1. (10 pts) Compute the following integrals:

(a) $\int_C (x+1)\,ds$ where $C$ is the line segment from $(1,0)$ to $(-2,4)$.

PARAMETERIZE

$x = 1-3t$ $\quad 0 \le t \le 1$ $\Big\}$ $\vec{r}(t) = \langle 1-3t, 4t \rangle$

$y = 0+4t$

$ds = \sqrt{(-3)^2+(4)^2}\,dt = 5\,dt$

$$\int_C (x+1)\,ds = \int_0^1 (1-3t+1)\,5\,dt$$

$$= 5\int_0^1 2-3t\,dt$$

$$= 5\left[2t - \tfrac{3}{2}t^2\Big|_0^1\right]$$

$$= 5\left[2-\tfrac{3}{2}\right] = 5\cdot\tfrac{1}{2} = \boxed{\tfrac{5}{2} = 2.5}$$

(b) $\iint_S 15z\,dS$, where $S$ is the surface of the sphere $x^2+y^2+z^2=1$ in the **first octant**.

PARAMETERIZE

$\vec{r}(\phi,\theta) = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi \rangle$

$|\vec{r}_\phi \times \vec{r}_\theta| = 1^2\sin\phi$ $\leftarrow$ from class!

$0 \le \phi \le \pi/2$

$0 \le \theta \le \pi/2$

$$\iint_S 15z\,dS = \int_0^{\pi/2}\int_0^{\pi/2} 15\cos\phi\sin\phi\,d\phi\,d\theta$$

$$= 15\int_0^{\pi/2} d\theta \int_0^{\pi/2}\cos\phi\sin\phi\,d\phi$$ $\quad u = \sin\phi$, $du = \cos\phi\,d\phi$

$$= 15\,\tfrac{\pi}{2}\int_0^1 u\,du$$

$$= \tfrac{15\pi}{2}\;\tfrac{1}{2}u^2\Big|_0^1 = \boxed{\tfrac{15\pi}{4}}$$

2. (10 pts) Compute $\iint_S \langle xz, yz, 3z\rangle \cdot d\mathbf{S}$ where $S$ is the part of the cone $z = \sqrt{x^2+y^2}$ that is between $z=1$ and $z=2$ with **downward** orientation.

PARAMETERIZE

OPTION 1:

$\vec{r}(x,y) = \langle x, y, \sqrt{x^2+y^2}\rangle$

$\vec{r}_x \times \vec{r}_y = \left\langle \frac{-\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}}, \frac{-\cancel{2}y}{\cancel{2}\sqrt{x^2+y^2}}, 1\right\rangle$ upward

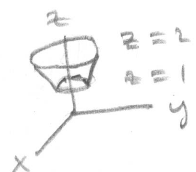
$\iint_S \langle xz, yz, 3z\rangle \cdot d\vec{S}$

$= \iint_D \langle x\sqrt{x^2+y^2}, y\sqrt{x^2+y^2}, 3\sqrt{x^2+y^2}\rangle \cdot \left\langle \frac{-x}{\sqrt{x^2+y^2}}, \frac{-y}{\sqrt{x^2+y^2}}, 1\right\rangle dA$

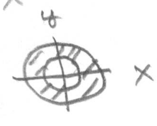
$= \iint_D -x^2-y^2+3\sqrt{x^2+y^2}\, dA$

Domain

Polar




$\int_0^{2\pi}\int_1^2 (-r^2+3r)\, r\, dr\, d\theta$




$\int_0^{2\pi} d\theta \int_1^2 -r^3+3r^2\, dr$

$2\pi \left[-\frac{1}{4}r^4 + r^3\Big|_1^2\right]$

$2\pi \left[(-4+8)-(-\frac{1}{4}+1)\right]$

$2\pi \left[4 - \frac{3}{4}\right]$

$2\pi \frac{13}{4}$

check orientation

DOWNWARD

$\boxed{-\frac{13\pi}{2}}$

OPTION 2:

$\vec{r}(u,v) = \langle v\cos u, v\sin u, v\rangle$

$0 \le u \le 2\pi$

$1 \le v \le 2$

$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -v\sin u & v\cos u & 0 \\ \cos u & \sin u & 1 \end{vmatrix}$

