Coupled Multi-species Bose-Einstein Condensates in a Generalized Periodic Potential

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Abstract

This paper first outlines a brief history of Bose-Einstein condensation and discusses the experimental production of Bose-Einstein condensates, or BECs. The dynamics of coupled multi-species BECs trapped in a generalized periodic potential are numerically examined. Basic ideas about BECs, quantum mechanics in general, and some numerical methods are discussed in order to make this paper more usable for the general audience. A certain acquaintance with mathematics and physics is assumed, but references to more detailed information are suggested when possible. Appendices contain general information about Jacobi elliptic functions and Fourier transforms. Numerical simulations show that it is possible to have more than one condensate coupled in a periodic potential that persist in time. Repulsive condensates with sufficient offset were seen to persist indefinitely under certain circumstances. Other solutions were found that succumbed to instability, but only after many time units. These solutions may be stable enough to be observed in the laboratory.

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1 Introduction and Overview

With the creation of the first Bose-Einstein condensate in 1995 the usually sub-microscopic world of quantum physics has been dragged into the world of phenomena observable with the naked eye. The creators of this first condensate, Eric A. Cornell and Carl E. Wieman of the University of Colorado at Boulder, and Wolfgang Ketterle of the Massachusetts Institute of Technology were awarded the 2001 Nobel prize in physics. This is well justified, as a Bose-Einstein condensate, or BEC, must be considered as a new state of matter, quite separate from gas, liquid, solid, and plasma. Like the photons in laser light, the atoms in a BEC are coherent. That is to say they all have the same wavelength, the same energy, and are in fact completely indistinguishable from one another. The atoms in this state come to exist as one object. In this way, the wave properties can not only be studied, but directly observed. In order to study and eventually make use of these macro-quantum objects, it is necessary to form larger and more stable condensates. To this end, this paper will first discuss Bose-Einstein condensation in general, and then present some recent numerical findings of the dynamics of coupled BECs in a generalized periodic potential.

1.1 History

Bose-Einstein Condensates, or BECs are so named as they were first postulated by Satyendra Nath Bose and Albert Einstein. In 1921, Bose began working as a reader in physics at Dacca University in Dacca, Bangladesh. Bose formulated Planck’s law of radiation from first principles. Planck’s law, named for Max Plank, governs the relationship between the peak frequency of light leaving a radiating blackbody and the temperature of the blackbody, and was a starting point for much of the ensuing revolution of quantum physics. Planck had essentially “guessed” the solution by combining Rayleigh’s Law, which is valid for low temperatures, and Wien’s Law, which works for high temperatures. Bose was able to validate the theory from postulates of physics. One prediction of Bose’s results was that of a condensed state of photons. He had predicted the laser. Bose was unable to get his results published. In 1924 he sent Einstein an article presenting his new results on Planck’s law [1]. Einstein extended the idea to matter in general, translated the paper from English to German and assured that these ideas were published in Zeitschrift für Physik.

In 1926, Fermi-Dirac statistics were developed by Enrico Fermi, an Italian physicist, and Paul Dirac, a British physicist. Fermi-Dirac statistics, which are obeyed by fermions, stand in direct
1 INTRODUCTION AND OVERVIEW

contrast to Bose-Einstein statistics, which are obeyed by bosons, in a dramatic way. Any number of bosons may exist in the same quantum state at the same time, whereas fermions may not. From this comes the Pauli exclusion principle, which forbids two electrons in the same atomic “orbit” from having the same spin state. Paul Dirac is credited with having first called particles with integer spin “bosons” [1]. All stable particles, sub-atomic and otherwise, are strongly believed to be either fermions or bosons. Thus with these two sets of statistical laws, much of the behavior of the universe is captured, at least in a sense of bulk, statistical behavior. Fermions have an intrinsic half-integer spin, like $1/2$, $3/2$, or $-1/2$, whereas bosons have integer spin. Quarks are sub-atomic particles that have third-integer spins, but these are not stable alone. These particles usually are found in threes making up protons and neutrons, which are themselves fermions.

In 1938, Fritz London invoked Bose-Einstein condensation to explain the superfluidity of $^4$He. Both superfluidity and superconductivity had been seen earlier in the century by H. Kamerlingh Onnes. It has also been postulated that superconductivity can also be explained as a Bose-Einstein condensation phenomenon. Both these phenomena occur at very low temperatures, as does Bose-Einstein condensation. Helium does not solidify, even at absolute zero, however, below a certain temperature a sample of $^4$He loses all of it’s viscosity, or resistance to flow. A superfluid sample, once set in circular motion, will continue to spin forever. A superconductor is a piece of matter that offers no resistance to the flow of electricity. If high temperature superconductors can be created, it could lead to many interesting developments in industry.

In 1995, Cornell and Wieman formed a BEC out of about 2000 $^{87}$Rb atoms [3]. In the same year, Ketterle was able to form a significantly larger BEC using sodium (Na) atoms. In 1999, Lene Hau, of Harvard University and the Rowland Institute for Science, and Steve Harris of Stanford University used a BEC to slow light down to about 38 miles per hour [4]. In 2001, Cornell, Wieman, and Ketterle were awarded the Nobel Prize for Physics. In March 2002, Carl Wieman reported the ability to make molecular BECs with $^{85}$Rb atoms to the American Physical Society making. This was created by rapidly changing the magnetic field in the trap to cause “Feshbach resonance,” which leads to some atoms joining to form molecules. The changing magnetic field also causes the condensate to first implode, and then explode. This has been called a “Bosenova,” and it’s dynamics are as yet not understood. Also in March 2002, Ketterle announced that he had been able to create a “degenerate Fermi sea” (DFS) of lithium-6 ($^6$Li) atoms by cooling them with a BEC [1]. The $^6$Li atoms are fermions, and will not condense like bosons, but “stack” in the lowest possible energy states. It is hoped that $^6$Li atoms can be induced to form Cooper pairs, which are bosonic, and so form a superfluid sample of $^6$Li [2]. Ketterle also reported channeling BECs through a waveguide on a printed circuit board. Part of the goal of this experiment is to determine whether one BEC can be split into two BECs on a chip. As of yet this has not been accomplished [1]. Bose-Einstein condensation is an exciting and rewarding field of research, and is continuing to make history.

1.2 Applications

Bose-Einstein condensates are nothing short of a new form of matter, totally distinct from the known solid, liquid, plasma, and gas phases of matter. It is possible that 1995 is the first time such a state of matter has ever been created in the history of the universe. Certainly it was the first time a BEC was seen by human beings. BECs offer a novel and unique window on the
world of physical phenomena. Purely for this reason, the study of BECs is a worthwhile scientific endeavor. The study of BECs goes far beyond the desire merely to understand what is physically possible, although this is a worthwhile pursuit in and of itself. Quantum mechanics is less than one hundred years old, but has nonetheless revolutionized our picture of the universe. The world of quantum mechanics is at base counterintuitive and in actuality, hard to observe. A major tenet of quantum mechanics is the Heisenberg uncertainty principle. This principle states that one may not simultaneously know where a particle is, and how it is moving. As BECs are composed of millions of particles, they offer a way to circumvent certain constraints imposed by the uncertainty principle. One may directly observe quantum phenomena that would otherwise be invisible in the context of Bose-Einstein condensates. Researchers may make "movies" of BECs without destroying the dynamics they wish to observe. Thus, further study of BECs can be expected to extend the understanding of the fundamental laws of nature.

Additionally, BECs are useful in other experiments. As stated above, a BEC was used to slow light down to the speed of a fast cyclist. This experiment uses the incredibly high index of refraction of the condensate. The researches obtained non-linear indices of refraction a million-fold more than any ever measured. This property of BECs may have direct applications to quantum computing [4]. BECs can also be used to examine and quantify the elementary wave nature of matter. Everyone is well acquainted with the scattering of light off of matter. This is the way we see the world around us. With Bose-Einstein condensates, this relationship may be reversed; we can observe the scattering of matter off of light! Ketterle's creation of a DFS by means of BEC cooling is a further example of how BECs can be used.

Lasers were considered an interesting but useless phenomena when were first developed. Time has shown this to be a monumental misunderstanding. Today there are few people in the developed world whose lives are not touched by the advent of lasers. Telecommunications are driven by the coherent delivery of electro-magnetic waves, or lasers. Compact discs are burned and read by the use of lasers. BECs could easily create another such technological revolution. At this point the usefulness of this new form of matter is unexplored, but this in no way discounts possible usefulness. A BEC can be used to make a maser, or matter laser. This could be used, for example, to deliver coherent atoms to a substrate, and so greatly increasing the purity and consistency of industrial products. It has also been supposed that BECs may prove very useful to the development of quantum computing. Quantum computing would increase computational power and therefore reduce computational time tremendously. As such it would allow for much better simulation of processes, such as the weather and chaotic airflow, that are now intractable without significant simplification. To this end, Jakob Reichel of the Max Plank Institute has been working to create a single atom detector from a Rb BEC [5].

This paper is concerned with the dynamics and stability of multi-species condensates. This is motivated by a desire to understand the potentially stabilizing effect of the interaction between condensates. The interference of matter with matter can be studied by observing two condensates held stable in the same trap. This is a very exciting prospect. The researchers at MIT, and presumably elsewhere as well, have successfully created multiple species condensates, and have placed them in standing light-wave traps in order to observe their dynamics.
2 Theoretical Background

2.1 Cooling and Condensing

The creation of Bose-Einstein condensates requires cooling very dilute atomic gases to incredibly low temperatures. In fact, the atomic gases used for the creation of BECs are much less dense than air at sea level. The 1995 experiment cooled approximately 2000 $^8$Rb atoms in a strong vacuum to within a billionth of a degree of absolute zero. This low temperature was achieved by means of laser cooling in combination with a magneto-optical trap. This paper will use the specific example of $^8$Rb atoms in order to illustrate the process of Bose-Einstein condensation. The specific method used by Wieman and Cornell is outlined and explained in order to give the reader more insight, and hopefully some appreciation of the experimental triumphs that have been achieved. Methods used by other investigators differ in a few fundamental areas. In addition to Rb, BECs have also been achieved with Na, spin polarized hydrogen (H), He-4, and several varieties of Coney Island brand hot dogs. Cooling these different atoms require different techniques, the differences manifesting mostly in the form of magneto-optical trap used. Wieman and Cornell used a linearly varying magneto-optical trap (MOT). Other traps used are SPOT, dark SPOT, and FIDO. See [9, 12] for more details. Figure 3 shows a very basic schematic of many BEC experiments performed today. Before discussing the creation of BECs, some quantum mechanics are in order.

