\}}, \\
\pi_3 & \in S_{\{10,11\}}, \\
\pi_4 & \in S_{\{12\}}. 
\end{align*}$$

As we move toward some sort of object that the elements of $S_n$ can act on, it becomes necessary to be able to permute the cells of a Ferrers diagram, and hence to be able to differentiate between them. Therefore we label them.
Definition 3.5. Take a Ferrers diagram of a partition $\lambda$ of $n$. A Young tableau of shape $\lambda$, call it $t$, is the Ferrers diagram of $\lambda$ with the cells bijectively labeled with the numbers $1, 2, \ldots, n$.

There is a clear way to separate the Young tableau of shape $\lambda$ into equivalence classes that will be useful later.

Definition 3.6. Let $T_1$ and $T_2$ be Young tableau of shape $\lambda$. We call $T_1$ and $T_2$ row equivalent if for every $x \in \{1, 2, \ldots, n\}$, $x$ is in the same row in $T_1$ and $T_2$. A tabloid of shape $\lambda$ is then defined to be all the Young tableau of shape $\lambda$ that are row equivalent to some $T$. We denote a tabloid by $\{T\}$.

It is a trivial process to show that tabloids partition the set of Young tableaux of shape $\lambda$ into equivalence classes. Furthermore, with Young tableaux we have an object upon which the elements of $S_n$ can act. So we define

$$\pi T = (\pi T_{(i,j)})$$

Meaning that in cell $(i, j)$, $\pi T$ has the element $\pi T_{(i,j)}$, where $\pi$ acts upon the number $T_{(i,j)}$ in the standard way.

We can define the action of the elements of $S_n$ on tabloids also. Let

$$\pi \{T\} = \{\pi T\}.$$ 

It turns out that this process is well defined. Let $T_1$ and $T_2$ both belong to the same tabloid equivalence class. $\{\pi T\}$ is the same regardless of whether we let the representative of $\{T\}$ be $T_1$ or $T_2$. Take a number $a \in \{1, 2, \ldots, n\}$. If $a$ is in the $i$th row in $T_1$, then it also must be in the $i$th row in $T_2$. Similarly, if $b \in \{1, 2, \ldots, n\}$ is in the $m$th row in $T_1$, then it is also in the $m$th row in $T_2$. So if $\pi(a) = b$, then $\pi T_1$ has $a$ in the $m$th row, as does $\pi T_2$. Hence the set of all tableaux row equivalent to $\pi T_1$ is identical to the set of all tableaux that are row equivalent to $\pi T_2$. So the action of a $\pi$ on a tabloid is well defined.

Now we make a further step toward a $S_n$-module by creating the following vector space.

Definition 3.7. Let $\lambda$ be a partition of $n$, then we define $M^\lambda$ to be the vector space of linear combinations of $\{t_1\}, \{t_2\}, \ldots, \{t_k\}$ over the field of the complex numbers, where $\{t_1\}, \{t_2\}, \ldots, \{t_k\}$ is the complete list of tabloids of shape $\lambda$. We call $M^\lambda$ the permutation module corresponding to $\lambda$.

Example 3.8. Let $\lambda$ be the partition of $3$, $(2, 1)$. Then the permutation module corresponding to $(2, 1)$ is

$$M^{(2,1)} = \mathbb{C}\{t_1\}, \{t_2\}, \{t_3\}\}.$$ 

Where

$$t_1 = \begin{bmatrix} 1 & 2 \\ 3 \\ 2 \end{bmatrix}, t_2 = \begin{bmatrix} 1 & 3 \\ 2 \\ 1 \end{bmatrix}, t_3 = \begin{bmatrix} 2 & 3 \\ 1 \end{bmatrix}.$$
Clearly if we are given a tabloid of shape $\lambda$, then we can reach any other tabloid of shape $\lambda$ by applying some permutation. If we have the tabloid
\[
\begin{array}{c}
1 & 3 \\
2 & 4 \\
\end{array}
\]
and we want the tabloid
\[
\begin{array}{c}
1 & 2 \\
3 & 4 \\
\end{array}
\]
then we need only apply the permutation $(2,3)$
\[
(23) \begin{array}{c}
1 & 3 \\
2 & 4 \\
\end{array} = \begin{array}{c}
1 & 2 \\
3 & 4 \\
\end{array}
\]
We call a G-module with this property cyclic. In other words, in a cyclic G-module $M$, one can take a particular generating element $v$ and by applying various elements of $G$, get any other desired generating element in $M$. From above we already saw that $M^\lambda$ is cyclic, since any tabloid can be generated by any other tabloid by applying the appropriate permutation $\pi$.

It is a simple counting problem to determine how many tabloids there are for a given shape $\lambda$, this will give us the dimension of $M^\lambda$. We begin by considering the number of tableaux of shape $\lambda$. We have $n$ unique boxes in which we need to place $n$ unique objects. This suggests that there $n!$ different tableaux. Now choose a given tableau $t$, we can ask how many tableaux belong to the same tabloid as $t$. For each ith row we have $\lambda_i$ numbers which we may permute. Since the total number of ways to permute each row is $\lambda_i!$, then the total number tableaux in one tabloid is $\lambda_1! \times \lambda_2! \times \cdots \times \lambda_k!$. Dividing the total number of tableaux by this, we find the number of tabloids and hence the dimension of $M^\lambda$ is $n!/(\lambda_1! \times \lambda_2! \times \cdots \times \lambda_k!)$.

### 3.2 Ordering Partitions

In working with partitions, it will become invaluable to have an ordering that we can follow (those interested in more background in partial and total orders should consult [2]). We will discuss two of these orders. One is a partial order and the other is a total order. We begin with the partial order.

**Definition 3.9.** Suppose that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_l)$ are partitions of $n$. Then we say that $\lambda$ *dominates* $\mu$ if for all $i \geq 1$

\[
\lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i.
\]

If $i > k$ or $i > l$, then we take $\lambda_i$ or $\mu_i$ to be equal to zero.
Example 3.10. Take the two partitions of 8, (4, 3, 1) and (3, 3, 2). We can say that (4, 3, 1) dominates (3, 3, 1, 1) since

\[ 4 \geq 3, \quad 4 + 3 \geq 3 + 3, \quad 4 + 3 + 1 \geq 3 + 3 + 1, \quad 4 + 3 + 1 + 0 \geq 3 + 3 + 1 + 1. \]

The following example shows why this partial order is not a total order.

Example 3.11. Take the two partitions (3, 3) and (4, 1, 1). Then

\[ 4 \geq 3, \text{ but } 4 + 1 < 3 + 3. \]

Hence (3, 3) does not dominate (4, 1, 1) and (4, 1, 1) does not dominate (3, 3).

The next partial order is a total order. It is called the lexicographic ordering. It should seem familiar since it is essentially the same ordering that a dictionary of the English language uses.

Definition 3.12. Suppose that \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \) and \( \mu = (\mu_1, \mu_2, \ldots, \mu_l) \) are partitions of \( n \). Then \( \lambda \) is greater than \( \mu \) (or \( \lambda \succ \mu \)) in lexicographic order if, for some index \( i \),

\[ \lambda_j = \mu_j \text{ for all } j < i \text{ and } \lambda_i < \mu_i. \]

Example 3.13. Take the two partitions of 7, \( \lambda = (3, 3, 1) \) and \( \mu = (3, 2, 2) \). Then \( \lambda \succ \mu \) in lexicographic order since \( \lambda_1 = 3 = \mu_1 \) and \( \lambda_2 = 3 = \mu_2 \).

It turns out that there is a useful relationship between these two orderings.

Theorem 3.14. If \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \) and \( \mu = (\mu_1, \mu_2, \ldots, \mu_l) \) are two partitions of \( n \), then if \( \lambda \) dominates \( \mu \), \( \lambda \) is also greater than \( \mu \) in lexicographic order.

Proof. Suppose that \( \lambda \) dominates \( \mu \). Then for any \( i \geq 1 \),

\[ \lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i. \]

In particular, let \( i \) be the first index for which \( \lambda_i \neq \mu_i \). Then since \( \lambda_1 = \mu_1, \lambda_2 = \mu_2, \ldots, \lambda_{i-1} = \mu_{i-1} \) we can subtract \( \lambda_1, \lambda_2, \ldots, \lambda_{i-1} \) from both sides of the inequality above to get that \( \lambda_i \geq \mu_i \). Since by assumption \( \lambda_i \neq \mu_i \), then \( \lambda_i > \mu_i \) and hence \( \lambda \succ \mu \) as desired. \( \square \)

3.3 Specht Modules

We are now prepared to construct the irreducible modules of \( S_n \), the Specht modules, \( S^\lambda \). Recall above that we were able to associate to each partition of \( n \) a subgroup of \( S_n \). We can take this idea further with tableaux.

Definition 3.15. Let \( T \) be a tableau, \( R_i \) be the set of all integers in the \( i \)th row of \( T \), and \( C_j \) be the set of all integers in the \( j \)th column of \( T \). Then we define

\[ R^T = S_{R_1} \times S_{R_2} \times \cdots \times S_{R_l} \]

and

\[ C^T = S_{C_1} \times S_{C_2} \times \cdots \times S_{C_m}. \]

We call \( R^T \) the row-stabilizer and \( C^T \) the column-stabilizer of \( T \).
Example 3.16. Take the tableau
\[ t = \begin{array}{c c c c c}
1 & 5 & 6 & 3 \\
2 & 4 \\
7 & 8 
\end{array} \]
then \( R_T = S_{\{1,3,5,6\}} \times S_{\{2,4\}} \times S_{\{7,8\}} \) and \( C_T = S_{\{1,2,7\}} \times S_{\{4,5,8\}} \times S_{\{6\}} \times S_{\{3\}} \).

Notice that \( R_T T \) is the same as \( \{T\} \), since \( R_T T \) contains all the different permutations of the elements in each row.

Definition 3.17. Let \( T \) be a tableau. Then the associated polytabloid is
\[ e_T = k_T \{T\} \]
where,
\[ k_T = \sum_{\pi \in C_T} (\sgn\pi) \pi. \]

Example 3.18. Let
\[ T = \begin{array}{c c c c c}
1 & 3 & 4 \\
2 & 5 
\end{array} \]
Then \( k_T = -(12) - (35) + (12)(35) + \epsilon \)

So, \[ e_T = -\{\begin{array}{c c c c c}
2 & 3 & 4 \\
1 & 5 
\end{array}\} - \{\begin{array}{c c c c c}
1 & 5 & 4 \\
2 & 3 
\end{array}\} + \{\begin{array}{c c c c c}
2 & 5 & 4 \\
1 & 3 
\end{array}\} + \{\begin{array}{c c c c c}
1 & 3 & 4 \\
2 & 5 
\end{array}\} . \]

Example 3.19. Next let
\[ T = \begin{array}{c c c c c}
4 & 3 & 1 \\
2 & 5 
\end{array} \]
Then \( k_T = -(24) - (35) + (24)(35) + \epsilon \)

So, \[ e_T = -\{\begin{array}{c c c c c}
2 & 3 & 1 \\
4 & 5 
\end{array}\} - \{\begin{array}{c c c c c}
4 & 5 & 1 \\
2 & 3 
\end{array}\} + \{\begin{array}{c c c c c}
2 & 5 & 1 \\
4 & 3 
\end{array}\} + \{\begin{array}{c c c c c}
4 & 3 & 1 \\
2 & 5 
\end{array}\} . \]

We are finally ready to define the Specht modules, \( S^\lambda \).

Definition 3.20. Let \( \lambda \) be a partition, then the Specht module associated with \( \lambda \) is the submodule of \( M^\lambda \) spanned by the polytabloids \( e_T \), where \( T \) is of shape \( \lambda \).