$= \langle v\cos u, v\sin u, -v\rangle$

DOWNWARD

$\iint_S \langle xz, yz, 3z\rangle \cdot d\vec{S}$

$\int_0^{2\pi}\int_1^2 \langle v^2\cos(u), v^2\sin(u), 3v\rangle \cdot \langle v\cos(u), v\sin(u), -v\rangle\, dv\, du$

$v^3(\cos^2(u)+\sin^2(u))$

$= \int_0^{2\pi}\int_1^2 v^3 - 3v^2\, dv\, du$

$= \int_0^{2\pi} du \int_1^2 v^3-3v^2\, dv$

$= 2\pi \left[\frac{1}{4}v^4 - v^3\Big|_1^2\right]$

$= 2\pi \left[(4-8)-(\frac{1}{4}-1)\right]$

$= 2\pi \left[-4 + \frac{3}{4}\right]$

$= 2\pi \cdot -\frac{13}{4}$

$= \boxed{-\frac{13\pi}{2}}$

3. (10 pts) Consider the vector field $\mathbf{F}(x, y, z) = \langle y^2 + 2, 2xy, 3z^2 \rangle$ on $\mathbb{R}^3$. Note that $\operatorname{curl} \mathbf{F} = \mathbf{0}$. Let $C$ be the curve parameterized by $\mathbf{r}(t) = \langle 5t^{10}, \cos(\pi t), 2t^3 - t - 1 \rangle$ for $0 \leq t \leq 1$. Compute $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$.

(Please use the consequences of the fact that $\operatorname{curl} \mathbf{F} = \mathbf{0}$).

① $f_x(x,y,z) \overset{?}{=} y^2 + 2$

$f(x,y,z) = \int y^2 + 2\, dx = y^2 x + 2x + g(y,z)$

② $f_y(x,y,z) \overset{?}{=} 2xy$

$\underbrace{2yx + 0 + g_y(y,z)}_{f_y(x,y,z)} \overset{?}{=} 2xy$

$\Rightarrow \quad g_y(y,z) = 0$

$g(y,z) = h(z)$ $\qquad \Rightarrow f(x,y,z) = y^2x + 2x + h(z)$

③ $f_z(x,y,z) \overset{?}{=} 3z^2$

$0 + 0 + h'(z) \overset{?}{=} 3z^2$

$\Rightarrow h(z) = \int 3z^2 dz = z^3 + k$ $\qquad$ a constant

$$\boxed{f(x,y,z) = y^2x + 2x + z^3 + k}$$

$\int_C \vec{F} \cdot d\vec{r} = f(\vec{r}(1)) - f(\vec{r}(0))$

$= f(5,-1,0) - f(0,1,-1)$

$= [(-1)^2(5) + 2(5) + (0)^3] - [(1)^2(0) + 2(0) + (-1)^3]$

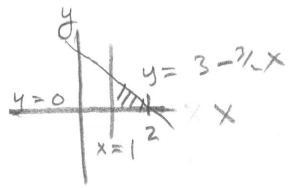
$= \quad 15 + 1 = \boxed{16}$

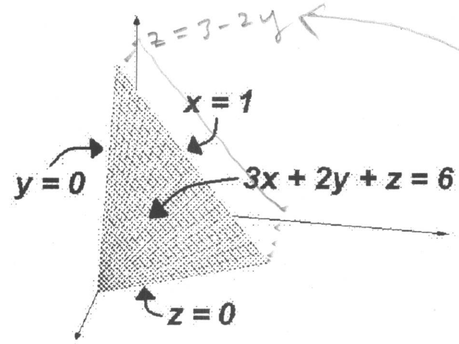
4. (10 pts) Set up (**DO NOT EVALUATE**) two triple integrals that represent the volume of the solid bounded by the planes $3x + 2y + z = 6$, $z = 0$, $y = 0$, and $x = 1$. You must give two answer in the orders specified.

