Sample Solutions for Practice Problems on Convergence of Sequences and Series, Pointwise vs. Uniform Convergence

1. Test each of the following series for convergence.

   (a) $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$

   Ratio test: $\left|\frac{u_{n+1}}{u_n}\right| = \frac{(n+1)^2}{(2n+2)(2n+1)} \to \frac{1}{4} < 1$, so the series converges.

   (b) $\sum_{n=1}^{\infty} \frac{n!}{3^n}$

   Ratio test: $\left|\frac{u_{n+1}}{u_n}\right| = \frac{n+1}{3} \to \infty$, so the series diverges.

   (c) $\sum_{n=1}^{\infty} \frac{n^{n+1/n}}{(n+1/n)^n}$

   Terms do not go to 0, so series cannot converge. $\frac{n^{n+1/n}}{(n+1/n)^n} > \frac{n^n}{(n+1)^n} = \left(\frac{1}{1+1/n}\right)^n \to 1/e \neq 0$.

   (d) $\sum_{n=1}^{\infty} \frac{(n!)^2}{2^{n^2}}$

   Ratio test: $\left|\frac{u_{n+1}}{u_n}\right| = \frac{(n+1)^2}{2^{2n+1}} \to 0 < 1$, so the series converges.

   (e) $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^n}$

   For $n > e^2$, $\ln n > 2$, so $(\ln n)^n > 2^n$, and $\frac{1}{(\ln n)^n} < \frac{1}{2^n}$. Since the geometric series $\sum \frac{1}{2^n}$ converges, this series does also by comparison test.

2. If $\sum a_n$ and $\sum b_n$ are absolutely convergent, show that $\sum(a_n + b_n)$ is also. If $\sum a_n$ is absolutely convergent, show that $\sum a_n^2$ and $\sum a_n/(1 + a_n)$ (where $a_n \neq -1$ for any n) are as well.

   Let $A = \lim_{n\to\infty} \sum_{k=1}^{n} |a_k|$ and $B = \lim_{n\to\infty} \sum_{k=1}^{n} |b_k|$. These limits exist since $\sum a_k$ and $\sum b_k$ are absolutely convergent. Since the limit of a sum of two sequences is the sum of the limits, $A+B = \lim_{n\to\infty} \left(\sum_{k=1}^{n} |a_k| + \sum_{k=1}^{n} |b_k|\right) = \lim_{n\to\infty} \sum_{k=1}^{n}(|a_k| + |b_k|)$. By the triangle inequality, each term $|a_k + b_k| \leq |a_k| + |b_k|$ and since $\sum(|a_k| + |b_k|)$ is convergent, $\sum |a_k + b_k|$ must be as well; that is $\sum(a_k + b_k)$ is absolutely convergent.

   If $\sum a_n$ is absolutely convergent, then $a_n \to 0$ as $n \to \infty$, so for n sufficiently large $|a_n| < 1$. For such n, $|a_n^2| < |a_n|$, so by the comparison test $\sum a_n^2$ converges absolutely. Also, for n sufficiently large $|a_n| < \frac{1}{2}$ so that $|1 + a_n| > \frac{1}{2}$. For such n, $|a_n/(1+a_n)| < 2|a_n|$ and since $\sum 2|a_n| = 2\sum |a_n|$ is convergent, $\sum |a_n/(1 + a_n)|$ is as well.

3. Discuss the convergence of the following series. State when the series converges absolutely and when it converges conditionally.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n - \ln n}$

The series does not converge absolutely since $\frac{1}{n-\ln n} > \frac{1}{n}$, and $\sum \frac{1}{n}$ diverges. To see if it converges conditionally, apply the alternating series test. First note that $|u_n| = \frac{1}{n-\ln n} \to 0$ as $n \to \infty$. Then check that $|u_{n+1}| \leq |u_n|$. To see this, note that

$$\frac{1}{n+1-\ln(n+1)} \leq \frac{1}{n-\ln n} \iff n+1-\ln(n+1) \geq n - \ln n \iff$$

$$1 > \ln(n+1) - \ln(n) = \ln\left(\frac{n+1}{n}\right).$$

Since this last quantity approaches $\ln(1) = 0$ as $n \to \infty$, for n large enough it will certainly be less than 1. (In fact, even for $n = 1$, $\ln 2 < 1$.) Therefore, by the alternating series test, the series converges conditionally.