Theorem 3.21. The Specht modules \( S^\lambda \) are cyclic and are generated by all polytabloids.

To see this we will have to examine how elements of \( S_n \) act upon polytabloids.

Lemma 3.22. Let \( T \) be a tableau with \( n \) cells and fix \( \pi \in S_n \). Then the follow hold

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1. $C^{\pi T} = \pi C^T \pi^{-1}$.
2. $R^{\pi T} = \pi R^T \pi^{-1}$.
3. $k_{\pi T} = \pi k_T \pi^{-1}$.
4. $e_{\pi T} = \pi e_T$.

Proof. We will here prove 1 and 3, the proof of 2 and 4 can be found in [8].

For part 1, suppose $\sigma \in C^{\pi T}$. Then $\sigma$ permutes the elements of $\pi T$ so that each element remains in its original column. Hence $\sigma \pi T$ has the same column elements as $\pi T$. Multiplying both $\sigma \pi T$ and $\pi T$ by $\pi^{-1}$ we get that $\pi^{-1} \sigma \pi T$ has the same columns as $T$. This is true for the following reason. Suppose $\pi^{-1}$ sends some number $a \in \{1, 2, \ldots, n\}$ to $b \in \{1, 2, \ldots, n\}$. We know that $a$ is in the same columns in $\sigma \pi T$ and $\pi T$ and the same is true for $b$. Therefore $\pi^{-1}$ will send $a$ to the same column in both $\sigma \pi T$ and $\pi T$. This implies that $\pi^{-1} \sigma \pi \in C^T$. So $\sigma \in C^{\pi T} \pi^{-1}$. Therefore

$$C^{\pi T} \subseteq \pi C^T \pi^{-1}.$$

Next suppose that $\sigma \in \pi C^T \pi^{-1}$. Then $\pi^{-1} \sigma \pi \in C^T$. By definition $\pi^{-1} \sigma \pi$ permutes the elements of $C^T$ so that each element remains in its original column. Multiplying both $C^T$ and $\pi^{-1} \sigma \pi C^T$ by $\pi$ one gets $\sigma \pi C^T$ and $\pi C^T$. Because $C^T$ and $\pi^{-1} \sigma \pi C^T$ have the same elements for each column, then by the argument in the paragraph above $\sigma \pi C^T$ and $\pi C^T$ will have the same column elements. This implies that $\sigma \in C^{\pi T}$. Therefore

$$\pi C^T \pi^{-1} \subseteq C^{\pi T}$$

and hence

$$\pi C^T \pi^{-1} = C^{\pi T}.$$

For part 3 we begin with

$$k_{\pi T} = \sum_{\sigma \in C^{\pi T}} (\text{sgn}\sigma) \sigma.$$

From Lemma 3.21.1

$$\sum_{\sigma \in C^{\pi T}} (\text{sgn}\sigma) \sigma = \sum_{\sigma \in \pi C^T \pi^{-1}} (\text{sgn}\sigma) \sigma.$$

Substituting $\sigma = \pi \sigma' \pi^{-1}$ where $\sigma \in C^T$, we get

$$\sum_{\sigma \in \pi C^T \pi^{-1}} (\text{sgn}\sigma) \sigma = \sum_{\sigma \in \pi C^T \pi^{-1}} (\text{sgn}\pi \sigma' \pi^{-1}) \sigma \pi' \pi^{-1}.$$

By bringing out the $\pi$'s and $\pi^{-1}$'s we have that

$$\sum_{\sigma \in \pi C^T} (\text{sgn}\pi \sigma' \pi^{-1}) \pi \sigma' \pi^{-1} = (\text{sgn}\pi) \text{sgn}(\pi^{-1}) \pi k_T \pi^{-1} = \pi k_T \pi^{-1}$$

which completes the proof. □
It is now clear why the Specht modules are cyclic. We know that for any tableau $T'$ of shape $\lambda$ with $n$ cells, we can get any other tableau $T$ of shape $\lambda$ by multiplying $T'$ by some $\pi$ in $S_n$. Therefore, if we want some polytabloid $\mathbf{e}_T$ and we have a polytabloid $\mathbf{e}_{T'}$, we need only perform the operation $\pi \mathbf{e}_{T'} = \mathbf{e}_{\pi T'} = \mathbf{e}_T$.

**Example 3.23.** We will here show that the Specht modules give all the irreducible representations of $S_3$. We begin with the partition $(3)$. Then $\mathbf{e}_{\{123\}} = \{123\}$. Since there is only one row, $S_{\{123\}}$ behaves like the identity when elements of $S_3$ act on it. Thus $S^{(3)}$ carries the trivial representation. This is not surprising since $S^{(3)}$ is a submodule of $M^{(3)}$ and $M^{(3)} = \mathbb{C}\{123\}$ and so itself is the trivial representation.

Next we consider the partition $(1, 1, 1)$. So

$$T = \begin{array}{c}
1 \\
2 \\
3 \end{array}$$

and $C^T = S_3$. This implies that

$$k_T = \sum_{\sigma \in S_n} \text{sgn}(\sigma)\sigma.$$

So take some $\pi$. From the Lemma 3.21.4

$$\mathbf{e}_{\pi T} = \pi \mathbf{e}_T = \sum_{\sigma \in S_n} \text{sgn}(\sigma)\pi \sigma \{T\}.$$

If we replace $\pi \sigma$ by $\tau$, then we have that

$$\mathbf{e}_{\pi T} = \sum_{\tau \in S_3} \text{sgn}(\pi^{-1})\tau \{T\} = \text{sgn}(\pi^{-1}) \sum_{\tau \in S_3} \text{sgn}(\tau)\tau \{T\} = \text{sgn}(\pi)\mathbf{e}_T.$$

Because $\text{sgn}(\pi^{-1}) = \text{sgn}(\pi)$. So every polytabloid is a scalar multiple of $\mathbf{e}_T$, in particular, they are equal to $\mathbf{e}_T$ or $-\mathbf{e}_T$. So,

$$S^{(1,1,1)} = C\mathbf{e}_T.$$

Thus $S^{(1,1,1)}$ is the sign representation.

Finally, we look at the partition $(2,1)$. Clearly we can identify any tabloid by the one element in the second row. For example

$$T = \begin{array}{c}
1 \\
2 \\
3 \end{array} = 3$$

will be denoted as 3. Then $\mathbf{e}_T = 3 - 1$. The full list of tableaux of shape $(2,1)$ is

$$T_1 = \begin{array}{c}
1 \\
2 \\
3 \end{array}, T_2 = \begin{array}{c}
2 \\
1 \\
3 \end{array}, T_3 = \begin{array}{c}
1 \\
1 \\
2 \end{array}, T_4 = \begin{array}{c}
3 \\
1 \\
2 \end{array}, T_5 = \begin{array}{c}
2 \\
3 \\
1 \end{array}, T_6 = \begin{array}{c}
3 \\
2 \\
1 \end{array}.$$
The full number of polytabloids is

\[
\begin{align*}
\mathbf{e}_{T_1} &= \begin{array}{c} 1 \ 2 \\ 3 \ 2 \end{array} - \begin{array}{c} 3 \\ 1 \ 2 \end{array} = 3 - 1 \\
\mathbf{e}_{T_2} &= \begin{array}{c} 2 \ 1 \\ 3 \ 2 \end{array} - \begin{array}{c} 3 \\ 1 \ 2 \end{array} = 3 - 2 \\
\mathbf{e}_{T_3} &= \begin{array}{c} 1 \ 3 \\ 2 \ 1 \end{array} - \begin{array}{c} 2 \\ 3 \ 1 \end{array} = 2 - 1 \\
\mathbf{e}_{T_4} &= \begin{array}{c} 3 \ 1 \\ 2 \ 3 \end{array} - \begin{array}{c} 2 \\ 1 \ 3 \end{array} = 2 - 3 \\
\mathbf{e}_{T_5} &= \begin{array}{c} 2 \ 3 \\ 1 \ 2 \end{array} - \begin{array}{c} 1 \\ 3 \ 2 \end{array} = 1 - 2 \\
\mathbf{e}_{T_6} &= \begin{array}{c} 3 \ 2 \\ 1 \ 3 \end{array} - \begin{array}{c} 1 \\ 2 \ 3 \end{array} = 1 - 3.
\end{align*}
\]

So,

\[
S^{(2,1)} = \{c_1(3 - 1) + c_2(3 - 2) + c_3(2 - 1) + c_4(2 - 3) + c_5(1 - 2) + c_6(1 - 3) : \\
c_1, c_2, c_3, c_4, c_5, c_6 \in \mathbb{C}\}
\]

\[
= \{1(-c_1 - c_3 + c_5 + c_6) + 2(-c_2 + c_3 + c_4 - c_5) + 3(c_1 + c_2 - c_4 + c_6) : \\
c_1, c_2, c_3, c_4, c_5, c_6 \in \mathbb{C}\}.
\]

Since

\[
(-c_1 - c_3 + c_5 + c_6) + (-c_2 + c_3 + c_4 - c_5) + (c_1 + c_2 - c_4 + c_6) = 0
\]

one can rewrite \(S^{(2,1)}\) as

\[
S^{(2,1)} = \{k_1(1) + k_2(2) + k_3(3) : k_1 + k_2 + k_3 = 0 : k_1, k_2, k_3 \in \mathbb{C}\}.
\]

So the dimension of \(S^{(2,1)}\) is 2, and a basis is

\[
B = \{2 - 1, 3 - 1\}.
\]

It is left to the reader to show that the characters for these three \(S_3\)-modules match the known character table for \(S_3\) (See Appendix A). Once this is done, it will be the clear that the set of all \(S^\lambda\), where \(\lambda\) ranges over all partitions of 3, is the full set of irreducible representations of \(S_3\).
3.4 The Irreducible Representations $S^\lambda$

We would like to established that in general the full set of $S^\lambda$ give all the irreducible representations of $S_n$. To do this we need an inner product on $M^\lambda$.

**Definition 3.24.** We define the inner product on $M^\lambda$ to be

$$\langle \{T\}, \{T'\} \rangle = \delta_{\{T\},\{T'\}}$$

where $\delta_{\{T\},\{T'\}}$ is the Kronecker delta and $\{T\}, \{T'\} \in M^\lambda$.

We will also need to make use of the group algebra sums,

$$H^+ = \sum_{\pi \in H} \pi$$

$$H^- = \sum_{\pi \in H} (\text{sgn}(\pi))\pi.$$

Where $(\text{sgn}(\pi))$ is the the familiar sign of the permutation. Then we have the following lemma,

**Lemma 3.25.** Let $H$ be a subgroup of $S_n$. Then the follow are true.

1. If $\pi \in H$, then

$$\pi H^- = H^- \pi = \text{sgn}(\pi)H^-.$$

2. For any $u, v \in M^\lambda$,

$$\langle H^- u, v \rangle = \langle u, H^- v \rangle.$$

3. If the transposition $(b, c) \in H$, then we can factor

$$H^- = k(e - (b, c))$$

where $k \in \mathbb{C}[S_n]$.

4. If $T$ is a tableau with $b, c$ in the same row of $T$ and $(b, c) \in H$, then

$$H^- (t) = 0.$$

**Proof.** We omit proof of the first part since it is almost identical to what we did for the partition $(1, 1, 1)$ in the Example 12.

For the second part we assume that $u$ and $v$ are basis elements of $M^\lambda$, we leave it as an exercise for reader to extend the proof to general $u, v \in M^\lambda$. So we begin with,

$$\langle H^- u, v \rangle = \langle \sum_{\pi \in H} \text{sgn}(\pi)\pi u, v \rangle = \sum_{\pi \in H} \langle \text{sgn}(\pi)\pi u, v \rangle.$$
Now if for some $\pi \in H$, $\pi u = v$, then certainly $u = \pi^{-1}v$, and $\langle \pi u, v \rangle = \langle u, \pi^{-1}v \rangle$. Similarly, if $\pi u \neq v$, then $u \neq \pi^{-1}v$ and $\langle \pi u, v \rangle = \langle u, \pi^{-1}v \rangle = 0$.

