An Introduction to Tropical Varieties

Ravi Shroff*

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1 Introduction

We motivate our study of tropical varieties by considering the tropical semiring $\mathbb{R}_{\text{trop}}$. This object is defined as $\mathbb{R} \cup \{\infty\}$ equipped with the tropical addition $\oplus$ and the tropical multiplication $\odot$. Tropical addition corresponds to minimization in euclidean space, and tropical multiplication is euclidean addition. Here is an example:

$$5 \oplus 7 = \min\{5, 7\} = 5 \quad \quad 5 \odot 7 = 5 + 7 = 13.$$

Notice that additive inverses do not exist in $\mathbb{R}_{\text{trop}}$ and the additive identity is $\infty$. The lack of additive inverses is what makes $\mathbb{R}_{\text{trop}}$ a semiring rather than a ring. The adjective "tropical" was coined in honor of the Brazilian mathematician Imre Simon who did important initial work in this area.

Recall that a (normal) algebraic variety of a set of polynomials is the collection of common zeros. Thus an algebraic variety may be interpreted as a generalization to higher dimensions of an algebraic curve. A tropical variety is then in some sense the analogue of a normal variety, but in tropical $n$-space.

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There are four generally equivalent definitions of the term tropical variety, three algebraic in flavor and one more geometric. The first definition may be roughly described as the image of the standard variety under a certain class of maps into tropical space. This definition coincides with the notion of non-archimedean amoeba from dynamical systems. The second definition of a tropical variety is the set of points where the “tropicalizations” of all polynomials in the corresponding ideal of the variety fail to be differentiable. The third definition follows a slight variation of Gröbner basis theory, and the fourth is of historical importance but will not be discussed in this paper. We will give exact definitions of three of these descriptions, and outline parts of the proof of their equivalence.

Tropical varieties in one dimension have been completely classified. They coincide exactly with the set of rational graphs satisfying a “zero tension condition” at each vertex. We will discuss this classification, proved in [2] and provide several examples. We will also discuss an algorithm based on the Newton polytope for drawing tropical varieties. There are many open questions in this field, including classification of tropical varieties in higher dimensions. One open question that we will outline is that of finding a “tropical basis” for an ideal, that is, a generating set that provides enough information to draw the tropical variety.

Tropical varieties are used in several settings. They have applications to mathematical biology, especially in the construction of phylogenetic trees [1]. In addition, tropical varieties coincide with the nonarchimedean amoebas studied by Einsiedler, Lind, and Kapranov in [3]. Tropical varieties are also computed using Gröbner bases, a powerful set of algebraic tools. One of the most useful facts about tropical varieties is that they are polyhedral sets, so the theorems in combinatorics and polytope theory can be used to answer questions from distant branches of math.

Tropical varieties have proven useful in other circumstances, including normal algebraic geometry and mathematical physics. It turns out that tropical varieties display important topological and combinatorial facts about their corresponding complex varieties, and are easier to work with. Additionally, G. Mikhalkin has been using tropical varieties in connection with Gromov-Witten invariants, which is a topic in mathematical physics.
2 Preliminaries

2.1 Nonarchimedean fields

Let $F$ be a field with an absolute value $|\cdot|$ which satisfies the ultrametric inequality $|a+b| \leq \max\{|a|, |b|\}$ for every $a, b \in F$. Then $F$ is called nonarchimedean and the map $v : F \to \mathbb{R} \cup \{\infty\}$ defined by $v(a) = -\log|a|$ is called a valuation on $F$. The map $v$ satisfies the following properties:

1. $v(a) = \infty \iff a = 0$
2. $v(ab) = v(a) + v(b)$
3. $v(a + b) \geq \min\{v(a), v(b)\}$

for all $a, b \in F$. Note that the multiplicativity of the absolute value implies that $v(-a) = v(a)$. The axioms also imply that $v(1) = 0$, from which we can conclude that $v(1) = -v(a)$. We define the value group of $v$ to be $v^\mathbb{R} = \{b \in \mathbb{R} : b = v(a) \text{ for } a \in F\}$.

