Finite Element Methods in One Dimension

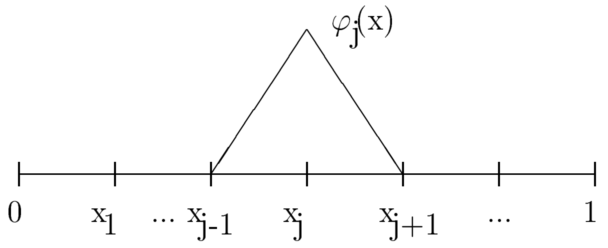
$$\mathcal{L}u(x) \equiv -\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u(x) = f(x) \tag{1}$$

$$x(0) = 0, \quad x(1) = 0 \tag{2}$$

Divide the interval $[0,1]$ into subintervals and approximate $u(x)$ by a *continuous piecewise linear* function:

$$u(x) \approx \sum_{j=1}^{n-1} c_j \varphi_j(x), \tag{3}$$

where $\varphi_1(x), \ldots \varphi_{n-1}(x)$ form a basis for the set of continuous piecewise linear functions with value 0 at the endpoints.



Note the $\varphi_1(x), \ldots, \varphi_{n-1}(x)$ are linearly independent and that any continuous piecewise linear function $\ell(x)$ on this grid with value 0 at the endpoints can be written as a linear combination of these functions: $\ell(x) = \sum_{j=1}^{n-1} \ell(x_j)\varphi_j(x)$.

We want to choose the coefficients $c_1, \ldots, c_{n-1}$ in (3) so that the function there approximately satisfies the differential equation (1), but note that while each $\varphi_j(x)$ is continuous, its first derivative is discontinuous at the nodes, and its second derivative is undefined in the usual sense. Instead of trying to satisfy the differential equation (1) in a pointwise sense, we will express this equation in the *weak form*:

$$\langle \mathcal{L}u, v\rangle = \langle f, v\rangle \tag{4}$$

for all functions $v(x)$, where the inner product of two functions is defined by

$$\langle v, w\rangle = \int_0^1 v(x)w(x)\,dx.$$

Substituting for $\mathcal{L}$ the differential operator defined in (1), and integrating by parts, equation (4) becomes:

$$\int_0^1 \left(-\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u(x)\right) v(x)\,dx =$$

$$-p(x)\frac{du}{dx}v(x)\,\Big|_0^1 + \int_0^1 p(x)u'(x)v'(x)\,dx + \int_0^1 q(x)u(x)v(x)\,dx = \int_0^1 f(x)v(x)\,dx.$$

If we require also that $v(x)$ satisfy the given boundary conditions (2), then this becomes

$$\int_0^1 p(x)u'(x)v'(x)\,dx + \int_0^1 q(x)u(x)v(x)\,dx = \int_0^1 f(x)v(x)\,dx. \tag{5}$$

The exact solution $u(x)$ satisfies equation (5) for all functions $v(x)$ that vanish at the endpoints of the interval. We cannot expect our approximate solution to satisfy equation (5) for *all* such functions $v(x)$, but perhaps we can force it to satisfy (5) for all continuous piecewise linear functions; i.e., for all linear combinations of $\varphi_1(x), \ldots, \varphi_{n-1}(x)$. This gives us $n-1$ equations for the $n-1$ unknowns, $c_1, \ldots, c_{n-1}$:

$$\int_0^1 p(x)\left(\sum_{j=1}^{n-1} c_j\varphi_j'(x)\right)\varphi_i'(x)\,dx + \int_0^1 q(x)\left(\sum_{j=1}^{n-1} c_j\varphi_j(x)\right)\varphi_i(x)\,dx =$$

$$\sum_{j=1}^{n-1} c_j \int_0^1 \Big(p(x)\varphi_j'(x)\varphi_i'(x) + q(x)\varphi_j(x)\varphi_i(x)\Big)\,dx = \int_0^1 f(x)\varphi_i(x)\,dx, \quad i = 1, \ldots, n-1. \tag{6}$$

We can write this set of equations in matrix form as $A\mathbf{c} = \mathbf{f}$, where $\mathbf{c}$ is the vector of unknown coefficients $(c_1, \ldots, c_{n-1})^T$, and

$$A_{ij} = \int_0^1 \Big(p(x)\varphi_j'(x)\varphi_i'(x) + q(x)\varphi_j(x)\varphi_i(x)\Big)\,dx, \quad \mathbf{f}_i = \int_0^1 f(x)\varphi_i(x)\,dx. \tag{7}$$

In order to determine the entries of the matrix A and right-hand side vector $\mathbf{f}$, we must first write down explicit expressions for $\varphi_i(x)$ and $\varphi_i'(x)$:

$$\varphi_i(x) = \begin{cases} (x - x_{i-1})/(x_i - x_{i-1}), & x \in [x_{i-1}, x_i] \\ (x_{i+1} - x)/(x_{i+1} - x_i), & x \in [x_i, x_{i+1}] \\ 0 & \text{otherwise} \end{cases}$$

$$\varphi_i'(x) = \begin{cases} 1/(x_i - x_{i-1}), & x \in [x_{i-1}, x_i] \\ -1/(x_{i+1} - x_i), & x \in [x_i, x_{i+1}] \\ 0 & \text{otherwise} \end{cases}$$

If $p(x)$, $q(x)$, and $f(x)$ are simple enough functions, we can now compute the integrals in (7) exactly. If not, then we can use quadrature formulas, such as a one-point Gauss quadrature formula (the midpoint rule) over each subinterval. Note that if $|i-j| > 1$, then the subintervals over which φ_i and φ_j are nonzero do not overlap, and the same holds for φ_i' and φ_j'. It follows that $A_{ij} = 0$ for $|i-j| > 1$; i.e., A is a *tridiagonal* matrix. Note also that for this problem $A_{ij} = A_{ji}$ for all i, j, so A is a *symmetric* matrix.

Let us consider the simplest case in which $p(x) \equiv 1$, $q(x) \equiv 0$, and the nodes are equally spaced: $x_i - x_{i-1} \equiv h$. Then

$$\begin{aligned} A_{ii} &= \int_0^1 (\varphi_i'(x))^2\,dx \\ &= \int_{x_{i-1}}^{x_i} \left(\frac{1}{x_i - x_{i-1}}\right)^2 dx + \int_{x_i}^{x_{i+1}} \left(\frac{-1}{x_{i+1} - x_i}\right)^2 dx = \frac{2}{h}, \end{aligned}$$

$$
\begin{aligned}
A_{i,i+1} &= \int_0^1 \varphi_i'(x)\varphi_{i+1}'(x)\,dx \\
&= \int_{x_i}^{x_{i+1}} \left(\frac{-1}{x_{i+1}-x_i}\right)\left(\frac{1}{x_{i+1}-x_i}\right)dx = -\frac{1}{h}.
\end{aligned}
$$

If we divide each side by h, the linear system becomes

$$
\frac{1}{h^2}\begin{pmatrix} 2 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix} = \frac{1}{h}\begin{pmatrix} \langle f, \varphi_1 \rangle \\ \vdots \\ \vdots \\ \langle f, \varphi_{n-1} \rangle \end{pmatrix}.
$$

Note that the matrix on the left is the same one that would arise from a centered finite difference approximation. Finite element schemes often turn out to be almost the same as finite difference formulas. Sometimes, however, what appear to be minor differences in the way boundary conditions are handled or in the way the right hand side vector is formed turn out to be important for accuracy.