Basic quantum mechanics: De Broglie wavelength

Waves and particles would seem to be fundamentally different entities. Waves propagate through a medium, have a wavelength, and can constructively and destructively interfere with each other. Particles are, or at least are thought to be, pieces of “stuff.” Particles have mass, a position, and spin. A wave has no mass, per se. A particle, as thought of before this century, has no wavelength. Physicists in the 1800’s and early 1900’s were concerned with the nature of light and matter. Experiments had shown that light indeed displayed both particle and wave behavior. Light had been known to constructively and destructively interfere since 1802, when Thomas Young had performed a primitive double slit experiment. In a brief aside, Thomas Young was a British physician and scientist, who not only made important discoveries in optics, but was also instrumental in the translation of the Rosetta stone [6]. Heinrich Hertz, working in the late 1800’s, discovered the photoelectric effect. This effect showed that light sometimes behaved like a particle. Einstein, in the annum mirabilis, 1905 proposed what Planck had tried to avoid years earlier, namely that light comes in bundles, or quanta. By the 1920’s, the dual wave/particle nature of light, and electromagnetic radiation in general, was a widely accepted physical fact, but matter was still assumed to be purely particle-like. In 1924, a French graduate student and prince by the name of Louis de Broglie made an outrageous claim in his doctoral thesis. He proposed that matter in general, and the electron in particular, also had a wave nature. De Broglie simply allowed matter to also have a wave nature, demanded consistency with known physics and saw what fell out. What fell out was the De Broglie relation

$$\lambda_{dB} = \frac{h}{p},$$

where $\lambda_{dB}$ is the de Broglie wavelength of the particle, and $p$ is the particles momentum. The de Broglie relation shows that as a particle loses momentum, it’s wavelength increases. There are two
Figure 2: The most famous picture in all of physics, the first Solvay conference in Brussels, Belgium, 1911. Many of the physicists discussed in this paper are pictured here.

ways to lose momentum, which is mass times velocity, one is to lose mass, the other is to go slower. One immediate result of de Broglie’s assumption was to explain an anomaly in the Bohr model of the atom [6]. De Broglie was unable to get his results published, and in a familiar manner to Bose, he sent his results to Einstein, who declared them brilliant, and de Broglie received his doctorate, got his results published, and won the Nobel Prize in 1929 for this work [7]. The take home lesson is, if you can’t get your work published, send it to Einstein!

**Basic quantum mechanics:** *Schrödinger equation*

* A few days ago I read with great interest the ingenious thesis of Louis de Broglie, which I finally got hold of..*

---

Erwin Schrödinger, in a letter to Einstein, 1925 [7]
Erwin Schrödinger was an Austrian physicist who, in 1926, revolutionized a revolution. Max Planck called his wave mechanics "epoch-making work." Inspired by de Broglie's bold assertions of the wave nature of matter, Schrödinger put forth an equation for the time evolution of a particle behaving as wave packet. In one spatial dimension and in the absence of an external potential, the Schrödinger wave equation is

\[ i\hbar \frac{\partial \psi(x, t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x, t)}{\partial x^2}, \]  

where \( \hbar \) is Plank's constant divided by \( 2\pi \), and \( m \) is the mass of the particle. This is a wave equation, which gives rise to wave behavior. The most general solution to this equation is

\[ \psi(x, t) = \frac{1}{\sqrt{2\pi\hbar}} \int dp \phi(p) e^{i\left(px - \frac{p^2}{2m} t\right)}. \]  

In Eq. (3), \( p \) is the particles momentum, and \( \phi(p) \) is some function of that momentum that depends on the initial state of the particle. From this there is no obvious way to say what \( \psi(x, t) \) or \( \phi(p) \) are physically. Schrödinger himself did not know at first how to interpret it. The function \( \psi(x, t) \) is complex-valued in general, and it is difficult to assign physical meaning to a complex-valued function, although there are notable exceptions. It was Max Born [7] who made the suggestion that the interpretation of \( \psi \) be that

\[ P(x, t) dx = |\psi(x, t)|^2 dx = \psi(x, t)^* \psi(x, t) dx, \]

where \( dx \) is a differential, or a little piece of the domain, and \( P(x, t) dx \) is the probability of finding the particle in some little \( dx \) around \( x \) at time \( t \). The asterisk denotes complex conjugation.

\[ f^* f = \left[ \Re(f) + i\Im(f) \right] \left[ \Re(f) - i\Im(f) \right] = \Re(f)^2 + \Im(f)^2, \]

where \( \Re(f) \) and \( \Im(f) \) stand for the real and imaginary part of \( f \) respectively. In order for \( P(x, t) \) to be a valid probability, it is necessary that the total probability be one, or

\[ \int_{-\infty}^{\infty} dx |\psi(x, t)|^2 = 1. \]

When \( \psi \) is such that Eq. (5) is true, then the wavefunction \( \psi \) is said to be normalized. Notice that Eq. (3) looks similar to the definition of the Fourier transform. This is not accidental, and will be important in this paper.

The Schrödinger equation is a wave equation, and as such generates solutions that satisfy de Broglie's postulates regarding the wave nature of matter. As the equation is first order in time, the equation can be solved for known spatial initial conditions without knowing initial velocity conditions. If Eq. (2) were second order in time, this would not be the case. This is a very important fact, as the Heisenberg uncertainty principle forbids simultaneous knowledge of both the position and velocity of a particle.

**Basic quantum mechanics: **Heisenberg uncertainty principle

*The more precisely the position is determined, the less precisely the momentum is known in this instant, and vice versa.*
Werner Heisenberg, in his uncertainty paper, 1927 [7]

Werner Heisenberg was born in Germany and worked in München, Göttingen, and in Copenhagen, Denmark. Heisenberg is generally credited with inventing quantum mechanics. A paper he wrote in 1925 attempted to formulate quantum mechanics on quantities that are observable, as opposed to Neils Bohr’s electron orbits, which cannot be observed. The results began what was to be known as matrix mechanics. His contributions to modern physics did not stop there. He is probably best known for the Heisenberg Uncertainty Principle. The uncertainty principle states

\[ \Delta p \Delta x \geq \frac{\hbar}{2}, \]  
\[ \Delta E \Delta t \geq \frac{\hbar}{2}, \]  

where \( \Delta p \) denotes the uncertainty in a particle’s momentum, \( \Delta x \), the uncertainty in the particle’s position, \( \Delta E \), in the particle’s energy, and \( \Delta t \) is the imprecision in the time that the energy was observed. In quantum mechanics, \( \Delta \tau \) is the standard deviation of \( \tau \) about its expected mean value. The quantity \( \hbar \) is a very small, fixed constant. As \( \hbar \) is constant, if \( \Delta p \) gets small, then \( \Delta x \) must increase accordingly. It is not possible to know exactly where a particle is, and how fast it is moving at the same time. Likewise, if \( \Delta E \) is to be made small, then \( \Delta t \) must be allowed to grow. It is not possible to know exactly how much energy a particle had and what time it had that energy at the same time. Heisenberg had rigorous mathematical justification for these relations. The arguments have to do with the Fourier transform of a particle-like “wave-packet.” Very roughly, a wavefunction highly localized in the spatial domain will be highly spread out in the frequency domain. (See Appendix B for the relation between the spatial domain and the frequency domain). The spatial domain contains information about the particle’s velocity. This information must be gotten from the frequency domain. Therein lies the reciprocity between the position and the momentum, which is a function of velocity. The relation for energy and time follows similarly.

**General experimental method**

A general method of making BECs is to boil atoms off of a sample in an oven, pre-cool them, trap, and further cool and condense the resulting cloud with a magnetic field. Once the cloud forms a BEC the condensate can be studied. The behavior of BECs in the presence of a potential can
be examined by illuminating the condensate with far off-resonance lasers. In [9], Ketterle et al. write "blue light is used to add blips and red light to add dips to the trapping potential. So we have all the tools to shape, kick, slice, shake and stir the condensates." Many techniques can be used to probe the condensate. For the purposes of this paper it is most important that probability density information can be extracted from the condensate in a non-destructive way. The interested reader is directed to [9, 12] for a very good overview of the many techniques used to understand Bose-Einstein condensates.

The name "rubidium" comes from the Latin word "rubidius" meaning "deepest red." This element is an alkali metal and was first discovered via a spectroscope by Bunsen and Kirchoff in 1861. It is now considered to be the 16th most abundant element in the earth's crust. Rubidium is a soft silvery-white metal that may be solid or liquid at room temperature. It spontaneously ignites in air, and reacts violently with water. Rubidium is the 37th element, and the most common isotope is of atomic weight 85. Thus this isotope, when electrically neutral, has 37 protons, 37 electrons and 48 neutrons. As the total number of protons, neutrons, and electrons is even, neutral rubidium-85 is a boson. In the ground state, rubidium has the electron structure of krypton, a noble gas, except for one unpaired electron in the 5s shell. The unpaired electron makes rubidium atoms susceptible to magnetic fields [14].