So noting that $\text{sgn}(\pi) = sgn(\pi^{-1})$ we may rewrite our sum as,

$$\sum_{\pi \in H} \langle \text{sgn}(\pi)u, v \rangle = \sum_{\pi \in H} \langle u, \text{sgn}(\pi^{-1})\pi^{-1}v \rangle = \langle u, H^{-1}v \rangle.$$  

For the third part, suppose that $(b, c) \in H$. Then let $K$ be the subgroup of $H$, $\{e, (b, c)\}$. Take a transversal of the cosets of $K$ in $H$; $k_1, k_2, \ldots, k_j$. One can then write

$$H = k_1 \{e, (b, c)\} \bigcup k_2 \{e, (b, c)\} \bigcup \cdots \bigcup k_j \{e, (b, c)\}.$$  

This implies that

$$H^{-} = \sum_{i} k_i (e - (b, c)).$$

Which is what we were looking for.

For the final part, begin by recalling from Lemma 3.24.3 that we can write

$$H^{-} \{T\} = \sum_{i} k_i (e - (b, c))\{T\}.$$  

Since $b$ and $c$ are in the same row of $\{T\}$, $(b, c)\{T\} = \{T\}$. So

$$\sum_{i} k_i (e - (b, c))\{T\} = \sum_{i} k_i (\{T\} - \{T\}) = 0.$$

\[
\square
\]

Before we can establish the important Submodule Theorem we need the following corollaries.

**Corollary 3.26.** Let $T$ be a $\lambda$-tableau and $T'$ be a $\mu$-tableau, where $\mu$ and $\lambda$ are partitions of $n$. If $k_T\{T'\} \neq 0$, then $\lambda \geq \mu$. If $\lambda = \mu$, then $k_T\{T'\} = \pm e_T$.

*Proof.* [8] Suppose that $b$ and $c$ are two elements in the same row of $T'$. Then they cannot be in the same column of $T$. If they were in the same column of $T$, then we could write $k_T = k(e - (b, c))$ and $k_T\{T'\} = 0$ by the Lemma 3.24.4. Then by the Dominance Lemma it must be the case that $\lambda \geq \mu$.

If $\lambda = \mu$, then we must have $\{T'\} = \pi\{T\}$ for some $\pi \in C^T$. Then by the first part of this lemma,

$$k_T \{T'\} = k_T \pi\{T\} = sgn(\pi)k_T\{T\} = \pm e_T.$$  

\[
\square
\]

**Corollary 3.27.** If $u \in M^\mu$ and let $T$ be a tableau of shape $\mu$, then $k_Tu$ is a multiple of $e_T$.
Proof. We can write \( u = \sum_i c_i \{Y_i\} \) where the \( Y_i \) are \( \mu \)-tableaux. By Corollary 3.25, \( k_Tu = \sum \pm c_i e_T \).

We can now prove the Submodule Theorem.

**Theorem 3.28.** Let \( U \) be a submodule of \( M^\mu \). Then
\[
U \supseteq S^\mu \quad \text{or} \quad U \subseteq S^\mu_{\perp}.
\]
Where \( S^\mu_{\perp} = \{ u \in M^\mu : \langle u, v \rangle = 0 \text{ for all } v \in S^\mu \} \). When the field is \( \mathbb{C} \), the \( S^\mu \) are irreducible.

**Proof.** Let \( u \in U \) and take some \( \mu \)-tableau \( T \). From the corollary above we have that \( k_Tu = fe_T \) for some field element \( f \).

Now suppose that there exists a \( u \) and a \( T \) with the relation above such that \( f \neq 0 \). Then since \( u \) is in the submodule \( U \), we have that \( fe_T = k_Tu \in U \). Thus \( e_T \in U \), which implies that \( \pi e_T \in U \) for all \( \pi \in S_n \) since \( U \) is the submodule. Thus \( S^\mu \subseteq U \) since \( S^\mu \) is cyclic.

Otherwise suppose that for all \( u \in U \), \( k_Tu = 0 \). For any \( u \in U \), and an arbitrary \( \mu \)-tableau \( T \), we use Lemma 3.24.2 to get that
\[
\langle u, e_T \rangle = \langle u, k_T \{t\} \rangle = \langle k_Tu, \{t\} \rangle = \langle 0, \{t\} \rangle = 0.
\]

Since the \( e_T \) span \( S^\mu \), we have \( u \in S^\mu_{\perp} \), as desired.

Since we are here only concerned with working over the complex numbers, it would be good to consider this particular field. This leads us to the following theorem.

**Theorem 3.29.** Suppose that the field of scalars is \( \mathbb{C} \). If \( \theta \) is a homomorphism from \( S^\mu \) to \( M^\lambda \) that is nonzero, then \( \lambda \supseteq \mu \). If \( \lambda = \mu \), then \( \theta \) is multiplication by some scalar.

**Proof.** Now since \( \theta \) is a nonzero homomorphism, then for some basis vector \( e_T \) of \( S^\mu \), \( \theta(e_T) \neq 0 \). Because \( \langle \cdot, \cdot \rangle \) is an inner product with complex scalars, \( M^\mu = S^\mu \oplus S^\mu_{\perp} \). Thus we can extend \( \theta \) to all of \( M^\mu \) by letting \( \theta(u) = 0 \) for any \( u \in S^\mu_{\perp} \). So we have that
\[
0 \neq \theta(e_T) = \theta(k_T \{T\}) = k_T \theta(\{T\}) = k_T \sum_i c_i \{Y_i\}
\]
where the \( Y_i \) are \( \mu \)-tableaux. Then by Corollary 3.25 above we have that \( \lambda \supseteq \mu \).
Next suppose that $\mu = \lambda$. Now by Corollary 3 for any $u \in M^\mu$ where the shape of $T$ is equal to $\mu$, $k_T u$ is a multiple of $e_T$. Then

$$0 \neq \theta(e_T) = \theta(k_T \{ T \}) = k_T \theta(\{ T \}) = k_T \sum_i c_i \{ Y_i \} = c e_T$$

where the $Y_i$ are $\mu$-tableaux and $c$ is some constant. So for any permutation $\pi$ we have

$$\theta(e_{\pi T}) = \theta(\pi e_T) = \pi \theta(e_T) = \pi(c e_T) = c e_{\pi T}.$$ 

So $\theta$ certainly is equal to multiplication by a constant $c$. $\square$

Now we can verify the theorem that we have been working toward.

**Theorem 3.30.** The $S^\mu$, where $\mu$ is a partition of $n$, form a complete list of irreducible $S_n$ - modules over the complex numbers.

**Proof.** We know that the $S^\lambda$ are irreducible from the Submodule Theorem above and from the fact that $S^\lambda \cup S^\lambda^\perp = \emptyset$ for the field $C$. Also, we know that the number of $S^\lambda$ is equal to the number of conjugacy classes of $S_n$. Next, since the number of irreducible $S_n$ modules is equal to the number of conjugacy classes of $S_n$, we need only show that the $S^\lambda$ are pairwise inequivalent if we want to show that they form a complete list of irreducible $S_n$-modules. If there are two that are isomorphic, say $S^\lambda$ and $S^\mu$, then there exists a homomorphism from $S^\lambda$ to $M^\mu$, and also a homomorphism from $S^\mu$ to $S^\lambda$. By Theorem 3.28 this implies that both $\lambda$ dominates $\mu$ and $\mu$ dominates $\lambda$, which implies that $\lambda = \mu$. This shows that the $S^\lambda$ are pairwise inequivalent, and hence provide a full list of irreducible representations of $S_n$. $\square$

From this we get the following easy corollary.

**Corollary 3.31.** The permutation modules decompose as

$$M^\mu = \bigoplus_{\lambda \geq \mu} m_{\lambda \mu} S^\lambda$$

with diagonal multiplicity $m_{\lambda \lambda} = 1$.

**Proof.** This follows from Theorem 3.28. If $S^\lambda$ appears in $M^\mu$ with a nonzero coefficient, then $\lambda \geq \mu$. If $\lambda = \mu$, then since $S^\lambda$ is irreducible, $\dim \text{Hom}(S^\lambda, M^\mu)$ is the multiplicity of $S^\lambda$ in $M^\mu$. This implies that $m_{\lambda, \mu} = 1$, since there is only one homomorphism from $S^\lambda$ to $M^\mu$, which is multiplication by a scalar. $\square$

### 3.5 Standard Tableaux and a Basis for $S^\lambda$

We saw above that $S^\lambda$ is generated by polytabloids. We might ask though whether these polytabloids are independent. For the purposes of finding matrix representations it would also be convenient if we could find a basis for each $S^\lambda$. It turns out that there is an easy way to do this.
Definition 3.32. A tableau \( T \) is called standard if both its columns and rows are increasing sequences. We say that the corresponding tabloid and polytabloid are also standard.

Example 3.33. \( T \) is a standard tableau

\[
T = \begin{array}{c}
1 & 3 & 6 \\
2 & 4 \\
5 
\end{array}
\]

while the tableau \( T' \) is not

\[
T' = \begin{array}{c}
1 & 3 & 6 \\
2 & 4 & 5 
\end{array}
\]

since the third column is not increasing.

We will now work toward proving that the set of polytabloids generated by the standard tableaux for a partition \( \lambda \) do in fact form a basis for \( S^\lambda \). Before we do this we will need to study the ways in which we might order tabloids.

Definition 3.34. A composition is an ordered set of integers

\[
\lambda = \{\lambda_1, \lambda_2, ..., \lambda_k\}
\]

such that \( \lambda_1 + \lambda_2 + ... + \lambda_k = n \).

A composition then differs from a partition in that there is no requirement that the parts be decreasing. We form Ferrers diagrams and tableau from compositions in the same way that we formed them from partitions. Also the dominance order on compositions is defined in the same way that it is for partitions, except now when we pick the parts of the composition in the order that they are listed, we may not be picking the largest values first.

Now let \( \{T\} \) be a tabloid with shape \( \lambda \), where \( \lambda \) partitions \( n \). Then for any \( 1 \leq i \leq n \) we let \( \{T^i\} \) be the tabloid formed by placing all the elements in \( \{T\} \) less than \( i \) in the same positions that they have in \( \{T\} \). Then we say that \( \{T^i\} \) has shape \( \lambda^i \).

Example 3.35. Let

\[
\{T\} = \begin{array}{c}
3 & 4 & 6 \\
1 & 2 & 5 
\end{array}
\]

Then

\[
\{T^1\} = \{\begin{array}{c} \emptyset \\
1 \end{array}\}, \{T^2\} = \{\begin{array}{c} \emptyset \\
1 & 2 \end{array}\}, \{T^3\} = \{\begin{array}{c} \emptyset \\
3 & 1 & 2 \end{array}\}, \{T^4\} = \{\begin{array}{c} \emptyset \\
3 & 4 \end{array}\}
\]

\[
\{T^5\} = \{\begin{array}{c} \emptyset \\
3 \end{array}\}, \{T^6\} = \{\begin{array}{c} \emptyset \\
3 & 4 & 6 \end{array}\}
\]

The associated compositions are

\[
\lambda^1 = (0, 1), \lambda^2 = (0, 2), \lambda^3 = (1, 2), \lambda^4 = (2, 2), \lambda^5 = (2, 3), \lambda^6 = (3, 3).
\]
Definition 3.36. If \( \{T\} \) and \( \{T'\} \) are two tabloids with composition sequences \( \lambda^i \) and \( \mu^i \) respectively, then we say that \( \{T\} \) dominates \( \{T'\} \) if \( \lambda^i \geq \mu^i \) for all \( i \).

