The Forward Operator for a Seismic Inverse Problem

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ABSTRACT. We study the forward map associated with a one-dimensional forward and inverse problem in reflection seismology. In a stratified elastic half space, an impulsive plane wave is applied at the surface and the resulting particle velocity at the surface is recorded. The forward problem is to find the response at the surface given the characteristic impedance of the medium. In contrast, the inverse problem is to determine the characteristic impedance from the response at the surface. Using energy estimates, we will prove continuous differentiability of the forward map. This will allow us to conclude existence of a solution to the inverse problem.

1. INTRODUCTION AND FORMULATION

In this paper, we will be concerned with an inverse problem in reflection seismology in one-dimension. Using arguments adapted from Symes [7], we will show that the Forward map is Fréchet differentiable.

Consider a stratified elastic medium in the half space \( \{ z > 0 \} \). Let \( \rho \) denote the density of the medium and \( \lambda, \mu \) denote the Lamé constants. We assume that \( \rho, \lambda, \mu \) depend on the depth \( z \) only. A plane wave \( u(z, t) \) will satisfy the following equation

\[
(\rho \partial_t^2 - \partial_z((\lambda + 2\mu)\partial_z))u = 0.
\]

The wave speed associated with (1) is

\[
c = \left(\frac{\lambda + 2\mu}{\rho}\right)^{1/2}.
\]

If we regard \( \rho, \lambda, \mu \) as unknown, by imposing initial and boundary conditions, \( u(z, t) = 0 \) for \( t < 0 \) and \( \partial_z u(0, t) = \delta(t) \), for the inverse problem, using the known impulse response \( u(0, t) \), we would like to find \( \rho, \lambda, \mu \). However, it turns out that we cannot find \( \rho, \lambda, \mu \). We make a change of variable, changing the space variable to travel time, in order to pose an inverse problem that we can solve. Define the travel time variable \( x \) by

\[
x = \int_0^z \frac{dz'}{c(z')}.
\]

Then (1) becomes

\[
(\eta(x)\partial_t^2 - \partial_z(\eta(x)\partial_z))u(x, t) = 0
\]

where \( \eta(x) = \rho(z)c(z) \). This equation is known as Webster's Horn Equation. We call the coefficient \( \eta \) the impedance of the medium and \( \eta \) is always positive. Note that in more than one dimension, a similar global change of variable to travel time cannot be made.
Now, we consider the following initial boundary value problem

\begin{align}
(2a) \quad (\eta(x) \partial_t^2 - \partial_x(\eta(x) \partial_x))u(x, t) &= 0 \\
(2b) \quad \partial_x u(0, t) &= -\delta(t), \\
(2c) \quad u(x, t) &= 0, \quad t < 0.
\end{align}

We will normalize \( \eta \) so that \( \eta(0) = 1 \). Observe that by finite speed of propagation, \( u(x, t) = 0 \) for \( x > t \).

For a locally integrable \( f : [a, b] \to \mathbb{R} \), the Sobolev norm of \( f \) is defined by

\[ \|f\|_{H^1[a, b]} = \left( \|f\|_{L^2[a, b]}^2 + \|f'\|_{L^2[a, b]}^2 \right)^{1/2}, \]

which is equivalent to

\[ \|f\|_{L^\infty[a, b]} + \|f'\|_{L^2[a, b]}. \]

Here, the derivatives are taken in the weak sense.

The problem, (2) is well posed in the distributional sense and has a unique distribution solution for \( \eta \in H^1 \) (see [3] or [4]).

Let

\[ f_0(z) = H(z), \quad f_1(z) = zH(z), \ldots, \quad f_k(z) = \frac{1}{k!}z^kH(z), \ldots, \]

where \( H \) is the unit step function. Observe that \( f_k' = f_{k-1} \) for \( k \geq 0 \) and we set \( f_{-1} = \delta \).

The boundary conditions and the progressing wave expansion described in Courant and Hilbert [2] show that \( u \) is of the following form

\[ u(x, t) = \sum_{k=0}^{N} u_k(x) f_k(t - x) + s(x, t), \]

where \( s \) is a solution of (2) as smooth as \( f_{N+1}(t - x) \). Plugging the above expression for \( u \) into the differential equation, we get that the \( u_k \)'s must satisfy

\[ \eta_x((u^{k-1})' - u^k) + \eta((u^{k-1})'' - 2(u^k)') = 0. \]

In particular, using the convention that \( u^{(k)} = 0 \) for \( k < 0 \),

\[ \eta_x u^0 + 2\eta(u^0)' = 0. \]

By (4) and using the boundary conditions,

\[ -\delta(t) = u_x(0, t) = -u^0(0)f_{-1}(t) + (u^0)'(0)f_0(t) + \sum_{k=1}^{N}((u^k)'(0)f_k(t) - u^0(0)f_{k-1}(t)) + s_x(0, t) \]

\[ = -u^0(0)\delta(t) + (u^0)'(0)H(t) + \sum_{k=1}^{N}((u^k)'(0)f_k(t) - u^0(0)f_{k-1}(t)) + s_x(0, t), \]

This forces \( u^0(0) = 1 \) and all other terms to be zero. Combined with (5), \( u^0(x) = \eta^{-1/2}(x) \).

Then using (4) and (3),

\[ \lim_{\tau \to 0^+} u(x, x + \tau) = u^0(x) \lim_{\tau \to 0^+} f_0(\tau) = u^0(x). \]
Thus, the solution $u$ of the initial boundary value problem satisfies
\[
\lim_{\tau \to 0^+} u(x, x + \tau) = \eta^{-1/2}(x)
\]
and is smooth outside the characteristic $\{x = t\}$.

We can rewrite (2) as
\[
(\partial_t^2 - \partial_x^2 - (\partial_x \sigma)\partial_x)u = 0
\]
where $\sigma(x) = \log \eta(x)$. By finite speed of propagation, the boundary values $u(0, t)$ for $0 \leq t \leq 2T$ depend on $\eta(x)$ (or $\sigma(x)$) for $0 \leq x \leq T$ (by similar argument as seen in [2]).

