Assignment 1.

Due Wed., Apr. 18.

Objectives: To solve Laplace's equation using a boundary integral equation.

(1) Let S be a C^2 simple closed curve. Show that

$$\lim_{\substack{Q \to \hat{Q} \\ Q \in S}} \frac{\partial}{\partial \mathbf{n}_Q} \left(\log |Q - \hat{Q}| \right) = \frac{1}{2} \kappa(\hat{Q}),$$

where $\kappa(\hat{Q})$ denotes the curvature of S at the point $\hat{Q}$, and $\partial / \partial \mathbf{n}_Q$ denotes the derivative in the direction of the outward pointing normal at Q.

(2) Write a code in MATLAB or another language to solve $\Delta u = 0$ on the region D inside the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

for given Dirichlet data prescribed on the boundary. Use an integral equation formulation. I recommend the equation:

$$\mu(\hat{Q}) + \frac{1}{\pi} \int_S \mu(Q) \frac{\partial}{\partial \mathbf{n}_Q} \left(\log |Q - \hat{Q}| \right) \, ds(Q) = 2 f(\hat{Q}), \quad \hat{Q} \in S,$$

where $\mu(Q)$ is the jump in the solution u at the point Q. The solution at any point P within D is then given by:

$$u(P) = \frac{1}{2\pi} \int_S \mu(Q) \frac{\partial}{\partial \mathbf{n}_Q} \left(\log |Q - P| \right) \, ds(Q).$$

Test your code by taking the exact solution to be

$$u(x, y) = 1 + 2 \left(\frac{x - x_0}{(x - x_0)^2 + (y - y_0)^2} \right)$$

for some point (x_0, y_0) outside the ellipse. Once you have computed the density function $\mu(Q)$, use the integral representation to evaluate $u(P)$ at several points inside the domain D. Try to determine how the error decreases with the mesh spacing h between the boundary points. Try ellipses with different eccentricities to see how the eccentricity of the ellipse affects the accuracy of the approximate solution.

Finally, try using the integral representation for $u(P)$ to evaluate the solution at points near the boundary of the ellipse. Are your results very accurate? Try to explain why or why not?