A Quantitative Version of Marstrand’s Theorem

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1 Introduction

I began this project in the Winter of 1999 under the direction of Professor Toro, with the intention of remaining involved in mathematical research (the first taste of which I had had during the previous summer in the REU program here at the University of Washington). The project initially consisted of constructing Cantor sets in \( n \)-dimensional Euclidean space (as presented in Section 4). I then began studying material related to Marstrand's Theorem, a result in geometric measure theory. Professor Toro believed that the theorem could be extended and provided me with an outline of the possible proof. I filled in the details over the next several months. This work culminated in the quantitative extension of Marstrand's Theorem presented here.

I recommend the experience to anyone who enjoys mathematics and is willing and able to devote enough time to make it worthwhile. If you are an undergraduate giving serious consideration to being a research mathematician, writing a senior thesis is a great way to see how well the occupation suits you before actually going to graduate school.

The paper itself is structured as follows. In the next section, we state Marstrand's theorem and establish notation and definitions. Then in Section 3, we state and prove the quantitative version. Finally, we define some Cantor sets and use them to establish bounds on constants relevant to our theorem.

2 Preliminaries

In [2], Marstrand proved the following result, as stated in [3]:

**Theorem 2.1** (Marstrand's Theorem) Let \( s \) be a positive number. Suppose that there exists a Radon measure \( \mu \) on \( \mathbb{R}^n \) such that the density \( \Theta^s(\mu, a) \) exists and is positive and finite in a set of positive \( \mu \) measure. Then \( s \) is an integer.

In this paper, we show that it is possible to weaken the condition that the density exists in a set of positive \( \mu \) measure. A pinching of the upper and lower densities is sufficient. By this we mean

\[
\frac{1}{1 + \varepsilon} \leq \frac{\Theta^s(\mu, a)}{\Theta^s(\mu, a)} \leq 1, \tag{2.1}
\]

where \( \Theta^s(\mu, a) \) and \( \Theta^s(\mu, a) \) are the lower and upper densities, respectively, and \( \varepsilon \) is a positive real number that depends only on \( s \) and \( n \). Although the set where (2.1) holds must still have positive \( \mu \) measure, the density need not exist on such a set for the conclusion of the theorem to be valid.

Before we proceed, it is prudent to establish some definitions (following [3]).

**Notation** Let \( B(x, r) \) denote the open ball of radius \( r \) about the point \( x \) in \( \mathbb{R}^n \), and similarly let \( \overline{B}(x, r) \) denote the closed ball. Let \( d(x, A) \) denote the standard Euclidean
distance between a point \( x \in \mathbb{R}^n \) and a set \( A \subset \mathbb{R}^n \). Let \( C_0(\mathbb{R}^n) \) be the set of continuous functions on \( \mathbb{R}^n \) with compact support. Let \( \text{diam}(A) \) denote the diameter of a set \( A \subset \mathbb{R}^n \).

**Definition 2.1** Let \( \mu \) be a measure on \( \mathbb{R}^n \). The support of the measure \( \mu \) is defined by

\[
\text{spt}(\mu) = \{ x \in \mathbb{R}^n : \mu(B(x,r)) > 0 \ \forall r > 0 \}. \tag{2.2}
\]

Note that in particular \( \text{spt}(\mu) \) is a closed set.

**Definition 2.2** Let \( 0 \leq s < \infty \) and let \( \mu \) be a measure on \( \mathbb{R}^n \). The upper \( s \)-density of \( \mu \) at \( a \in \mathbb{R}^n \) is defined by

\[
\Theta^s(\mu, a) = \limsup_{r \downarrow 0} (2r)^{-s} \mu(B(a,r)) \tag{2.3}
\]

and the lower \( s \)-density is defined by

\[
\Theta^s_*(\mu, a) = \liminf_{r \downarrow 0} (2r)^{-s} \mu(B(a,r)). \tag{2.4}
\]

If they agree, their common value is called the \( s \)-density of \( \mu \) at \( a \) and is denoted by \( \Theta_s(\mu,a) \).

**Definition 2.3** Let \( 0 \leq s < \infty \). A non-zero Radon measure \( \mu \) is called \( s \)-uniform if there exists a positive number \( c \) such that

\[
0 < \mu(B(x,r)) = cr^s < \infty \tag{2.5}
\]

for every \( x \in \text{spt}(\mu) \) and every \( 0 < r < \infty \).

**Definition 2.4** Let \( \mu, \mu_1, \mu_2, \ldots \) be Radon measures on \( \mathbb{R}^n \). We say that the sequence \( \{ \mu_i \} \) converges weakly to \( \mu \),

\[
\mu_i \rightharpoonup \mu \tag{2.6}
\]

if

\[
\lim_{i \to \infty} \int \phi d\mu_i = \int \phi d\mu \quad \text{for all } \phi \in C_0(\mathbb{R}^n). \tag{2.7}
\]

**Definition 2.5** Let \( \mu \) be a Radon measure on \( \mathbb{R}^n \). We say that \( \nu \) is a tangent measure of \( \mu \) at a point \( a \in \mathbb{R}^n \) if \( \nu \) is a non-zero Radon measure on \( \mathbb{R}^n \) and if there exist sequences \( \{ r_i \} \) and \( \{ c_i \} \) of positive numbers such that \( r_i \downarrow 0 \) and

\[
c_i T_{a,r_i} \mu \rightharpoonup \nu, \tag{2.8}
\]

where \( T_{a,r_i} \mu(A) = \mu(r_i A + a) \). The set of all such tangent measures is denoted by \( \text{Tan}(\mu,a) \).
Definition 2.6 Let \( A, B \subseteq \mathbb{R}^n \). The *Hausdorff distance* between \( A \) and \( B \) is

\[
D[A, B] = \sup\{d(x, B) : x \in A\} + \sup\{d(y, A) : y \in B\}. \tag{2.9}
\]

\( D \) defines a metric on the class of closed sets in \( \mathbb{R}^n \).

**Notation** If \( A, A_1, A_2, \ldots \) are sets in \( \mathbb{R}^n \) we write that \( A_i \to A \) if \( A_i \) converges to \( A \) in the Hausdorff distance sense as \( i \to \infty \).

The following two propositions will prove useful.

**Proposition 2.1** Let \( A, B, A_1, A_2, \ldots \) be closed sets in \( \mathbb{R}^n \). Assume that \( A_i \to A \) in the Hausdorff distance sense. Let

\[
B = \{x \in \mathbb{R}^n : \exists\{x_i\} : x_i \in A_i \text{ and } x = \lim_{i \to \infty} x_i\}. \tag{2.10}
\]

Then \( A = B \).

**Proof.** We show that \( D[A, B] = 0 \). The proof proceeds by contradiction. Assume \( D[A, B] > 0 \). Either there is an \( \alpha \in A \) with \( d(\alpha, B) > 0 \) or a \( \beta \in B \) with \( d(\beta, A) > 0 \).