We now present two standard facts about valuations on $F$.

**Lemma 1**: If $v(a) > v(b)$ then $v(a + b) = v(b)$ for all $a, b \in F$.

*Proof*: By definition, $v(a + b) \geq \min\{v(a), v(b)\}$, so $v(a + b) \geq v(b)$. On the other hand, $v(b) \geq \min\{v(a + b), v(-a)\} = \min\{v(a + b), v(a)\} = v(a + b)$.

Therefore $v(a + b) = v(b)$, as desired.

**Lemma 2**: For all $a, b, c \in F$ with $a + b + c = 0$, at least two of $v(a), v(b), v(c)$ are equal and minimal.

*Proof*: If $a + b + c = 0$ then $a + b = -c$, so $v(a + b) = v(-c) = v(c)$. If $v(a) > v(b)$ then $v(a + b) = v(b)$ by Lemma 1, so $v(b) = v(c)$ as desired.

Three examples of nonarchimedean fields are below. We will generally use the first and second fields to compute explicit examples of tropical varieties. However, the third example is standard in the current literature, such as [1] and [2].

**Example 1**: $\mathbb{Q}$ with the $p$-adic valuation. That is, let $p$ be a prime integer and write $a \in \mathbb{Q}$ as $a = p^k \frac{b}{c}$ with $k \in \mathbb{Z}$ and $b, c$ relatively prime integers not divisible by $p$. Then the map $v(a) = -k$ is a valuation on $\mathbb{Q}$. We will mainly be concerned with $\mathbb{Q}_p$, the algebraic closure of $\mathbb{Q}_p$. We also use the important fact from algebra that a valuation on a field $F$ extends uniquely to a valuation on the field $\bar{F}$.

**Example 2**: Any field $F$ with the trivial valuation. To obtain the trivial valuation, define $|0| = 0$ and $|a| = 1$ for all $a \in F$. This is an absolute value which satisfies the ultrametric inequality. Therefore we may define $v(a) = -\log|a|$, so this valuation maps zero to infinity, and every other element to zero.

**Example 3**: Consider the algebraic closure $K$ of the field $\mathbb{C}(t)$, the rational functions in one variable over $\mathbb{C}$. We define a valuation on $K$ to be the unique extension of the order map on $\mathbb{C}(t)$, the order
of the zero or pole of an element at the origin. This particular field is more complicated to work with than the other two cases, so to illustrate general phenomena we will look to \( \bar{\mathbb{Q}}_p \) and \( \mathbb{C} \) with the trivial valuation.

### 2.2 Tropicalizations and tropical hypersurfaces

We now define the notion of tropicalization of a polynomial and a tropical hypersurface. Let \( F \) be a nonarchimedean field with valuation \( v \) and let

\[
f(x_1, \ldots, x_n) = \sum_{\alpha \in K} c_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n}
\]

be a polynomial in \( F[x_1, \ldots, x_n] = F[x] \) where \( c_\alpha \in F \setminus \{0\} \) and \( K \) is a collection of vectors in \( \mathbb{Z}^n \). The general setting for tropical mathematics is the ring of Laurent polynomials over \( F \), that is, where \( K \) may consist of any collection of elements of \( \mathbb{Z}^n \). In this paper we will restrict our attention to standard polynomials, where \( K \) is a collection of elements in the nonnegative \( n \) dimensional integers.

The *tropicalization* of \( f \) is denoted by

\[
trop(f) = \min_{\alpha \in K} (v(c_\alpha) + a_1 x_1 + \cdots + a_n x_n).
\]

The tropicalization is essentially the function obtained by replacing multiplication and addition by their tropical counterparts and then taking the valuation of the coefficients. The tropicalization is therefore a piecewise linear function from \( \mathbb{R}^n \rightarrow \mathbb{R} \).