For $\eta \in H^1[0, T]$ with $\eta > 0$, we will now consider solutions to the problem
\[
\begin{align*}
(6a) \quad (\partial_t^2 - \partial_x^2 - (\partial_x \sigma)\partial_x)u &= 0 \quad \text{in} \quad \{(x, t) : 0 \leq x \leq t \leq 2T - x\}, \\
(6b) \quad \lim_{\tau \to 0^+} u(x, x + \tau) &= \eta^{-1/2}(x) \quad 0 \leq x \leq T, \\
(6c) \quad u_x(0, t) &= 0, \quad 0 < t \leq 2T,
\end{align*}
\]
where $\eta$ is a function of $x$ and $u$ is a function of $x, t$. The solution $u$ will be in $H^1$. The inverse problem can be reformulated as follows:

Given $g(t) \in H^1[0, 2T]$, find $u$ and $\eta := e^\sigma$ such that
\[
\begin{align*}
(7a) \quad (\partial_t^2 - \partial_x^2 - (\partial_x \sigma)\partial_x)u &= 0 \quad \text{in} \quad \{(x, t) : 0 \leq x \leq t \leq 2T - x\}, \\
(7b) \quad \lim_{\tau \to 0^+} u(x, x + \tau) &= \eta^{-1/2}(x) \quad 0 \leq x \leq T, \\
(7c) \quad u_x(0, t) &= 0, \quad 0 < t \leq 2T, \\
(7d) \quad u(0, t) &= g(t) \quad 0 \leq t \leq 2T,
\end{align*}
\]
From now on, define $\overline{u}(x) = \lim_{\tau \to 0^+} u(x, x + \tau)$.

**Definition 1.1.** Define the Forward map $F : H^1[0, T] \to H^1[0, 2T]$ by $F(\sigma)(t) = g(t)$ where $\sigma$ and $g$ are from the system (7) above.

The inverse problem has been solved using several methods. By deriving related integral equations and considering the propagation of singularities as a result of the impulsive boundary condition, the inverse problem can be solved (see Bube and Burridge [1] or Symes [5]). Another way to solve the inverse problem is to derive a priori estimates for the log of the impedance, $\sigma$, in terms of the data and use a contraction mapping argument to get that the solution of the inverse problem will correspond to a fixed point of a certain map (Symes [6]).

By using arguments seen in Symes [7], the inverse problem will be solved by examining the Forward map. First, we will derive energy estimates as seen in Symes [7]. These energy estimates will let us show that $F$ is differentiable. Note that if $F$ is differentiable then so is the map from the impedance $\eta$ to the response. Finally, using differentiability, we will apply the Inverse Function theorem to conclude of an existence of a local inverse. That is, we can theoretically solve the inverse problem.
2. Energy Estimates

Let \( \Omega = \{(x,t) : 0 \leq x \leq t \leq 2T - x\} \). Suppose \( u, w \in C^\infty(\mathbb{R}^2) \) and \( v, \psi, \sigma \in C^\infty(\mathbb{R}) \). We will consider energy estimates for solutions of the following problem

\[
\begin{align*}
(8a) & \quad (\partial_t^2 - \partial_x^2 - (\partial_x \sigma) \partial_x) u = vw \quad \text{in} \ \Omega, \\
(8b) & \quad \partial_x u(0,t) = 0, \quad 0 < t \leq 2T, \\
(8c) & \quad u(x) = \psi(x), \quad 0 \leq t \leq T.
\end{align*}
\]

Figure 1. \( \Omega = \{(x,t) : 0 \leq x \leq t \leq 2T - x\} \) is the shaded region above

First, we will derive two identities that will be used in the proofs of the energy estimates. Define the following form

\[
\omega_E = \frac{1}{2} \eta [(\partial_t u)^2 + (\partial_x u)^2] dx + \eta (\partial_x u)(\partial_t u) dt
\]

where \( \eta = e^\sigma \). By Stokes' Theorem,

\[
\int_{\partial \Omega} \omega_E = \int_{\Omega} d\omega_E.
\]
Define \[ \bar{u}(x) = u(x, 2T - x). \]

Multiplying (8a) by \( \eta \) we have \( \eta \partial_x^2 - \eta \partial_t^2 - (\partial_x \eta) \partial_x u = \eta \nu w. \) Calculating this form of the equation, we get

\[
\begin{align*}
    d\omega_E &= \left( \frac{\partial}{\partial x} (\eta \partial_x u \partial_t u) - \frac{\partial}{\partial t} \left( \frac{1}{2} \eta [ (\partial_x u)^2 + (\partial_t u)^2 ] \right) \right) dx \wedge dt \\
    &= \left( (\partial_x \eta)(\partial_x u)(\partial_t u) + \eta (\partial_t^2 u)(\partial_t u) + \eta (\partial_x u)(\partial_x \partial_t u) \right) dx \wedge dt \\
    &\quad - \eta ((\partial_t u)(\partial_t^2 u) + (\partial_x u)(\partial_x \partial_t u)) \\
    &= \eta (x) v(x) w(x, t) \partial_t u(x, t) dt \wedge dx.
\end{align*}
\]

Since \( u \) is smooth, \( \bar{u}(x) = u(x, x) \) and \( \bar{u}'(x) = \partial_t u(x, x) + \partial_x u(x, x). \) Then using that \( \partial_x u(0, t) = 0, \) we have the first identity

\[
\frac{1}{2} \int_0^T \eta (\bar{u})^2 dx - \frac{1}{2} \int_0^T \eta (\bar{u}')^2 dx = \int_{\partial \Omega} \omega_E = \int_{\Omega} \int d\omega_w
\]

\[
= \int_{0}^{T} \int_{x}^{2T-x} \eta(x) v(x) w(x, t) \partial_t u(x, t) dt \ dx.
\]

Now, define the following form

\[ \omega_Q = \frac{1}{2} [(\partial_t u)^2 + (\partial_x u)^2] dt + (\partial_t u)(\partial_x u) dx. \]

For the following argument and lemmas we will define

\[ Q(x) = \frac{1}{2} \int_{x}^{2T-x} ((\partial_t u)^2(x, t) + (\partial_x u)^2(x, t)) dt \]

and \( \Omega_{a,b} = \{(x, t) : x_a \leq x \leq x_b, x \leq t \leq 2T - x\}. \) By Stokes' Theorem,

\[
\int_{\partial \Omega_{a, b}} \omega_Q = \int_{\Omega_{a, b}} d\omega_Q
\]

Calculating using (8) we get

\[
\begin{align*}
    d\omega_Q &= ((\partial_t u)(\partial_x \partial_t u) + (\partial_x u)(\partial_t^2 u)) - ((\partial_t u)(\partial_x^2 u) + (\partial_t u)(\partial_x \partial_t u)) dx \wedge dt \\
    &= -[\nu w + (\partial_x \sigma)(\partial_x u)](\partial_x u)dx \wedge dt.
\end{align*}
\]

Then we arrive at the second identity

\[
\frac{1}{2} \int_{x_a}^{x_b} ((\bar{u})^2 + (\bar{u}')^2) dx + Q(x_b) - Q(x_a) = \int_{\partial \Omega_{a, b}} \omega_Q = \int_{\Omega_{a, b}} d\omega_Q
\]

\[
= -\int_{x_a}^{x_b} \int_{x}^{2T-x} (\partial_x \sigma)(\partial_x u)^2 dt \ dx \\
- \int_{x_a}^{x_b} \int_{x}^{2T-x} \nu w(\partial_x u) dt \ dx.
\]

From now on \( C \) will denote a constant depending on \( \| \sigma \|_{H^1} \) and \( T. \) Also, the constant \( C \) may change from line to line.
Lemma 2.1. Suppose $v = 0$ in (8). Then
\[ Q(x) \leq C \| \psi' \|_2^2 \]
for $0 \leq x \leq T$.