Note that if \( \alpha \in A \), there is a sequence of points \( a_i \in A_i \) with \( a_i \to \alpha \) since \( A_i \to A \). This is immediate from the definition of the Hausdorff distance. Therefore we conclude that \( \alpha \in B \).

Assume now that there is a \( \beta \in B \) with \( d(\beta, A) > 0 \). By the definition of \( B \) there exists a sequence of points \( \{a_i\} \), \( a_i \in A_i \), with \( a_i \to \beta \). Since \( A_i \to A \) in the Hausdorff distance sense and \( d(a_i, A) \leq D[A_i, A] \), it follows that \( \beta \in \overline{A} = A \), which contradicts our assumption.

Therefore we conclude that \( D[A, B] = 0 \), and thus \( A = B \) since they are both closed. \( \blacksquare \)

**Proposition 2.2** Let \( CL_r \) be the collection of all closed sets in \( \overline{B(0, r)} \). The metric space \((CL_r, D)\) is compact.

3 The Result

3.1 Quantitative Version of Marstrand’s Theorem

Using the density condition (2.1) we have the following generalized form of Marstrand’s Theorem:

**Theorem 3.1** Let \( s \) be a positive number and let \( n \) be a positive integer. There exists a positive number \( \varepsilon(s, n) \) depending only on \( s \) and \( n \) so that if \( \mu \) is a Radon measure on \( \mathbb{R}^n \) satisfying (2.1) in a set of positive \( \mu \) measure, then \( s \) is an integer.

The key ingredient in the proof of Theorem 3.1 is the following lemma, which is proved in the next section.
Lemma 3.1 (Main Lemma) Let $s$ be a positive number, and let $n$ be a positive integer. There exists a positive number $\varepsilon(s,n)$ such that if $\mu$ is a non-zero Radon measure on $\mathbb{R}^n$ satisfying

$$\frac{1}{1+\varepsilon(s,n)} \leq \frac{\mu(B(x,r))}{r^s} \leq 1 + \varepsilon(s,n)$$  \hspace{1cm} (3.1)

for all $r > 0$ and all $x \in \text{spt}(\mu)$, where $B(x,r) \subset \mathbb{R}^n$ is the open ball of radius $r$ about $x$, then $s$ is an integer.

We now indicate how Theorem 3.1 follows from the Main Lemma. Recall that (see [3], Lemma 14.7):

Theorem 3.2 Let $s$ be a positive number, $\mu$ a Radon measure on $\mathbb{R}^n$, and $A$ the set of the points $a \in \mathbb{R}^n$ such that

$$0 < \Theta^s_1(\mu,a) \leq \Theta^{**s}(\mu,a) < \infty. \hspace{1cm} (3.2)$$

Then the following holds at $\mu$ almost all points $a \in A$:

For every $\nu \in \text{Tan}(\mu,a)$ there is a positive number $c$ such that

$$tc^{cs} \leq \nu(B(x,r)) \leq cr^{s} \hspace{1cm} (3.3)$$

for all $x \in \text{spt}(\nu)$ and $r > 0$, where $t = \frac{\Theta^s_1(\mu,a)}{\Theta^{**s}(\mu,a)}$.

Proof. Assume that $\mu$ is a non-zero Radon measure $\mu$ which satisfies (2.1) for $\varepsilon(s,n)$ as in the Main Lemma on a set $A$ of positive $\mu$ measure. Theorem 2.5 in [4] ensures that $\text{Tan}(\mu,a) \neq \emptyset$ for $\mu$ almost all $a \in \mathbb{R}^n$ because $\mu$ is a locally finite measure (the measures of balls are finite). Theorem 3.2 and the fact that $\mu(A) > 0$ guarantee that there is a tangent measure $\nu$ such that

$$cr^{s} \frac{\Theta^s_1(\mu,a)}{\Theta^{**s}(\mu,a)} \leq \nu(B(x,r)) \leq cr^{s} \hspace{1cm} (3.4)$$

for all $r > 0$ and all $x \in \text{spt}(\nu)$, where $c$ is some positive real number. Thus we may apply our bound on the ratio of the upper and lower densities of $\mu$ to obtain

$$\frac{1}{1 + \varepsilon(s,n)} \leq \frac{\nu(B(x,r))}{cr^{s}} \leq 1. \hspace{1cm} (3.5)$$

$\frac{1}{c}\nu$ is a Radon measure satisfying the hypothesis of the Main Lemma, thus $s$ is an integer.
3.2 Proof of the Main Lemma

Our theorem now depends on the proof of the Main Lemma. The proof proceeds by contradiction. Therefore we assume that there are \( s \in \mathbb{R}^+ \setminus \mathbb{N} \) and \( n \in \mathbb{N} \) such that for every \( \varepsilon > 0 \) there exists a non-zero Radon measure \( \mu_\varepsilon \) on \( \mathbb{R}^n \) with

\[
\frac{1}{1 + \varepsilon} \leq \frac{\mu_\varepsilon(B(x,r))}{r^s} \leq 1 + \varepsilon
\]  

for all \( r > 0 \) and all \( x \in \text{spt}(\mu_\varepsilon) \). Using these measures, we are able to produce an \( s \)-uniform measure that satisfies the conditions of Marstrand's theorem. This \( s \)-uniform measure is obtained as the limit of a sequence of weakly convergent Radon measures. We define these Radon measures as follows.

Let \( \varepsilon_i \downarrow 0 \) be a sequence of positive real numbers with \( \varepsilon_i \leq 1 \). Let \( \mu_i = \mu_{\varepsilon_i} \) for \( i \in \mathbb{N} \), so that

\[
\frac{1}{1 + \varepsilon_i} \leq \frac{\mu_i(B(x,r))}{r^s} \leq 1 + \varepsilon_i
\]  

for all \( r > 0 \) and \( x \in \Sigma_i = \text{spt}(\mu_i) \). Without loss of generality we may assume that \( 0 \in \Sigma_i \) for each \( i \) (apply an appropriate translation to each measure). Now let \( \{r_i\} \) be a sequence of positive real numbers such that \( r_i \to 0 \). Form the sequence \( \{\nu_i\} \) with \( \nu_i(E) = \frac{\mu_i(E)}{r_i^s} \) for each Borel set \( E \subset \mathbb{R}^n \). It is from this sequence that we extract an \( s \)-uniform measure.