Rubidium atoms are boiled off of a sample in the oven. Rubidium boils at 961 Kelvin. The root mean square velocity of atoms in an ideal gas as a function of temperature is

\[ v_{rms} = \sqrt{\frac{3k_B T}{m}}, \]

where \( T \) is the temperature and \( k_B \) is Boltzmann's constant. This means that atoms leaving the oven are traveling at a root mean square velocity (\( v_{rms} \)) of 529 meters per second. The \( rms \) velocity is very close to the average velocity, but directly formulated from first principles. The atoms that leave the oven travel about one meter down a tube that sits in an inhomogeneous magnetic field. The atoms are bombarded with photons of wavelength appropriate to cool them before entry to the magneto-optical trap, or MOT. This process is called pre-cooling. Pre-cooling is further explained below. The MOT, used by Wieman and Cornell has a minimum of the magnetic field at the center, and increases linearly away from that minimum. Other traps may have other magnetic variability. Quadratic and harmonic traps have also been constructed. The optical part of the trap is the so-called optical molasses formed by three orthogonally oriented sets of two counter-propagating lasers [9, 12]. The MOT is also further explained below.

**Pre-cooling**

Although it may seem counter-intuitive, matter can indeed be cooled by laser light. Consider the atom in Fig. 4 moving with a certain velocity in the positive \( x \) direction. Suppose further that the atom is struck by a photon moving in the negative \( x \) direction. Remember that the energy of a photon is given by \( E_\gamma = h \nu \), where \( \nu \) is the frequency of the photon. If the photon is of a frequency corresponding to the energy difference between two excitation states of the atom, the atom can absorb the photon. This will briefly excite an electron to a higher orbit. Conservation of momentum requires that the atom now be traveling slower in the \( x \) direction, and thus the system it is part of is cooler. However, this new excited state is unstable, and the electron will soon fall
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Figure 4: A completely elastic collision between an atom and a photon. The atom absorbs the energy of the photon, thus exciting to a higher state. Momentum must be conserved, thus the atom/photon system must have reduced velocity in the initial direction of travel. The photon is re-emitted in a random direction, virtually guaranteeing a net loss of velocity. The arrows are meant to denote the momenta in a particular direction.

back to the less excited state, and in so doing emit a photon. Due to conservation of momentum this will also change the velocity, but as the direction of the re-scatter is random it is almost a statistical certainty that the \( x \) component of the velocity will be reduced. Due to the Doppler effect, slower atoms "see" photons differently from faster atoms. If the laser send photons of a color corresponding to the energy difference between two states of a \(^{87}\text{Rb} \) atom traveling at \( v_{\text{rms}} \), those photons will be invisible to atoms that have been slowed. Steven Chu and others at Bell Labs in New Jersey elegantly solved this problem by careful laser tuning and exploitation of the Doppler effect [11]. They received a Nobel Prize for their work in 1997 [8]. They developed a technique in which the pre-cooling guide sits in an inhomogeneous magnetic field. The changing magnetic field causes Zeeman splitting. This changes the separation between the atomic energy levels, and thus the frequency of photon that slowed atoms can absorb. Careful tuning of the magnetic field can insure that the atoms are well cooled. It is also very important to ensure that the slowest atoms are not sped up. For this reason, the whole experimental apparatus must be held in a very low pressure vacuum. Typical speeds after pre-cooling are on the order of 30 meters per second. At the end of the pre-cooling phase the atoms reach the MOT [9].

Evaporative magnetic cooling

Laser/Doppler cooling is thus far insufficient to remove enough energy to cause runaway Bose-Einstein condensation. In order to achieve nano-Kelvin, it is necessary to remove the more energetic atoms from the trap, and allow the remaining atomic cloud to rethermalize. This is done in the MOT in a process called evaporative magnetic cooling. The magnetic trap can be set by many different methods [9]. Cornell and Wieman used two Helmholtz coils, and Ketterle used two anti-Helmholtz coils. The magnetic field induces a preferential spin-state, which allows experimenters to remove only the most energetic atoms from the trap and cool the cloud. The magnetic field also serves to compress the condensate, which increases the collision rate. The increased collision rate allows the cloud to rethermalize as it evaporates. Evaporative magnetic cooling is analogous to common evaporation. The temperature of a sample is a bulk function of the momenta, and thus
the velocity, of the particles in that system. For example, a cup of hot coffee feels hot because the water molecules in the coffee are moving more quickly than the water molecules in the drinker. In order to cool the sample of coffee, the molecules in the coffee must be slowed down on the whole. One way to do this is by evaporative cooling. A water molecule will leave the liquid coffee and enter the gas phase, escaping the coffee, if it has enough kinetic energy to break free of the inter-molecular attractions in the coffee. In order to cool the coffee, those energetic atoms must not be allowed to re-enter the liquid state. If the fast molecule re-enters the cup it will potentially bring all the kinetic energy it left with back to the cup. Blowing on a cup of coffee carries away the most energetic molecules. In order for the coffee to cool, and not just evaporate, the average time between escapes should be less than the rethermalization time. Rethermalization essentially means to re-equilibrate. The coffee must come to the new equilibrium temperature before another atom leaves the liquid phase, or else the evaporation rate will exceed the cooling rate, and the coffee will evaporate.

Most of the same is true of cooling atoms in a magnetic trap. Similar processes must be undergone to form a BEC. The magnetic field holds the atoms in the trap, but those with enough momentum can partially escape the trap. This would be fine if such atoms would completely escape the trap. They do not, and must be “blown” out. In this case the "blowing" is done by inducing spin flips with radio frequency, or rf, photons. The magnetic field induces a Zeeman split of the ground state energy. Without the magnetic field there exist two degenerate ground states for the atom. One state has the unpaired electron in the 5s shell in a spin-up state, the other has it in a spin-down state. These states have the same energy in the absence of a magnetic field, but with the field applied, the state with the electron aligned with the field has less energy than the anti-aligned state. The aligned state is magnetically trapped, whereas the anti-aligned state is not. Atoms with enough energy to reach parts of the trap with larger magnetic fields will also
have larger Zeeman splitting. The cloud is bathed with rf photons at the frequency appropriate to induce spin-flips in only the most energetic atoms. The remaining atoms in the cloud then have time to reach a new, cooler equilibrium. Without the rethermalization time, the sample will simply evaporate, and not reach runaway condensation. The magnetic field is slowly turned down, thus causing atoms of lower and lower energies to be removed from the trap [9, 16]. Figure 5 shows a simple diagram of evaporative cooling. After a time in the magnetic field, the sample will contain only atoms below the threshold kinetic energy induced by the magnetic field. Lower overall kinetic energy means lower temperature. The magnetic field strength is lowered slowly enough to allow for rethermalization, and the sample is cooled to barely above absolute zero.

As the atoms cool and lose their kinetic energy, the Heisenberg uncertainty principle dictates that they become delocalized in space. As an atom slows down, its velocity, and thus its momentum (mass times velocity) is assured to be ever closer to zero. If the atom's momentum is very close to zero, the uncertainty in it's momentum, $\Delta p$, is also very close to zero. As $\Delta p$ gets very small the uncertainty principle requires that $\Delta x$, the uncertainty in the position, get very large. Thus there is a larger region of space over which it is reasonable to find the atom. What this says is that the atom is delocalized in space. A delocalized atom can be thought of as one that is smeared over a region of space. Atoms begin to overlap each other. The lower energy atoms also can only occupy only the lowest possible quantum energy states. Atoms with an even number of protons, neutrons and electrons are bosons. As such, any number of $^{85}$Rb atoms can occupy the same quantum state. When these atoms have lost almost all their kinetic energy, they are all in the same state, and delocalized in space, they become one object. This is a superatom or BEC.

Detailed mathematics have been avoided up to this point. The paper now makes a rather abrupt switch into the mathematical formalism and physical justification needed to continue. Those who are not interested in the detailed formalism may skip to Sec. (3) without much loss of continuity. In the intervening sections this paper attempts to justify and derive a system of differential equations that govern the behavior of coupled Bose-Einstein condensates trapped in a generalized periodic potential. Further, assumptions are proposed that allow for numerical simulation of such BECs. Various types and sub-types of coupled systems of BECs are outlined. Thus the goal of the paper from here to section (3) is to derive physically relevant and mathematically tractable forms of BECs in a periodic potential. A further goal is to determine which of these forms may be experimentally achieved. In general what is found illustrates that coupled BECs can be expected in traps that are sufficiently dense, and in which the coupling strengths balance appropriately. Please excuse the detailed nature of what follows, but it is meant to be useful for a particular audience. It should be said that the following sections do attempt to make the concepts understandable, albeit with consultation of references.

### 2.2 $N$-Body Equation and Mean-Field Theory

The $N$-body Schrödinger Equation has been a postulate of quantum mechanics for most of this century. A derivation of Eq. (8) from first principles has recently been forwarded by Micheal J. Hall and Marcel Reginatto [18], although in this paper the equation is taken as a postulate. This equation governs the behavior of $N$ interacting particles, hence $N$-body, in the presence of an
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external potential. It has the form:

\[ i\hbar \partial_t \Psi = -\frac{\hbar^2}{2m} \Delta^N \Psi + \sum_{k=1}^{N} W(x_k - x_j) \Psi + \sum_{k=1}^{N} V(x_k) \Psi, \tag{8} \]

where

\[ x_k = (x_{e_1}, x_{e_2}, x_{e_3})_k, \tag{9a} \]
\[ \Psi = \Psi(x_1, x_2 \ldots x_N, t), \tag{9b} \]
\[ \Delta^N \Psi(x) = \sum_{k=1}^{N} (\partial^2_{x_k,e_1} + \partial^2_{x_k,e_2} + \partial^2_{x_k,e_3}). \tag{9c} \]

Further, \( W(x_k - x_j) \) is the interaction of particle \( k \) on particle \( j \), \( V(x) \) is a time independent external potential, \( i \) is \( -1 \), and \( \hbar \) is the ratio of the Planck constant to \( 2\pi \). Here \( e_1, e_2, \) and \( e_3 \) denote orthogonal unit vectors, the most common of which are the Cartesian coordinates \( \hat{x}, \hat{y}, \) and \( \hat{z} \). \( \Psi(x, t) \) takes on complex values in general. In and of itself, \( \Psi \) has no physical meaning. It only becomes physically meaningful as \( |\Psi|^2 \Psi(x_k, t) \Psi^*(x_k, t) \), which is interpreted as the probability of finding particle \( k \) at position \( x_k \). Note that \( \Psi^* \) denotes the complex conjugate of \( \Psi \).