(a) In the order $dzdydx$:

INNER BOUNDS: $0 \le z \le 6 - 3x - 2y$

PROJECTION:
$z = 0$
$\Rightarrow 3x + 2y = 6$
$y = 3 - \frac{3}{2}x$



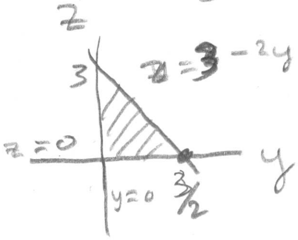

$0 \le y \le 3 - \frac{3}{2}x$
$1 \le x \le 2$

$$\boxed{\int_1^2 \int_0^{3-\frac{3}{2}x} \int_0^{6-3x-2y} 1 \, dz\,dy\,dx} = 3/4$$

(b) In the order $dxdzdy$:

INNER BOUNDS: $1 \le x \le \frac{6 - 2y - z}{3}$

PROJECTION:
$x = 1$
$\Rightarrow 3 + 2y + z = 6$
$z = 3 - 2y$

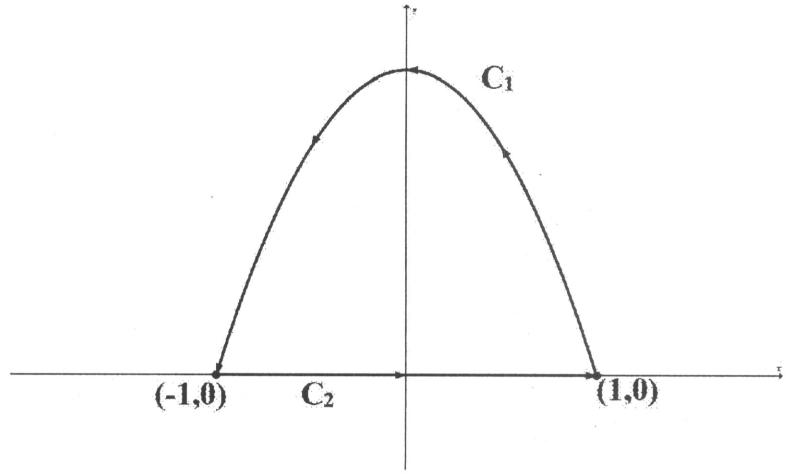



$0 \le z \le 3 - 2y$
$0 \le y \le 3/2$

$$\boxed{\int_0^{3/2} \int_0^{3-2y} \int_1^{\frac{6-2y-z}{3}} 1 \, dx\,dz\,dy} = 3/4$$

5. (10 pts) Consider the vector field $\mathbf{F}(x,y,z) = \langle x^4 + 3x, x^3 - \cos(y)\rangle$ on $\mathbf{R}^2$. Let $C$ be the positively oriented **CLOSED** curve that consists of the curve $C_1$ which is the arc of parabola $y = 1 - x^2$ from $(1,0)$ to $(-1,0)$ followed by the curve $C_2$ which is the line segment from $(-1,0)$ to $(1,0)$.

Compute $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$.




GREEN'S THEOREM

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \int_{-1}^{1}\int_0^{1-x^2} (3x^2 - 0)\, dy\, dx$$

$$= \int_{-1}^{1} 3x^2(1-x^2)\, dx$$

$$= \int_{-1}^{1} 3x^2 - 3x^4\, dx$$

$$= x^3 - \frac{3}{5}x^5 \Big|_{-1}^{1}$$

$$= (1 - 3/5) - (-1 + 3/5)$$

$$= 2/5 - -2/5 = \boxed{4/5}$$

6. (10 pts) You impose a coordinate system on a ~~hand~~ hot sand beach and find the temperature at each point is given by $T(x,y) = x^2 + y^2 + 4y + 90$ degrees Fahrenheit, where $x$ and $y$ are in feet.

Assume you walk barefoot half way around a circular path, $C$, from $(3,0)$ to $(-3,0)$ in such a way that your motion is parameterized by $\mathbf{r}(t) = \langle 3\cos(t), 3\sin(t)\rangle$ where $t$ is in seconds with $0 \le t \le \pi$.

GIVE UNITS FOR ALL YOUR ANSWERS.

(a) Give the direction and magnitude of the greatest rate of change at the point (3,0). (This question has nothing to do with $C$).