The extraction proceeds as follows. First we prove that our sequence of Radon measures \( \{\nu_i\} \) is uniformly bounded on compact sets. Then we apply a compactness theorem to show that there is a subsequence that converges weakly to a Radon measure which we now call \( \nu_\infty \). Then we prove that there is a further subsequence for which the supports of the Radon measures converge to a limit \( \Lambda_\infty \) in the Hausdorff distance sense uniformly over compact sets. We also show that \( \Lambda_\infty = \text{spt}(\nu_\infty) \). Finally we show that \( \nu_\infty \) is indeed an \( s \)-uniform measure and that \( \nu_\infty(\Lambda_\infty) > 0 \).

3.2.1 Weak Convergence of the Measures

Recall the compactness theorem for Radon measures (see [3] for more details):

**Theorem 3.3** If \( \mu_1, \mu_2, \ldots \) are Radon measures in \( \mathbb{R}^n \) with

\[
\sup\{\mu_i(K) : i = 1, 2, \ldots \} < \infty
\]

for all compact sets \( K \subset \mathbb{R}^n \), then there is a weakly convergent subsequence of \( \{\mu_i\} \).

We must prove that \( \sup\{\nu_i(K) : i = 1, 2, \ldots \} < \infty \). This is almost immediate from the way each \( \nu_i \) is defined. For if we let \( K \subset \mathbb{R}^n \) be a compact set, we know \( K \subset B(0, \rho) \) for some \( \rho > 0 \). But since \( 0 \in \text{spt}(\nu_i) \), this means that

\[
\nu_i(K) \leq \nu_i(B(0, \rho)) = \frac{\mu_i(B(0, r_i \rho))}{r_i^s} \leq (1 + \varepsilon_i) \rho^s < 2\rho^s.
\]  

(3.9)
Thus \( \sup\{\nu_i(K) : i = 1, 2, \ldots\} \leq 2p^* < \infty \). Hence there is a subsequence of \( \{\nu_i\} \) (reindexed for convenience) such that

\[
\nu_i \rightharpoonup \nu_{\infty}, \tag{3.10}
\]

where \( \nu_{\infty} \) is a Radon measure (weakly convergent limits are Radon in Definition 2.4).

### 3.2.2 Convergence of the Supports

Let \( \Lambda_i = spt(\nu_i) \). We now show that there exists a closed set \( \Lambda_{\infty} \) and a subsequence of \( \{\Lambda_i\} \), denoted by \( \{\Lambda'_j\} \), so that

\[
D[\Lambda'_j \cap \overline{B(0,j)}, \Lambda_{\infty} \cap \overline{B(0,j)}] \leq \frac{1}{2^j}. \tag{3.11}
\]

Recall that \( CL_r \) is the collection of all closed sets in \( \overline{B(0,r)} \).

We use a diagonal argument to show that there is a subsequence of \( \{\Lambda_i\} \) with the desired properties. \( \{\Lambda_i \cap \overline{B(0,1)}\} \) is a sequence of closed sets. Since \( (CL_1,D) \) is compact, there is a subsequence \( \{\Lambda'_i\} \) of \( \{\Lambda_i\} \) such that \( \Lambda'_i \cap \overline{B(0,1)} \rightarrow \Lambda'_{\infty} \) in \( (CL_1,D) \), where \( \Lambda'_{\infty} \) is a closed set in \( \overline{B(0,1)} \). Furthermore, there is a convergent subsequence \( \{\Lambda^1_i\} \) of \( \{\Lambda'_i\} \) such that \( \Lambda^1_i \cap \overline{B(0,2)} \rightarrow \Lambda^1_{\infty} \), where \( \Lambda^1_{\infty} \) is a closed set in \( \overline{B(0,2)} \). Continuing in this way, we construct sequences \( \{\Lambda^j_i\} \) in \( \{\Lambda_i\} \) such that the following hold:

1. \( \Lambda^j_i \cap \overline{B(0,j)} \rightarrow \Lambda^j_{\infty} \), where \( \Lambda^j_{\infty} \) is a closed set in \( \overline{B(0,j)} \)

2. \( \{\Lambda^j_i\} \) is a subsequence of \( \{\Lambda^{j-1}_i\} \) for \( j > 1 \) (\( \{\Lambda^1_i\} \) is a subsequence of \( \{\Lambda_i\} \))

Let

\[
\Lambda_{\infty} = \cup_{k=1}^{\infty} \Lambda^k. \tag{3.12}
\]

We show that for \( k \geq j \),

\[
\Lambda^k_{\infty} \cap B(0,j) = \Lambda^j_{\infty} \cap B(0,j). \tag{3.13}
\]

Let \( y \in \Lambda^k_{\infty} \cap B(0,j) \). By Proposition 2.1 there exists a sequence \( y_i \in \Lambda^k_i \cap B(0,k) \) so that \( \lim_{i \to \infty} y_i = y \). Since \( y \in B(0,j) \) there is \( i_0 \in \mathbb{N} \) so that for \( i \geq i_0 \), \( y_i \in B(0,j) \). Moreover, since \( \{\Lambda^k_i\} \) is a subsequence of \( \{\Lambda^j_i\} \), modulo relabeling the sequence we may assume that \( y_i \in \Lambda^j_i \). Since \( \Lambda^j_i \cap \overline{B(0,j)} \rightarrow \Lambda^j_{\infty} \) as \( i \to \infty \), \( \Lambda^j_i \cap \overline{B(0,j)} \rightarrow \Lambda^j_{\infty} \) as \( i \to \infty \). Again by Proposition 2.1, \( y \in \Lambda^j_{\infty} \cap B(0,j) \). Thus

\[
\Lambda^k_{\infty} \cap B(0,j) \subset \Lambda^j_{\infty} \cap B(0,j). \tag{3.14}
\]

Let \( y \in \Lambda^j_{\infty} \). There exists \( y_i \in \Lambda^j_i \cap \overline{B(0,j)} \) so that \( y = \lim_{i \to \infty} y_i \). Since \( \{\Lambda^k_i\} \) is a subsequence of \( \{\Lambda^j_i\} \) there is a subsequence of \( \{y_i\} \) denoted by \( \{y_{i_i}\} \) so that \( y_{i_i} \in \Lambda^k_i \cap B(0,j) \)
with \( y = \lim_{i \to \infty} y_i \). By Proposition 2.1 we have that \( y \in \Lambda^k_\infty \cap \overline{B(0,j)} \). Note that this shows that
\[
\Lambda^j_\infty \subset \Lambda^k_\infty \cap \overline{B(0,j)},
\] (3.15)

and therefore for \( k \geq j \)
\[
\Lambda^k_\infty \cap \overline{B(0,j)} = \Lambda^j_\infty \cap \overline{B(0,j)}.
\] (3.16)