Proof. Letting $x_a = x, x_b = T$ and using that $v = 0$, applying (10) we have
\[ \frac{1}{2} \int_x^T (\psi')^2 \, dz + \frac{1}{2} \int_x^T (\tilde{\psi}')^2 \, dz = Q(x) - \int_x^T \int_z^{2T-x} (\partial_x \sigma)(\partial_x u)^2 \, dt \, dz. \]
By an easy estimation,
\[ Q(x) \leq \frac{1}{2} \int_x^T |\psi'|^2 \, dz + \frac{1}{2} \int_x^T (\tilde{\psi}')^2 \, dz + 2 \int_x^T |\partial_x \sigma| Q(z) \, dz. \]
Let $\alpha(x) = \frac{1}{2} \int_x^T |\psi'|^2 \, dx + \frac{1}{2} \int_x^T (\tilde{\psi}')^2 \, dx$. By Gronwall’s inequality and estimating,
\[ Q(x) \leq \alpha(x) + 2 \int_x^T \alpha(z) |\partial_x \sigma| e^{2 \int_z^T |\partial_x \sigma|} \, dz \]
\[ \leq \left( \frac{1}{2} \| \psi' \|_2^2 + \frac{1}{2} \| \tilde{\psi}' \|_2^2 \right) \left( 1 + 2 \int_x^T |\partial_x \sigma| e^{2 \int_z^T |\partial_x \sigma|} \, dz \right) \]
\[ = \left( \frac{1}{2} \| \psi' \|_2^2 + \frac{1}{2} \| \tilde{\psi}' \|_2^2 \right) \exp \left( 2 \int_0^T |\partial_x \sigma| \, dx \right). \]
Let $M = \sup \eta(x)$ and $m = \inf \eta(x)$. Observe that $M \leq \exp(|\sigma(x)|) \leq \exp(|\sigma|_{H^1})$ and $m \geq \exp(-|\sigma|_{H^1})$. Then by (9),
\[ \frac{M}{2} \| \psi' \|_2^2 \geq \frac{1}{2} \int_0^T \eta(\psi')^2 \, dx = \frac{1}{2} \int_0^T \eta(\tilde{\psi}')^2 \, dx \geq \frac{m}{2} \| \tilde{\psi}' \|_2^2 \]
so $\| \tilde{\psi}' \|_2^2 \leq (M/m) \| \psi' \|_2^2$. Therefore,
\[ Q(x) \leq C \| \psi' \|_2^2. \]

Lemma 2.2. Suppose $\psi = 0$. Then
\[ Q(x) \leq CT \| v \|_2^2 w^* \]
where
\[ w^* = \sup_{0 \leq x \leq T} \left( \int_x^{2T-x} w^2(x, t) \, dt \right). \]

Proof. By (10),
\[ Q(x) = \frac{1}{2} \int_x^T (\tilde{\psi}')^2 \, dz + \int_x^T \int_z^{2T-x} (\partial_x \sigma)(\partial_x u)^2 \, dt \, dz + \int_x^T \int_z^{2T-x} \nu w(\partial_x u) \, dt \, dz \]
and so by estimating,
\[ Q(x) \leq \frac{1}{2} \int_x^T (\tilde{\psi}')^2 \, dz + \int_x^T \int_z^{2T-x} \left| (\partial_x \sigma)(\partial_x u)^2 + |v||w||\partial_x u| \right| \, dt \, dz. \]
By applying (9),
\[ -\frac{1}{2} \int_0^T \eta(u)^2 \, dx = \int_0^T \int_x^{2T-x} \eta w \partial_t u \, dt \, dz. \]
Then using the notation of the proof of the previous lemma,
\[ \frac{m}{2} \int_0^T (\tilde{u})^2 \, dx \leq M \int_0^T \int_x^{2T-x} |v||w||\partial_t u| \, dt \, dz. \]
For all \( \alpha > 0 \),
\[ \int_x^{2T-x} |w||\partial_x u| \, dt = \int_x^{2T-x} (\alpha|w|) \left( \frac{1}{\alpha} |\partial_x u| \right) \, dt \]
\[ \leq \frac{1}{2} \int_x^{2T-x} (\alpha^2|w|^2 + \alpha^{-1}|\partial_x u|^2) \, dt \]
\[ \leq \frac{\alpha^2 w^*}{2} + \frac{\alpha^{-1}}{2} \int_x^{2T-x} |\partial_x u|^2 \, dt \]
and similarly
\[ \int_x^{2T-x} |w||\partial_x u| \, dt \leq \frac{\alpha^2 w^*}{2} + \frac{\alpha^{-2}}{2} \int_x^{2T-x} |\partial_x u|^2 \, dt. \]
Let \( Q^* = \sup_{0 \leq x \leq T} Q(x) \). Then using (12) and (13),
\[ \frac{1}{2} \int_x^T \tilde{u} \, dz \leq \frac{M}{m} \int_0^T \int_t^T \left( \frac{\alpha^2 w^*}{2} + \alpha^{-2} Q^* \right) \, dz. \]
By (14),
\[ \int_x^T \int_x^T \left( |\partial_x \sigma(z)(\partial_x u)^2 + |v||w||\partial_x u| \right) \, dt \, dz \leq 2 \int_0^T |\partial_x \sigma|Q \]
\[ + \int_0^T |v| \left( \frac{\alpha^2 w^*}{2} + \alpha^{-2} Q^* \right) \, dz. \]
Therefore by (11),
\[ Q(x) \leq \left( 1 + \frac{M}{m} \right) \int_0^T |v(z)| \left( \frac{\alpha^2 w^*}{2} + \alpha^{-2} Q^* \right) \, dz + 2 \int_0^T |\partial_x \sigma|Q \, dz. \]
By Gronwall’s inequality,
\[ Q \leq C \left( \frac{\alpha^2 w^*}{2} + \alpha^{-2} Q^* \right) \left( \int_0^T |v(z)| \, dz \right) \exp \left( 2 \int_0^T |\partial_x \sigma| \, dx \right). \]
If \( v = 0 \) almost everywhere, \( u = 0 \) since \( \psi = 0 \). By setting
\[ \alpha^2 = 2 \left( \int_0^T |v| \, dx \right) \exp \left( 2 \int_0^T |\partial_x \sigma| \, dx \right). \]
Thus, we can simplify (15) and use Holder’s inequality to get
\[ Q \leq C \left( \int_0^T |v| \, dx \right)^2 w^* \exp \left( 4 \int_0^T |\partial_x \sigma| \, dx \right) + \frac{1}{2} Q^* \]
\[ \leq CT w^* |v|^2 + \frac{1}{2} Q^*. \]
The result follows. \( \square \)
Lemma 2.3. The solution $u$ of (8) satisfies
\[ Q(x) \leq C \left( \| \psi' \|_2^2 + \| v \|_2^2 w^* \right). \]