The *tropical hypersurface* of \( f \) is the set of points where \( \trop(f) \) achieves its minimum value at least twice (where two of the linear functions have the same minimal value). Equivalently, the tropical hypersurface is the set of points where \( \trop(f) \) fails to be differentiable. The tropical hypersurface is the intersection of closed sets and is hence closed. The tropical hypersurface of \( f \) is denoted by \( T(f) \).

**Example 1:** Let our nonarchimedean field be \( \mathbb{C} \) with the trivial valuation, and \( f(x) = 2 + 3x + 5x^2 + 6x^3 \). Then

\[
trop(f) = \min(1, x, 1 + 2x, 1 + 3x)
\]

and \( T(f) = \{-\frac{1}{2}, 1\} \).

**Example 2:** Let \( f(x) = 2 + 3x + 4x^2 + \frac{1}{2}x^4 \) over \( \bar{\mathbb{Q}}_2 \). Then

\[
trop(f) = \min(1, x, 2 + 2x, -2 + 4x)
\]

and \( T(f) = \{\frac{3}{4}, 1\} \).

### 2.3 Other algebraic terminology

We now define the notion of the initial ideal of an ideal in \( F[x] \). This terminology is similar to that of basic Gröbner basis theory, and is necessary for the third definition of a tropical variety. We present
the general definition from [1], but a variation of this theory used for drawing tropical varieties is less complicated.

We first fix an element \( t \) in the nonarchimedean field \( F \) that has \( v(t) = 1 \). For instance, this may be any nonzero element in \( \mathbb{C} \) with the trivial valuation, or the element \( p \) in \( \mathbb{Q}_p \). We let \( R_F \) denote the set \( \{ a \in F : v(a) \leq 0 \} \) and \( M_F \) denotes the set \( \{ a \in F : v(a) > 0 \} \). The set \( R_F \) is called a local ring and the set \( M_F \) is a maximal ideal in \( R_F \). Since \( M_F \) is maximal, the quotient \( \mathbb{k} = R_F / M_F \) is a field.

Now fix a polynomial \( f(x) \) in a polynomial ring \( F[x] \) and a vector \( \omega \in \mathbb{R}^n \). The weight of a monomial \( c_{\alpha} x_1^{\alpha_1} \cdots x_n^{\alpha_n} \) in \( f(x) \) is defined to be \( v(c_{\alpha}) + \alpha_1 \omega_1 + \cdots + \alpha_n \omega_n \), that is, the tropicalization of the monomial evaluated at \( \omega \). Let \( \bar{f}(x_1, \ldots, x_n) = f(t^{\alpha_1} x_1, \ldots, t^{\alpha_n} x_n) \) and denote the smallest weight of any term of \( f \) by \( \bar{w} \). Define the initial form \( \text{in}_\omega(f) \) as the image of \( t^{-w} f \) in \( \mathbb{k}[x] \). The initial form \( \text{in}_\omega(0) = 0 \).

Now given any ideal \( I \) in \( F[x] \), we define its initial ideal to be \( \langle \text{in}_\omega(f) : f \in I \rangle \).

**Example:** Consider the principal ideal generated by the polynomial \( f(x, y) = 4 + \frac{1}{2} x + y \) in \( \mathbb{Q}_2[x, y] \). Let \( \omega \) be the vector \((3, 5)\). Then the weights of the terms 4, \(\frac{1}{2}x, y \) are respectively 2, 2, and 5. Now \( \bar{f} = 4 + 2^3(\frac{1}{2}x) + 2^5 y = 4 + 4x + 32y \). Then \( \text{in}_\omega(f) = 1 + x \) in \( \mathbb{k}[x] \).

### 3 Tropical varieties and their equivalence

We are now ready to present the exact definitions of a tropical variety, and prove part of the theorem on their equivalence.

Fix an ideal \( I \) in a polynomial ring \( F[x] \) where \( F \) is an algebraically closed nonarchimedean field with a valuation \( v \). Denote the variety of \( I \) in \( \mathbb{A}^n \) by \( V(I) \). We work in \( \mathbb{A}^n \) to avoid taking the valuation of zero, which is infinity.

