Partial Answers to Practice Problems on Limits

3. **(a)** $\lim_{n\to\infty} \frac{n^3-1}{3n^3+n-4} = \frac{1}{3}$

(b) $\lim_{n\to\infty} \frac{n\cos n}{n^2+24} = 0$

(c) $\lim_{n\to\infty} \frac{2^n+1}{2^n-5} = 1$

(d) $\lim_{n\to\infty}[(n+1)^{1/3} - n^{1/3}] = 0$

(e) $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^2}{n^3} = \lim_{n\to\infty}[\frac{1}{n^3}\frac{n(n+1)(2n+1)}{6}] = \frac{1}{3}$

(f) $\lim_{n\to\infty} \left(\frac{n^3}{2n^2-1} - \frac{n^2}{2n+1}\right) = \frac{1}{4}$

4. **(a)** $n!$ unbounded, monotone increasing

(b) $\sin\frac{n\pi}{2}$ bounded (between -1 and 1), not monotone

(c) $(-1)^n + \frac{1}{n}$ bounded (between -1 and $3/2$), not monotone

(d) r^n if $|r| > 1$ unbounded; monotone increasing if $r > 0$, otherwise not

5. We have always spoken of *the* limit of a sequence as though it were impossible for a sequence to have more than one limit. Prove that this is so.

Suppose the sequence $\{s_n\}$ has two limits, A_1 and A_2. Then given any $\epsilon > 0$, there is a number N_1 and a number N_2 such that $|s_n - A_1| < \epsilon$ whenever $n > N_1$ and $|s_n - A_2| < \epsilon$ whenever $n > N_2$. Suppose, in order to show a contradiction, that $A_1 \neq A_2$. Let $\epsilon = |A_1 - A_2|/4 > 0$. Then for $n > \max\{N_1, N_2\}$,

$$|s_n - A_1| < \frac{|A_1 - A_2|}{4} \Longrightarrow -\frac{|A_1 - A_2|}{4} < A_1 - s_n < \frac{|A_1 - A_2|}{4}$$

$$|s_n - A_2| < \frac{|A_1 - A_2|}{4} \Longrightarrow -\frac{|A_1 - A_2|}{4} < s_n - A_2 < \frac{|A_1 - A_2|}{4}$$

Adding these two inequalities, we find

$$-\frac{|A_1 - A_2|}{2} < A_1 - A_2 < \frac{|A_1 - A_2|}{2} \Longrightarrow |A_1 - A_2| < \frac{|A_1 - A_2|}{2}.$$

For $A_1 \neq A_2$ we can divide the above inequality by $|A_1 - A_2|$ to obtain $1 < \frac{1}{2}$, which is a contradiction. Therefore the two limits must be the same. $\square$