Let \( k_0 \geq 1 \). Then
\[
\Lambda^k_\infty \cap \overline{B(0,k_0)} = \bigcup_{k=1}^{k_0} (\Lambda^k_\infty \cap \overline{B(0,k_0)}),
\] (3.17)

but for \( k < k_0 \)
\[
\Lambda^k_\infty \cap \overline{B(0,k_0)} = \Lambda^k_\infty \cap \overline{B(0,k)} = \Lambda^k_\infty \subset \Lambda^{k_0}_\infty \cap \overline{B(0,k)} \subset \Lambda^{k_0}_\infty \cap \overline{B(0,k_0)} = \Lambda^{k_0}_\infty.
\] (3.18)

Thus
\[
\Lambda^k_\infty \cap \overline{B(0,k_0)} = \Lambda^{k_0}_\infty.
\] (3.19)

We now show that \( \Lambda_\infty \) is closed. Let \( x \) be an accumulation point of \( \Lambda_\infty \), and let \( x_i \in \Lambda_\infty \) be a sequence with \( x_i \to x \). Choose \( i_0 \) so that \( |x_i - x| < 1 \) for all \( i \geq i_0 \). Set \( k_0 = |x| + 1 \). Then \( x_i \in \overline{B(0,k_0)} \) for \( i \geq i_0 \). Equation (3.19) guarantees that \( x_i \in \Lambda^{k_0}_\infty \) for \( i \geq i_0 \). Thus \( x \) is an accumulation point of \( \Lambda^{k_0}_\infty \). But \( \Lambda^{k_0}_\infty \) is a closed set. Therefore \( x \in \Lambda^{k_0}_\infty \). Hence \( x \in \Lambda_\infty \), and we conclude that \( \Lambda_\infty \) is a closed set.

Since
\[
\Lambda^j_i \cap \overline{B(0,j)} \to \Lambda^j_\infty = \Lambda_\infty \cap \overline{B(0,j)}
\] (3.20)

for each \( j \geq 1 \), there exists \( i_j \geq i_{j-1}(i_0 = 1) \) so that for \( i \geq i_j \)
\[
D[\Lambda^j_i \cap \overline{B(0,j)}, \Lambda^j_\infty] = D[\Lambda^j_i \cap \overline{B(0,j)}, \Lambda_\infty \cap \overline{B(0,j)}] \leq \frac{1}{2j}.
\] (3.21)

Let \( \Lambda'_j = \Lambda^j_i \). Then \( \{\Lambda'_j\} \) is a subsequence of \( \{\Lambda_i\} \) and
\[
D[\Lambda'_j \cap \overline{B(0,j)}, \Lambda_\infty \cap \overline{B(0,j)}] \leq \frac{1}{2j}.
\] (3.22)

At this point, we relabel our sequences such that \( \Lambda_i = \Lambda'_j \) and note that
\[
D[\Lambda_i \cap \overline{B(0,i)}, \Lambda_\infty \cap \overline{B(0,i)}] \leq \frac{1}{2i}.
\] (3.23)

The labels for \( \{\nu_i\} \) and \( \{\mu_i\} \) are changed similarly. We also relabel the sequence \( \{\varepsilon_i\} \). \( \varepsilon_i \downarrow 0 \), so our relabeling does not affect our prior arguments.
3.2.3 \( \Lambda_\infty \) is the Support of the Limit Measure \( \nu_\infty \)

We now prove that \( \Lambda_\infty = \text{spt}(\nu_\infty) \). The first step is to show that

\[
\text{spt}(\nu_\infty) = \{ x \in \mathbb{R}^n : \exists \{x_i\}; x_i \in \Lambda_i \text{ and } x = \lim_{i \to \infty} x_i \}. \tag{3.24}
\]

Take \( x \in \text{spt}(\nu_\infty) \). We now show that there is a sequence \( \{x_i\} \) with \( x_i \in \Lambda_i \) such that \( x_i \to x \). The proof proceeds by contradiction, so assume no such sequence exists. Then there are \( \epsilon_0 > 0 \) and a subsequence \( \{i_k\} \) such that \( d(x, \Lambda_{i_k}) > \epsilon_0 \). Note that \( B(x, \frac{\epsilon_0}{2}) \cap \Lambda_{i_k} = \emptyset \). Let \( \varphi \in C_0(B(x, \frac{\epsilon_0}{2})) \). Then

\[
\int \varphi d\nu_\infty = \lim_{k \to \infty} \int \varphi d\nu_{i_k}. \tag{3.25}
\]

Since \( \text{spt}(\nu_{i_k}) \cap \text{spt}(\varphi) = \emptyset \), \( \int \varphi d\nu_{i_k} = 0 \) for all \( k \). Thus \( \int \varphi d\nu_\infty = 0 \) for all \( \varphi \in C_0(B(x, \frac{\epsilon_0}{2})) \). This implies that \( x \) is not in the support of \( \nu_\infty \), which is a contradiction.

Now we claim that if \( \{x_i\} \) is a sequence with \( x_i \in \Lambda_i \) such that \( x_i \to x \), then \( x \in \text{spt}(\nu_\infty) \). Let \( \delta > 0 \) and let \( \varphi_\delta \in C_0(B(x, \delta)) \). We then see that

\[
\int \varphi_\delta d\nu_i \geq \nu_i(B(x, \frac{\delta}{2})), \tag{3.26}
\]

and thus

\[
\int \varphi_\delta d\nu_\infty \geq \lim_{i \to \infty} \nu_i(B(x, \frac{\delta}{2})). \tag{3.27}
\]

We have assumed that \( x_i \to x \), so for large \( i \) we see that \( x_i \in B(x, \frac{\delta}{4}) \). We know that \( \nu_i(B(x_i, \frac{\delta}{4})) = \frac{\mu_i(r_i B(x_i, \frac{\delta}{4}))}{r_i} = \frac{\mu_i(B(r_i x_i, r_i \frac{\delta}{4}))}{r_i} \). The fact that \( x_i \in \text{spt}(\mu_i) \) implies that \( r_i x_i \in \text{spt}(\mu_i) \), so

\[
\frac{\mu_i(B(r_i x_i, r_i \frac{\delta}{4}))}{r_i} > (\frac{\frac{\delta}{4}}{\frac{\delta}{2}})^s \frac{1}{1+\varepsilon_i} > \frac{1}{2} \cdot (\frac{\delta}{4})^s. \tag{3.28}
\]

Therefore \( \int \varphi_\delta d\nu_\infty \geq \frac{1}{2} (\frac{\delta}{4})^s \). Thus \( \nu_\infty(B(x, \delta)) > 0 \). This is true for all \( \delta > 0 \) so we may conclude that \( x \in \text{spt}(\nu_\infty) \).

