The Brouwer fixed point theorem

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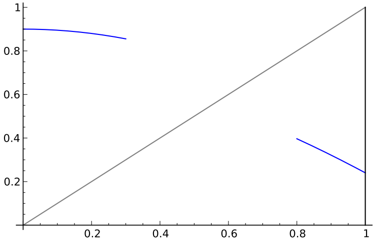
Suppose that S is a set. A function $f : S \to S$ has a fixed point if there is an element $x \in S$ so that $f(x) = x$. A fixed point theorem is a theorem like this: with some conditions on S or f or both, f must have a fixed point.

Examples

- Any contraction from $\mathbb{R}$ to $\mathbb{R}$ has a fixed point.
- The intermediate value theorem implies that every continuous function $f : [0,1] \to [0,1]$ has a fixed point.



My topic: the Brouwer fixed point theorem, which generalizes both of these.

A function $f : [0,1] \to [0,1]$ is a contraction if f contracts distances: for all $x_1, x_2 \in [0,1]$,

$$|x_1 - x_2| > |f(x_1) - f(x_2)|.$$

Theorem

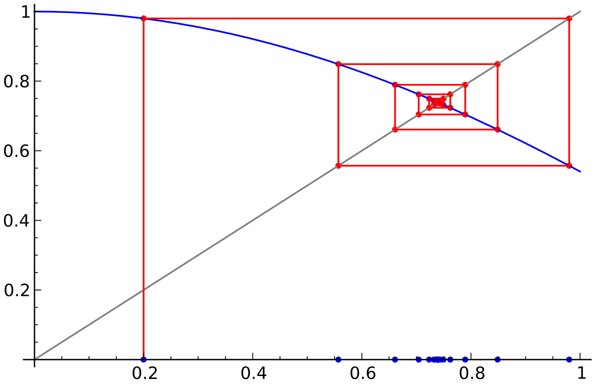
Any contraction $f : [0,1] \to [0,1]$ has a unique fixed point.

Outline of proof.

Pick any $x_0 \in [0,1]$. Define a sequence $\{x_0, x_1, x_2, \dots\}$ by

$$\begin{aligned} x_1 &= f(x_0) \\ x_2 &= f(x_1) = f(f(x_0)) \\ x_3 &= f(x_2) = f(f(f(x_0))) \\ &\vdots \end{aligned}$$

Then $\lim_{n\to\infty} x_n$ is the (unique) fixed point of f. □



Consider $f(x) = \cos x$ on $[0, 1]$. Let $x_0 = 1/5$ and let $x_1 = f(x_0)$. Let $x_2 = f(x_1)$. Let $x_3 = f(x_2)$. Iterate 20 more times. . . (The actual fixed point is approximately 0.739.)

The same theorem holds in 2 (and higher) dimensions

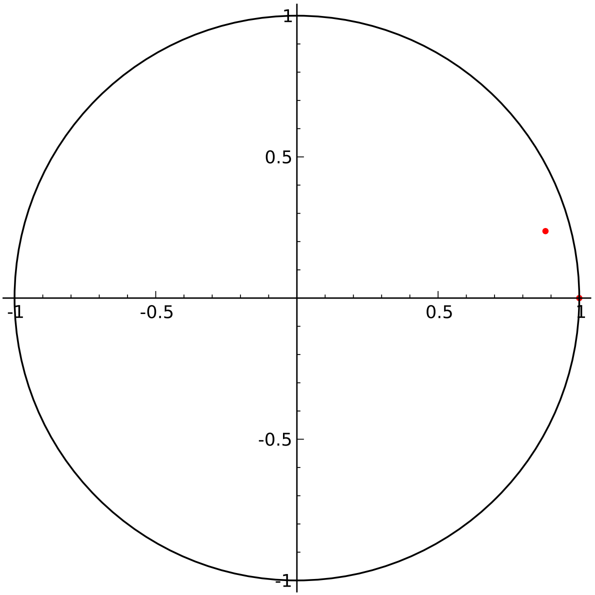
Examples

1. Maps
2. Disks

```
P = circle((0,0), 1, aspect_ratio=1)
s = 0.9  # scale factor
phi = pi/15 # rotation angle
rot = matrix([[cos(phi), -sin(phi)], [sin(phi), cos(ph
shift = vector([0, (1-s)/2]) # vertical translation

v = vector((1, 0))
P += point(v, color='red')
v = s*rot*v + shift
v = vector((n(v[0]), n(v[1])))
P += point(v, color='red')
```