Proof. Write $u$ as $u = u_1 + u_2$ where $u_1$ solves (8) with $v = 0$ and $u_2$ solves (8) with $\psi = 0$ with corresponding energy $Q_1, Q_2$. Then using that
\[
\int_{x}^{2T-x} 2(\partial_x u_1)(\partial_x u_2)dt \leq \int_{x}^{2T-x} (\partial_x u_1)^2 + (\partial_x u_2)^2 dt,
\]
we have
\[ Q(x) \leq 2(Q_1(x) + Q_2(x)) \]
By applying Lemma 2.1 and Lemma 2.2, we get the result. \qed

Lemma 2.4. Suppose there exists a constant $k > 0$ such that for $0 \leq x \leq T$,
\[ \int_{0}^{x} |v(z)|^2 dz \leq k \int_{0}^{x} |\psi'(z)|^2 dz. \]
Then there exists a constant $C > 0$ such that
\[ \| \psi' \|_2^2 \leq CQ(0). \]

Proof. By estimating we get
\[
Q(x) + \frac{1}{2} \int_{0}^{x} (\psi'(z))^2 dz \leq Q(0) + \int_{0}^{x} \int_{z}^{2T-x} |\partial_x \sigma(z)| (\partial_x u)^2 (z, t) dt dz
\]
\[ + \int_{0}^{x} \int_{z}^{2T-x} v(z) w(z, t) \partial_x u(z, t) dt dz \quad \text{(by (10))} \]
\[ \leq Q(0) + \int_{0}^{x} \partial_x \sigma(z) Q(z) dz \]
\[ + w^* \int_{0}^{x} |v(z)| \left( \int_{z}^{2T-x} (\partial_x u)^2 (z, t) dt \right)^{\frac{1}{2}} dz \quad \text{(by Holder)} \]
\[ \leq Q(0) + \delta w^* \int_{0}^{x} v^2(z) dz + \int_{0}^{x} \left( \partial_x \sigma(z) + \frac{w^*}{2\delta} \right) Q(z) dz \]
for any $\delta > 0$, (for all $a, b, (\sqrt{3a})(b/\sqrt{\delta}) \leq \delta a^2 + b^2/\delta$). By choosing $\delta$ small enough so that $\delta w^* k < \frac{1}{2}$ and by using the hypothesis,
\[
Q(x) + C \int_{0}^{x} (\psi'(z))^2 dz \leq Q(0) + \int_{0}^{x} \left( \partial_x \sigma(z) + \frac{w^*}{2\delta} \right) \left( Q(z) + C \int_{0}^{x} (\psi'(y))^2 dy \right) dz \]
where $C > 0$. Then by Gronwall's inequality,
\[
C \int_{0}^{x} (\psi'(z))^2 dz \leq Q(x) + C \int_{0}^{x} (\psi'(z))^2 dz \leq Q(0) \cdot \exp \left( \int_{0}^{x} \left( \partial_x \sigma(z) + \frac{w^*}{2\delta} \right) dz \right) .
\]
The conclusion follows. \qed
Proposition 2.5. Let \( \sigma, \eta, u \) be smooth and satisfy (6). Define
\[
Q(x) = \frac{1}{2} \int_{x}^{2T-x} ((\partial_t u)^2(x, t) + (\partial_x u)^2(x, t)) \, dt
\]
Then
\[
Q(x) \leq C \|\partial_x \sigma\|_2^2.
\]

Proof. Observe that
\[
\|\partial_x \eta^{-1/2}\|_2 = \frac{1}{\sqrt{2}} \| (\partial_x \sigma) e^{-\sigma/2} \|_2 \leq C \|\partial_x \sigma\|_2.
\]
By applying Lemma 2.3, we get the result. \(\square\)

Proposition 2.6. Let \( u_i \) and \( \sigma_i \) be smooth and satisfy (6), \( i = 1, 2 \). Let \( v = u_2 - u_1 \) and define
\[
Q(x) = \frac{1}{2} \int_{x}^{2T-x} ((\partial_t v)^2(x, t) + (\partial_x v)^2(x, t)) \, dt.
\]
Then for \( 0 \leq x \leq T \),
\[
Q(x) \leq C \|\partial_x (\sigma_1 - \sigma_2)\|_2^2.
\]

Proof. The function \( v \) satisfies the following boundary value problem
\[
(\partial_t^2 - \partial_x^2 - (\partial_x \sigma_1) \partial_x) v = (\partial_x (\sigma_1 - \sigma_2)) \partial_x u_2 \quad \text{in} \ \Omega,
\]
\[
\partial_x v(0, t) = 0, \quad 0 \leq t \leq 2T,
\]
\[
\bar{v}(x) = \eta_2^{-1/2}(x) - \eta_1^{-1/2}(x), \quad 0 \leq x \leq T,
\]
where \( \eta_i = e^{\sigma_i} \) for \( i = 1, 2 \).

Observe that
\[
\eta_2^{-1/2}(x) - \eta_1^{-1/2}(x) = \eta_2^{-1/2}(1 - \exp \left( -\frac{1}{2}(\sigma_2 - \sigma_1) \right)).
\]
Then
\[
\|\partial_x (\eta_2^{-1/2} - \eta_1^{-1/2})\|_2 \leq \|\partial_x \eta_2^{-1/2}\|_2 \|1 - \exp \left( -\frac{1}{2}(\sigma_2 - \sigma_1) \right)\|_\infty
\]
\[
\leq \frac{1}{2} \|\partial_x (\sigma_2 - \sigma_1)\|_2 \|\eta_2^{-1/2} \exp \left( -\frac{1}{2}(\sigma_2 - \sigma_1) \right)\|_\infty.
\]
Since \( |1 - e^x| \leq |x| e^{|x|} \) for all \( x \), \( (\sigma_2 - \sigma_1)(x) = \int_0^x \partial_x (\sigma_2 - \sigma_1)(z) \, dz \), Cauchy-Schwarz implies
\[
\|1 - \exp \left( -\frac{1}{2}(\sigma_2 - \sigma_1) \right)\|_\infty \leq \frac{1}{2} \|\sigma_2 - \sigma_1\|_\infty \|\sigma_2 - \sigma_1\|_\infty
\]
\[
\leq C \|\partial_x (\sigma_2 - \sigma_1)\|_2 \|\exp \left( -\frac{1}{2}|\sigma_2 - \sigma_1| \right)\|_\infty.
\]
Then using that \( \|\partial_x \eta_i^{-1/2}\|_2 \leq C \|\partial_x \sigma_i\|_2 \) for \( i = 1, 2 \),
\[
\|\partial_x (\eta_2^{-1/2} - \eta_1^{-1/2})\|_2 \leq C \|\partial_x (\sigma_2 - \sigma_1)\|_2,
\]
where the constant depends on $\sigma_1, \sigma_2$. By Proposition 2.5,
\[ \int_x^{2T-x} |\partial_x u_2|^2 \leq C\|\partial_x \sigma_2\|_2^2. \]

By applying Lemma 2.3, we get the result. □

3. DIFFERENTIABILITY

Lemma 3.1. The forward map $F : H^1[0, T] \to H^1[0, 2T]$ is continuous.