We use a similar argument to that of Proposition 2.1 to show that \( \Lambda_\infty = \text{spt}(\nu_\infty) \). Let \( x \in \text{spt}(\nu_\infty) \) and let \( j_0 \geq |x| + 1 \). There exists a sequence \( \{x_i\} \), \( i \geq j_0 \), so that \( x_i \in \Lambda_i \), and \( \lim_{i \to \infty} x_i = x \). There exists \( j_1 \geq j_0 \) so that \( |x - x_j| < 1 \) for \( j \geq j_1 \). Hence for \( i \geq j_1 \), \( x_i \in \Lambda_i \cap B(0, i) \), and there exists \( y_i \in \Lambda_\infty \cap B(0, i) \) so that \( |x_i - y_i| \leq \frac{1}{2i} \). Thus \( \lim_{i \to \infty} y_i = x \), and since \( \Lambda_\infty \) is closed we conclude that \( x \in \Lambda_\infty \). This proves that \( \text{spt}(\nu_\infty) \subseteq \Lambda_\infty \).

Now let \( x \in \Lambda_\infty \), and let \( j_0 \geq |x| + 1 \). For \( i \geq j_0 \) there is \( x_i \in \Lambda_i \cap B(0, i) \) so that \( |x_i - x| \leq \frac{1}{2i} \). Hence

\[
x = \lim_{i \to \infty} x_i \in \text{spt}(\nu_\infty), \tag{3.29}
\]

that is, \( \Lambda_\infty \subseteq \text{spt}(\nu_\infty) \), which shows that \( \Lambda_\infty = \text{spt}(\nu_\infty) \).
3.2.4 \( \nu_\infty \) Satisfies Marstrand’s Hypothesis

In order to complete the proof of the Main Lemma, we simply need to see that \( \nu_\infty \) satisfies the hypothesis of Marstrand’s theorem, that is, \( \nu_\infty \) is a Radon measure on \( \mathbb{R}^n \) such that the density \( \Theta^s(\nu_\infty, \mathcal{A}) \) exists and is positive and finite in a set of positive \( \nu_\infty \) measure. First note that \( \nu_\infty \) is a Radon measure because it is the limit of a weakly convergent sequence of Radon measures. In order to obtain the density condition, we prove that \( \Theta^s(\nu_\infty, \mathcal{A}) \) exists and is positive for all \( r > 0 \) and all \( x \in \Lambda_\infty \) \( (\nu_\infty \) is an \( s \)-uniform measure) and that \( \Theta^s(\nu_\infty, \mathcal{A}) > 0 \). Then

\[
0 < \Theta^s(\nu_\infty, x) = 2^{-s} < \infty \quad \text{for all} \quad x \in \Lambda_\infty, \quad \text{a set of positive} \quad \Lambda_\infty \quad \text{measure, and we are done.}
\]

First we prove that \( \nu_\infty \) is an \( s \)-uniform measure. Recall that (see [3], Theorem 1.24):

**Theorem 3.4** Let \( \mu_1, \mu_2, \ldots \) be Radon measures on \( \mathbb{R}^n \). If \( \mu_i \to \mu \), \( K \subset \mathbb{R}^n \) is compact and \( G \subset \mathbb{R}^n \) is open, then

\[
\mu(K) \geq \limsup_{i \to \infty} \mu_i(K)
\]

and

\[
\mu(G) \leq \liminf_{i \to \infty} \mu_i(G).
\]

We use this theorem to prove that \( \nu_\infty(\mathbf{B}(x, r)) = r^s \) for all \( r > 0 \) and \( x \in \Lambda_\infty \). So let \( r > 0 \) and \( x \in \Lambda_\infty \). As we proved above, there is a sequence \( x_i \to x \) with \( x_i \in \Lambda_i \). We know that for each \( i \geq 1 \)

\[
\nu_i(\mathbf{B}(y, \rho)) = \frac{\mu_i(\mathbf{B}(y, r_i, \rho))}{r_i^s}
\]

for all \( \rho > 0 \) and \( y \in \Lambda_i \). So if we consider the open ball \( \mathbf{B}(x, r) \subset \mathbb{R}^n \), we know that

\[
\nu_\infty(\mathbf{B}(x, r)) \leq \liminf_{i \to \infty} \nu_i(\mathbf{B}(x, r)) = \liminf_{i \to \infty} \frac{\mu_i(\mathbf{B}(r_i x, r_i))}{r_i^s}.
\]

Now some simple manipulation yields the desired result:

\[
\liminf_{i \to \infty} \frac{\mu_i(\mathbf{B}(r_i x, r_i))}{r_i^s} \leq \liminf_{i \to \infty} \frac{\mu_i(\mathbf{B}(r_i x, r_i r + r_i |x_i - x|))}{r_i^s}
\]

\[
\leq \liminf_{i \to \infty} (1 + \epsilon_i) \frac{(r_i r + r_i |x_i - x|)^s}{r_i^s}
\]

\[
\leq \liminf_{i \to \infty} (1 + \epsilon_i) (r + |x_i - x|)^s
\]

\[
= r^s.
\]

So

\[
\nu_\infty(\mathbf{B}(x, r)) \leq r^s.
\]

9
In order to prove that $\nu_\infty$ is an $s$-uniform measure, all that remains to be seen is that

$$\nu_\infty(B(x, r)) \geq r^s. \quad (3.35)$$

Let $r > 0$ and $x \in \Lambda_\infty$ as before. Let $\gamma > 0$. We have that

$$\nu_\infty(B(x, r)) \geq \limsup_{i \to \infty} \nu_i(B(x, r)). \quad (3.36)$$

So

$$\nu_\infty(B(x, r)) \geq \limsup_{i \to \infty} \frac{\mu_i(B(x, r, \gamma))}{r_i^s}. \quad (3.37)$$

Since $x_i \to x$ there is $i_0 \geq 1$ so that for $i \geq i_0$ we have $|x - x_i| < \frac{r}{2}$. Then

$$\limsup_{i \to \infty} \frac{\mu_i(B(x, r, \gamma))}{r_i^s} \geq \limsup_{i \to \infty} \frac{\mu_i(B(x, r, \gamma - r_i|x - x_i|))}{r_i^s} \geq \limsup_{i \to \infty} \left( \frac{1}{1 + \varepsilon_i} \right) \frac{(r_i \gamma - r_i|x - x_i|)^s}{r_i^s} = \limsup_{i \to \infty} \left( \frac{1}{1 + \varepsilon_i} \right)(\gamma - |x - x_i|)^s = \gamma^s.$$ 

So we have proven that $\nu_\infty(B(x, r)) \geq \gamma^s$ for any $\gamma > 0$. This implies that $\nu_\infty(B(x, r)) \geq r^s$. For assume that $\nu_\infty(B(x, r)) < r^s$. Then there is $r_1 < r$ such that $\nu_\infty(B(x, r)) = r_1^s$. But then we can choose $\gamma_1$ with $r_1 < \gamma_1 < r$. Thus we see that

$$\nu_\infty(B(x, r, \gamma_1)) \geq \gamma_1^s > r_1^s = \nu_\infty(B(x, r)). \quad (3.38)$$

But $B(x, r_1) \subseteq B(x, r)$, and so we have a contradiction.