The \( N \)-body Schrödinger equation is Hamiltonian, or energy conserving. This means that a system governed by this equation maintains a constant amount of energy. The energy in the system may switch between kinetic and potential energy, but the equation contains no mechanism by which energy may enter or leave the system. To get an intuitive feeling for how energy may switch back and forth between kinetic and potential, consider a much simpler system, that of a mass bouncing, undamped, on an ideal spring. As the mass bounces up and down on the spring, the energy shifts between kinetic and potential. When at the top of its motion, the system is has the maximum gravitational potential energy, but it is not moving, and so has no kinetic energy. At the bottom of the motion the potential energy of the spring is maximized, but the system has minimum gravitational potential. At any other point in the motion the mass has some kinetic energy, some fraction of the energy is gravitational potential, and some fraction of the energy is tied up in the potential energy of the spring. Without a damping or pumping force, energy neither enters or leaves the system. For such a system it is useful to examine an object known as the Lagrangian, denoted as \( L \). The Lagrangian is defined as the kinetic energy minus the potential energy in a system, and as such characterizes where the energy in a system is. The Lagrangian for the system \([25]\) of Eq. (8) can be written as

\[ L = \int_{-\infty}^{\infty} dx^N \left[ \frac{i\hbar}{2} (\Psi \Psi_t^* - \Psi^* \Psi_t) + \frac{\hbar^2}{2m} |\Delta^N \Psi|^2 - \sum_{k=1}^{N} (W - V)|\Psi|^2 \right]. \tag{10} \]

The Lagrangian is constructed such that minimizing the variations \( \delta L/\delta \Psi \) will return the governing equation \([24]\). That is, Eq. (10) represents the weak form \([17]\) of Eq. (8). At this point an ansatz is introduced in the form of some assumptions.

\[ \Psi = \psi_1 \psi_2 \ldots \psi_N, \tag{11a} \]
\[ \psi_1 = \psi_2 = \psi_3 = \ldots = \psi_N = \psi. \tag{11b} \]
Theoretical Background

The particles in a one-species BEC are all identical. Furthermore, Bose-Einstein statistics require that the total wavefunction $\Psi$ for a group of bosons must be totally symmetric under interchange of particles. This ansatz, known as the Hartree-Fock approximation, satisfies these conditions. Equation (11a) states that the wavefunction of the whole $N$-body condensate, $\Psi$, is a product of individual one-particle wavefunction contributions, $\psi_k = \psi$, from each of the $N$ particles. Equation (11b) states that each of the $\psi_k$ are identical.

As stated above, the wavefunction $\Psi$ will be such that it will minimize the Lagrangian, $L$. This condition can be stated as

$$\frac{\delta L}{\delta \Psi} = 0,$$

where $\delta$ is the variational operator, and $L$ is given in Eq. (10). Remembering that $\Psi$ is the product of the separated $\psi_k$, the calculation of Eq. (12) is much simplified. If the spatial domain is restricted to one dimension, as is appropriate in the quasi 1-dimensional case under consideration later, the variation of $L$ with respect to $\Psi$ is

$$\frac{\delta L}{\delta \Psi} = \frac{\partial}{\partial t} \frac{\partial L}{\partial \psi_{t}} + \frac{\partial}{\partial x} \frac{\partial L}{\partial \psi_x} - \frac{\partial L}{\partial \Psi} = 0. \quad (13)$$

The result of taking the variational derivative is the following Euler-Lagrange equation (14). More details on the calculus of variations can be found in [24].

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x,t)}{\partial x^2} + V(x)\psi(x,t) + (N - 1)\psi(x,t) \int_{-\infty}^{\infty} dy W(x-y) |\psi(y,t)|^2. \quad (14)$$

Note that Eq. (14) holds for each of the $N$ particles. Equation (14) gives the nonlinear, non-local mean-field dynamics of the wavefunctions $\psi_j(x,t)$ under the usual assumptions of Hartree-Fock theory [25, 21]. The equation is non-linear and non-local due to the last term. Not only is the term cubic in $\psi$, but it is an integral term, taking the whole domain into account. This term captures the interaction of one particle with all the rest of the particles. This is a mean-field approach as the $N$-body wavefunction is obtainable via only a single one-particle wavefunction. Interactions need not be calculated pair-wise, as suggested by Eq. 8, making analytic progress possible. The interaction term from the Euler-Lagrange equation is much simplified by the mean-field approximation, however it is still problematic. The next section begins with some further simplification of Eq. (14).

2.3 Governing Equations and Coupled Condensate Solutions

Local Interaction

In order to make further analytic progress, a purely local interaction is often assumed. That is, the interaction function from Eq. (14), $W$, is assumed to behave like a Dirac delta function.

$$W(x-y) = \kappa \delta(x-y). \quad (15)$$

The properties of the delta function make the integration of Eq. (14) a trivial matter. The result is well known as both the Gross-Pitaevskii (GP) equation, and also the nonlinear Schrödinger
equation (NLS).

\[ i \frac{\partial \psi}{\partial t} = -\frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} + V(x)\psi + \alpha \psi|\psi|^2, \quad \alpha = \kappa (N - 1). \] (16)

The interaction of the particle with itself is characterized by the parameter \(\alpha\). For \(\alpha > 0\), the interaction is repulsive, and for \(\alpha < 0\), the interaction is attractive. If \(\alpha = 0\), the particle neither attracts or repels itself. It may be odd to think of a particle interacting with itself, however recall that these particles are not assumed to be classical point-like particles, but wave-like particles spread out in space. The parameter \(\alpha\) captures the tendency of the particle to spread out or coalesce in the absence of an external potential.

The assumption given by Eq. (15) does, however, imply particles in the condensate are non-interacting with each other. Each particle in this description interacts only with itself, and not with neighboring particles. The physical justification of this approximation is questionable, but seems reasonable in the limit of a dilute gas of weakly interacting atoms, such as those in alkali gas BECs. Further, such assumptions make analytic progress possible. Also, as the assumption has already been made that the \(\psi_k\) are all identical, this purely local interaction may not be a stretch. Recent work addresses these problems by omitting the delta function interaction approximation [22].

**Coupled Condensates**

Up to this point, only single species condensates have been discussed. One-species condensates are ones in which there is one wavefunction, \(\Psi\), of which all the atoms in the condensate are a part. There may be other atoms in the trap which are not condensed, but these are ignored. In runaway condensation the fraction of non-condensed atoms is negligible. Now BECs containing more than one species are addressed. Bose-Einstein condensates with multiple species are condensates with more than one wavefunction to which the atoms in the trap contribute.

In quantum mechanics, the state of the electrons in orbit about the nucleus of an atom is completely characterized by a set of quantum numbers. Mathematically, these quantum numbers arise as eigenvalues when solving the Schrödinger equation. These eigenvalues are usually constrained to certain values by imposed boundary conditions. For example, if the solution is sinusoidal, and required to be zero at the end points, then the solution is required to have a period that is some half-integer fraction of the length of the domain. These eigenvalues are also separation constants, and as such, relate different "parts" of a solution to each other [17]. Physically, the eigenvalues represent quantities such as the energy, the spin, and the component of the spin in a particular direction [13].

A "ground state" is the electron configuration about an atom that minimizes the electrical potential energy. In quantum mechanics it is possible for atoms to have many ground states, and sometimes these states may be of the same energy. This is known as degeneracy. The degenerate states are not identical; they are characterized by different sets of quantum numbers, aside from the energy quantum number. Experiments have been performed with two condensates in a trap where the two states differed in their hyperfine states. As the states are not identical, they may have different self-interactions. One ground state may even be self attractive, while another is self-repulsive. If two quantum states both have the ground state energy, and are both weak-field seeking magnetically, then, as there is no preferential state, a two-species condensate would be expected.
A very thorough treatment of two-species condensates and the quantum number thereof can be found in [26]. It is also possible that atoms with different nuclei could be simultaneously condensed. Perhaps it is possible to condense both rubidium and sodium in the same trap. There would be no reason to assume that atoms in different quantum states or atoms with different nuclei would have the same self-interaction coefficient, \( \alpha \), as given in Eq. (16). Thus, it is reasonable to imagine there could be some number \( M \) different condensates in a trap. This paper is concerned with the dynamics of \( M \) interacting condensates trapped in a one dimensional potential. The approximation of one-dimensional condensates is valid for the cigar-shaped condensates first made in some types of MOTs. A system of \( M \) interacting condensates in a one dimension potential is governed by the physically rescaled equation:

\[
\frac{i}{\partial t} \psi_j = \frac{1}{2} \frac{\partial^2 \psi_j}{\partial x^2} + V_j(x) \psi_j + \psi_j \sum_{k=1}^{M} \alpha_{jk} |\psi_k|^2,
\]

\( j = 1 \ldots M. \) (17)

This equation is known as the \( N \)-body non-linear Schrödinger equation, or more simply as NLS.

**Solutions**

Analytic solutions to Eq. (17) can be found by judicious choice of external potential. Deconinck and Kutz have proposed, in [20], a generalized periodic potential of the form

\[
V_j(x) = V_{0j} \text{sn}^2(mx, k),
\]

(18)

where \( \text{sn}(x, k) \) is the elliptic sine function, \( m \) scales the domain, and \( k \) is the elliptic modulus. It is important to note that for \( k = 0 \), \( \text{sn}(mx, k) = \sin(x) \), and that this is known as the trigonometric limit. For more details on the elliptic sine function, please see Appendix A. It is further suggested that the wavefunctions \( \psi_j \) be decomposed as [20]

\[
\psi_j(x, t) = r(x)e^{-i(\omega t - \theta_j(x))}.
\]

(19)

The factor \( r(x) \) is required to be real-valued, and may be thought of as an *envelope* in the \( x \) direction within which the condensate may modulate. The exponential factor causes both time dependent phase oscillation with frequency \( 2\pi/\omega \) and spatially dependent phase modulation as given by \( \theta(x) \). The phase of a wavefunction is a difficult concept, and luckily may be largely ignored in what follows. Richard Feynman, renowned physicist and funnyman once said, "anyone who tells you he understands why quantum mechanics works is lying." All physically measurable information is contained in the envelope, and the exponential factor contains only phase information. Recall from above that \( \psi^* \psi \), once properly normalized, is interpreted as the probability density.