Proof. We show the result first for smooth functions. Let $\epsilon > 0$. By Proposition 2.5, for $\|\sigma_1 - \sigma_2\|_{H^1[0,T]} < \min(\epsilon/(2\sqrt{2C}), \epsilon/(4\sqrt{CT}))$,
\[ 2Q(x) = \int_x^{2T-x} ((\partial_x v)^2(x,t) + (\partial_t v)^2(x,t)) dt \leq 2C\|\partial_x (\sigma_1 - \sigma_2)\|_2^2 \leq \frac{\epsilon^2}{4}, \frac{\epsilon^2}{8T}, \]
where $v = u_1 - u_2$ with $u_1, u_2$ being solutions of (6) corresponding to $\sigma_1, \sigma_2$. In particular,
\[ ||(F(\sigma_1) - F(\sigma_2))'||_{L^2[0,2T]} = \sqrt{2Q(0)} < \frac{\epsilon}{2}. \]

Also, for every $0 \leq t \leq 2T$, by Cauchy-Schwarz,
\[ |F(\sigma_1)(t) - F(\sigma_2)(t)| = |v(0,t)| = \int_0^t \partial_t v(0,\tau) d\tau \leq \sqrt{t}||\partial_t v(0,\cdot)||_2 \leq \sqrt{4TQ(0)} < \frac{\epsilon}{2}. \]

Therefore, $||F(\sigma_1) - F(\sigma_2)||_{H^1[0,2T]} < \epsilon$. In fact, we have established that $F$ on smooth functions is locally uniformly continuous. Combined with denseness of smooth functions in $H^1$, we can extend $F$ to $H^1[0,T]$ such that $F$ is continuous as a map on $H^1[0,T]$. □

Let $\sigma_0 \in H^1[0,T]$ with $\eta_0 = e^{\sigma_0}$ and let $u_0$ be the corresponding solution to (6). Define a map $J_{\sigma_0} : H^1[0, T] \to H^1[0, 2T]$ by $J_{\sigma_0}(\delta \sigma) = \delta u(0,t)$ where $\delta u$ solves
\[ (16a) \quad (\partial_t^2 - \partial_x^2 - \partial_x (\sigma_0) \partial_x) \delta u = (\partial_x (\delta \sigma)(\partial_x u_0), \]
\[ (16b) \quad \delta u(x) = -\frac{\delta \sigma}{2}\frac{\eta_0}{\eta_0^{1/2}}, \quad 0 \leq t \leq 2T, \]
\[ (16c) \quad (\delta u)_x(0,t) = 0, \quad 0 < t < 2T. \]

Theorem 3.2. The map $J_{\sigma_0}$ is a bounded linear operator.

Proof. Linearity is obvious. Assume $\delta \sigma \in H^1[0,T]$ is smooth and let $||\delta \sigma||_{H^1} = 1$. Then $||\delta \sigma||_\infty, ||\partial_x \delta \sigma||_2$ are bounded. Estimating we get
\[ \left\| \frac{d}{dx} \delta u(x) \right\|_2 = \left\| \frac{\partial_x \delta \sigma}{2} \frac{\eta_0}{\eta_0^{1/2}} + \frac{\delta \sigma}{4} (\partial_x \sigma_0) \eta_0^{1/2} \right\|_2 \]
\[ \leq \left\| \frac{\partial_x \delta \sigma}{2} \right\|_2 \left\| \frac{\eta_0}{\eta_0^{1/2}} \right\|_\infty + \left\| \frac{\delta \sigma}{4} \right\|_\infty \left\| \partial_x \sigma_0 \right\|_2 \left\| \frac{\eta_0}{\eta_0^{1/2}} \right\|_\infty \]
\[ \leq C < \infty \]

for every $\delta \sigma \in H^1[0,T]$ with $||\delta \sigma||_{H^1} = 1$ where the constant $C$ depends only on $||\sigma_0||_{H^1}$. Since $u_0 \in H^1, ||\partial_x u_0||_2 < \infty$. By applying Lemma 2.3, we get that sup{||$\partial_t \delta u(0,\cdot)$||_2 : $||\delta \sigma||_{H^1} = 1$} < \infty.
For every $t, 0 \leq t \leq 2T$,
\[
|\delta u(0,t)| = |\delta u(0,0) + \int_0^t \partial_t(\delta u(0,\tau))d\tau| \leq \frac{1}{2} |\delta \sigma(0)| + \sqrt{2T} ||\partial_t \delta u(0,\cdot)||_2.
\]
Using this, we get that sup\{||\delta u(0,\cdot)||_\infty : ||\delta \sigma||_{H^1} = 1\} < \infty. Therefore, sup\{||\delta u(0,\cdot)||_{H^1[0,2T]} : ||\delta \sigma||_{H^1} = 1\} < \infty and so $J_{\delta \sigma}$ is bounded.

**Theorem 3.3.** Let $\sigma_0 \in H^1[0,T]$. Then for $\sigma^1 \in H^1[0,T]$ with $||\sigma^1||_{H^1} = 1$,
\[
\left| J_{\delta \sigma}(\sigma^1) - \frac{1}{\varepsilon}[F(\sigma_0 + \varepsilon \sigma^1) - F(\sigma_0)] \right|_{H^1[0,2T]} \to 0
\]
as $\varepsilon \to 0$.

**Proof.** First, assume $\sigma_0, \sigma^1$ are smooth. Let $\sigma = \sigma_0 + \varepsilon \sigma^1$, $\eta(x) = \exp(\sigma(x))$, and $\eta_0(x) = \exp(\sigma_0(x))$. Let $u_0, u$ solve (6) with $\sigma_0, \sigma$ respectively, and let $u^1$ solve (16) with $\delta \sigma$ replaced by $\sigma^1$. We will first show $||\partial_t u(0,\cdot)||_2 = O(\varepsilon)$ where $v = u^1 - \frac{1}{\varepsilon}(u - u_0)$. Observe that $v$ satisfies the following problem
\begin{align}
(17a) \quad & (\partial_t^2 - \partial_x^2 - \partial_x(\sigma_0)\partial_x)u = (\partial_x \sigma^1)(\partial_x(u_0 - u)), \\
(17b) \quad & \bar{v}(x) = -\frac{\sigma^1}{2} \eta_0^{-1/2}(x) - \frac{1}{\varepsilon}(\eta^{-1/2}(x) - \eta_0^{-1/2}(x)), \quad 0 \leq t \leq 2T, \\
(17c) \quad & v_x(0,t) = 0, \quad 0 < t \leq 2T.
\end{align}