Combining our two inequalities, we find that $\nu_\infty$ is an $s$-uniform measure with

$$\nu_\infty(B(x, r)) = r^s. \quad (3.39)$$

Now it is easy to show that $\nu_\infty(\Lambda_\infty) > 0$. For since $0 \in \Lambda_\infty$, we know that $\nu_\infty(B(0, 1)) = 1$. Since $\Lambda_\infty = spt(\nu_\infty)$, $\Lambda_\infty$ is the smallest closed set such that $\nu_\infty(R^n \setminus \Lambda_\infty) = 0$. Therefore $\nu_\infty(\Lambda_\infty) \geq 1$. We conclude that $\nu_\infty$ satisfies the hypothesis of Marstrand's theorem, and thus $s$ is an integer.

4 Examples

In this concluding section, we present some examples of sets and measures that provide upper bounds for $\varepsilon(s, n)$. Specifically, we examine probability measures on Cantor sets. Following
[1] (see Section 2.3, page 6), we construct the Cantor sets as collections of sequences and note that they support measures satisfying condition (3.1) for \( r \) small and some \( \varepsilon > 0 \). Then we use the fact that these sequence collections are bilipschitz equivalent to the Euclidean Cantor sets to construct measures supported on the Euclidean Cantor sets that satisfy condition (3.1) for \( r \) small and some \( \varepsilon > 0 \).

Before we proceed, some more definitions are required.

**Definition 4.1** Let \( \mu \) be a measure on \( \mathbb{R}^n \). We say that \( \mu \) is **Ahlfors regular of dimension** \( d \) (\( d > 0 \)) if there is a positive constant \( K \) such that

\[
K^{-1}r^d \leq \mu(B(x, r)) \leq Kr^d \tag{4.1}
\]

for all \( x \in \text{spt}(\mu) \) and all \( r \leq \text{diam}(\text{spt}(\mu)) \).

**Definition 4.2** Let \((M, d(x, y))\) and \((N, \rho(x, y))\) be metric spaces. A mapping \( f : M \to N \) is said to be **bilipschitz** if there is a \( C > 0 \) such that

\[
C^{-1}d(x, y) \leq \rho(f(x), f(y)) \leq Cd(x, y) \tag{4.2}
\]

for all \( x, y \in M \). Two spaces are said to be **bilipschitz equivalent** if there is a bilipschitz mapping of one onto the other.

### 4.1 Cantor Sets as Collections of Sequences

We follow the construction in [1]. Let \( F \) be a finite set with \( k \geq 2 \) elements. Let \( F^\infty \) denote the set of sequences \( \{x_i\} \) with \( x_i \in F \) for each \( i \). This is our Cantor set.

Now we put a metric on \( F^\infty \). Given \( x = \{x_i\} \) and \( y = \{y_i\} \) in \( F^\infty \), let \( L(x, y) = l \) if \( l \) is the largest integer such that \( x_i = y_i \) when \( 1 \leq i \leq l \), and set \( L(x, y) = \infty \) when \( x = y \). Given \( 0 \leq a \leq 1 \) set

\[
d_a(x, y) = a^{L(x, y)}, \tag{4.3}
\]

where the right side is interpreted to be 0 when \( L(x, y) = \infty \). This defines a metric on \( F^\infty \). Note that in this metric \( F^\infty \) has diameter 1.

We now construct a probability measure on \( F^\infty \), which we later use to construct the measure that satisfies condition (3.1) for \( r \) small and some \( \varepsilon > 0 \). Let \( \mu_0 \) denote the probability measure on \( F \) that assigns mass \( 1/k \) to each element of \( F \). Take \( \mu \) to be the infinite product of copies of \( \mu_0 \).

Using these definitions (see [1]), we find that \( \mu(B_a(x, a^j)) = k^{-j} \) where \( B_a(x, a^j) \) is the open ball around \( x \in F^\infty \) with respect to the metric \( d_a(x, y) \).

**Proposition 4.1** \( \mu \) is a **Ahlfors regular measure of dimension** \( s \), where \( s \) satisfies \( a^s = k^{-1} \).
Proof. Let $s$ be a positive real number such that $a^s = k^{-1}$. Let $x \in F^\infty$ and let $r \leq 1$. Then there is $j \geq 0$ with $a^{j+1} \leq r \leq a^j$.

\begin{align*}
k^{-j-1} \leq \mu(B_a(x, r)) & \leq k^{-j} \quad (4.4) \\
\frac{k^{-j-1}}{r^s} \leq \frac{\mu(B_a(x, r))}{r^s} & \leq \frac{k^{-j}}{r^s} \quad (4.5) \\
(a^{j+1})^s \leq \frac{\mu(B_a(x, r))}{r^s} & \leq (a^j)^s \quad (4.6) \\
a^s \leq \frac{\mu(B_a(x, r))}{r^s} & \leq \frac{1}{a^s} \quad (4.7) \\
k^{-1} \leq \frac{\mu(B_a(x, r))}{r^s} & \leq k \quad (4.8)
\end{align*}

Note that the standard "middle-thirds" Cantor set corresponds to $(F^\infty, d_a)$ with $k = 2$ and $a = 1/3$.

4.2 The Euclidean Representations

Now we construct some sets in $\mathbb{R}^n$ and show that they are bilipschitz equivalent to some of the collections of sequences described above. We then define a measure $\nu$ on each set and use it to find an upper bound for $\varepsilon(s, n)$.

Let $n$ be a positive integer. Let $k = 2^n$. Let $s \in (0, n)$. Since we want $s$ to satisfy $a^s = k^{-1}$, we set $a = 2^{-\frac{n}{s}}$. Note that $a < \frac{1}{2}$. We use $k$ and $a$ to define a sequential Cantor set $F^\infty$ as above. We inherit the notation from the previous subsection.