**Definition 1:** The tropical variety of \( I \) is the topological closure of the set \( \{(v(u_1), \ldots, v(u_n)) : (u_1, \ldots, u_n) \in V(I)\} \).

**Definition 2:** The tropical variety of \( I \) is the set \( \cap \mathcal{T}(f) : f \in I \).

**Definition 3:** The tropical variety of \( I \) is the set of all vectors \( \omega \) in \( \mathbb{R}^n \) such that \( \text{in}_\omega(I) \) does not contain a monomial.

The fourth definition of a tropical variety is known as a Bieri-Groves set. The proof of the equivalence of these four definitions is done in two papers. We have

\[
1 \implies 2 \implies 3 \implies 1 \quad \text{proven in [1]}
\]

\[
3 \iff \text{(negative of the Bieri-Groves set)} \quad \text{proven in [3]}
\]

We will outline the proof of the first two implications, that Definition 1 implies Definition 2, and Definition 2 implies Definition 3. However, the proofs that Definition 3 implies Definition 1 and that
Definition 1 (the nonarchimedean amoeba) is equivalent to the negative of the Bieri-Groves set are beyond the scope of this paper. We mention that it is in the proof of the first omitted implication that the requirement that $F$ be algebraically closed is necessary.

**Proposition 1**: If $\omega$ is an element in the tropical variety as defined in Definition 1, then $\omega$ is in the intersection of all tropical hypersurfaces of all polynomials in $I$.

**Proof**: Since a tropical variety as in Definition 2 is a closed set, it suffices to let $\omega = (v(u_1), \ldots, v(u_n))$ where $(u_1, \ldots, u_n) \in V(I)$. This is because if such $\omega$ are contained in the Definition 2 tropical variety then their topological closure must also be contained in the tropical variety. By definition any $f$ in $I$ must vanish at $(u_1, \ldots, u_n)$, so we may write

$$f(u_1, \ldots, u_n) = \sum_{a \in K} c_a u_1^{a_1} \cdots u_n^{a_n} = 0$$

By Lemma 2 from Section 2, that means that the valuation of at least two monomials in $f(u_1, \ldots, u_n)$ attain a minimal value. That is, there exist monomials $A = c_{a_1} x_1^{a_{11}} \cdots x_n^{a_{1n}}$ and $B = c_{a_2} x_1^{a_{21}} \cdots x_n^{a_{2n}}$ such that

$$v(c_{a_1}) + a_{11}v(u_1) + \cdots + a_{1n}v(u_n) = v(c_{a_2}) + a_{21}v(u_1) + \cdots + a_{2n}v(u_n)$$

and this value is minimal. However this means exactly for every $f$ in $I$ the tropicalization of $f$ attains its minimal value twice at $\omega$. Hence $\omega$ is in the intersection of all tropical hypersurfaces of all polynomials in $I$, as desired.

**Proposition 2**: If $\omega$ is an element in the tropical variety as defined in Definition 2, then $in_\omega(I)$ does not contain a monomial.

**Proof**: Let $f \in I$ and assume that trop$(f)$ attains its minimum at least twice at $\omega$. Since the weight of a monomial of $f$ is the same as the tropicalization of that monomial evaluated at $\omega$, this means $f$ has two monomials with smallest weight $\omega$. Denote the monomials by $A$ and $B$ as in the proof of Proposition 1 and their weights by

$$\omega = v(c_{a_1}) + a_{11}v(u_1) + \cdots + a_{1n}v(u_n) = v(c_{a_2}) + a_{21}v(u_1) + \cdots + a_{2n}v(u_n).$$

In $\tilde{f}$, these two monomials become

$$\tilde{A} = c_{a_1} t^{a_{11}}v(u_1) + \cdots + a_{1n}v(u_n)x_1^{a_{11}} \cdots x_n^{a_{1n}}$$

and

$$\tilde{B} = c_{a_2} t^{a_{21}}v(u_1) + \cdots + a_{2n}v(u_n)x_1^{a_{21}} \cdots x_n^{a_{2n}}.$$ 