By Taylor expansion,
\[
\eta^{-1/2}(x) - \eta_0^{-1/2}(x) = \eta_0^{-1/2}(\exp(-\varepsilon \sigma^1(x)/2) - 1) = -\frac{\varepsilon \sigma^1(x)}{2} \eta_0^{-1/2}(x) + r(\varepsilon, x).
\]
Since $||\sigma^1||_{H^1} = 1$, $\frac{1}{2}r(\varepsilon, x)$ is uniformly bounded in $\varepsilon$ by some function of $||\sigma_0||_{H^1}$ and $T$ by Taylor’s Theorem. Then
\[
\frac{d}{dx} r(\varepsilon, x) = \left( \frac{d}{dx} \eta_0^{-1/2}(x) \eta_0^{-1/2}(x) \right) \left( \exp \left( -\frac{\varepsilon \sigma^1(x)}{2} \right) - 1 + \frac{\varepsilon \sigma^1(x)}{2} \right)
+ \eta_0^{-1/2}(x) \left( \frac{\varepsilon}{2} \partial_x \sigma^1 \right) \left( 1 - \exp \left( -\frac{\varepsilon \sigma^1(x)}{2} \right) \right).
\]
Estimating, we get
\[
\left| \frac{d}{dx} r(\varepsilon, x) \right| \leq \left| \frac{d}{dx} \eta_0^{-1/2}(x) \right| \left| \exp \left( -\frac{\varepsilon \sigma^1(x)}{2} \right) - 1 + \frac{\varepsilon \sigma^1(x)}{2} \right|_{\infty}
+ \left| \eta_0^{-1/2} \right| \left| \frac{\varepsilon}{2} \partial_x \sigma^1 \left( 1 - \exp \left( -\frac{\varepsilon \sigma^1(x)}{2} \right) \right) \right|_{\infty}.
\]
Then using that $\left| (d/dx) \eta_0^{-1/2}(x) \right| \leq C ||\partial_x \sigma_0||_{\infty}, \exp(-\varepsilon \sigma^1/2) - 1 + \varepsilon \sigma^1/2 = O(\varepsilon^2), \exp(-\varepsilon \sigma^1/2) - 1 = O(\varepsilon)$, and $||\sigma^1||_{H^1} = 1$,
\[
(19) \quad \left| \frac{d}{dx} r \right|_2 = O(\varepsilon^2),
\]
where the constant in the $O(\varepsilon^2)$ term depends on $||\sigma_0||_{H^1}$ and $T$. By (17b), (18), (19),
\[
(20) \quad \left| \partial_x \bar{v} \right|_2 = O(\varepsilon).
\]
Let \( w = \partial_x (u_0 - u) \). Then by Proposition 2.5,
\[
(21) \quad \int_0^{2T} w(x, t)^2 dt = O(\varepsilon^2).
\]
Therefore by applying Lemma 2.3, (20), and (21),
\[
(22) \quad \int_0^{2T} (\partial_x v)^2(0, t) dt \leq C \left( \| \partial_x \bar{v} \|^2_2 + \| \sigma^1 \|^2_2 \right) \int_0^{2T} w(x, t)^2 dt = O(\varepsilon^2),
\]
so
\[
\| \partial_x v(0, \cdot) \|^2_2 = O(\varepsilon).
\]
Note that by Cauchy-Schwarz
\[
\| v(0, \cdot) \|^2_\infty \leq |v(0, 0)| + \sqrt{2T} \| \partial_t v(0, \cdot) \|^2_2.
\]
By (18) and (17b), \( v(0, 0) = O(\varepsilon) \). Then
\[
\| v(0, \cdot) \|^2_\infty = O(\varepsilon),
\]
and so \( \| v(0, \cdot) \|^2_{H^1} = O(\varepsilon) \). Now, by Lemma 3.1 and Theorem 3.2, we can extend the result to \( \sigma, \sigma^1 \in H^1[0, T] \). \( \square \)

**Definition 3.4.** Let \( U, V \) be Banach Spaces. A function \( F : U \rightarrow V \) is Frechet differentiable on \( U \) if for every \( x \in U \), there exists a bounded linear operator \( J_x : U \rightarrow V \) such that
\[
\lim_{\delta x \rightarrow 0} \frac{\| F(x + \delta x) - F(x) - J_x(\delta x) \|_V}{\| \delta x \|_U} = 0.
\]

**Corollary 3.5.** The map \( F \) is Frechet differentiable.

**Proof.** For \( \delta \sigma \in H^1[0, T] \) with \( \| \delta \sigma \|_{H^1} \) small, let \( \delta \sigma = \varepsilon \sigma^1 \), where \( \varepsilon = \| \delta \sigma \|_{H^1} \) and \( \sigma^1 = (\delta \sigma)/\| \delta \sigma \|_{H^1} \). Then by the previous theorem
\[
\frac{\| F(\sigma_0 + \delta \sigma) - F(\sigma_0) - J_{\sigma_0}(\delta \sigma) \|_{H^1}}{\| \delta \sigma \|_{H^1}} = \frac{\| J_{\sigma_0}(\sigma^1) - \frac{1}{\varepsilon} [F(\sigma_0 + \varepsilon \sigma^1) - F(\sigma_0)] \|_{H^1[0, 2T]}}{\| \delta \sigma \|_{H^1}}
\]
\[
\rightarrow 0 \quad \text{as} \quad \varepsilon \rightarrow 0. \quad \square
\]

Note that the derivative of \( F \) is then map \( DF : H^1[0, T] \rightarrow H^1[0, 2T] \) defined by \( DF(\sigma) = J_\sigma \).