$\nabla T(x,y) = \langle 2x, 2y+4\rangle$ $\quad \boxed{\nabla T(3,0) = \langle 6, 4\rangle}$ Direction

$|\nabla T(3,0)| = \sqrt{6^2 + 4^2} = \boxed{\sqrt{52}\ \frac{{}^\circ F}{ft}}$

(b) As you walk along the curve $C$, what is the rate of change of temperature with respect to time at $t = \pi/4$ seconds?

$$\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} = 2x(-3\sin(t)) + (2y+4)(3\cos(t))$$
$$= 2\cdot\frac{3\sqrt{2}}{2}\left(-3\tfrac{\sqrt{2}}{2}\right) + \left(2\cdot\tfrac{3\sqrt{2}}{2}+4\right)\left(3\tfrac{\sqrt{2}}{2}\right)$$
$$= -9 + (3\sqrt{2}+4)\,3\sqrt{2}/2$$
$$= -9 + 9 + 6\sqrt{2}$$
$$\boxed{6\sqrt{2}\ {}^\circ F/sec}$$

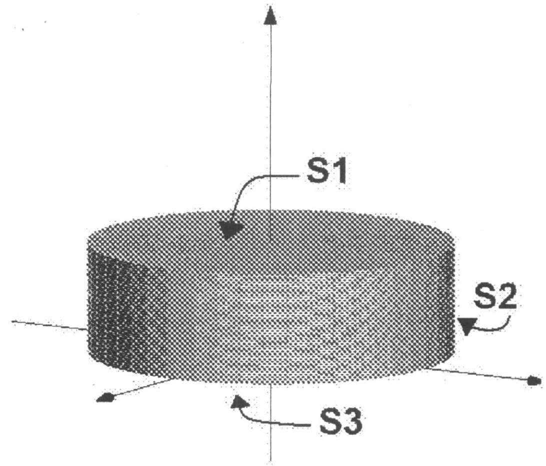
(c) Compute $\frac{1}{3\cancel{6}\pi}\int_C T(x,y)ds$. (This is the average temperature along $C$).

$$\frac{1}{3\pi}\int_0^\pi (9 + 12\sin(t) + 90)\sqrt{(-3\sin(t))^2 + (3\cos(t))^2}\,dt$$
$$\frac{3}{3\pi}\int_0^\pi 99 + 12\sin(t)\,dt$$
$$\frac{1}{\pi}\left[99t - 12\cos(t)\Big|_0^\pi\right] = \frac{1}{\pi}[(99\pi - -12) - (0 - 12)]$$
$$= \frac{1}{\pi}[99\pi + 24]$$
$$= \boxed{99 + \frac{24}{\pi}}\ {}^\circ F$$

7. (10 pts) Consider the vector field $\mathbf{F}(x, y, z) = \langle 1 + 3y^3, -6x, -3z^2 + x\rangle$ on $\mathbb{R}^3$. Let $S$ be the **CLOSED** surface that consists of the cylinder $x^2 + y^2 = 9$ for $0 \le z \le 1$ and the parts of the planes $z = 0$ and $z = 1$ that are inside the cylinder. Find the flux of $\mathbf{F}$ across $S$.

That is, compute $\displaystyle\iint_S \mathbf{F} \cdot d\mathbf{S}$.

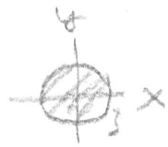
You may pick either the outward or inward orientation for $S$, but in the end I want you to tell me if the net flux of $\mathbf{F}$ across $S$ is outward or inward.




Divergence Thm

outward orientation

$$\iiint_E \operatorname{div} \vec{F}\, dV$$




$$\iiint_E 0 + 0 - 6z \, dV$$

$$\int_0^{2\pi}\int_0^3\int_0^1 -6z \; r\,dz\,dr\,d\theta$$

$$\int_0^{2\pi} d\theta \int_0^3 r\,dr \int_0^1 -6z\,dz$$

$$2\pi \quad \tfrac{1}{2}r^2\Big|_0^3 \quad -3z^2\Big|_0^1$$

$$2\pi \quad \frac{9}{2} \quad (-3) \quad = -27\pi$$

NET FLUX OF **F** ACROSS $S$ (circle one): INWARD OR OUTWARD