Solving Eq. (17) using Eqs. (18) & (19) leads to two distinct solution types. These are referred to, as per Deconinck et al, types A and B [21].

**An Example Calculation**

A detailed derivation of the type A solutions is presented below for the case of a single species BECs in order to show the way the calculations to obtain the restrictions on the parameters proceed for
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all types and sub-types. The calculations for multi-species condensates proceed in an analogous, but messier, way. For a one species condensate, set \( M = 1 \) in Eq. (17) to see

\[
\frac{i \partial \psi}{\partial t} = -\frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} + V(x) \psi + \alpha \psi |\psi|^2,
\]

(20)

Equation (19), the assumed form of the solution, can be written as

\[
\psi(x, t) = r(x) e^{-i(\omega t - \theta(x))} = r(x) \phi(x, t),
\]

(21)

where \( \phi(x, t) \) is meant to denote the exponential part of \( \psi \). Equation (20) requires the time derivative, the second spatial derivative, and modulus squared, of \( \psi \). Each of these will be found in turn. The envelope, \( r(x) \), has no time dependence, and taking the time of \( \phi(x, t) \) derivative brings down factor of \( = i \omega \), so

\[
\frac{\partial \psi}{\partial t} = -i \omega r \phi.
\]

(22)

The first spatial derivative of \( \psi = r(x) \phi(x, t) \) is

\[
\frac{\partial \psi}{\partial x} = r' \phi + r \phi',
\]

(23)

where \( f' \) denotes the derivative of \( f \) with respect to \( x \). The second spatial derivative is

\[
\frac{\partial^2 \psi}{\partial x^2} = r'' \phi + 2 r' \phi' + r \phi''.
\]

(24)

The function \( \theta \) is an unknown function of \( x \), so

\[
\phi' = i \theta' \phi,
\]

(25)

and

\[
\phi'' = i \theta'' \phi + i^2 \theta' \phi = i \theta'' \phi - \theta' \phi.
\]

(26)

Thus

\[
\frac{\partial^2 \psi}{\partial x^2} = r'' \phi + 2 i r' \theta' \phi + i r \theta'' \phi - r \theta'^2 \phi.
\]

(27)

Due to phase cancellation, \( |\psi|^2 = r^2(x) \). Plugging the results obtained above, and dividing out \( \phi \neq 0 \), Eq. (20) becomes

\[
\omega r = -\frac{1}{2} r'' - i r' \theta' - \frac{1}{2} i \theta'' + r^3 + V(x) r
\]

(28)

Equation 28 must hold for both the real and imaginary parts. Therefore, from the imaginary part

\[
0 = -r' \theta' - \frac{1}{2} r \theta'',
\]

(29)

or equivalently

\[
r' \theta' = \frac{1}{2} r \theta''.
\]

(30)
Divide Eq. (30) by $r\theta'$ to see

$$\frac{r'}{r} = -\frac{1}{2} \frac{\theta''}{\theta'}. \quad (31)$$

Note that

$$\frac{\partial}{\partial x} \ln f = \frac{f'}{f}.$$  

in order to see that Eq. (31) may be rewritten as

$$\frac{\partial}{\partial x} \ln r = -\frac{1}{2} \frac{\partial}{\partial x} \ln \theta'.$$  

(32)

Integrate both sides of Eq. (32) with respect to $x$ to obtain the requirement that

$$\theta(x) = c \int_0^x \frac{d\tau}{r^2(\tau)}. \quad (33)$$

The real part of Eq. (28) becomes

$$\omega r = -\frac{1}{2} r'' + \frac{1}{2} r \theta'^2 + r^3 + V(x) r \quad (34)$$

Taking the derivative with respect to $x$ of Eq. (33) reveals that

$$\theta' = \frac{c}{r^2}, \quad \text{so} \quad \theta'^2 = \frac{c^2}{r^4}. \quad (35)$$

With this, and a little algebra, Eq. (34) may be written as

$$\omega r = -\frac{1}{2} r'^2 + \frac{c^2}{r^2} + r^4 + V r^4. \quad (36)$$

At this point, the properties of the elliptic functions become useful. Please see Appendix A for some basic information about these functions. Recall that the potential is assumed to be of the form $V(x) = -V_0 \text{sn}^2(x, k)$. For the type A solutions, the envelope $r(x)$ is assumed to given by

$$r^2(x) = A \text{sn}^2(x, k)^2 + b. \quad (37)$$

The following are also used to proceed.

$$r^3(x) = (A \text{sn}^2(x, k) + B)^{3/2}, \quad (38a)$$

$$r^4(x) = A^2 \text{sn}^2(x, k)^4 + 2AB \text{sn}^2(x, k) + B^2, \quad (38b)$$

$$r^6(x) = A^3 \text{sn}^6(x) + 3A^2B \text{sn}^4(x, k) + 3AB^2 \text{sn}^2(x, k) + B^3, \quad (38c)$$

$$r'(x) = A \left( A \text{sn}^2(x, k) + B \right)^{-1/2} \text{sn}(x, k) \text{sn}'(x, k). \quad (38d)$$

Further,

$$r''(x) = -A^2 \left( A \text{sn}^2(x, k) + B \right)^{-3/2} \text{sn}^2(x, k) \text{sn}^2 + A \left( A \text{sn}^2(x, k) + B \right)^{-1/2} \left( \text{sn}(x, k) \text{sn}'(x, k) \right)' \quad (39)$$
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Note also that the properties of the elliptic functions allow \((\text{sn}(x)\text{sn}'(x))'\) to be written as

\[1 - (1 + k^2)\text{sn}^2(x, k) + k^2\text{sn}^4(x, k).\]

The following four equations are obtained by plugging these results into Eq. (36) and equating powers of \(\text{sn}(x, k)\).

\[
\begin{align*}
\omega B^2 &= -\frac{AB}{2} + \frac{c^2}{2} + B^3, \\
2AB\omega &= AB(1 + k^2) + 3AB^2 - V_0B^2, \\
\omega A^2 &= -\frac{1}{2} \left(3ABk^2 - A^2(1 + k^2)\right) + 3A^2B - 2ABV_0, \\
0 &= -A^2k^2 + A^3 - V_0A^2.
\end{align*}
\]

(40a) (40b) (40c) (40d)

From these it follows that

\[
\begin{align*}
\omega &= \frac{1}{2} \left(1 + k^2 + 3B - \frac{BV_0}{V_0 + k^2}\right), \\
c^2 &= B \left(1 + \frac{B}{V_0 + k^2}\right) (V_0 + k^2 + Bk^2), \\
A &= V_0 + k^2.
\end{align*}
\]

(41a) (41b) (41c)

Calculations for the type B solutions are almost identical. The squared envelopes, \(r^2(x)\), of the type B solutions are linear in the elliptic functions. There are restrictions on some of the parameters that serve to define a space of possible exact solutions. The sections that directly follow show the forms of the envelope, \(r(x)\), for both the type A and B solutions, and the constraints that follow from these solution types. Three sub-types of solution, based on the type of elliptic function in the envelope, are shown for the type B solutions.

**Type A**

The type A solutions are given by:

\[r_j^2 = A_j \text{sn}^2(m_jx, k) + B_j.\]

(42)

These solution must satisfy

1. \(\sum_{k=1}^{M} \alpha_{jk}A_k = V_{0j} + m_j^2k^2\)
2. \(\omega_j = \frac{1}{2}m_j^2(1 + k^2) + \frac{1}{2}m_j^2k^2B_j + \sum_{k=1}^{M} \alpha_{jk}B_k\)
3. \(c_j^2 = m_j^2B_j \left(A_j + B_j\right)(A_j + k^2B_j)\)
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Type B

The type B solutions arise from a potential choice

\[ V(x) = -\frac{3}{8} m^2_j k^2 \text{sn}^2(mx, k). \]  

(43)

These solutions are of three different sub-types, Sn, Cn and Dn. The Sn solutions have the form

\[ r_j^2 = A_j \text{sn}(m_j x, k) + B_j, \]  

(44)

and must further satisfy

1. \[ B_j = \frac{-4 A_j}{m^2_j k^2} \sum_{k=1}^{M} \alpha_{jk} A_k \]

2. \[ \omega_j = \frac{1}{8} m^2_j (1 + k^2) - \frac{1}{8} m^2_j k^2 \frac{B_j^2}{A_j^2} + \sum_{k=1}^{M} \alpha_{jk} B_k \]

3. \[ c_j^2 = \frac{m^2_j}{4 A_j^2} (B_j^2 - A_j^2) ((A_j^2 - k^2 B_j^2)) \]

The Cn solutions have the form:

\[ r_j^2 = A_j \text{cn}(m_j x, k) + B_j, \]  

(45)

and must further satisfy

1. \[ B_j = \frac{-4 A_j}{m^2_j k^2} \sum_{k=1}^{M} \alpha_{jk} A_k \]

2. \[ \omega_j = \frac{1}{8} m^2_j (1 + k^2) - \frac{1}{8} m^2_j k^2 \frac{B_j^2}{A_j^2} + \sum_{k=1}^{M} \alpha_{jk} B_k \]

3. \[ c_j^2 = \frac{m^2_j}{4 A_j^2} (B_j^2 - A_j^2) ((1 - k^2) A_j^2 + k^2 B_j^2)) \]