**Theorem 3.6.** The forward map \( F \) is continuously differentiable

**Proof.** By the previous corollary, \( F \) is differentiable. Fix a \( \sigma_0, \sigma_1 \in C^\infty[0, T] \) and a bounded set \( U \in H^1[0, T] \) containing \( \sigma_0 \) and \( \sigma_1 \). Let \( u_0, u_1 \) be solutions of (6) corresponding to \( \sigma_0, \sigma_1 \). Let \( \delta \sigma \in C^\infty[0, T] \) and \( w_0, w_1 \) be corresponding solutions to (16) with \( \sigma_0, \sigma_1 \). Note that by definition, \( DF(\sigma_0)(\delta \sigma) = w_0(0, t) \) and \( DF(\sigma_1)(\delta \sigma) = w_1(0, t) \). Then the difference \( v := w_0 - w_1 \) solves
\[
\begin{align*}
(23a) \quad & (\partial_t^2 - \partial_x^2 - \partial_x(\sigma_0) \partial_x) v = (\partial_x \delta \sigma)(\partial_x (u_0 - u_1)) + (\partial_x (\sigma_0 - \sigma_1))(\partial_x w_1) \quad \text{in} \quad \Omega, \\
(23b) \quad & \overline{v}(x) = \frac{\delta \sigma}{2}(\eta_1^{-1/2} - \eta_0^{-1/2}) \quad 0 \leq t \leq 2T, \\
(23c) \quad & v_x(0, t) = 0 \quad 0 < t \leq 2T,
\end{align*}
\]
where \( \eta_0 = e^{\sigma_0}, \eta_1 = e^{\sigma_1} \). By Proposition 2.5, for every \( x \),
\[
\int_{x}^{2T-x} (\partial_x(u_0 - u_1))^2(x, t) dt \leq C\|\partial_x(\sigma_0 - \sigma_1)\|_2^2.
\]
By an argument similar to that in Theorem 3.2,
\[
\int_{x}^{2T-x} (\partial_x w_1)^2(x, t) dt \leq C\|\delta \sigma\|_{H^1},
\]
where the constant \( C \) is dependent on \( \|\sigma_1\|_{H^1} \). Also, it can be shown, using an argument similar to the proof of Prop. 2.6, that
\[
\|\partial_x[\delta \sigma(\eta_1^{-1/2} - \eta_0^{-1/2})]\|_2 \leq \|\partial_x(\delta \sigma)\|_2\|\eta_1^{-1/2} - \eta_0^{-1/2}\|_\infty + \|\delta \sigma\|_\infty \|\partial_x(\eta_1^{-1/2} - \eta_0^{-1/2})\|_2 \\
\leq C\|\partial_x(\sigma_0 - \sigma_1)\|_{H^1}\|\delta \sigma\|_{H^1}.
\]
A similar result to Lemma 2.3 will hold when the right side of equation 7(a) is replaced by \( v_1 w_1 + v_2 w_2 \) where \( v_1 \in C^\infty \) and \( w_i \in C^\infty(\mathbb{R}^2) \). Then by this extension of Lemma 2.3 and the previous three inequalities,
\[
\|DF(\sigma_0)(\delta \sigma) - DF(\sigma_1)(\delta \sigma)\|_2 = \|\partial_t v(0, \cdot)\|_2 \leq C\|\sigma_0 - \sigma_1\|_{H^1}\|\delta \sigma\|_{H^1}.
\]
Here the constant \( C \) depends only on the bounded set \( U \) and not directly on \( \sigma_0, \sigma_1 \). By using Cauchy-Schwarz as in the previous proofs, we can control \( \|v(0, \cdot)\|_\infty \) just like \( \|\partial_t v(0, \cdot)\|_2 \).
Then
\[
\|DF(\sigma_0) - DF(\sigma_1)\|_{L((H^1), H^1)} \leq C\|\sigma_0 - \sigma_1\|_{H^1}.
\]
Thus, \( DF \) is continuous. □

**Lemma 3.7.** For \( \sigma \in H^1[0, T] \),
\[
\|\partial_x(\delta \sigma)\|_2 \leq C\|\partial_t(DF(\sigma)(\delta \sigma))\|_2
\]
and the range of \( DF(\sigma) \) is closed in \( H^1 \).

**Proof.** By definition, \( DF(\sigma)(\delta \sigma) = \delta u(0, t) \) where \( \delta u \) solves (16). By an easy calculation,
\[
(24) \quad -\frac{1}{2} \eta^{-1/2} \partial_x(\delta \sigma) = \partial_x \overline{\delta u} - \partial_x(\sigma) \overline{\delta u}.
\]
By using a Poincaré's inequality type argument,
\[
\int_0^x (\overline{\delta u}(z))^2 dz \leq C \int_0^x (\partial_x \overline{\delta u}(z))^2 dz.
\]
Then estimating using (24),
\[
\int_0^x (\partial_x(\delta \sigma))^2(z) dz \leq C \int_0^x (\partial_x \overline{\delta u}(z))^2 dz.
\]
Then by Lemma 2.4,
\[
\|\partial_x(\delta \sigma)\|_2 \leq C\|\partial_x \overline{\delta u}\|_2 \leq C'Q(0) = C'\|\partial_t(\delta u)(0, \cdot)\|_2 = C'\|\partial_t(DF(\sigma)(\delta \sigma))\|_2.
\]
By linearity of \( DF(\sigma) \) and the inequality above, we get that the range of \( DF(\sigma) \) is closed. □
Corollary 3.8. For \( \sigma \in H^1[0,T] \), \( DF(\sigma) \) is injective.

Proof. Suppose \( DF(\sigma)(\tau) = 0 \). More precisely this implies that \( u_{\tau}(0,t) = 0 \) for \( 0 \leq t \leq 2T \), where \( u_{\tau} \) solves

\begin{align*}
(25a) & \quad (\partial^2_t - \partial^2_x - \partial_x(\sigma) \partial_x) u_{\tau} = (\partial_x \tau)(\partial_x u) \quad \text{in } \Omega, \\
(25b) & \quad \bar{u}_{\tau}(x) = -\frac{\tau}{2} \eta^{-1/2}, \quad 0 \leq t \leq 2T; \\
(25c) & \quad (u_{\tau})_x(0,t) = 0, \quad 0 < t \leq 2T,
\end{align*}

where \( \eta = e^\sigma \) and \( u \) is a solution of (6).

By the previous lemma, \( \partial_x \tau = 0 \). Therefore, by finite speed of propagation, \( u_{\tau} = 0 \), which implies that \( \tau = 0 \), and we conclude that \( DF(\sigma) \) is injective. \( \square \)

Lemma 3.9. For \( \sigma \in H^1[0,T] \), \( DF(\sigma) = J_\sigma \) is surjective.

Proof. Let \( f \in H^1[0,2T] \). We want to find \( \tau \in H^1[0,T] \) such that \( DF(\sigma)(\tau) = f \). That is, we want to find \( \tau \) such that

\begin{align*}
(26a) & \quad (\partial^2_t - \partial^2_x - \partial_x(\sigma) \partial_x) v = (\partial_x \tau)(\partial_x u) \quad \text{in } \Omega, \\
(26b) & \quad \bar{v}(x) = -\frac{\tau}{2} \eta^{-1/2}, \quad 0 \leq t \leq 2T, \\
(26c) & \quad v_x(0,t) = 0, \quad 0 < t \leq 2T, \\
(26d) & \quad v(0,t) = f, \quad 0 < t \leq 2T,
\end{align*}

where \( \eta = e^\sigma \) and \( u \) is a solution of (6).