Therefore in $t^{-\omega}\tilde{f}$, the monomials become

$$t^{-\omega}\tilde{A} = c_{a_1} t^{-v(c_{a_1})}x_1^{a_{11}} \cdots x_n^{a_{1n}}$$

and

$$t^{-\omega}\tilde{B} = c_{a_2} t^{-v(c_{a_2})}x_1^{a_{21}} \cdots x_n^{a_{2n}}.$$ 

Now neither $t^{-\omega}\tilde{A}$ nor $t^{-\omega}\tilde{B}$ vanish in $k[x]$ because

$$v(c_{a_1} t^{-v(c_{a_1})}) = v(c_{a_1}) + v(t^{-v(c_{a_1})}) = v(c_{a_1}) - v(c_{a_1}) = 0,$$

and similarly, $v(c_{a_2} t^{-v(c_{a_2})}) = 0$. That is, $t^{-\omega}\tilde{A}$ nor $t^{-\omega}\tilde{B}$ are in $R_F[x]$ but not in $M_F[x]$. This implies that $in_\omega(f)$ has at least two monomials in $k[x]$, so $in_\omega$ contains no polynomial consisting of a single monomial, as desired.

As might be expected, if $f$ is a polynomial in $F[x_1, \ldots, x_n]$, and if $a = (a_1, \ldots, a_n)$ is a point in $(F^\times)^n$ then the tropical hypersurface of $f_a = f(a_1x_1, \ldots, a_nx_n)$ is a translation of the tropical hypersurface of $f$. 

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**Proof:** Let \( V(f) = \{ p \in (F^x)^n : f(p) = 0 \} \) Then \( V(f_a) = \{ \frac{p}{a} \in (F^x)^n : f(p) = 0 \} \). That is, the standard algebraic variety of \( f_a \) is the algebraic variety of \( f \) scaled by \( \frac{1}{a} \). Then for every \( \frac{p}{a} \in V(f_a) \), we have \( v(\frac{p}{a}) = v(p) + v(\frac{1}{a}) \). Therefore \( T(f_a) = T(f) + (v(\frac{1}{a_1}), \ldots, v(\frac{1}{a_n})) = T(f) - (v(a_1), \ldots, v(a_n)) \).

### 4 One dimensional tropical varieties

#### 4.1 Drawing tropical varieties

We now discuss how to draw tropical varieties in the plane. For simplicity, our examples will be principal ideals in two variables. We start with the example \( f(x) = 2 + x + y \). Suppose we are working over \( R \) with the trivial valuation. Since \( f(x) \) is a linear function, we expect the tropical variety to produce a tropical line. How do we determine the shape of the tropical variety of \( f \)?

One method is to work as in Definition 1 of a tropical variety. For any point \((x_0, y_0)\) in \( V(I) \), we have \( 2 + x_0 + y_0 = 0 \). This means that the valuation of at least two of \( \{2, x_0, y_0\} \) must be equal and minimal. Therefore the tropical variety consists of the following three rays:

- **Ray 1:** \( v(2) = v(x_0) \leq v(y_0) \)
- **Ray 2:** \( v(2) = v(y_0) \leq v(x_0) \)
- **Ray 3:** \( v(x_0) = v(y_0) \leq v(2) \).

The tropical variety looks like the union of three rays:

![The tropical variety of 2 + x + y](image)

Now we consider a general quadratic polynomial in two variables over a nonarchimedean field \( F \), \( f(x) = a_1 + a_2 \cdot x + a_3 \cdot x^2 + a_4 \cdot xy + a_5 \cdot y + a_6 \cdot y^2 \) where \( a_i \in F \). We could follow the same procedure as before to draw the tropical variety, but this will be time-consuming. We therefore introduce an algorithm based on polytope theory for constructing the tropical variety of \( f(x) \). The algorithm is demonstrated in more detail in [3], and a slight variation in [2].

To introduce this algorithm, we fix a polynomial \( f(x, y) = \sum_{\alpha \in K} c_{\alpha} x^{a_1} y^{a_2} \) with \( K \in (Z^+)^2 \) and
$c_a \in F$ and make some preliminary definitions.