We can immediately prove the following lemma.

**Lemma 3.37.** If \( k < l \) and \( k \) appears in a lower row than \( l \) in \( \{T\} \) then \( (k, l) \{T\} \geq \{T\} \).

**Proof.** Suppose that \( \{T\} \) and \( (k, l) \{T\} \) have composition sequences \( \lambda^i \) and \( \mu^i \) respectively. Then for \( i < k \) and \( i \geq l \), \( \lambda^i = \mu^i \). Next consider when \( k \leq i < l \). Here the associated compositions are \( \lambda^i = \{\lambda^i_1, \lambda^i_2, \ldots, \lambda^i_k\} \) and \( \mu^i = \{\mu^i_1, \mu^i_2, \ldots, \mu^i_k\} = \{\lambda^i_1, \lambda^i_2, \ldots, \lambda^i_j + 1, \ldots, \lambda^i_m - 1, \ldots, \lambda^i_k\} \). Clearly, \( \mu^i \geq \lambda^i \), and hence \((k, l) \{T\} \geq \{T\}\). \( \square \)

**Definition 3.38.** If \( v \in M^\mu \), then \( v = \sum c_i \{T_i\} \). We say that \( \{T_i\} \) appears in \( v \) if \( c_i \neq 0 \).

As a corollary to Lemma 3.36 we have.

**Corollary 3.39.** If \( T \) is a standard tableau and \( \{Y\} \) appears in \( e_T \), then \( \{T\} \geq \{Y\} \).

**Proof.** We know that since \( T \) is standard, all of its columns are increasing, as are all of its rows. We also know that since \( \{Y\} \) appears in \( e_T \), then \( \{Y\} \) can be formed from \( \{T\} \) by some number of column inversions. All of these column inversions must be taking some \( l \) and \( k \) and switching them, where \( k < l \) and \( k \) is in a lower column than \( l \). It could not be otherwise since \( T \) is already standard. By induction, and using Lemma 3.36, this would indicate that \( \{T\} \) dominates \( \{Y\} \). \( \square \)

To characterize the leading terms in \( \{e_T\} \) we need the following definition.

**Definition 3.40.** Let \( (A, \leq) \) be a poset. Then an element \( b \in A \) is the maximum if \( b \geq c \) for all \( c \in A \). An element \( b \) is a maximal element if there is no \( c \in A \) with \( c > b \). Minimum elements and minimal elements are defined analogously.

**Lemma 3.41.** Let \( v_1, v_2, \ldots, v_m \) be elements of \( M^\mu \). Suppose, for each \( v_i \), we can choose a tabloid \( \{T_i\} \) appearing in \( v_i \) such that

1. \( \{T_i\} \) is maximum in \( v_i \).
2. The \( \{T_i\} \) are all distinct.

Then \( v_1, v_2, \ldots, v_m \) are independent in \( S^\lambda \).

**Proof.** We can choose the way that we label \( v_1, v_2, \ldots, v_m \) such that \( \{T_1\} \) is maximal. We can show that with these conditions \( \{T_1\} \) occurs only in \( v_1 \). Suppose that it also occurred in \( v_j \). If such was the case then by assumption there would be a \( \{T_j\} \) in \( v_j \) such that \( \{T_1\} \geq \{T_j\} \). Then \( \{T_1\} \) would not
be maximal among the \( \{T_2\}, \{T_3\}, \ldots, \{T_m\} \), which is a contradiction. Since \( \{T_1\} \) occurs only in \( v_1 \), then if we have the relation,

\[
c_1v_1 + c_2v_2 + \cdots + c_mv_m = 0.
\]

To cancel out the \( \{T_1\} \) in \( v_1 \) we need \( c_1 = 0 \). Proceeding by induction it is easy to show that the rest of the coefficients are zero. 

This leads us to a statement about the independence of the standard \( \lambda \)-tableaux.

**Theorem 3.42.** The set

\[ \{e_T : T \text{ is a standard } \lambda \text{-tableau} \} \]

is independent.

**Proof.** By Corollary 3.38, \( \{T\} \) is maximum in \( e_T \). Then by Lemma 3.39 and the assumption that all \( \{T\} \) are distinct, we have that the \( e_T \) for standard \( T \) are independent. 

Now all that we need to show is that the standard polytabloids \( e_T \) span all of \( S^\lambda \). We will omit the process that goes into proving this. It involves using a procedure called a strengthening algorithm [8]. We will skip straight to the result which is the following theorem.

**Theorem 3.43.** [8] The set

\[ \{e_T : T \text{ is a standard } \lambda \text{-tableau} \} \]

spans \( S^\lambda \).

We actually can go further and say the following.

**Theorem 3.44.** For any partition \( \lambda \) the following hold:

1. The set of vectors \( \{e_T : T \text{ is a standard } \lambda \text{-tableau} \} \) is a basis for \( S^\lambda \).

2. \( \dim(S^\lambda) = f^\lambda \).

3. \( \sum_{\lambda \vdash n} (f^\lambda)^2 = n! \), where \( f^\lambda \) is the number of standard \( \lambda \)-tableaux.

**Proof.** The first two of these statements come immediately from Theorems 3.41 and 3.42. The third part comes from Theorem 4.2.
3.6 The Branching Rule

In Section 2.2 we discussed restricted and induced representations of a general finite group. We now ask how these concepts might apply to the symmetric group. To do this we return to our Ferrers diagram. Notice that moving from a representation of $S_{n-1}$ to a representation of $S_n$ can be thought of as adding a cell to the right side of the one of the rows in the Ferrers diagram. Similarly, moving from a representation of $S_n$ to a representation of $S_{n-1}$ can be thought of as removing one of the far right cells of a row. With the above motivation, we make the following definition.

Definition 3.45. Take some Ferrers diagram which we call $\lambda$. We call a cell $(i, j)$ an inner corner of $\lambda$ if its removal leaves the Ferrers diagram of a partition. We call this new partition $\lambda^-$. Similarly we call a cell $(i, j)$ an outer corner of $\lambda$ if $(i, j) \not\in \lambda$ and if the addition of $(i, j)$ produces the Ferrers diagram of a partition. We call this new partition $\lambda^+$.

Example 3.46. Take the Ferrers diagram $\lambda$ for the partition $(5, 3, 2, 1)$

If we place a bullet \( \bullet \) at the inner corners of $\lambda$, then $\lambda$ looks like

If we place a cell with a circle \( \circ \) in it at the outer corners of $\lambda$, then $\lambda$ looks like

Clearly whether we are adding or subtracting a cell, we have several possibilities for where we will place or remove it respectively. Furthermore, the partitions we create by this process are exactly the partitions that we get when we preform restriction or reduction.

Example 3.47. Adding an outer corner cell to our $\lambda$ above we get the following values for $\lambda^+$.

$$\lambda^+ = \{ \text{ [Diagram representations]} \}.$$
It turns out that if we preform induction we get that
\[ S^{(5,3,2,1)} \uparrow S^{(6,3,2,1)} \oplus S^{(5,4,2,1)} \oplus S^{(5,3,3,1)} \oplus S^{(5,3,2,2)} \oplus S^{(6,3,2,1,1)}. \]

This leads us to the branching theorem for the Symmetric group.

**Theorem 3.48.** [8] If \( \lambda \) partitions \( n \), then

1. \( S^{\lambda} \downarrow S_{n-1} = \bigoplus_{\lambda^-} S^{\lambda^-} \).
2. \( S^{\lambda} \uparrow S_{n+1} = \bigoplus_{\lambda^n} S^{\lambda^n} \).

We now move on to focus specifically on the character table of the symmetric group \( S_n \).

4 The Character Table of \( S_n \)

4.1 The Murnaghan-Nakayama Rule

One of the most useful applications of combinatorics to symmetric group representation theory is the Murnaghan-Nakayama Rule. This algorithm tells us how to calculate any cell \( \chi^{\lambda}(\alpha) \) of the character table \( S_n \), in terms of a sum of characters for partitions strictly smaller than \( \lambda \) and \( \alpha \). Before we formally state the theorem however, we need a few definitions.

**Definition 4.1.** Let \( \mu \) and \( \lambda \) be Ferrers diagrams such that for every cell \( (i,j) \in \mu \), that same cell exists in \( \lambda \), \( (i,j) \in \lambda \). Then the skew diagram corresponding to this containment is defined as
\[ \lambda/\mu = \{ (i,j) \in \lambda : (i,j) \notin \mu \}. \]

**Example 4.2.** If \( \lambda \) and \( \mu \) have the following shapes
\[ \lambda = \begin{tabular}{|c|c|c|} \hline \text{ } & \text{ } & \text{ } \\ \hline \text{ } & \text{ } & \text{ } \\ \hline \text{ } & \text{ } & \text{ } \\ \hline \end{tabular}, \quad \mu = \begin{tabular}{|c|c|c|} \hline \text{ } & \text{ } & \text{ } \\ \hline \text{ } & \text{ } & \text{ } \\ \hline \text{ } & \text{ } & \text{ } \\ \hline \end{tabular} \]

then \( \lambda/\mu \) has the shape
\[ \mu/\lambda = \begin{tabular}{|c|c|c|} \hline \text{ } & \text{ } & \text{ } \\ \hline \text{ } & \text{ } & \text{ } \\ \hline \text{ } & \text{ } & \text{ } \\ \hline \end{tabular}. \]

**Definition 4.3.** A skew hook or rim hook is a skew diagram that has the property that if we start at the east most cell, we may move through every cell in the diagram once and only once, by making westward and northward moves.

For example, one skew hook might be
\[ \begin{tabular}{|c|c|c|} \hline \text{ } & \text{ } & \text{ } \\ \hline \text{ } & \text{ } & \text{ } \\ \hline \text{ } & \text{ } & \text{ } \\ \hline \end{tabular}. \]
Definition 4.4. Let $\xi$ be a rim hook. Then the leg length of $\xi$ is

$$\ell(\xi) = \text{(number of rows of } \xi) - 1.$$ 

We are now ready to state the Murnaghan-Nakayama Rule.

Theorem 4.5. [6, 7] Let $\lambda$ be a partition of $n$ and $\alpha = (\alpha_1, \alpha_2, ..., \alpha_k)$ be a composition of $n$. Then for the character $\chi^{\lambda}_{\alpha}$ associated with $\lambda$ and $\alpha$ in the character table of $S_n$ we have the following equality

$$x^{\lambda}_{\alpha} = \sum_{\xi} (-1)^{\ell(\xi)} \chi^{\lambda-\xi}_{\alpha'}$$

where $\alpha' = (\alpha_2, ..., \alpha_k)$ and where the sum runs over all rim hooks $\xi$ in $\lambda$ with $\alpha_1$ cells. Also, $\lambda - \xi$ is the shape found by removing all the cells of $\xi$ from $\lambda$.

By iterating this rule we can eventually reduce the value of $\chi^{\lambda}_{\alpha}$ to a sum of positive and negative 1’s. This can be seen in the following example where we calculate the value of $\chi^{(5,4,3,1)}_{(6,4,2,1)}$.