Let \( 0 < T_1 \leq T \). For \( \tau \in H^1[0,T_1] \), define \( S\tau \) by

\[ S\tau(x) = -2\eta^{1/2}(x)\bar{v}(x), \]

where \( \bar{v} \) solves

\[ \begin{align*}
(\partial^2_t - \partial^2_x - \partial_x(\sigma) \partial_x) v &= (\partial_x \tau)(\partial_x u) \quad \text{in } \{0 \leq x \leq T_1, x \leq t \leq T_1 - x\}, \\
v_x(0,t) &= 0, \quad 0 < t \leq 2T_1, \\
v(0,t) &= f, \quad 0 < t \leq 2T_1.
\end{align*} \]

A fixed point of \( S \) will give us \( \tau(x) \) on \([0,T_1]\). First, we will show \( S \) maps into \( H^1[0,T_1] \).

By the energy identity (10),

\[ \frac{1}{2} \int_0^{T_1} (\bar{v}')^2 dx \leq Q(T_1) + \frac{1}{2} \int_0^{T_1} (\bar{v})^2 dx \]

\[ \leq Q(0) - \int_0^{T_1} \int_x^{2T-x} (\partial_x \sigma)(\partial_x v)^2 dt dx - \int_x^{2T-x} (\partial_x \tau)(\partial_x u)(\partial_x v) dt dx. \]

Then by argument similar to that found in the proof of Lemma 2.4,

\[ \int_0^{T_1} (\bar{v}')^2 dx \leq C(\|f\|_{H^1} + \|\tau\|_{H^1}), \]

where the constant is dependent on \( \sigma, u, T_1 \). Thus, \( S \) maps \( H^1[0,T_1] \) into \( H^1[0,T_1] \). Now, our goal is to show that \( S \) is a contraction map for \( T_1 \) small. Let \( \tau_1, \tau_2 \in H^1[0,T_1] \). Then
\[ S(\tau_1 - \tau_2) = -2\tau^{-1/2}u_{\tau_1} - u_{\tau_2}, \] where \( u_{\tau_1} \) and \( u_{\tau_2} \) are solutions corresponding to \( \tau_1 \) and \( \tau_2 \). Then \( w := u_{\tau_1} - u_{\tau_2} \) solves

\[
\begin{align*}
(\partial_t^2 - \partial_x^2 - (\partial_x \sigma)(\partial_x))w &= (\partial_x(\tau_1 - \tau_2))(\partial_x u), \\
w_x(0, t) &= 0, \quad 0 < t \leq 2T_1, \\
w(0, t) &= 0, \quad 0 \leq t \leq 2T_1.
\end{align*}
\]

Define \( u^*(T) = \sup_{0 \leq z \leq T} \int_{x}^{2T-x} (\partial_x u(z, t))^2 dz \). Estimating using a similar argument as in the proof of Lemma 2.4, for \( 0 \leq z \leq T_1 \),

\[
Q(z) + \frac{1}{2} \int_{0}^{z} (\bar{w}'(x))^2 dx \leq u^*(T_1) \int_{0}^{z} (\partial_x(\tau_1 - \tau_2))^2(x) dx + \int_{0}^{z} \left( \partial_x \sigma(x) + \frac{u^*(T_1)}{2} \right) (Q(x) + \frac{1}{2} \int_{0}^{z} (\bar{w}')^2 dx) dx,
\]

so by Gronwall's inequality,

\[
\frac{1}{2} \int_{0}^{T_1} (\bar{w}')^2(x) dx \leq u^*(T_1) \left( \int_{0}^{T_1} (\partial_x(\tau_1 - \tau_2))^2(x) dx \right) \exp \left( \int_{0}^{T_1} \left( \partial_x \sigma(x) + \frac{u^*(T_1)}{2} \right) dx \right) \leq u^*(T_1) \exp \left( \int_{0}^{T_1} \left( \partial_x \sigma(x) + \frac{u^*(T_1)}{2} \right) dx \right) \| \tau_1 - \tau_2 \|_{H^1[0, T_1]}.
\]

Let \( M_{T_1} = 2u^*(T_1) \exp \left( \int_{0}^{T_1} \left( \partial_x \sigma(x) + \frac{u^*(T_1)}{2} \right) dx \right) \) so \( \| \bar{w}' \|_2 \leq \sqrt{2MT_1} \| \tau_1 - \tau_2 \|_{H^1[0, T_1]} \). By Cauchy-Schwarz and the fact that \( \bar{w}(0) = \lim_{t \to 0^+} w(0, t) = 0 \) by the initial conditions, \( \| \bar{w} \|_\infty \leq \sqrt{T_1} \| \bar{w}' \|_2 \). Hence,

\[
\| S\tau_1 - S\tau_2 \|_{H^1[0, T_1]} \leq C(\sqrt{2M_{T_1}} + \sqrt{2T_1M_{T_1}}) \| \tau_1 - \tau_2 \|_{H^1[0, T_1]},
\]

where the constant \( C \) depends only on \( \sigma \) and \( f \). Then for \( T_1 \) small enough, \( S \) is a contraction mapping on \( H^1[0, T_1] \). The choice of \( T_1 \) is determined using only \( \sigma \) and \( u \). Then \( S \) has a fixed point by the Banach Fixed Point theorem. Thus, we have constructed \( \tau, v \) on \([0, T_1] \). We can repeat the argument above to extend \( \tau, v \) to \([T_1, 2T_1] \) so after finitely many steps we have \( \tau, v \) on \([0, T] \). Therefore, \( DF(\sigma) \) is surjective. \( \square \)

**Lemma 3.10.** Let \( \sigma \in H^1[0, T] \). Then \( DF(\sigma) \) is a linear isomorphism.

**Proof.** By Theorem 2.2, \( DF(\sigma) \) is a bounded linear operator. By Corollary 2.8, \( DF(\sigma) \) is injective. By the previous lemma, \( DF(\sigma) \) is also surjective.

Then by the Bounded Inverse Theorem, the inverse of \( DF(\sigma) \) is bounded. Therefore, \( DF(\sigma) \) is a linear isomorphism. \( \square \)

**Theorem 3.11.** (Inverse Function Theorem) Let \( X, Y \) be Banach spaces and \( U \subset X \) be an open set. Let \( F : U \to Y \) be continuously differentiable. Suppose there exists \( \bar{x} \) such that \( DF(\bar{x}) : X \to Y \) is a linear isomorphism. Then there exist neighborhoods \( V \subset U \) of \( \bar{x} \) and \( W \subset Y \) of \( F(\bar{x}) \) along with a continuously differentiable function \( g : W \to V \) such that \( f(g(x)) = x \) for \( x \in W \).
Thus, we can apply the inverse function theorem to get a local inverse of the forward map $F$ and solve the inverse problem locally for any impulse response in the range. In addition, the inverse map, bringing impulse response to impedance $\eta$, is continuously differentiable. Also, we have that the range of the forward map is open. Using arguments in this paper (Lemma 2.4 and the proof of Proposition 2.6 in particular) or Theorem 2 of [6], one can show every impulse response $g$ there is at most one impedance $\eta$ such that $F(\log \eta) = g$. Assuming we take an impulse response in the range, we have global existence.

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