Starting with $(1, 0)$:



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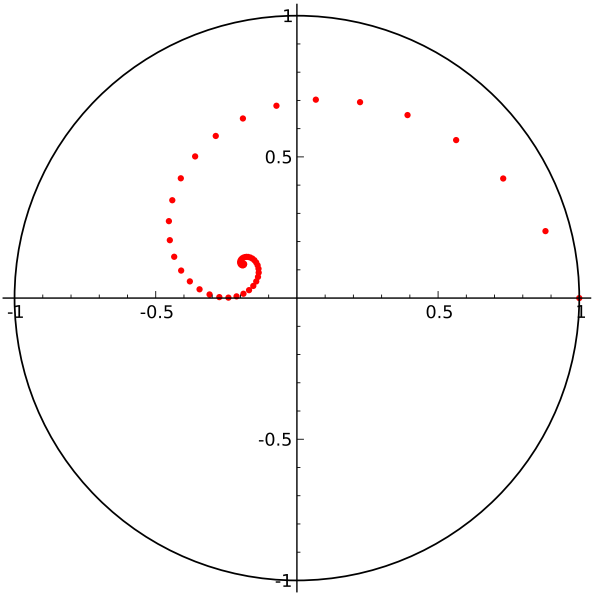
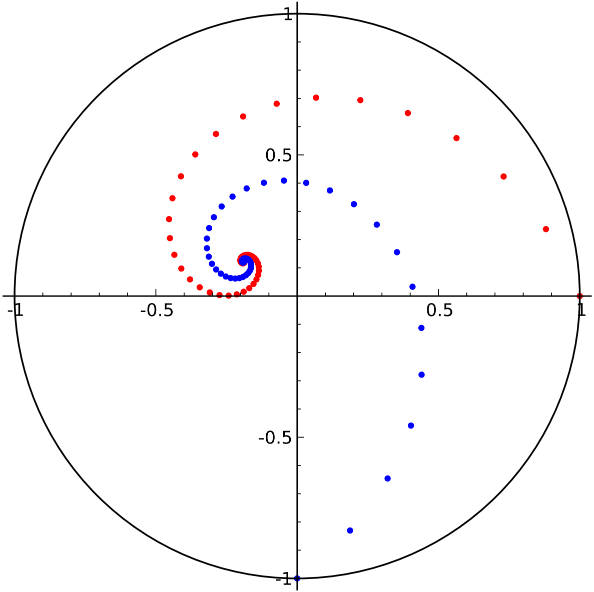
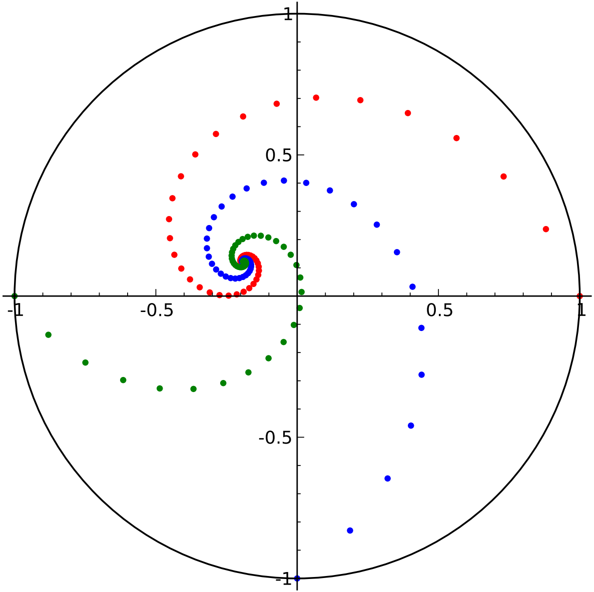


Starting with $(0, -1)$:



Starting with $(-1, 0)$:

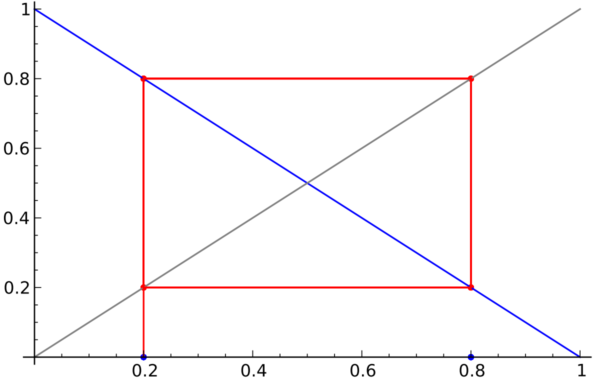


The Brouwer fixed point theorem removes the contraction condition.

Theorem

Let S be the unit interval $[0,1]$ or the unit square or the unit cube or . . . Let $f : S \to S$ be a continuous function. Then f has a fixed point.

- The fixed point need not be unique.
- There is no obvious limit for finding it: consider rotations or the following example.



Consider $f(x) = 1 - x$ on $[0,1]$. Let $x_0 = 1/5$, let $x_1 = f(x_0)$, let $x_2 = f(x_1)$, etc. Then

$$x_0 = x_2 = x_4 = \cdots = 1/5,$$
$$x_1 = x_3 = x_5 = \cdots = 4/5.$$

Two subsets X and Y of $\mathbb{R}^2$ are topologically equivalent (or homeomorphic) if there are continuous bijections $g : X \to Y$ and $g^{-1} : Y \to X$.

Examples

For example, the following are all topologically equivalent to each other:

- disk of radius 1: $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$
- disk of radius 2
- any disk
- triangle
- quadrilateral
- pentagon
- polygon

Theorem (The Brouwer fixed point theorem in $\mathbb{R}^n$)

Fix an integer $n \geq 0$ and let $D \subset \mathbb{R}^n$ be the unit disk:

$$D = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| \leq 1\}.$$

Then any continuous function $f : D \to D$ has a fixed point.

Proposition

Suppose that X and Y are topologically equivalent, and that every continuous function $f : X \to X$ has a fixed point. Then every continuous function $h : Y \to Y$ has a fixed point.

Proof of proposition.

Suppose that X and Y are topologically equivalent, and suppose that $h : Y \to Y$ is continuous. Consider

$$\begin{CD} Y @>h>> Y \\ @AgAA @VVg^{-1}V \\ X @>g^{-1}\circ h\circ g>> X \end{CD}$$

Since the bottom function will have a fixed point, so will the top one: if $g^{-1} \circ h \circ g(x) = x$, then $g(x)$ is a fixed point for h: applying g to both sides gives $h(g(x)) = g(x)$. □

Examples

- contractions
- maps
- salad dressing, cocktails
- the fundamental theorem of algebra
- every real $n \times n$ matrix with all positive entries has a positive eigenvalue
- Hex can't end in a draw
- existence of Nash equilibrium

Theorem (The Brouwer fixed point theorem in $\mathbb{R}^2$)

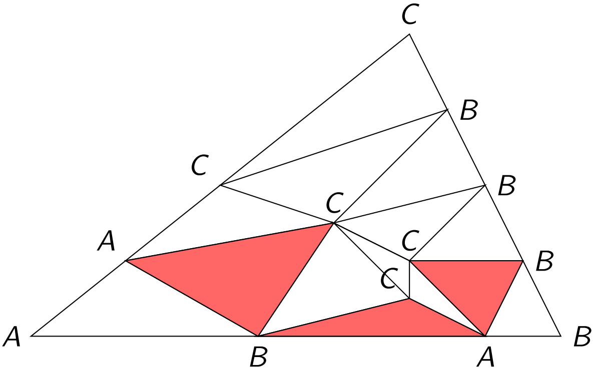
Let $D \subset \mathbb{R}^2$ be the unit disk. Then any continuous function $f : D \to D$ has a fixed point.

There are many proofs of the Brouwer fixed point theorem, and I'm going to describe one using a result from combinatorics. First, because of the proposition, we can replace D with anything topologically equivalent to it. We will work with a triangle, instead.