Example 4.6. The first iteration, looking for rim hooks with 6 cells, gives us

The resulting calculations are

$$\chi^{(5,4,3,1)}_{(6,4,2,1)} = (-1)^2 \chi^{(3,2,1,1)}_{(4,2,1)} + (-1)^2 \chi^{(5,2)}_{(4,2,1)}$$

Notice that we multiply each sum by positive one $(-1)^2$ because the leg length of each rim hook is two. Continuing to iterate we have

and

For the second of these shapes, there are no rim hooks with 4 cells, so $\chi^{(53)}_{(421)} = 0$. Continuing the expansion of $\chi^{(321)}_{(421)}$ we have

$$\chi^{(3,2,1,1)}_{(4,2,1)} = (-1)^2 \chi^{(3)}_{(2,1)}; \chi^{(5,2)}_{(4,2,1)} = 0$$

therefore,

$$\chi^{(5,4,3,1)}_{(6,4,2,1)} = (-1)^2 (-1)^2 \chi^{(3)}_{(2,1)} + (-1)^2 (0).$$
Next note that the first branch has one rim hook with two cells
\[ \begin{array}{ccc}
\Box & \Box & \Box \\
\end{array} \quad \Rightarrow \quad \begin{array}{ccc}
\Box & \Box & \Box \\
\end{array} \]
\[ \chi^{(3)}_{(2,1)} = (-1)^0 x^{(1)}_{(1)} \]
which implies that,
\[ \chi^{(3,2,1,1)}_{(4,2,1)} = (-1)^{2+2}(-1)^0 x^{(1)}_{(1)} \]
Finally,
\[ \chi^{(1)}_{(1)} = 1 \Rightarrow \chi^{(3,2,1,1)}_{(4,2,1)} = (-1)^{2+2+0}(-1)^0 = 1. \]
So \( \chi^{(5,4,3,1)}_{(6,4,2,1)} \) has character \((-1)^2(-1)^2(-1)^0(-1)^0 = 1.\)

The proof of the Murnaghan Nakayama rule can be sketched out as follows [8]. Let \( m = \alpha_1 \) and consider \( \pi \sigma \in S_{n-m} \times S_m \subseteq S_n \), where \( \pi \) has type \((\alpha_2, \alpha_3, \ldots, \alpha_k)\) and \( \sigma \) is an \( m \)-cycle. It turns out that
\[ \chi^\lambda_{\alpha} = \chi^\lambda_{\pi \sigma} = \chi^\lambda_{\downarrow S_{n-m} \times S_m} (\pi \sigma) = \sum_{\mu+n-m, \nu+m} m^\lambda_{\mu \nu} \chi^\mu (\pi) \chi^\nu (\sigma). \]
By Theorem 2.22 and a map called the characteristic map, one can show that \( m^\lambda_{\mu \nu} = c^\lambda_{\mu \nu}. \) The coefficient \( c^\lambda_{\mu \nu} \) is called the a Littlewood-Richardson coefficient and is related to Schur functions. So we can write
\[ \chi^\lambda (\pi \sigma) = \sum_{\mu+n-m} \chi^\mu (\pi) \sum_{\nu+m} c^\lambda_{\mu \nu} \chi^\nu (\sigma). \]
Using something called Jacobi-Trudi determinants it can be shown that
\[ \chi^\nu = \begin{cases} 
(-1)^{m-r} & \text{if } \nu = (r, 1^{m-r}) \\
0 & \text{otherwise} 
\end{cases} . \]
So the only case we need to consider is when \( \nu = (r, 1^{m-r}) \). One can prove that \( c^\lambda_{\mu \nu} = 0 \) unless each edge-wise connected component of \( \lambda/\mu \) is a rim hook. If this is the case then,
\[ c^\lambda_{\mu \nu} = \binom{k-1}{c-r} . \]
Here \( k \) is the number of component hooks spanning \( c \) columns. Putting all this together we have that,
\[ \chi^\lambda (\pi \sigma) = \sum_{\mu} \chi^\mu (\pi) \sum_{r=1}^{m} \left( \frac{k-1}{c-r} \right) (-1)^{m-r} . \]
By some elementary arguments about binomial coefficients,
\[ \sum_{r=1}^{m} \left( \frac{k-1}{c-r} \right) (-1)^{m-r} = \begin{cases} 
(-1)^{m-r} & \text{if } k-1 = 0 \\
0 & \text{otherwise} 
\end{cases} . \]
If $k = 1$, then $\lambda / \mu$ is a single skew hook $\xi$ with $m$ cells and $c$ columns. Which gives us $m - c = l\ell(\xi)$, so we finally have,

$$\chi^\lambda(\pi\sigma) = \sum_{\xi|m} (-1)^{l\ell(\xi)} \chi^{\lambda - \xi}(\pi).$$

Which completes the proof.

One somewhat surprising aspect of the Murnaghan-Nakayama rule is that for a character $x_\alpha^\lambda$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ need not be a partition. The order of the numbers in $\alpha$ does not affect the value of the character that we find by using the rule. This begins to make sense when we note that because each $\alpha$ corresponds to a conjugacy class of $S_n$, which can be viewed as multiplication of elements of cycle type $\alpha_1, \alpha_2, \ldots, \alpha_k$, and these independent cycles commute, then the conjugacy class defined by (cycle type $\alpha_1$)(cycle type $\alpha_2$) ... (cycle type $\alpha_k$) may just as well be represented as (cycle type $\alpha_k$)(cycle type $\alpha_4$) ... (cycle type $\alpha_2$) or any other ordering of $\alpha_1, \alpha_2, \ldots, \alpha_k$.

### 4.2 An Open Problem

In Richard Stanley's 1999 paper "Positivity problems and conjectures in algebraic combinatorics" [9] he put forth a collection of important unsolved problems in Algebraic Combinatorics. Problem 12 in this paper asks for a "...combinatorial interpretation of the row sums of the character table of $S_n$, thereby combinatorially reproving that they are non-negative." At the time of writing this paper, this problem remains unsolved. Stanley says "reproving" because this fact can be easily proved using some of the machinery of representation theory discussed at the beginning of this paper. We will now do this.

**Theorem 4.7.** The row sums of the character table of $S_n$ are non-negative.

**Proof.** First we will show that if $\chi$ is the character of a group $G$ acting on itself by conjugation, and $\chi_i$ is the character of an irreducible representation in $\chi$, then the multiplicity $m_{\chi_i}$ of $\chi_i$ in $\chi$ is equal to

$$\langle \chi, \chi_i \rangle.$$

By Corollary 2.15, we know that $\chi = m_1 \chi_1 + \cdots + m_k \chi_k$, the sum of its irreducible representations, each multiplied by its multiplicity. Therefore

$$\langle \chi, \chi_i \rangle = \langle \sum_j m_j \chi_j, \chi_i \rangle = \sum_j m_j \langle \chi_j, \chi_i \rangle = m_i.$$

Next we want to show that

$$\langle \chi, \chi_i \rangle = \sum_K \chi_i(K)$$

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where \( K \) ranges over all conjugacy classes of \( G \), and where \( \chi(K) \) denotes \( \chi(g) \) for any \( g \in K \). Now we consider the character \( \chi(g) \). Since we define \( \chi(g) \) as the trace of the matrix associated with \( g \) acting by conjugation on the basis \( \{ e, g_1, \cdots, g_j \} \), then \( \chi(g) \) is equal to the centralizer of \( g \), \( |Z(g)| \), since the diagonal entry of a column associated with \( g_j \) will contain a 1 if \( gg_jg^{-1} = g_j \) and 0 if not. By the Counting Formula, \( |G| = |C_g||Z(g)| \) where \( C_g \) is the conjugacy class of \( g \). So

\[
\langle \chi, \chi_k \rangle = \frac{1}{|G|} \sum_K |C_g| \chi(g) \chi_k(g) = \frac{1}{|G|} \sum_K |C_g| \frac{|G|}{|C_g|} \chi_k(g) = \sum_K \chi_k(g)
\]

as desired. Since the multiplicity of \( \chi_k \) is always non-negative, then the row sums of the character table of the symmetric group are always non-negative. \( \square \)

The remainder of this paper will discuss some of the results of my search for a combinatorial proof of Theorem 4.7.

4.3 Toward a Solution

As stated above, one possible route to a combinatorial proof of row sum non-negativity might be by showing that the rows associated with specific shapes are non-negative. With this in we mind, will move toward proving positivity for the row sum associated with the shape \((n-1,1)\). First though, we need the following result.

**Theorem 4.8.** Let \( \alpha \) be a composition of \( n \), \( \alpha = (\alpha_1, \alpha_2, \cdots, \alpha_i) \), and \( \phi = (n-1,1) \) be a partition of \( n \). Then

\[
\chi^\phi_\alpha = b - 1
\]

where \( b \) is equal to the number of \( \alpha_1, \alpha_2, \cdots, \alpha_i \) that are equal to 1.

**Proof.** First suppose that no \( \alpha_j \) is equal to one. Then all \( (\alpha_1, \alpha_2, \cdots, \alpha_i) \) are greater than or equal to two, including \( \alpha_1 \).

Proceeding with the Murnaghan-Nakayama rule we see that the first \( i-1 \) rim hooks must be removed from the first row. Otherwise, we would have to take the one cell in the second row, and since each \( \alpha_j \geq 2 \), we would also have to take the top left cell, which we cannot do since we still have at least one more rim hook to remove. Note that since each of the first \( i-1 \) rim hooks are in the first row only, they all have leg length 0.

Finally, we remove the last rim hook of \( \alpha_i \) cells, one of which must be the second row cell and another of which must be the top-left corner cell. So this last rim hook will have leg length 1. Therefore, the sum of the leg lengths of all the rim hooks will be 1. So

\[
\chi^\phi_\alpha = \sum_{\xi} (-1)^{\omega} = (-1)^1 = -1
\]
Note that there is only one rim hook \( \xi \) since the first \( i - 1 \) rim hooks must be taken only from the first row and therefore are simply taken as sequential sets of \( \alpha_j \) cells, causing there to be only one way that one may take them.

Next we suppose that there is only one \( \alpha_j = 1 \). Now since we can reorder \( \alpha \) without changing the value of the resulting character, we reorder \( \alpha \) so that it is weakly decreasing and call it \( \alpha' = (\alpha'_1, \alpha'_2, \ldots, \alpha'_t) \). Then it must be the case that \( \alpha_i = 1, \alpha_{i-1} \geq 2 \). Again using the Murnaghan-Nakayama rule, we begin removing rim hooks from the first row. Before we remove the \( i - 1 \)th rim hook, we have \( \alpha_{i-1} + 1 \) cells remaining in the shape. If we remove cells only from the first row, then we will be left with the cell in the second row only, which does not work. If we remove the cell in the second row, we will be left with one cell in the first row which is not the top left cell, and that clearly does not work. So there is no sequence of rim hooks that completely reduce the shape \((n - 1, 1)\). Hence \( \chi_\alpha^\delta = 0 \) by the Murnaghan-Nakayama rule.

We will use induction on \( b \) to show that the theorem holds when \( b \geq 2 \). We will again use the weakly decreasing reordering of \( \alpha, \alpha' \). As our base case let \( b = 2 \). Because of our ordering, we must begin by removing all the rim hooks with more than one cell. These all must be taken successively from the far right of the first row, and each will have leg length 0, since they have cells in only one row. After \( i - 2 \) rim hook removals we will have only three one cell rim hooks left to remove from the shape.

There is only one way to do this: take the first one-cell rim hook as the bottom cell and the second one as the top cell. This can be shown as

\[
\begin{array}{c}
\boxed{} \\
\boxed{} \\
\boxed{2} \\
\boxed{1}
\end{array}
\]

where the number indicates the order in which we take each cell.