Finally the Dn solutions have the form:

\[ r_j^2 = A_j \text{dn}(m_j x, k) + B_j, \]  

(46)

and must further satisfy

1. \[ B_j = \frac{-4 A_j}{m^2_j k^2} \sum_{k=1}^{M} \alpha_{jk} A_k \]

2. \[ \omega_j = \frac{1}{8} m^2_j (1 + k^2) - \frac{1}{8} m^2_j k^2 \frac{B_j^2}{A_j^2} + \sum_{k=1}^{M} \alpha_{jk} B_k \]

3. \[ c_j^2 = \frac{m^2_j}{4 A_j^2} (A_j^2 - B_j^2) ((1 - k^2) A_j^2 - k^2 B_j^2)) \]
2.4 Parameter Spaces

The constraints from the previous section, along with the requirement that $r(x)$ be real, and analytic, describe "parameter spaces." There are categorically different parameter spaces for the type A and B solutions. The spaces are also different for each of the three sub-types of the type A and B solutions, although the sub-type spaces of a given type are very similar in structure to each other. There is little insight to be gained from intimate knowledge of the parameter spaces. However, it is important to know something about the parameters being adjusted. For the most part the solutions were tested for two condensates in a trap with a standing light-wave potential. This is the case for both the type A and the type B solutions. All solutions were tested as trivial phase solutions. For these, $\theta = 0$. This means that the condensates are independent of phase across the spatial dimension. The reader interested in more detail on the parameter spaces is referred to Deconinck, et al [22].

The governing system of equations, Eq. (17), captures all interaction information in the $\alpha_{jk}$. That is, $\alpha_{ab}$ is the influence felt by condensate $a$ as exerted by condensate $b$. The diagonal elements$^1$ of $\alpha$ are the self-interaction coefficients. $\alpha_{aa} < 0$ implies that the $a$th condensate is self-attractive. $\alpha_{aa} > 0$ implies that the $a$th condensate is self-repulsive. If $\alpha_{aa} = 0$, then the $a$th condensate does not self-interact. The off diagonal elements of $\alpha$ capture the cross-interaction of the condensates. Thus if $\alpha_{bc} > 0$, then condensate $b$ is repelled by condensate $c$, and so forth. In all that follows, $\alpha$ is assumed, and constructed to be, symmetric. This means that the influence felt by $b$ from $c$ is the same as that felt by $c$ from $b$. The forces simply form a third-law pair.

As was stated earlier, type A solutions have the form $r^2_j = A_j \text{sn}^2(m_j x, k) + B_j$. By correct choice of $A$ and $B$, $r^2(x)$ may take on either a $\text{cn}(x,k)$ or $\text{dn}(x,k)$ form. These forms will also be called $\text{Sn}$, $\text{Cn}$, and $\text{Dn}$ subtypes. The elliptic modulus $k$ can exist on the interval $k = [0, 1]$. The potentials strengths $V_k$ for each condensate can be picked from intervals that are parameterized by the self interactions. The captions of the figures in Section 3.2 include the relevant parameters. For example, Fig. 6 shows that the figure was generated with $V^1_0 = V^2_0 = -1$, and $k = \frac{1}{2}$. The off diagonal interaction strengths are fixed by choice of the other parameters.

Type B solutions have the form $r^2_j = A_j \Phi(m_j x, k) + B_j$, where $\Phi$ stands for one of $\text{sn}(x,k)$, $\text{cn}(x,k)$, or $\text{dn}(x,k)$. Type B solutions were only modeled for two condensates in a trap. For this case, there are only two off-diagonal elements of $\alpha$, and they are equal. For a selected value of the off diagonal interactions, and a selected subtype for each of the $M$ condensates, there exist possible values for the $A_j$ and $B_j$. The potential is fixed for all type B solutions. See Eq. (43). The solutions with these possible values are checked for stability. Mixing of types and sub-types is not allowable in general. The parameters as given will only balance the governing system if the elliptic functions are of the same kind. Recall that in the example calculation it was necessary to equate powers of $\text{sn}(x,k)$. This would not be possible in general for mixed-type condensates.

3 Results

BECs are ephemeral. They are difficult to create, and once a BEC has been created, it is short-lived. Condensates that last for only seconds are considered ancient. In order to experimentally

$^1$The $\alpha_{jk}$ can be thought of as forming a matrix. The diagonal elements are those for which $j = k$. 
3 RESULTS

probe BECs, those that are stable to perturbations must be created. If a condensate falls apart with the slightest change of the initial conditions, it is unlikely that such a condensate would ever be observed in the laboratory. It is important for the further development of BECs that researchers understand the properties of a system of condensates that make that system either robust or susceptible to instability. Toward this end, many researchers are attempting to determine the stability of particular configurations of condensates. In particular, this work with Nathan Kutz, Bernard Deconinck, and Brandon Warner has focused on the stability of condensates trapped in an \( \text{sn}^2(x, k) \) standing electro-magnetic potential. We consider this a generalized potential in the sense that \( \text{sn}(x, k) \) is a generalization of the sine function.

A solution to a differential equation is considered "stable" if small changes to that solution do not destroy the viability or structure of that solution. The purpose of this research is to determine the stability of exact solutions to Eq. (17) of the forms discussed. In some cases, as in the case of one condensate with trivial phase dependence, it is possible to obtain analytic stability results [20]. For the most part, including all cases in this paper, stability must be established numerically. This is done by creating initial conditions, \( \psi_j \), that solve Eq. (17), adding a small amount of numerical noise to those \( \psi_j \), and evolving the \( \psi_j \) in time according to Eq. (17). The initial solutions are exact solutions to the governing equations. That is, if the initial conditions were plugged into Eq[ governing ] with \( \partial_t \psi = 0 \), the equality would be true. Noise is added to determine if a particular initial condition is a robust exact solution, or one that is somewhat accidental. As an illustration, consider a simple pendulum. The "solution" with the pendulum straight up is an exact solution of the equations of motion, but any small deviation from absolutely straight up results in the pendulum falling. The exact solution with the pendulum at the bottom is totally stable. If the pendulum is perturbed from this equilibrium, it returns to the bottom. Here "small amount of noise" means random perturbations whose amplitude are small relative to the amplitude of the initial conditions. If the initial conditions persist indefinitely through time, even with the perturbation, those initial conditions are considered stable. Some solutions were found that did go unstable, but did so after many time units. Some of these are labeled "experimentally stable." This is meant to point out that such initial conditions could be observed in the lab, as the modeled condensates persisted before deterioration. Before presenting the numerical results, a brief description of the numerical method used is outlined.

3.1 Algorithm

The governing system given by Eq. (17) is used to evolve exact initial conditions in time to determine the stability of those solutions. Initial conditions of a certain type and sub-type are created using particular values of the adjustable parameters. This is done with MATLAB script. These initial conditions are then passed to compiled Fortran code for evolution. The Fortran programs uses a fourth order Runge-Kutta method in the temporal dimension and a filtered pseudo-spectral method in the spatial dimension. The fourth order Runge-Kutta method is a very well known numerical method for solving differential equations. If the reader is unfamiliar with this method, but interested in the details, please see any edition of Boyce and DiPrima [23]. The filtered pseudo-spectral method is less well known, and harder to find explained in standard texts. As such, a brief introduction to this powerful method is presented here. Those unfamiliar with the Fourier transform and its properties are invited to consult Appendix B.
Figure 6: Stable repulsive type A Dn-Dn solution. Here $V_1 = V_2 = -1$, and $k = \frac{1}{2}$.

Consider some function $u$ of one spatial dimension $x$ and of time $t$, so that $u = u(x, t)$. Further let $u$ satisfy the partial differential equation

$$\frac{\partial u}{\partial t} = Lu + N(u), \quad (47)$$

where $L$ is a differential operator that is linear in $u$ and involving only spatial derivatives, and $N(u)$ is some arbitrary non-linear function of $u$. Notice that Eq. (17) is of this form for each of the $\psi_j$. Now Fourier transform Eq. (47) in $x$ to see

$$\hat{u}_t = \beta(k)\hat{u} + \hat{N}(u), \quad (48)$$

where $\beta(k)$ is some polynomial in the wavenumber $k$, as determined by the operator $L$, and $\hat{f}$ denotes the Fourier transform of $f$ in the $x$ direction. This is now an ordinary differential equation for $\hat{u}$. Next, by way of analogy with the method of integrating factors, rearrange Eq. (48), and multiply by an unknown function $\mu = \mu(k, t)$ to obtain

$$\mu \hat{u}_t - \beta(k)\mu \hat{u} = \mu\hat{N}(u). \quad (49)$$
Figure 7: Stable repulsive type B Dn-Dn solution. Here $\alpha = 0$, $k = \frac{1}{2}$. As $\alpha = 0$, these two condensates are uncoupled.

As $(uv)_x = u_x v + uv_x$, Eq. (49) may be written as a perfect derivative if

$$\mu_t = -\beta(k) \mu.$$  \hspace{1cm} (50)

Equation (50) is easily solved by letting

$$\mu = e^{-\beta(k) t}.$$  \hspace{1cm} (51)

Thus Eq. (48) may be written as

$$(\mu \hat{u})_t = \mu \hat{N}(u).$$  \hspace{1cm} (52)

Now make the substitution $U = \mu \hat{u}$ to find

$$U_t = e^{-\beta(k) t} \hat{N}(u),$$  \hspace{1cm} (53)

and note that $u = F^{-1} [u, \mu]$. Now Eq. (53) may be evolved in time more easily and accurately than Eq. (47). The added accuracy of this scheme comes from both the exact calculation of the spatial
3 RESULTS

Figure 8: Unstable repulsive type B Dn-Dn solution. Here \( \alpha = -0.4, k = \frac{1}{2} \).

derivatives and the global nature of the Fourier transform. As the following section shows, this problem is well suited to a spectral solution. The periodic nature of the problem is well captured by solution in the frequency domain. Instabilities of the solutions invariably stem from modes close to the natural frequency of the domain. Instabilities starting from somewhere other than frequencies close to the principle mode are a sign of numerical error. Note that in order to avoid global error, it is important to make sure that enough Fourier modes are included to capture all relevant behavior. In practice this is accomplished by verifying all results against results obtained with twice as many Fourier modes. If the evolution of a solution with \( 2^5 \) Fourier modes identical to an evolution using \( 2^6 \) Fourier modes, it is reasonable to assume that the former evolution has converged to the correct evolution.