**Definition 1:** The *support* of $f$ is the set $K$.

**Definition 2:** The *convex hull* of a set of points $P$ in $n$ dimensions is the intersection of all convex sets containing $P$.

**Definition 3:** The *Newton polytope* of $f$ is the convex hull in $\mathbb{R}^3$ of the set $\{(a, t) : a \in K, t \geq v(c_a)\}$. This will generally be a 3 dimensional polytope.

To find the directions of the rays in the tropical variety, we project the lower hull of the Newton polytope of $f$ onto $\mathbb{R}^2$, which will give a convex polygon in the plane and a subdivision of that polygon. The dual graph to the subdivision of the planar polygon will be a combinatorial description of the tropical variety of $f(x, y)$ reflected about the line $y = -x$.

To completely determine the tropical variety, we must find the location of the vertices in the above dual graph. Each vertex corresponds to a face in the lower hull of the Newton polytope. To obtain the vertex corresponding to a face in the lower hull, we look at the outward unit normal vector $d$ to the face. We then form an infinite ray from the origin in the direction of $d$. The intersection of this ray with the plane $z = -1$ in $\mathbb{R}^3$ is the desired vertex.

**Example 1:** Our first example is the polynomial $f(x) = 2 + x + y$ over $\mathbb{R}$ with the trivial valuation. We hope to obtain the same tropical variety by the algorithm as by the procedure in the beginning of the section. The support of $f$ is the the set $\{(0, 0, 0), (1, 0, 0), (0, 1, 0)\}$. The dual graph therefore consists of the three infinite rays pointing in the directions $(0, -1), (-1, 0), (1, 1)$. The vertex of the graph is the origin, since the Newton polytope of $f$ has a flat bottom. This is exactly the tropical variety shown in the figure at the beginning of the section, except reflected about the line $y = -x$.

**Example 2:** Our second example is the quadratic $f(x) = 2 + x + 2x^2 + y + xy + 2y^2$ over $\mathbb{Q}_2$. The tropical variety of this polynomial should give an idea of what tropical quadratic curves look like. We will omit the calculation of the vertices and just present the combinatorial description of the variety. The support of $f$ is the set $\{(0, 0, 1), (1, 0, 0), (2, 0, 1), (0, 1, 0), (1, 1, 0), (0, 2, 1)\}$. The corresponding subdivision and dual graph are shown below. The tropical variety of this quadratic
looks similar to a line but it has six unbounded edges and three bounded edges.

Example 3: For our third example we draw the combinatorial description of the tropical variety in $\mathbb{Q}_p$ of the cubic

$$f(x) = p + x + px^2 + p^2 x^3 + y + px^2 y + y^2 + xy^2 + py^3.$$  

The subdivision looks like:

Hence the reflection about the line $y = -x$ of the tropical variety looks like:
4.2 Classifying one-dimensional tropical varieties

The three preceding examples demonstrate certain conditions that a one-dimensional tropical variety must satisfy. Namely, there is a "balance" or "zero tension" condition at each node in the variety. To introduce this condition, we fix a nonarchimedean field and make some preliminary definitions.

**Definition 1:** A planar graph is *rational* if it is a finite union of rays and segments whose directions have coordinates in the rational numbers, whose vertices are in the value group, and where each ray or segment has positive integral multiplicity.

**Definition 2:** Every ray in a one-dimensional tropical variety has a *multiplicity*, which is the lattice length of the corresponding edge in the subdivision formed by the Newton polytope algorithm.

**Definition 3:** Let $p$ be a vertex of a rational graph, and $v_1, \ldots, v_k$ are the lattice vectors of shortest length in the directions of the rays emanating from $p$. Let $m_1, \ldots, m_k$ be the multiplicities of the corresponding edges. Then the graph is said to be *balanced* if the condition

$$m_1 v_1 + \cdots + m_k v_k = 0$$

holds for all $p$.