Now we make the inductive hypothesis that this is true up to \( b = k \). We will show that it is also true for the \( b = k + 1 \) case. If \( b = k + 1 \) we proceed exactly as in the base case until we are left with \( k + 1 \) cells in the form \((k, 1)\),

\[
\begin{array}{c}
\boxed{\ldots} \\
\boxed{\ldots} \\
\boxed{2} \\
\boxed{1}
\end{array}
\]

Now we can begin by either removing the \( k + 1 \)th one-cell rim hook from the second row cell or the far right cell on the first row. If we remove the cell on the second row that we are left with,

\[
\begin{array}{c}
\boxed{\ldots} \\
\boxed{\ldots} \\
\boxed{\ldots} \\
\boxed{\ldots}
\end{array}
\]

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and there is now only one way to remove the \( k \) remaining one-cell rim hooks, taking them successively from the far right of the first row. If we remove the
far right cell, then we are left with the shape \((k - 1, 1)\).

\[
\begin{array}{c}
\square \\
\square \\
\end{array}
\]

By our inductive hypothesis there are \( b - 1 \) ways to remove the removing the remaining cells, since the character \( \chi_{\phi_{(\alpha_1, \alpha_2, \ldots, \alpha_{n-1})}} \) \( = b - 1 \) will be equal to the number of ways of removing the \( b - 1 \) one-cell rim hooks. Adding these two values together we get that there are \((b - 1) + 1 = b\) ways of removing the one cell rim hooks from this shape. Since each of these have leg length 0, the character \( \chi_{\phi} = b \). Which completes the proof.

To make the process of counting the 1’s of a partition easier to deal with, we will define the following function.

**Definition 4.9.** Let \( \mu \) be the map from the partitions of some \( n \) to \( \mathbb{Z} \) defined as

\[
\mu(p) = b.
\]

Where \( p \) is a partition, and \( b \) is the number of 1’s in \( p \).

Below we give some of the partial sums

\[
\sum_{j=q}^{i} [\mu(p_j) - 1]
\]

over the partitions of small \( n \), \( P_n = (p_1, p_2, \ldots, p_i) \). \( P \) is listed in lexicographic order, and \( 1 \leq q \leq i \).

**Example 4.10.** For \( n = 1 \) there is only the partition \((1)\) and only is partial sum is therefore \( \mu((1)) - 1 = 0 \)

**Example 4.11.** When \( n = 2 \), then there are two partitions, \((2)\) and \((1, 1)\). Then the partial sums are

\[
[\mu((2)) - 1] + [\mu((1, 1)) - 1] = 0
\]

\[
\mu((1, 1)) - 1 = 1.
\]

**Example 4.12.** For the \( n = 3 \) case, there are three partitions: \((3)\), \((2, 1)\), \((1, 1, 1)\). These have partial sums:

\[
[\mu((3)) - 1] + [\mu((2, 1)) - 1] + [\mu((1, 1, 1)) - 1] = 1
\]

\[
[\mu((2, 1)) - 1] + [\mu((1, 1, 1)) - 1] = 2
\]

\[
[\mu((1, 1, 1)) - 1] = 2.
\]
Example 4.13. Finally, for the \(n = 4\) case we have five partitions: \((4), (3, 1), (2, 2), (2, 1, 1), (1, 1, 1, 1)\). Then we get the following partial sums:

\[
\sum_{j=1}^{5} |\mu(p_j) - 1| = 2
\]

\[
\sum_{j=2}^{5} |\mu(p_j) - 1| = 3
\]

\[
\sum_{j=3}^{5} |\mu(p_j) - 1| = 3
\]

\[
\sum_{j=4}^{5} |\mu(p_j) - 1| = 4
\]

\[
\sum_{j=5}^{5} |\mu(p_j) - 1| = 3.
\]

With the help of these examples we can prove the following theorem.

Theorem 4.14. For any \(n \geq 3\), let \(P_n = (p_1, p_2, \cdots, p_i)\) be the set of partitions of \(n\) in lexicographic order. Then for any \(1 \leq q \leq i\),

\[
\sum_{j=q}^{i} x^{(n-1, 1)}_{p_j} = \sum_{j=q}^{i} |\mu(p_j) - 1| > 0
\]

where \(\mu(p_j)\) gives the number of 1's in the partition \(p_j\). In particular the row sums for shape \((n-1, 1)\) are positive.

Proof. We will show this by strong induction on \(n\). As our base case we note from the examples above that the theorem holds true for \(n = 1\) through \(n = 4\). Next we make the inductive hypothesis that our theorem is true up to \(n = k\). We will show that it is also true then for \(n = k + 1\). To do this we write the partitions of \(k + 1\) in lexicographic order. If \(n + 1\) is even, then

\((n + 1)\)

\((n, 1)\)

\((n - 1, 2)\)

\((n - 1, 1, 1)\)

\(\vdots\)

\((\frac{n + 1}{2} - 1, \frac{n + 1}{2} - 1, 2)\)
\[
\left( \frac{n+1}{2} - 1, \frac{n+1}{2} - 1, 1, 1, 1 \right)
\]
\[
\vdots
\]
\[
(2, 1, 1, 1, \cdots, 1)
\]
\[
(1, 1, 1, 1, \cdots, 1, 1).
\]

If \( n + 1 \) is odd, then

\[
(n+1)
\]
\[
(n, 1)
\]
\[
(n-1, 2)
\]
\[
(n-1, 1, 1)
\]
\[
\vdots
\]
\[
(\frac{n}{2} - 1, \frac{n}{2} - 1, 2)
\]
\[
(\frac{n}{2} - 1, \frac{n}{2} - 1, 1, 1)
\]
\[
\vdots
\]
\[
(2, 1, 1, 1, \cdots, 1)
\]
\[
(1, 1, 1, 1, \cdots, 1, 1).
\]

Either way it should be clear that except for the first partition \((n + 1)\), the rest can be grouped as either full or partial sets of partitions for some integer \(r\), where \(n + 1 - r\) is appended as the first element of the partition. Furthermore, if the partition of \(r\) is only partial, then it is missing all its beginning terms up to the first term where the first term in the partition is equal to \(n + 1 - r\) (otherwise the partition would not be weakly decreasing when \(n + 1 - r\) is appended as the first element of the partition).

Since adding the beginning \(n + 1 - r\) term does not change their value under \(\mu\), except in the last case, then we can view

\[
\sum_{j=2}^{i-1} [\mu(p_j) - 1]
\]

for \(P_{n+1} = (p_1, p_2, \cdots, p_i)\) simply as a sum of partial or full sums for \(n \leq k + 1\). Since we know from our inductive hypothesis that partial sums are all positive or 0, then

\[
\sum_{j=2}^{i-1} [\mu(p_j) - 1] \geq 0.
\]

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We know that \([\mu(p_1) - 1] = [\mu((n+1)) - 1] = -1\), and since \(p_i = (1, 1, 1, \cdots, 1)\), then \([\mu(p_1) - 1] = k\). So

\[
\sum_{j=1}^{4} [\mu(p_j) - 1] \geq k - 1 \geq 1
\]

since \(k \geq 3\). From Theorem 19 we know that \(\chi_{\rho_j}^{(n-1,1)} = b - 1 = \mu(p_j) - 1\). So

\[
\sum_{j=1}^{4} \chi_{\rho_j}^{(n-1,1)} \geq 1.
\]

Thus we have that the row sum for the row of shape \((n-1,1)\) is positive. □

Those interested in sequence of integers produced by the row sums of the row of shape \((n-1,1)\) should consult [3].

4.4 Symmetries in the Character Table of \(S_n\)

It makes sense to continue our examination of the character table of \(S_n\) by looking for symmetries in it. For instance we might wonder whether there is a relationship between two characters that are in the same column but whose row shapes are mirror reflections of each other across the diagonal. We define this relationship explicitly below.

**Definition 4.15.** Let \(\lambda\) and \(\alpha\) be partitions of \(n\). We call \(\lambda\) and \(\alpha\) **conjugate** when the Ferrers diagram of \(\lambda\) has cell \((i,j)\) if and only if the Ferrers diagram of \(\alpha\) has cell \((j,i)\).

**Lemma 4.16.** Let the leg length of a rim hook \(\xi\) be \(\ell(\xi)\). Then the arm length of \(\xi\) is equal to

\[
\partial \partial(\xi) = \alpha - \ell(\xi) - 1
\]

where \(\xi\) has \(\alpha\) cells.

**Proof.** We will prove this by induction on \(\alpha\). If we have a rim hook \(\xi\) with only one cell, \(\alpha = 1\), then clearly \(\ell(\xi) = 0\) and \(\partial \partial(\xi) = 0\). So the base case is valid.

Next we make the inductive hypothesis that the arm length formula is valid up to \(\alpha = n\). We will show that it is also valid for \(\alpha = n + 1\). Take a rim hook with \(n + 1\) cells. We remove the cell that is the furthest north and to the right. Now we know that the arm length formula is valid for rim hooks with \(n\) cells. So the arm length for this rim hook \(\xi'\) missing a cell is

\[
\partial \partial(\xi') = (n) - \ell(\xi') - 1.
\]

Now we add the cell back that we had removed. We can either add it directly to the right of the cell that is furthest northward and eastward, or directly above
this cell. If we add it to the right, then the total number of cells increases by one and so does the arm length. Then since \( \ell(\xi') = \ell\xi \) we have that,

\[
\partial\xi = \partial\xi + 1 = (n - \ell(\xi') - 1) + 1 = (n + 1) - \ell(\xi') - 1 = (n + 1) - \ell\xi - 1.
\]

So in this case the formula holds. If we instead place the cell above the top-right existing cell, then then the total number of cells increases by one and so does the leg length. So since \( \ell\ell(\xi') + 1 = \ell\xi \),

\[
\partial\xi = \partial\xi + 1 = n - \ell(\xi') - 1 = n - (\ell\xi + 1) - 1 = n + 1 - \ell\xi - 1.
\]

So in this case also the formula holds. Hence by induction we have that our formula is true for all rim hooks. \( \square \)

**Theorem 4.17.** Let \( \lambda' \) be the conjugate to a shape of \( \lambda \). Then if \( \alpha \vdash n \), \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_i) \) we can say the following about the \( \chi^\lambda_\alpha \) and \( \chi^{\lambda'}_\alpha \).

1. If \( n + i \) is even \( \chi^\lambda_\alpha = \chi^{\lambda'}_\alpha \).
2. If \( n + i \) is odd \( \chi^\lambda_\alpha = -\chi^{\lambda'}_\alpha \).

**Proof.** We know from the Murnaghan-Nakayama Rule that

\[
\chi^\lambda_\alpha = \sum_{\lambda_\alpha(\alpha_1, \alpha_2, \ldots, \alpha_i)} (-1)^{\eta}
\]

where \( \lambda_\alpha(\alpha_1, \alpha_2, \ldots, \alpha_i) \) is the set of all rim hook decompositions of \( \lambda \), beginning by removing a rim hook with \( \alpha_1 \) cells and ending by removing a rim hook with \( \alpha_i \) cells. Also, \( \eta = \sum_{j=1}^{i} \ell(\xi_j) \) is the sum of leg lengths for the rim hooks of a given combination. We know that if \( \xi \) is a rim hook for a shape \( \beta \), then \( \xi' \) (the conjugate of \( \xi \)) is a rim hook for \( \beta' \). Since the conjugate exchanges rows for columns, then \( \partial\xi = \ell\xi' \), because the number of rows that \( \xi \) occupies will be the number of rows that \( \xi' \) occupies. Consider

\[
\chi^{\lambda'}_\alpha = \sum_{\lambda_\alpha'(\alpha_1, \alpha_2, \ldots, \alpha_i)} (-1)^{\eta'}
\]

where \( \eta' = \sum_{j=1}^{i} \ell(\xi'_j) \). Given any decomposition \( \xi_1, \xi_2, \ldots, \xi_i \) in \( \lambda_\alpha'(\alpha_1, \alpha_2, \ldots, \alpha_i) \), there is a decomposition sequence \( \xi_1, \xi_2, \ldots, \xi_i \) in \( \lambda_\alpha(\alpha_1, \alpha_2, \ldots, \alpha_i) \) such that,

\[
\sum_{j=1}^{i} \ell(\xi_j) = \sum_{j=1}^{i} \ell(\xi'_j) = \sum_{j=1}^{i} \partial(\xi_j) = \sum_{j=1}^{i} \alpha_j - \ell(\xi'_j) - 1 = n + 1 + \sum_{j=1}^{i} \ell(\xi_j).
\]