Let $I_0$ be the interval $[0, 1] \subset \mathbb{R}$. Let $I_1 \subset I_0$ be the union of the intervals $[0, a]$ and $[1-a, 1]$. Construct $I_j$ in the standard Cantor fashion. We call those intervals of $I_j$ containing the left endpoint of a subinterval of $I_{j-1}$ left components and those containing right endpoints we call right components. For example, $[0, a]$ is a left component of $I_1$, and in $I_2$, $[0, a^2] \subset [0, a]$ is a left component while $[a - a^2, a] \subset [0, a]$ is a right component. Now we take products of the interval $I_j$ to construct sets in $\mathbb{R}^n$. Let $E_j = \prod_{i=1}^n I_j$. Our Euclidean Cantor set $E \subset \mathbb{R}^n$ is defined to be

\begin{equation}
E = \cap_{j=0}^\infty E_j.
\end{equation}
4.3 Bilipschitz Equivalence of the Cantor sets

In order to obtain a bilipschitz map from $E$ to $F^\infty$, we need to assign a sequence $F_x \in F^\infty$ to every $x$ in our set $E$. The set $F$ from which we constructed $F^\infty$ has $2^n$ elements. For convenience, consider these elements to be the integers from 0 to $2^n - 1$. These correspond to the $2^n$ subcomponents made from each component of the set $E_j$ during the construction of $E_{j+1}$. Set for $x = (x^1, \ldots, x^n) \in E$ and $j \geq 1$

$$F_{x,j} = \sum_{i=1}^{n} A_j(l, x)2^{l-1},$$  \hspace{1cm} (4.10)

where $A_j(l, x) = 0$ if $x^l$ is in a left component of $I_j$, and $A_j(l, x) = 1$ if $x^l$ is in a right component of $I_j$. Note that since $x^l \in \cap_{j=1}^{\infty} I_j$ for each $l = 1, \ldots, n$ by the standard construction of Cantor sets, the $A_j(l, x)$, and therefore $F_{x,j}$, are well defined. We define the map $\varphi : E \to F^\infty$ by $\varphi(x) = F_x = \{F_{x,j}\}$, where $F_{x,j}$ denote the $j$-th element of the sequence $F_x$.

**Proposition 4.2** $E$ is bilipschitz equivalent to $F^\infty$.

**Proof.** We prove this by showing that the map $\varphi$ is a surjective bilipschitz mapping.

We show that there is a constant $C$ so that for $x, y \in E$

$$C^{-1}|x - y| \leq d_a(F_x, F_y) \leq C|x - y|.$$  \hspace{1cm} (4.11)

Recall that $d_a(F_x, F_y) = a^{L(F_x, F_y)}$, where $L(F_x, F_y)$ is the largest index $L$ such that $F_{x,i} = F_{y,i}$ for all $1 \leq i \leq L$. Set $L = L(F_x, F_y)$. If $x = y$, (4.11) is trivial, thus we assume $x \neq y$, so that $L < \infty$. We now show that $x$ and $y$ are in the same component of $E_L$. This is obvious for $L = 0$. Otherwise, this is the case if

$$A_j(l, x) = A_j(l, y) \quad \forall j = 1, \ldots, L, l = 1, \ldots, n.$$  \hspace{1cm} (4.12)

By Equation (4.10) and the fact that $F_{x,i} = F_{y,i}$ for all $1 \leq i \leq L$, we see that (4.12) holds if every non-negative integer has a unique decomposition into a sum of powers of 2. That is, for every non-negative integer $N$ there exists a sequence $\{\delta_{l,N}\}$ where $\delta_{l,N} \in \{0, 1\}$ for all $l \geq 0$ such that

$$N = \sum_{l=0}^{\infty} \delta_{l,N}2^l,$$  \hspace{1cm} (4.13)

and if $\{\delta'_{l,N}\}$ is another such sequence satisfying (4.13) then $\delta'_{l,N} = \delta_{l,N}$ for all $l \geq 0$. Note that since $N < \infty$, there is a least integer $l_N$ depending only on $N$ such that

$$2^{l_N+1} > N.$$  \hspace{1cm} (4.14)

and that $\delta_{l,N} = 0$ for all $l > l_N$. 

13
Recall the proof of the existence and uniqueness of such decompositions. First we prove uniqueness. Recall that

\[ 2^n - 1 = \sum_{l=0}^{n-1} 2^l \quad (4.15) \]

for all \( n \geq 1 \). Assume that \( \{ \delta_{l,N} \} \) and \( \{ \delta'_{l,N} \} \) are two sequences satisfying (4.13) with \( \delta_{l,N} \in \{0,1\} \) and \( \delta'_{l,N} \in \{0,1\} \) for all \( l \geq 0 \). Then \( \sum_{l=0}^{l_N} \delta_{l,N} 2^l = \sum_{l=0}^{l_N} \delta'_{l,N} 2^l \). We argue by contradiction. Recall that \( l_N \) is the least integer such that \( \delta_{l,N} = 0 \) for all \( l > l_N \) and assume that there is a greatest integer \( m \leq l_N \) such that \( \delta_{m,N} \neq \delta'_{m,N} \). Without loss of generality let \( \delta_{m,N} = 1 \) and \( \delta'_{m,N} = 0 \). Then \( \sum_{l=0}^{m} \delta_{l,N} 2^l = \sum_{l=0}^{m} \delta'_{l,N} 2^l \) and thus \( 2^m \leq \sum_{l=0}^{m} \delta_{l,N} 2^l \). If \( m = 0 \), then we have a contradiction since \( \delta'_{m,N} = 0 \). If \( m \geq 1 \), then because \( \delta'_{m,N} = 0 \) we have \( 2^m \leq \sum_{l=0}^{m-1} \delta_{l,N} 2^l \). This is a contradiction to Equation (4.15), so we conclude that \( \{ \delta_{l,N} \} \) and \( \{ \delta'_{l,N} \} \) are identical. Now we prove the existence of decompositions. The proof proceeds by induction on \( N \). For \( N = 0 \), the sequence \( \delta_{l,0} = 0 \) for all \( l \geq 0 \) is sufficient. Let \( N \) be a positive integer, and assume that \( \{ \delta_{l,m} \} \) exists for all non-negative integers less than \( N \). If \( N \) is even, then \( N \) has the decomposition defined by \( \delta_{l,N} = \delta_{l-1,N/2} \) for every \( l \geq 1 \), with \( \delta_{0,N} = 0 \). If \( N \) is odd, then \( N \) has the decomposition defined by \( \delta_{l,N} = \delta_{l-1,N-1/2} \) for every \( l \geq 1 \), with \( \delta_{0,N} = 1 \). Therefore every non-negative integer has a unique decomposition into a sum of powers of 2, and we conclude that \( x \) and \( y \) are in the same component of \( E_L \).

Every component of \( E_L \) is an \( n \)-dimensional cube of side length \( a^L \). An easy way to see this is to look at the component of \( E_L \) containing the origin. The cube containing the origin is the set \( \prod_{l=1}^{n} [0, a^L] \), and all of the components of \( E_L \) are cubes of the same side length. As the greatest distance between \( x \) and \( y \) in such a cube is \( n^{1/2} a^L \), we may conclude that

\[ n^{1/2} |x - y| \leq a^L = d_a(F_x, F_y). \quad (4.16) \]

Note that this relationship is satisfied for \( L = 0 \).