An important fact pertaining to the classification of one-dimensional tropical varieties appears as Theorem 3.3 in [2]. The theorem states that any purely $(n - 1)$ dimensional tropical variety in $\mathbb{R}^n$ is the tropical hypersurface of some tropicalization of a polynomial. This theorem is useful because it allows us to work with principal ideals when classifying varieties. That is, although a one-dimensional tropical variety may come from an ideal which has many generators, there is a tropical polynomial whose hypersurface coincides with the variety, so we may consider the single polynomial rather than the multiple generators of the ideal.

The classification of one-dimensional tropical varieties may then be stated as:

**Theorem:** The one-dimensional tropical varieties are exactly the balanced rational graphs in $\mathbb{R}^2$.

The proof of this theorem is discussed in [2].
5 Tropical bases and open questions

One major open question in the study of tropical varieties is how to find a tropical basis for an ideal. A tropical basis for an ideal is a generating set that provides enough information to draw the tropical variety. By "enough information," we mean that the tropical variety of the ideal should equal the intersection of the tropical hypersurfaces of the polynomials in the generating set. Here is an example of a set which is not a tropical basis for its ideal.

Example 1: We will consider this example in $\mathbb{C}[x,y]$ with the trivial valuation. Let $I = \langle x + y, 3 + x + y \rangle$. Since 3 is in $I$ and we are working over $\mathbb{C}$, $I$ must be $\mathbb{C}$ itself, and thus $V(I)$ is the empty set. This implies that the tropical variety is empty. However, the intersection of the tropical hypersurfaces of $x + y$ and $3 + x + y$ is the ray $x = y = t, t \leq 0$. Therefore the presentation of the ideal as $\langle x + y, 3 + x + y \rangle$ does not give us enough information to draw the tropical variety. We need to intersect hypersurfaces from additional polynomials in $I$ to intersect down to the empty set.

It was conjectured in [1] that a Universal Gröbner Basis would provide a tropical basis. Recall that a Universal Gröbner basis of an ideal is a Gröbner basis with respect to any term order. However, this conjecture was recently proven false. Here is the relevant example.

Example 2: Let $F$ be a nonarchimedean field with $v(1) = 0$, and consider the ideal $I = I_1 \cap I_2 \cap I_3$ in $F[x, y, z]$, where $I_1 = \langle x + y, z \rangle, I_2 = \langle x + z, y \rangle, I_3 = \langle y + z, x \rangle$. Now $I_1$ contains $z$, so $I_1$ contains the monomial $xyz$. Similarly both $I_2$ and $I_3$ contain $xyz$, so $I$ contains $xyz$. Now since $I$ contains a monomial, the tropical variety of $I$ must be empty. This follows from results in Section 3, namely that with respect to any vector $\omega \in \mathbb{R}^n$, we have $in_{\omega}(I)$ contains the monomial $xyz$ and hence no vector can be in the tropical variety of $I$.

On the other hand, the set $\{ x + y + z, x^2y + xy^2, y^2z + yz^2, x^2z + zz^2 \}$ is a Universal Gröbner Basis for $I$. We may tropicalize the generating polynomials to obtain the following four functions.

\[
\min(x, y, z) \quad \min(2x + y, 2y + x) \quad \min(2x + z, 2z + x) \quad \min(2y + z, 2z + y).
\]

Each linear function in every tropicalization is equal at any point of the form $(j, j, j)$ for $j \in \mathbb{R}$, so at any such point the tropical hypersurfaces of the four generating polynomials intersect. This implies
that the intersection of the tropical hypersurfaces of the generating polynomials is nonempty, so it cannot be the tropical variety of $I$.

Another open question in the study of tropical varieties is the "recognition problem" in higher dimensions. The problem is essentially given an object in $\mathbb{R}^k$ for $k \geq 3$ that is an intersection of polyhedral sets, what conditions determine whether the object is a tropical variety or not? The balanced rational graph condition solves the recognition problem for one-dimensional varieties, but there is currently no analogue of this criteria in higher dimensions.
References