So

\[
\chi^{\lambda'}_\alpha = \sum_{\lambda_\alpha'(\alpha_1, \alpha_2, \ldots, \alpha_i)} (-1)^{\eta'} = \sum_{\lambda_\alpha(\alpha_1, \alpha_2, \ldots, \alpha_i)} (-1)^{n+i+\eta}
\]

\[
= \sum_{\lambda_\alpha'(\alpha_1, \alpha_2, \ldots, \alpha_i)} (-1)^{n+1}(-1)^{\eta}
\]

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\[ (-1)^{n+1} \sum_{\lambda' \prec (\alpha_1, \alpha_2, \ldots, \alpha_i)} (-1)^{\gamma} = (-1)^{n+1} \lambda_{\alpha}^\lambda. \]

Clearly then if either both \( n \) and \( i \) are even then

\[ \chi_{\alpha}^\lambda = \chi_{\alpha}^{\lambda'}. \]

If either \( n \) is even and \( i \) is odd, or \( i \) is even and \( n \) is odd, then

\[ \chi_{\alpha}^\lambda = -\chi_{\alpha}^{\lambda'}. \]

This completes the proof. \( \square \)

From the theorem above we can draw the nice corollary

**Corollary 4.18.** Suppose that for some shape \( \lambda, \lambda' = \lambda \). Then when \( n \) is even and \( i \) is odd, or \( n \) is odd and \( i \) is even, \( \chi_{\alpha}^\lambda = \chi_{\alpha}^{\lambda'} = 0. \)

**Proof.** From above, \( \chi_{\alpha}^\lambda = -\chi_{\alpha}^{\lambda'} \). Under these conditions for \( i \) and \( n \). Also since \( \lambda' = \lambda \), \( \chi_{\alpha}^\lambda = \chi_{\alpha}^{\lambda'} \). It follows that \( \chi_{\alpha}^\lambda = 0. \) \( \square \)

### 4.5 Zeros in the Character Table of \( S_n \)

Another observation about the character table of \( S_n \) concerns its zeros. If we are using the Murnaghan-Nakayama rule to calculate characters, there are two ways that we can get a zero. The first, which we will call type A, happens when there are simply no ways to decompose our shape with the rim hook sizes from the given composition. The other possibility, which we call type B, is that we are able to remove rim hooks into an even number of branches and the value of these branches sum to zero (there are an equal number of positive and negative ones). Since the difference between these two types of zeros is found in the way they are gotten through the Murnaghan-Nakayama process, it would be useful to know whether altering the way that we perform the process might switch zeros back and forth between the two types. Here we will investigate whether it is possible to change the type of zero character by reordering our composition \( \alpha \) which determine the rim hook size and the order in which we remove them. It turns out that this is sometimes the case. Consider the following example.

**Example 4.19.** We will calculate the character of \( x_{(2,2,1)}^{(2,1,1,1)} \) and then reorder \( (2,2,1) \) into the composition \( (2,1,2) \) and then calculate \( x_{(2,1,2)}^{(2,1,1,1)} \). We begin below.

\[
\begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\downarrow \\
\end{array}
\]

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There is no rim hook on this shape (2, 1) that contains exactly two cells. Therefore we conclude that the character $x_{(2,1,1,1)}^{(2,1,1,1)} = 0$. Now we calculate $x_{(2,1,2)}^{(2,1,1,1)}$. 

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So we are left with the sum \((-1)(1)(1) + (-1)(1)(-1) = -1 + 1 = 0.\) Hence for a least some zero characters \(x^\lambda_\alpha\), reordering \(\alpha\) changes \(x^\lambda_\alpha\) from a type A zero to a type B zero.

Reordering \(\alpha\) does not always result in a zero character changing types. It is left to the reader to show that regardless of how we reorder \((2,1,1,1)\) in \(x^{(3,1,1)}_{(2,1,1,1)}\), we still have a type B zero.

5 Future Directions

One strategy to proving the row sum problem might be to find a map that takes a Murnaghan-Nakayama reduction scheme that yields a \(-1\), and maps it to a scheme that gives a \(1\). If we could then show that the map was injective we would have effectively combinatorial reproved the non-negativity of the row sums. As motivation to this strategy, look at the following example.

Example 5.1. Take the shape generated by the partition \(\lambda = (5,3,2,2)\) and also the partition \(\alpha = (5,4,2,1)\). One Murnaghan-Nakayama reduction scheme for this \(\lambda\) using rim hooks with size determined by \(\alpha\) is

\[
\begin{array}{cccc}
4 & 2 & 2 & 2 \\
3 & 1 & 1 \\
3 & 1 \\
1 & 1 \\
\end{array}
\]

Where the number in each cell denotes the step at which the cell was removed. Since the first rim hook, the set of cells with 1's in them, has leg length 2, and the second, third, and fourth rim hooks have leg lengths 0, 1, and 0 respectively, then this reduction scheme yields

\[(-1)^2 \times (-1)^0 \times (-1)^1 \times (-1)^0 = -1.\]

Now what is one way that we might alter our rim hook removal scheme so that we got a positive 1 rather than a negative one? One way would be to cut one of our rim hooks. We can do this in two ways. Firstly, we might cut the rim hook that has an odd leg length. Then we would have,

\[
\begin{array}{cccc}
5 & 2 & 2 & 2 \\
4 & 1 & 1 \\
3 & 1 \\
1 & 1 \\
\end{array}
\]

Which gives us,

\[(-1)^2 \times (-1)^0 \times (-1)^0 \times (-1)^0 \times (-1)^0 = 1.\]

This of course corresponds to a reduction of \(\lambda\) using rim hooks of size given by the partition \(\alpha' = (5,4,1,1,1)\). We could also cut one of the rim hooks that has even, non-zero leg length so that one of the resulting rim hooks has odd leg length.
This corresponds to a reduction of $\lambda$ using rim hook of size given by the composition $\beta = (3, 2, 4, 2, 1)$.

Now note that the only time that we cannot cut a rim hook in a reduction scheme to change the sign of the resulting 1 or $-1$, is when all rim hooks have leg length zero. This means that the reduction scheme already gives 1. This suggests that there might be some set of rules that would define a map from the reduction schemes that give $-1$ to the reduction schemes that give 1 which is injective. The difficulty comes from the fact that many cuts alter the ordering of the partition that we are using to determine the size of the rim hooks, as we saw in the last step of the example above.

Along with investigating the kind of cutting technique that I outline above, in the future I plan to do research on other ways of explicitly generating characters. In particular, I plan to read the paper "An Explicit Formula for the Characters of the Symmetric Group" by Michel Lassalle [5].

A Character Tables

Here we present character tables for $S_n$ from $n = 3$ to $n = 7$ generated in Matlab using the Murnaghan-Nakayama rule. The far right column consists of the row sums.

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</tr>
<tr>
<td>(2,1,1,1,1,1)</td>
<td>-4</td>
<td>6</td>
<td>5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(1,1,1,1,1,1)</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

B  The Character Table Generating Code

Here is the Matlab code used to generate the character tables above. The program consists of three files. Construction of a character table for some $S_n$ begins when the user specifies the value of $n$ in the file CharacterTable.m. This file then generates all the partitions for the given $n$. Each of these partitions is then sent to the function file Murnaghan.m which executes the Murnaghan-Nakayama algorithm. Murnaghan.m uses the file SkewReducer.m to find and remove each of the possible rim hooks, with SkewReducer.m returning the leg length of the removed rim hook as well as the new shape. Finally, Murnaghan.m returns the character to CharacterTable.m, and this second file organizes the table itself.

B.1  CharacterTable.m

clear all;

n = 8;

Walls = zeros(1,n - 1);

for i = 1 : n - 1
    Walls(i) = i;
end

Partitions = zeros(1, n - 1);

for j = 1 : n - 1
PartitionsTemp = nchoosek(Walls,j);
Transfer = zeros(length(Partitions(:,1)) + length(PartitionsTemp(:,1)), n)
for m = 1 : length(Partitions(:,1))
    Transfer(m,:) = Partitions(m,:);
end
for k = 1 : length(PartitionsTemp(:,1))
    Transfer(k + length(Partitions(:,1)),1:j) = PartitionsTemp(k,:);
end
Partitions = Transfer;
end
i = 0;
j = 0;
m = 0;
k = 0;
PartitionSums = zeros(length(Partitions(:,1)), length(Partitions(1,:)) + 1);
for i = 1 : length(Partitions(:,1))
    for j = 1 : (length(Partitions(1,:)) + 1)
        if (j == 1)
            PartitionSums(i,j) = Partitions(i,j);
        elseif (j == (length(Partitions(1,:)) +1))
            if (Partitions(i,j-1)>0)
                PartitionSums(i,j) = n - Partitions(i,j-1);
            else
                PartitionSums(i,j) = 0;
            end
        elseif (Partitions(i,j) == 0 && Partitions(i,j-1)>0 )
            PartitionSums(i,j) = n - Partitions(i,j-1);
        elseif (Partitions(i,j) == 0)
            PartitionSums(i,j) = 0;
        else
            PartitionSums(i,j) = Partitions(i,j) - Partitions(i,j-1);
        end
    end
end
PartitionSums(1,1) = n;
for m = 1 : length(PartitionSums(:,1))
    PartitionSums(m,:) = sort(PartitionSums(m,:), 'descend');
end
PartitionsFinal = zeros(length(PartitionSums(:,1)), length(PartitionSums(1,:)));
rowcount = 1;

for k = 1 : length(PartitionSums(:,1))
    count = 0;
    for l = 1 : k - 1
        if (PartitionSums(k,:) == PartitionSums(l,:))
            count = 1;
        end
    end
    if (count == 0)
        PartitionsFinal(rowcount,:) = PartitionSums(k,:);
        rowcount = rowcount + 1;
    end
end

ZerosCounter = 0;

for h = 1 : length(PartitionsFinal(:,1))
    if (PartitionsFinal(h,1) == 0)
        ZerosCounter = ZerosCounter + 1;
    end
end

PartitionsFinal = PartitionsFinal(1:(length(Partitions(:,1)) - ZerosCounter),:);

k = 0;
l = 0;

TheCharacterTable = zeros(length(PartitionsFinal(:,1)), length(PartitionsFinal(:,1)));

for k = 1 : length(PartitionsFinal(:,1))
    for l = 1 : length(PartitionsFinal(:,1))
        PartitionsFinal(k,:);
        TheCharacterTable(k,l) = Murnaghan(PartitionsFinal(k,:), PartitionsFinal);
    end
end

k = 0;
l = 0;

TheCharacterTableTemp = zeros(length(TheCharacterTable(:,1)), length(TheCharacterTable(:,1)));
TheCharacterTableTemp(:,1:length(TheCharacterTable(1,:))) = TheCharacterTable;

for k = 1 : length(TheCharacterTable(:,1))
    RowSums = 0;
end
for \( l = 1 : \text{length(TheCharacterTable}(1,:)) \)
    RowSums = RowSums + TheCharacterTable(k,1);
end
TheCharacterTable(k,length(TheCharacterTable(:,1)) + 1) = RowSums;
end

PartitionsFinal

TheCharacterTable

B.2 Murnaghan.m

function [ Character ] = Murnaghan( PartitionA , PartitionB )
ZeroCount = 0;
for \( l = 1 : \text{length(PartitionB)} \)
    if (PartitionB(l) == 0)
        ZeroCount = ZeroCount + 1;
    end
end
PartitionB = PartitionB(1:(length(PartitionB) - ZeroCount));