Now for some combinatorics. Suppose T is a triangle, and suppose it has been subdivided into smaller triangles. Label each vertex with A, B, or C according to these rules:

- the vertices of the original triangle all have different labels
- on the side AB, all of the vertices are either A or B
- similarly for the other sides
- (no restrictions on the labels of vertices in the middle)

This is called a Sperner labeling. In a Sperner labeling, a triangle is called complete if its vertices have all three labels.



Theorem (Sperner's lemma)

At least one triangle in a Sperner labeling is complete.

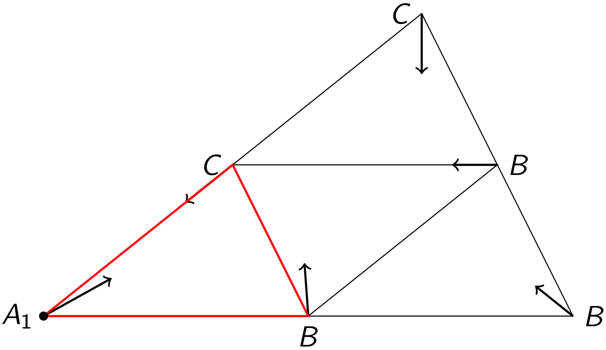
Proof of Sperner's lemma:

- by dimension. In dimension 1: easy.
- In dimension 1: actually prove that there are an odd number of "complete" edges. (Count vertices labeled A.)
- In dimension 2: count the number of edges labeled AB:

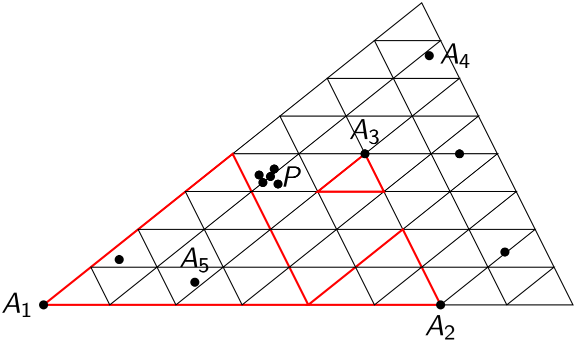
$$x = \#\{\text{triangles labeled } AAB \text{ or } ABB\}$$
$$y = \#\{\text{triangles labeled } ABC\}.$$

  Then we get $2x + y$.
- On the other hand, if i is the number of AB edges on the inside and o is the number on the outside, then we get $2i + o$.
- That is, $2x + y = 2i + o$.
- Since o is odd, y must be odd.

Proof of the Brouwer fixed point theorem in $\mathbb{R}^2$:
Let T be a triangle, as above, and suppose $f : T \to T$ is continuous.



For every point x in T, draw an arrow from x to $f(x)$.
Chop T into smaller triangles. Label each vertex: label A if the arrow points northeast, label B if the arrow points northwest, label C if any other direction. This produces a Sperner labeling.
Therefore there is a complete triangle with vertices A_1, B_1, C_1.



Chop T into even smaller triangles and find another, smaller, complete triangle $A_2B_2C_2$. Repeat, using smaller and smaller triangles, getting complete triangles $A_nB_nC_n$ for each $n \geq 1$. By a basic fact in topology (compactness of T), the set of points $\{A_1, A_2, A_3, \dots\}$ has a limit point P.

Since the triangles get smaller and smaller, P is also a limit point of $\{B_1, B_2, \dots\}$ and of $\{C_1, C_2, \dots\}$. So P is a point whose arrow must point northeast, northwest, and south, simultaneously. So the

In higher dimensions: the obvious generalization of Sperner's lemma holds in $\mathbb{R}^n$ for any n, and using it, one can prove the obvious generalization of the Brouwer fixed point theorem.

Theorem (The Brouwer fixed point theorem in $\mathbb{R}^n$)

Fix an integer $n \geq 0$ and let $D \subset \mathbb{R}^n$ be the unit disk:

$$D = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| \leq 1\}.$$

Then any continuous function $f : D \to D$ has a fixed point. The same is true if D is replaced by any space homeomorphic to it.

Brought to you by:

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