It is important to point out an assumption used in this research that may not be physically valid. In order to make use of a filtered pseudo-spectral method, it is necessary to assume periodic boundary conditions. This means that the left hand side of the solution and the potential is required to be completely identical to the right hand side. In effect this means the modeled condensates are not a cigar-shaped, but are condensates that live in a “doughnut.” As such, the modeled scenario
is not the same as the experiments to which this paper is drawing comparison. In defense of this assumption, there is no fundamental barrier to creating such a domain, and further, in the case of modeling many periods, say more than twelve, this approximation seems to be completely justified. Unfortunately, in order to obtain results within a reasonable time frame, it was often necessary to conduct numerical simulations on a small domain. In the end, I am concerned with the stability of a solution. In all the experiments thus far, the potentials modeled have many periods. Therefore, my advisors and I have adopted the following convention; a solution is deemed stable only if it is stable for a potential and solution twelve or more periods wide. It is very interesting to note that we have observed solutions that were stable, or quasi-stable, for potentials of few periods, but unstable for more periods. We hope to publish these results in the near future.

3.2 Numerical results of Two-species and Three-species Condensates

In this section, numerical results are presented and discussed. The figures in this section show the numerical computations for the $M$ interacting condensates in side by side stacks. Each stack shows,
Figure 10: Unstable type B mixed Dn-Dn solution. Here $\alpha = A$, $k = \frac{1}{2}$. The left condensate is a repulsive type $c$.

From top to bottom, the modulus of the solution squared, $|\Psi(x, t)|^2$, a contour plot of this modulus squared, the potential, $V(x)$, and the inverse tangent of the Fourier spectrum, $\tan^{-1} |\tilde{\Psi}(k)|$. The modulus squared of the wavefunction, once properly normalized to unity, may be interpreted as the probability of finding mass at a particular part of the domain. In the contour plots, the light areas indicate high likelihoods of finding mass. The plots of the solution and the potential have been scaled to occupy a constant vertical range, so proper interpretation of the figures requires careful attention to the vertical tick-marks.

For convenience we define repulsive to mean that all $M$ condensates are self-repelling, attractive requires all $M$ condensates be self-attracting, and mixed denotes some combination of self-repelling and self-attracting condensates. Of course in the case of two condensates mixed means one each of repulsive and attractive.

Figure 6 shows the dynamics for a type A repulsive dn-dn solution with $V_1 = V_2 = -1$. For these, the potential is attractive for each condensate. Figure 7 shows a type B repulsive dn-dn solution. These numeric results confirm the analytic proof of stability obtained for one condensate.
of trivial phase [20]. Notice that in Figs. 6 and 7, both condensates are localized in the troughs of the potential, as is the expected behavior for stable behavior in a linear system. It is expected that classical particles would gather in the troughs of an attractive potential and be stable, id est; marbles are expected to roll downhill and stay there! The stable solutions shown in these two figures illustrates one conclusion we have drawn. The primary condition for stability seems to be offset. The dn solutions are offset from zero, and oscillate about some nominal positive value. In the case of a probability density of a BEC, this is interpreted as the probability of finding mass in a particular region being zero nowhere. Such a configuration can be expected if the trap contains enough atoms. Thus what we see is that stability appears to be a strong function of trap density.

Figure 8 shows that not all exact repulsive dn-dn solutions are stable. However, this solution is stable for almost 500 time units, and thus should be considered experimentally stable. In this run, the condensates attract each other, as $\alpha = -0.4$. This solution shows behavior quite typical of unstable repulsive solutions. These solutions tend to break up and stay on the peaks of the potential. Notice in this case that the left hand condensate is peak on peak with respect to the
Figure 12: Unstable 3-species repulsive type A Dn-Dn-Dn solution. For this run $V_1 = V_2 = V_3 = \frac{1}{2}$, $k = \frac{1}{2}$.

potential, whereas the right hand condensate is peak on trough. Also note that the left condensate is not offset from zero. Figure 9 shows the dynamics of a typical attractive solution. The attractive instabilities tend to focus the condensates into peaks, which then wander about the domain. Note that the instability does not occur until after approximately 300 time units. This configuration may be stable enough to be realized experimentally. Again, this unstable solution is not offset from zero. These initial conditions model a trap that is not well loaded. Figure 10 shows an exact mixed dn-dn solution. This solution gain persists for approximately 300 time units before onset of instability. This result is quite exciting, as it suggests that two condensates with fundamentally different properties could be observed in a standing light-wave trap. Recent experimental results obtained by Ketterle et al. back up this statement. Mixed solutions were also found that persisted for over 3000 time units. This figure also shows the onset of instability. The Fourier spectrum shows new modes gradually being activated and very quickly reaching the amplitude of the starting modes. The spectrum shown quickly progress to one such as seen in Fig. 9.

Figure 11 shows the dynamics of a stable three-species repulsive dn-dn-dn condensate, and Fig.
12 shows the dynamics of an unstable three-species repulsive $dn$-$dn$-$dn$ condensate. These two scenarios are identical save for the sign of the potential. In Fig. 11, the potential is attractive. Figure 12 shows that a positive (or repelling) potential destabilizes the condensate. This is far from surprising. These figures are included mostly to show that the algorithm and method are working for more than two condensates. In principle, the method works for any number of interacting condensates. In reality, going beyond four becomes very time-intensive, and further, for each condensate added, the dimension of the associated parameter space doubles.

4 Summary and Conclusion

Bose-Einstein condensation is realized by cooling dilute gases to very near absolute zero, and in so doing, reducing the momenta of the constituent atoms to almost zero. As the momenta approach zero, the Heisenberg uncertainty principle demands that the atoms become delocalized. The delocalized atoms begin to overlap, and each atom tends toward its ground state. Many atoms, all in the same quantum state, begin to overlap each other, and occupy the same space at the same time. In this state there is no way to distinguish one atom from another, and a new state of matter has been achieved. This new state is known as a Bose-Einstein condensate, or BEC. This condensate can then be placed in an electro-magnetic potential of nearly arbitrary periodic structure, either alone or with other condensates. The dynamics of this system are governed by the $N$-body nonlinear Schrödinger (NLS) equation. This equation has exact solutions, the stability of which cannot be analytically ascertained at present. In order to be able to recommend stable experimental configurations, our group wishes to numerically simulate the time evolution of exact initial conditions according to the NLS using pseudo-spectral methods. We have made an extensive search of the parameter space of exact solutions, and found that the major factor in determining stability is the density of atoms in the trap.

Bose-Einstein condensation is an exciting and rewarding area of research, as evidenced by the incredible strides that have been made recently both in BEC research and in research making use of BECs. These condensates have allowed a strange and marvelous window on the quantum world. Researchers may make "quantum movies" of single and interacting condensates, and thus directly observe the quantum dynamics that govern the world of the very small and dictate the nature of the macroscopic world. Many applications have been proposed for BECs. Some of these applications are purely academic, while others are decidedly industrial in nature. Let it suffice to say the full extent of the possible utility of BECs is unknown.

Our group is pleased to have a numerical model based on the Hartree-Fock approximation that models the dynamics of many interacting Bose-Einstein condensates trapped in a periodic potential. With this model we have been able to explore many possible configurations of BECs, with the goal of determining the stability of these configurations. We hope that these results may be useful to experimentalists wishing to study BECs. It is much less time consuming, and immensely less costly to test these configurations through computer simulation than by actually conducting the experiment.

The numerical results from Sec. 3.2 predict that it is possible to have two or more condensates in a trap, and for the system to be stable. The important factor for stability proves to be the offset of the solution, hence stability was found only for $dn$ forms. The offset may be interpreted as having
above a certain nominal density in the trap. All stability results were found to be independent of $k$, the elliptic modulus in the following sense. If a solution was stable for a low value, $k < .5$, the solution was also stable for a high value, $k > .5$. However, a high $k$ value will cause the onset of instability to occur earlier than will a low $k$ value.

Experimental physicists have been able to create and observe the dynamics of multi-species, interacting condensates. In these experiments, the investigators were capable of varying the type and strength of self-interaction. Unfortunately, thus far no good comparison of our results with the experimental results has been performed.
A Jacobi Elliptic Functions

The Jacobi elliptic functions $\text{sn}(x, k)$, $\text{cn}(x, k)$ and $\text{dn}(x, k)$ are used in this paper. These functions can be thought of as generalizations of more common trigonometric functions. These functions and some interesting properties are presented in this appendix. For a more thorough treatment please see Abromowitz and Stegun.

The Jacobi elliptic functions arise from the inversion of the elliptic integral of the first kind,

$$u = F(\varphi, k) = \int_0^{\varphi} \frac{dt}{\sqrt{1 - k^2 \sin^2(t)}}$$  \hspace{1cm} (1)
A

Table 1: Relevant properties of the Jacobi elliptic functions.