PartitionATemp = zeros(2, length(PartitionA));
PartitionATemp(2,:) = PartitionA;
PartitionA = PartitionATemp;

while (PartitionB(1) > 0)
    PartitionCollector = [ 0 0 0 ; 0 0 0 ];
    for \( i = 1 : \text{length(PartitionA}(;,:)) \)
        PartitionAReduced, PartitionBReduced ] = SkewReducer(PartitionA(2*i,:))
        for \( j = 1 : \text{length(PartitionAReduced}(;,:)) \)
            PartitionAReduced(2*j - 1,1) = PartitionAReduced(2*j - 1,1) + PartitionBReduced(1)
        end
    end
    PartitionCollectorTemp = zeros(length(PartitionCollector(:,1)) + length(PartitionCollector(:,1)))
    PartitionCollectorTemp(m,1:length(PartitionCollector(1,:))) = PartitionCollectorTemp
    for \( n = 1 : \text{length(PartitionAReduced}(;,:)) \)
        PartitionCollectorTemp length(PartitionCollector(:,1)) + n,:) = Part
    end
end
PartitionCollectorTemp;
PartitionCollector = PartitionCollectorTemp;
end
PartitionCollectorTemp = zeros(length(PartitionCollector(:,1)) - 2, length)
for t = 3 : length(PartitionCollector(:,1))
    PartitionCollectorTemp(t - 2,:) = PartitionCollector(t,:);
end
PartitionA = PartitionCollectorTemp;
PartitionB = PartitionBReduced;
Summer = 0;
for r = 1 : length(PartitionA(:,1))
    for q = 1 : length(PartitionA(1,:))
        Summer = Summer + PartitionA(r,q);
    end
end
if (Summer == 0)
    PartitionB = [];
end
if (length(PartitionB) == 0)
    PartitionB = [0];
end
end
Sum = 0;
i = 0;

for i = 1 : length(PartitionA(:,1))/2
    Sum = Sum + (-1)^(PartitionA(2*i - 1));
end
Character = Sum;

B.3 SkewReducer.m

function [ PartitionAReduced , PartitionBReduced ] = SkewReducer( PartitionA ,

Tableaux = zeros(length(PartitionA(1,:)),PartitionA(1,1));

for j=1:length(PartitionA(1,:)),
    for i=(PartitionA(1,j) + 1):PartitionA(1,1),
        Tableaux(j,i)=1;
    end
end

%Creates the tableaux with 0's in the relevant boxes and 1's in the others
SkewBoxes = zeros(2, length(PartitionA(1,:)));  

for k=1:length(PartitionA(1,:)),  
    SkewBoxes(:,k) = [ k ; 1 ];  
end  

% Finds the boxes in the first column that don't have 1's.  
SkewBoxes = [ 0 ; 0 ];  
for m=1:length(PartitionA(1,:)),  
    for n=1:PartitionA(1,1),  
        if Tableaux(m,n)==0  
            SkewBoxes2 = zeros(2, length(SkewBoxes(:,1)) + 1);  
            for i=1:length(SkewBoxes(:,1))  
                SkewBoxes2(:,1) = SkewBoxes(:,1);  
            end  
            SkewBoxes2(:,length(SkewBoxes(:,1)) + 1) = [ m ; n ];  
            SkewBoxes = SkewBoxes2;  
        end  
    end  
end  
i = 0;  
SkewTemp = zeros(2, length(SkewBoxes(:,1))-1);  

for i = 2 : length(SkewTemp(:,1)) + 1  
    SkewTemp(:,i-1) = SkewBoxes(:,i);  
end  
SkewBoxes = SkewTemp;  
% Finds all other cells with a zero and notes their coordinates  
% in the vector SkewBoxes with dimensions (2,n).  
PossibleSkewPaths = nchoosek(1:length(SkewBoxes(:,1)), PartitionB(1));  
% A list of all possible selections of Partition B first value cells that do  
% not have 1's in them.  
Tableaux2 = zeros(length(Tableaux(:,1))+1, length(Tableaux(:,1)));  
for b=1:length(Tableaux(:,1))  
    Tableaux2(b,:) = Tableaux(b,:);  
end
Tableaux2(b,:) = Tableaux(b,:);
end

Tableaux2(length(Tableaux(:,1))+1,:) = ones(1,length(Tableaux(:,1)))
Tableaux3 = Tableaux2;
Tableaux2 = zeros(length(Tableaux3(:,1)),length(Tableaux3(1,:))+1);
for c=1:length(Tableaux3(1,:))
    Tableaux2(:,c) = Tableaux3(:,c);
end

Tableaux2(:,length(Tableaux3(1,:))+1) = ones(length(Tableaux3(:,1)),1);
Tableaux3 = Tableaux2;

ViableTableauxs = zeros(length(Tableaux3(:,1)),length(Tableaux3(1,:)));
for q=1:length(PossibleSkewPaths(:,1)),
    TableauxTemp = Tableaux3;
    Veracity = 0;
    for w=1:length(PossibleSkewPaths(1,:)),
        TableauxTemp(SkewBoxes(1,PossibleSkewPaths(q,w)),SkewBoxes(2,Possil
        end
    for t=1:length(Tableaux(:,1)),
        for r=1:length(Tableaux(1,:)),
            if (TableauxTemp(t,r)== 2 & TableauxTemp(t+1,r)== 0 || (Table
                Veracity = 1;
        end
    end
end
if Veracity == 0;
    ViableTableauxs2 = zeros(length(ViableTableauxs(:,1)) + length(Tableaux1
    for z=1:length(ViableTableauxs(:,1))
        ViableTableauxs2(z,:) = ViableTableauxs(z,:);
    end
    for u=1:length(TableauxTemp(:,1))
        ViableTableauxs2(u + length(ViableTableauxs(:,1)),:) = Tableaux
    end
    ViableTableauxs = ViableTableauxs2;
end

% Now we have a matrix that consists of all the 'viable' tableauxs stacked
% on top of eachother, with an empty matrix on top. We get rid of this
%below.

\[
\text{ViableTableauxsRev} = \text{zeros} \left( \text{length} \left( \text{ViableTableauxs}(:,1) \right) \right) - \text{length} \left( \text{Tableaux3}(:,1) \right);
\]

\[
i = 1;
\]
\[
j = 1;
\]
\[
k = 1;
\]
\[
m = 1;
\]
\[
n = 1;
\]
\[
p = 1;
\]
\[
r = 1;
\]
\[
t = 1;
\]

\[
\text{for } i = (\text{length} \left( \text{Tableaux3}(:,1) \right) + 1):(\text{length} \left( \text{ViableTableauxs}(:,1) \right)), \\
\text{ViableTableauxsRev} \left( i - \text{length} \left( \text{Tableaux3}(:,1),1 \right) \right) = \text{ViableTableauxs} \left( i,1 \right);
\]
\[
\text{end}
\]
\[
i = 1;
\]

%Next we need to weed out the matrices that do not have a path in them. To %do this we need to find the furthest element from the top and left in each %tableau.

\[
\text{FinalViabiles} = \text{zeros} \left( \text{length} \left( \text{Tableaux3}(:,1) \right), \text{length} \left( \text{Tableaux3}(1,:) \right) \right);
\]

\[
\text{for } j = 1:(\text{length} \left( \text{ViableTableauxsRev}(:,1) \right) / \text{length} \left( \text{Tableaux3}(:,1) \right)), \\
\text{TableauxTemp} = \text{zeros} \left( \text{length} \left( \text{Tableaux3}(,1) \right), \text{length} \left( \text{Tableaux3}(1,:) \right) \right); \\
\text{for } m = 1 : \text{length} \left( \text{Tableaux3}(,1) \right), \\
\text{TableauxTemp}(m,:) = \text{ViableTableauxsRev} \left( (j-1) * (\text{length} \left( \text{Tableaux3}(:,1) \right)) \right);
\]
\[
\text{end}
\]
\[
[r,c] = \text{find} \left( \text{TableauxTemp} > 1 \right); \\
\text{StepsMatrix} = \text{zeros} \left( \text{length} \left( r \right), 2 \right); \\
\text{for } n = 1 : \text{length} \left( r \right), \\
\text{StepsMatrix}(n,1) = r(n); \\
\text{end}
\]
\[
\text{for } p = 1 : \text{length} \left( r \right), \\
\text{StepsMatrix}(p,2) = c(p); \\
\text{end}
\]
\[
\text{FirstBlock} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}; \\
\text{for } t = 1 : \text{length} \left( \text{StepsMatrix}(,1) \right), \\
\text{if } \text{FirstBlock}(1,2) < \text{StepsMatrix}(t,2), \\
\text{FirstBlock} = \text{StepsMatrix}(t,:); \\
\text{else } \text{FirstBlock}(1,2) = \text{StepsMatrix}(t,2); \\
\text{if } \text{FirstBlock}(1,1) > \text{StepsMatrix}(t,1), \\
\text{FirstBlock} = \text{StepsMatrix}(t,:); \\
\text{end}
\end{bmatrix}
\]

55
end
end
counter = 0;
for r = 1 : length(StepsMatrix(:,1)) - 1
    if TableauxTemp(FirstBlock(1,1) + 1, FirstBlock(1,2)) == 2
        FirstBlock = [ FirstBlock(1,1) + 1, FirstBlock(1,2) ];
    elseif TableauxTemp(FirstBlock(1,1), FirstBlock(1,2) - 1) == 2
        FirstBlock = [ FirstBlock(1,1), FirstBlock(1,2) - 1 ];
    else
        counter = 1;
    end
end
if counter == 0
    FinalViabesTemp = zeros(length(FinalViabes(:,1)) + length(TableauxTemp);
    for i = 1 : length(FinalViabes(:,1))
        FinalViabesTemp(i,:) = FinalViabes(i,:);
    end
    for i = 1 : length(TableauxTemp(:,1))
        FinalViabesTemp(length(FinalViabes(:,1)) + i,:) = TableauxTemp;
    end
    FinalViabes = FinalViabesTemp;
end

i = 1;
j = 1;
k = 1;
m = 1;
n = 1;
p = 1;
r = 1;
t = 1;

FinalViabes2 = zeros(length(FinalViabes(:,1)) - length(TableauxTemp(:,1))
for i = 1 : (length(FinalViabes(:,1)) - length(TableauxTemp(:,1)))
    FinalViabes2(i,:) = FinalViabes(i + length(TableauxTemp(:,1)), :);
end
FinalViabes = FinalViabes2;
PrePartitionAReduced = zeros(2*length(FinalViabes(:,1))/length(TableauxTemp(:,1))
a = length(FinalViabes(:,1))/length(TableauxTemp(:,1));
for j = 1 : a
    LegLengthCounter = 0;
end
for k = 1 : length(TableauxTemp(:,1)) - 1
    [r] = find(FinalViables(length(TableauxTemp(:,1)) *(j-1) + k,:))
    [w] = find(FinalViables(length(TableauxTemp(:,1)) *(j-1) + k,:))
    PrePartitionAReduced(2*j, k) = length(r);
    if length(w) > 0
        LegLengthCounter = LegLengthCounter + 1;
    end
end
PrePartitionAReduced(2*j - 1, 1) = LegLengthCounter - 1;
end

PartitionAReduced = PrePartitionAReduced;

PrePartitionBReduced = zeros(1, length(PartitionB) - 1);

for p = 2 : length(PartitionB)
    PrePartitionBReduced(p-1) = PartitionB(p);
end

PartitionBReduced = PrePartitionBReduced;

% Now we have each skew hook for the given tableaux shape. These are in the %
% form of matrices stacked on top of one another. Next “erase” the twos in %
% the matrices, which effectively subtracts the skew plots from the tableaux %
% as specified in the rules of Murnaghan-Nakayama.

References


