A K-THEORETIC CLASSIFICATION OF TOTALLY REAL IMMERSIONS INTO $\mathbf{C}^n$.

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Abstract. Totally real immersions of an n-dimensional smooth manifold M into $\mathbf{C}^n$ exist, provided that the complexified tangent bundle of M is trivial. A bijection between the set of isotopy classes of such immersions and the complex K-group $K^1(M)$ is constructed.

Gromov [4] and Lees [7] have given a homotopy classification of totally real and Lagrangian immersions into complex and symplectic manifolds. Our aim here is to show that if the codomain in $\mathbf{C}^n$ then this classification has a simple K-theoretic discription.

We begin with a discussion of some elementary facts in linear algebra. Let V (resp W be an n-dimensional real (resp. complex) vector space. An $\mathbf{R}$-linear injection $h : V \to W$ is called *totally real* if its image $h(V)$ contains no non-trivial complex subspace. Let $V^{\mathbf{C}}$ denote the complexification of V and let $h^{\mathbf{C}} : V^{\mathbf{C}} \to W$ be the complex lienar map defined by the formula $h^{\mathbf{C}}(u + iv) = h(u) + ih(v)$. It is easily verified that h is a totally real injection if and only if $h^{\mathbf{C}}$ is a compelx vector space isomorphism. In fact, the correspondence $h \mapsto h^{\mathbf{C}}$ is a bijection between the set of totally real injections from V into W and the set of vector space isomorphisms from $V^{\mathbf{C}}$ into W. (An inverse is given by composition with the canonical totally real injection $V \to V^{\mathbf{C}}$.)

It follows that given a fixed totally real injection, say $k : V \to W$, the mapping $h \mapsto A = h^{\mathbf{C}} \circ (k^{\mathbf{C}})^{-1}$, which associates to each totally real injection an element in the group $\mathrm{GL}(W)$ of complex-linear automorphisms of W, is a bijection. Hence, each continuous family h_t, $0 \le t \le 1$, of totally real injections corresponds to a continuous family A_t, $0 \le t \le 1$, of automorphisms of W.

The above discussion extends to vector bundle maps in the obvious way. In particular, for M is a smooth n-dimensional manifold, an immersion $f : M \to \mathbf{C}^n$ is said to be *totally real* if the map $df^{\mathbf{C}} : TM^{\mathbf{C}} \to f^*(T\mathbf{C}^n) = M \times \mathbf{C}^n$ is an isomorphism of complex vector bundles. An *isotopy of totally real immersions* is a continuous family $f_t : M \to \mathbf{C}^n$, $0 \le t \le 1$, of smooth

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immersions which induces a homotopy $df_t^{\mathbf{C}} : TM^{\mathbf{C}} \to M \times \mathbf{C}^n$ of smooth bundle isomorphisms.

By Gromov [5, page 332], the set of isotopy classes of totally real immersion of M into $\mathbf{C}^n$ is in 1-1 correspondence with the set of homotopy classes of complex bundle isomorphisms from $TM^{\mathbf{C}}$ into $M \times \mathbf{C}^n$. It follows that a necessary and sufficient conditiaon for the existence of a totally real immersion of M into $\mathbf{C}^n$ is that the complexified tangent bundle $TM^{\mathbf{C}}$ be trivial. We can now state the main result of this note.

Theorem. *Let M be a smooth n-dimensional manifold with trivial complexified tangent bundle. Then there is a 1-1 correspondence between the set of isotopy classes of totally real immersion of M into $\boldsymbol{C}^n$ and the ocmplex K-group $K^1(M)$.*

Proof. Fix a totally real immersion, say $f : M \to \mathbf{C}^n$. It follows form the previous discussion that the bundle isomorphism $df^{\mathbf{C}} : TM^{\mathbf{C}} \to M \times \mathbf{C}^n$ can be used to define a 1-1 correspondence between the set of smooth bundle isomorphism from $TM^{\mathbf{C}}$ into $M \times \mathbf{C}^n$ and the set of smooth maps from M into $\mathrm{GL}(n, \mathbf{C})$.. This correspondence allous us to identify the set of isotopy classes of bundle isomorphisms with the underlying set of the group $[M, \mathrm{GL}(n, \mathbf{C})]$ of homotopy classes of continuous maps from M into the group $\mathrm{GL}(n, \mathbf{C})$.