(b) $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + (-1)^{n-1}\frac{x^n}{n} + \ldots$

For $|x| < 1$, this series converges absolutely since $\left|\frac{x^n}{n}\right| < |x|^n$ and $\sum |x|^n$ converges. For $|x| > 1$, the series diverges since the terms do not approach 0: $\left|\frac{x^n}{n}\right| \to \infty$ as $n \to \infty$. For $x = -1$, the series becomes $-1 - \frac{1}{2} - \frac{1}{3} - \ldots - \frac{1}{n} - \ldots$, which diverges. For $x = 1$, the series is $1 - \frac{1}{2} + \frac{1}{3} - \ldots + (-1)^{n-1}\frac{1}{n}$, which we showed converges to $\ln 2$. Hence for $x = 1$, the series is conditionally convergent.

4. Show that the sequence of functions $f_n(x) = \frac{x^{2n}}{1+x^{2n}}$ converges pointwise but not uniformly on the entire real line.

$$f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} 0 & \text{for } |x| < 1 \\ 1/2 & \text{for } |x| = 1 \\ 1 & \text{for } |x| > 1 \end{cases}.$$

The convergence cannot be uniform on the entire real line or on any interval containing ± 1, since if it were the limiting function $f(x)$ would have to be continuous there, and it is not.

5. Show that while the series $\sum_{n=1}^{\infty} \frac{\sin(2n\pi x)}{n^2}$ converges uniformly on the entire real line, the series cannot be differentiated term by term on any open interval.

Since $\left|\frac{\sin(2n\pi x)}{n^2}\right| \leq \frac{1}{n^2}$ for all x and $\sum \frac{1}{n^2}$ is a convergent series of positive numbers, it follows from the Weierstrass M-test (Theorem II in the text) that this series is absolutely convergent on the entire real line. If we differentiate term by term, however, then we obtain the series $\sum_{n=1}^{\infty} \frac{2\pi \cos(2n\pi x)}{n}$. There are some isolated points for which this series converges; for example if $x = 1/2$, then the series becomes $2\pi \sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n} = 2\pi \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, which converges conditionally. There are some other points at which it does not converge. If x is any integer then $\cos(2\pi nx) = 1$ for all n, and $\sum_{n=1}^{\infty} \frac{2\pi}{n}$ is divergent. The question of uniform convergence on some interval, which might justify term by term differentiation, is more delicate. You will not be responsible for this. Sorry, I didn't notice this when I posted the problem.

6. Use the fact that $\frac{1}{1+t} = 1 - t + t^2 - t^3 + \ldots$ if $-1 < t < 1$ and the fact that $\ln(1+x) = \int_0^x \frac{1}{1+t}\, dt$ to derive the series expansion

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots, \quad |x| < 1.$$

If $0 < r < 1$, then the series $1 - t + t^2 - t^3 + \ldots$ converges uniformly on $[-r, r]$, since the nth term is bounded in absolute value by r^n, which is the nth term in a convergent geometric series. Therefore, the series can be integrated term by term on any subinterval of $[-r, r]$. If $0 < |x| \le r$, then

$$\begin{aligned} \ln(1+x) = \int_0^x \frac{1}{1+t}\, dt &= \int_0^x 1\, dt - \int_0^x t\, dt + \int_0^x t^2\, dt - \int_0^x t^3\, dt + \ldots \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots. \end{aligned}$$

7. Find the interval of convergence of the following power series.

(a) $\sum_{n=1}^{\infty} n!(x-3)^n$

This converges only for $x = 3$, since if for any other x, the terms $n!(x-3)^n$ do not converge to 0, so the series cannot converge.

(b) $\sum_{n=1}^{\infty} \frac{x^n}{n(n+1)}$

Ratio test: $\left|\frac{x^{n+1}}{(n+1)(n+2)} \cdot \frac{n(n+1)}{x^n}\right| = \left|x \cdot \frac{n}{n+2}\right| \to |x|$. If $|x| < 1$, then the series converges. If $|x| > 1$, then the series diverges. Hence the interval of convergence is $(-1, 1)$. [Note that the series also converges if $x = \pm 1$, since $\frac{1}{n(n+1)} < \frac{1}{n^2}$, and $\sum \frac{1}{n^2}$ converges. Still the *interval of convergence* is defined as the open interval.