<table>
<thead>
<tr>
<th></th>
<th>sn(u, k)</th>
<th>cn(u, k)</th>
<th>dn(u, k)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(d/du)</td>
<td>(\text{cn} , \text{dn})</td>
<td>(-\text{sn} , \text{dn})</td>
<td>(-k^2 \text{sn} , \text{cn})</td>
</tr>
<tr>
<td>(d^2/du^2)</td>
<td>(2k^2 \text{sn}^3 - (1 + k^2) \text{sn} - 2k^2)</td>
<td>(-2 \text{cn}^3 + (2k^2 - 1) \text{cn})</td>
<td>(-2 \text{dn}^3 + (2 - k^2) \text{dn})</td>
</tr>
<tr>
<td>(k = 0) limit</td>
<td>sin</td>
<td>cos</td>
<td>1</td>
</tr>
<tr>
<td>(k = 1) limit</td>
<td>tanh</td>
<td>sech</td>
<td>sech</td>
</tr>
</tbody>
</table>

where \(k\) is known as the elliptic modulus, or simply the modulus, and \(k^2\) lies on the interval \([0, 1]\). \(\varphi\) is known as the amplitude

\[
\varphi = F^{-1}(u, k) = am(u, k) = am(u).
\]

Now define the Jacobi elliptic functions \(\text{sn}(u, k)\), \(\text{cn}(u, k)\) and \(\text{dn}(u, k)\) by

\[
\begin{align*}
\sin(\varphi) &= \sin(am(u)) \equiv \text{sn}(u, k), \\
\cos(\varphi) &= \cos(am(u)) \equiv \text{cn}(u, k), \\
\sqrt{1 - k^2 \text{sn}^2} &\equiv \text{dn}(u, k).
\end{align*}
\]

These are generalizations of the trigonometric functions. This can be seen by setting \(k = 0\). In this trigonometric limit:

\[
\begin{align*}
\text{sn}(u, 0) &= \sin(u), \\
\text{cn}(u, 0) &= \cos(u), \\
\text{dn}(u, 0) &= 1.
\end{align*}
\]

As \(k \to 1\), these functions approach a hyper-geometric limit:

\[
\begin{align*}
\text{sn}(u, 1) &= \tanh(u), \\
\text{cn}(u, 1) &= \text{sech}(u), \\
\text{dn}(u, 1) &= \text{sech}(u).
\end{align*}
\]

More properties of these functions are summarized in Tab. 1. The following squared function relations are also useful.

\[
\begin{align*}
\text{sn}^2(u, k) + \text{cn}^2(u, k) &= 1, \\
\text{dn}^2(u, k) - k^2 \text{cn}^2(u, k) &= 1 - k^2, \\
k^2 \text{sn}^2(u, k) + \text{dn}^2(u, k) &= 1.
\end{align*}
\]

All of the information for this section may be found in [30]
B Fourier Transforms and Spectrums

Figure 14: The elliptic functions plotted for increasing values of the elliptic modulus, $k$. On the left of each is the arctangent of it’s Fourier spectrum. The spectrum shows increasing numbers and amplitudes of the active Fourier modes. Notice that as $k$ is increased from 0.5 to 0.999999, modes that are harmonic to the pure modes continue to be activated. Thus the elliptic functions with increased $k$ values are built of ever increased numbers of purely sinusoidal components.

Fourier transforms are a powerful tool for solving differential equations. The basic idea is to transform the unknown function from a function of a spatial, or temporal, dimension, into a function in the frequency domain. This method decomposes a function into its frequency components. For example, the Fourier transform of a sine function is a single spike at the value of the frequency of the sine function. Due to properties of the Fourier transform, it is much easier to take the derivative of a transformed function. The Fourier transform can reduce some differential equations to algebraic equations. Difficulty often comes when one must transform back out of the frequency domain.

The Fourier and inverse Fourier transforms are denoted by $\mathcal{F}$ and $\mathcal{F}^{-1}$ respectively and are
given by

\[ F[f(r,t)] = C \int_{-\infty}^{\infty} dr \ f(r,t) e^{-ikr} = \hat{f}(k,t), \]  

(1a)

\[ F^{-1}[\hat{f}(k,t)] = \tilde{C} \int_{-\infty}^{\infty} dk \ \hat{f}(k,t) e^{ikr} = f(r,t), \]  

(1b)

with any normalization condition such that \( C \tilde{C} = 1/2\pi \). The function \( \hat{f}(k,t) \) is called the Fourier transform of \( f(r,t) \), and \( k \) is the wave number vector. For convenience, choose \( C = 1 \) and \( \tilde{C} = 1/2\pi \). As an example, take the Fourier transform of a Gaussian function \( f(x) = e^{-(x/a)^2} \), where \( a \) is an arbitrary parameter.

\[ \hat{f}(k) = \int_{-\infty}^{\infty} dx \ e^{-\left(\frac{x}{a}\right)^2} e^{-ikx}. \]  

(2)

Combine the exponents, and then complete the square.

\[ \hat{f}(k) = \int_{-\infty}^{\infty} dx \ e^{-\left(\frac{x}{a}\right)^2 - ikx - \frac{k^2}{4}}. \]  

(3)

Notice that the quantity in parentheses may be written as \(-\left(x + ik/2\right)^2\), and that the second exponential factor does not contain \( x \), and may be removed from the integral.

\[ \hat{f}(k) = e^{-\frac{k^2}{4}} \int_{-\infty}^{\infty} dx \ e^{-\left(\frac{x}{a} + ik/2\right)^2}. \]  

(4)

Now the integral in Eq. (4) cannot be explicitly calculated, but can be found, as the limit of an integral in the complex plane, to be \( \sqrt{\pi} \). So

\[ \hat{f}(k) = \sqrt{\pi} e^{-\frac{a^2 z^2}{4}}, \]  

(5)

which is also a Gaussian, but now as a function of \( k \). \( \hat{f} \), the Fourier transform of \( f \), is interpreted as representing the frequencies that make up \( f \). This function \( \hat{f} = \hat{f}(k) \) is also known as the spectrum of \( f \). Equation (5) points out a general property of the Fourier transform of a function. The parameter \( a \) scales both \( f \) and \( \hat{f} \), but in the function it narrows the Gaussian, and in the transform it widens the Gaussian. Generally, the narrower a function is the wider its Fourier transform will be. Another example will help clarify what the wave number \( k \) is. The Dirac delta generalized function is such that

\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \ \delta(k - \omega) e^{ikx} = \frac{1}{2\pi} e^{i\omega x}. \]  

(6)

So, by the definition of the inverse Fourier transform, \( \hat{f}(k) = \delta(k - \omega) \) is the Fourier transform of \( f(x) = 1/2\pi e^{i\omega x} \). Recall that \( e^{i\theta} = \cos(\theta) + i \sin(\theta) \), and so both the real and imaginary parts of \( f \) are periodic with period \( 2\pi/\omega \). The only frequency associated with \( f \) is \( \omega \), and this is reflected by the transform \( \hat{f} = \delta(k - \omega) \) which is zero for all \( k \neq \omega \) and integrates to one.

This paper makes much use of the Jacobi Elliptic functions. It is interesting to see the Fourier spectrums of these functions as \( k \) is increased. In the trigonometric limit, this is simply the Dirac delta function at the frequency, as well as opposite of the frequency, of the function. As the elliptic modulus increases, many more frequencies are needed. By adding up sine and cosine
functions of many frequencies, the sinusoidal shape is nudged into the more drastic curves of the elliptic functions with higher \( k \) values. This is shown in Fig. 14.

The usefulness of the Fourier transform comes from the way it transforms a derivative. To see this, take the Fourier transform of a derivative of a function \( f = f(x) \).

\[
\mathcal{F}\left[ \frac{d}{dx} f \right] = \int_{-\infty}^{\infty} \frac{d}{dx} f e^{-ikx} dx,
\]

and integrate by parts to get

\[
\mathcal{F}\left[ \frac{d}{dx} f \right] = f e^{-ikx}\big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f \frac{d}{dx} e^{-ikx} dx.
\]

In order for the Fourier transform to be defined, the first term on the right hand side must evaluate to zero. As \( d/dx(e^{ax}) = ae^{ax} \), the integral on the right hand side may be rewritten as

\[
-ik \int_{-\infty}^{\infty} f e^{-ikx} dx,
\]

but the underlined part is \( \hat{f} \), the Fourier transform of \( f \). Therefore

\[
\mathcal{F}\left[ \frac{d}{dx} f \right] = ik \hat{f},
\]

and taking further derivatives proceeds in the same way.

\[
\mathcal{F}\left[ \frac{d^n}{dx^n} f \right] = (ik)^n \hat{f}.
\]

Consider the following differential operator \( L \).

\[
L f = 3 \frac{d^3}{dx^3} f - 5 \frac{d^2}{dx^2} f + \frac{d}{dx} f.
\]

Taking the Fourier transform of Eq. (12) and using Eq. (11) gives

\[
\mathcal{F}[L f] = 3(ik)^3 \hat{f} - 5(ik)^2 \hat{f} + ik \hat{f},
\]

or

\[
\mathcal{F}[L f] = (-3ik^3 + 5k^2 + ik) \hat{f} = P(k) \hat{f},
\]

where \( P(k) \) denotes the polynomial in \( k \). Any linear differential operator transforms into a polynomial in the frequency domain. This is a very powerful method for dealing with partial differential equations. Consider a function \( g = g(x, t) \), and the function's derivative with respect to the time variable \( t \). Take the Fourier transform of \( \partial g(x, t)/\partial t \) with respect to the spatial variable \( x \).

\[
\mathcal{F}\left[ \frac{\partial}{\partial t} g(x, t) \right] = \int_{-\infty}^{\infty} dx \frac{\partial}{\partial t} g(x, t) e^{-ikx}.
\]
In polite company it is perfectly reasonable to exchange the order of the $t$ differentiation and the $x$ integration to obtain

$$\mathcal{F} \left[ \frac{\partial}{\partial t} g(x, t) \right] = \frac{\partial}{\partial t} \int_{-\infty}^{\infty} dx \, g(x, t) e^{-i k x} = \frac{\partial}{\partial t} \hat{g}(x, t).$$

(16)

Thus the $t$ derivative passes through the Fourier transform with respect to $x$. In this way, a linear partial differential equation for $g$ can be transformed into an ordinary differential equation for $\hat{g}$, the Fourier transform of $g$. This fact is what makes the technique so useful for partial differential equations.
References


[8] Cern Courier Online