The inclusion $\mathrm{U}(n) \subset \mathrm{GL}(n, \mathbf{C})$ is a homotopy equivalence, and the standard includions $\mathrm{U}(n) \subset \mathrm{U}(n+k)$ induce isomorphisms $\pi_k(\mathrm{U}(n)) \simeq \pi_k(\mathrm{U}(n+m)$ for all $k \leq 2n-1$ and all $m \geq 0$ (see [2, Theorem 4.1, p. 82]). Set $\mathrm{U} = \lim_{m\to\infty} \mathrm{U}(m)$. Because $\dim(M) < 2n-1$, it follows from [8, Cor. 14,p. 402] that there are group isomorphisms

$$[M, \mathrm{GL}(n, \mathbf{C})] \simeq [M, \mathrm{U}(n)] \simeq [M, \mathrm{U}] \,.$$

But $[M, \mathrm{U}]$ is the complex K-group $K^1(M)$ (see [1]). □

Remarks: (1) The above theorem holds if "totally real" is replaced by "Lagrangian" and $\mathbf{C}^n$ is equipped with the standard symplectic form. Indeed, every Lagrangian immersion into $\mathbf{C}^n$ is automatically totally real. Hence, the Gromov-Lees Theorem [7] shows that isotopy classes of Lagrangian immersions are in 1-1 corresponednce with isotopy classes of totally real immersions.

(2) Similarly, the results of [3] and [5], [4] show that the above thoerem holds with "totally real" replace by "Legendre" and $\mathbf{C}^n$ replaced by either of the contact manifolds $\mathbf{R}^{2n+1}$ or S^{2n+1} with their standard contact structures.

(3) If M is the n-sphere S^n, then $[S^n, \mathrm{U}] = \pi_n(\mathrm{U})$, which is $\mathbf{Z}$ for n odd an 0 for n even. In particular, there are $\mathbf{Z}$ distinct totally real immersions of S^3 into $\mathbf{C}^3$. This result was obtained by Stout and Zame [9] by slightly different means and was the motivation for the present note. Weinstein [10] has given an explicit construction of Lagrangian immersions of S^n into

$\mathbf{C}^n$. These immersions have non-trivial self-intersection numbers and are therefore not isotopic to embeddings. Therefore, even sphers cannot embed as totally real submanifolds of $\mathbf{C}^n$. In fact, it is stated in [5] and proved in both [6] and [9] that the only sphere which can embed as a totally real submanifold of $\mathbf{C}^n$ is the three-sphere.

(4) In many case $K^1(M)$ can be computed. For instance, in [1, Sec 2.5] it is shown that if $H^\bullet(M, \mathbf{Z})$ is torsion free then $K^1(M)$ is itself torsion free and have rank equal to the sum of the odd dimensionall Betti numbers of M.

References

[1] M. F. Atiyah and F. Hirzebruch. Vector bundles and homogeneous spaces. In *Proc. Sympos. Pure Math.*, volume 3, pages 7–38, Providence, R.I., 1961. Amer. Math. Soc.
[2] d. Husemoller. *Fibre Bundles*. McGraw-Hill, New York-London-Sydney, 1966.
[3] T. Duchamp. The classification of legendre immersions. 1982.
[4] M. F. Gromov. *Partial differential relations*, volume 9 of *Ergeb. Math. Grenzgeb.(3)*. Springer-Verlag, Berlin-New York, 1986.
[5] M. L. Gromov. Convex integration of differential relations i. *Math. USSR, Izvestija*, 7:329–343, 1973.
[6] Toshio Kawashima. Some remarks on lagrangian imbeddings. *J. Math. Soc. Japan*, 33:281–294, 1981.
[7] J. A. Lees. On the classification of lagrange immersions. *Duke Mathematics journal*, 43:217–224, 1976.
[8] E. H. Spanier. *Algebraic Topology*. McGraw-Hill, New York, 1966.
[9] E. L. Stout and W. R. Zame. A stein manifold topologically but not holomorphically equivalent to a domain in $\mathbf{c}^n$. *Adv. in Math.*, 60(2):154–160, 1986.
[10] A. Weinstein. *Lectures on Symplectic Manifolds*, volume 29 of *Regional Conference Series in Mathematics*. Amer. Math. Soc., Providence, 1977.

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