Now we want an upper bound on \( d_a(F_x, F_y) \). Since \( L \) is the largest index such that \( F_{x,i} = F_{y,i} \) for all \( 1 \leq i \leq L \), we know that \( F_{x,L+1} \neq F_{y,L+1} \). Therefore \( x \) and \( y \) are in different components of \( E_{L+1} \). The length of the intervals removed from \( I_L \) to construct \( I_{L+1} \) is \( a^L(1 - 2a) \). This is clear from the side lengths of our components. The length of \( I_0 \) is 1, and the length of each component of \( I_1 \) is \( a \). The following intervals are constructed in the standard way, so we conclude that the length of components of \( I_{j+1} \) is equal to \( a \) times the length of components of \( I_j \). The gaps between intervals must obey this rule as well, and the original gap in \( I_1 \) has length \( (1 - 2a) \). Therefore the gaps in \( I_{L+1} \) have length \( a^L(1 - 2a) \). We know that \( x \) and \( y \) are in different components of \( E_{L+1} \), thus there must be an \( l \) such that the \( l \)-th coordinate of \( x - y \), \( x^l - y^l \), satisfies

\[ |x^l - y^l| \geq a^L(1 - 2a). \quad (4.17) \]

We thereby conclude that

\[ |x - y| \geq a^L(1 - 2a) = d_a(F_x, F_y)(1 - 2a). \quad (4.18) \]
Note again that this relationship is satisfied for \( L = 0 \).

Combining estimates (4.16) and (4.18) we have the following

\[
n^{-\frac{1}{2}}|x - y| \leq d_a(F_x, F_y) \leq \frac{1}{1 - 2a}|x - y|.
\]  

(4.19)

Setting \( C = \max\{n^{\frac{1}{2}}, \frac{1}{1 - 2a}\} \), we see that

\[
C^{-1}|x - y| \leq d_a(\varphi(x), \varphi(y)) \leq C|x - y|.
\]  

(4.20)

This is precisely the statement that \( \varphi \) is bilipschitz.

To obtain a bilipschitz equivalence, we still need to demonstrate that \( \varphi \) is surjective. Let \( G \in F^\infty \), with

\[
G = \{G_i\}_{i=1}^{\infty}, G_i \in F = \{0, \ldots, 2^n - 1\}.
\]  

(4.21)

We know that each \( G_i \) has a unique decomposition into a sum of powers of 2 as in Equation (4.13), so

\[
G_i = \sum_{l=1}^{n} a_l 2^{l-1},
\]  

(4.22)

with \( a_l \in \{0, 1\} \) for all \( 1 \leq l \leq n \).

We construct \( x \in E \) as the limit of a sequence \( \{x_i\} \) with \( x_i = (x_i^1, \ldots, x_i^n) \in E_i \). We define \( x_i \) by induction as follows

\[
x_1 = (x_1^1, \ldots, x_1^n) \text{ where } \begin{cases} 
  x_1^l = 0 & \text{if } a_l^1 = 0 \\
  x_1^l = 1 - a_l^1 & \text{if } a_l^1 = 1 
\end{cases}
\]  

(4.23)

Note that 0 is the left hand point of the left component of \( I_1 \), and \( 1 - a \) is the left hand point of the right component of \( I_1 \).

Assume that \( x_j = (x_j^1, \ldots, x_j^n) \) has been constructed in such a way that for each \( l = 1, \ldots, n \), \( x_j^l \) is the left hand point of a component of \( I_j \). Call these components \( I^l_j \). We define \( x_{j+1} = (x_{j+1}^1, \ldots, x_{j+1}^n) \) as follows

\[
x_{j+1}^l = \begin{cases} 
  x_j^l & \text{if } a_{j+1}^l = 0 \\
  x_j^l + a_{j+1}^l (1 - a) & \text{if } a_{j+1}^l = 1 
\end{cases}
\]  

(4.24)

Note that each \( x_{j+1}^l \) is the left hand point of a component of \( I_{j+1} \).

Let \( C_j \) denote the connected component of \( E_j \) containing \( x_j \). From the construction it is clear that \( C_{j+1} \subset C_j \). Since each \( C_j \) is compact and \( \text{diam}(C_j) = a_3^n \) we have

\[
\cap_{j=1}^{\infty} C_j = \{x\}.
\]  

(4.25)

15
We claim that \( \varphi(x) = G \). Since \( x \in \bigcap_{j=1}^\infty C_j \), \( x \in C_j \) for each \( j \geq 1 \). Hence \( x \) and \( x_j \) belong to the same connected component of \( E_j \). \( A_j(l, x) = 0 \) if \( x^i \), and therefore \( x^i_j \), is in a left component of \( I_j \) and \( A_j(l, x) = 1 \) if \( x^i \), and therefore \( x^i_j \), is in a right component of \( I_j \). By the definition of \( x^i_j \), \( x^i_j \) is in a left component of \( I_j \) if \( a^i_j = 0 \) and \( x^i_j \) is in a right component of \( I_j \) if \( a^i_j = 1 \). Thus \( A_j(l, x) = a^i_j \). This implies that \( F_{x,j} = G_j \) and \( \varphi(x) = F_x = G \). This shows that \( \varphi \) is surjective.

4.4 Bounding the Constants

Finally, we construct a measure \( \nu \) on \( \mathbb{R}^n \) and use it to bound our constants \( \varepsilon(s, n) \). Let \( \nu(S) = \mu(\varphi(S \cap E)) \) for all \( S \subset \mathbb{R}^n \), where \( \mu \) is the measure supported on \( F^\infty \) defined in Section 4.1. We assume that \( s \) is not an integer.

Let \( B(x, r) \) be an open ball in \( \mathbb{R}^n \) with \( x \in E \) and \( r \leq 1 \). We immediately conclude that
\[
B_a(\varphi(x), C^{-1}r) \subset \varphi(B(x, r)) \subset B_a(\varphi(x), Cr)
\]
(4.26)
because \( \varphi \) is bilipschitz. Using our prior estimates on \( \mu \) (in the proof of Proposition 4.1), we see that
\[
\frac{n^{\frac{-s}{2}}}{k} \leq \frac{\nu(B(x, r))}{r^s} = \frac{\mu(\varphi(B(x, r)))}{r^s} \leq k \left( \frac{1}{1 - 2a} \right)^s.
\]
(4.27)
Thus we obtain the following expressions
\[
\Theta^{ss}(\mu, x) \leq k 2^{-s} \left( \frac{1}{1 - 2a} \right)^s
\]
(4.28)
\[
\Theta^s(\mu, x) \geq \frac{2^{-s} n^{\frac{s}{2}}}{k}.
\]
(4.29)
Note that \( a = 2^{-\frac{n}{s}} \) and \( k = 2^n \).

We know that \( \nu \) is a Radon measure. But \( s \) is not an integer, so these inequalities gives us a bound for \( \varepsilon(s, n) \) by Theorem 3.1:
\[
\varepsilon(s, n) < 2^{2n} \left( \frac{n^{\frac{s}{2}}}{1 - 2^{-\frac{n}{s} + 1}} \right)^s - 1.
\]
(4.30)

5 References

