Assignment 2.

Due Wed., Apr. 25.

Objectives: To implement the GMRES algorithm and to solve the exterior Dirichlet problem for Laplace's equation using a boundary integral equation.

(1) Replace the linear system solver in your code from Assignment 1 to use the GMRES algorithm (if you are not already using this). Report on how the number of GMRES iterations is affected by the mesh size (for a fixed ellipse) and how it is affected by the eccentricity of the ellipse.

(2) Convert your code from Assignment 1 (with GMRES as the linear system solver) to solve the *exterior* Dirichlet problem: $\Delta u = 0$ outside the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

for given Dirichlet data prescribed on the boundary. An appropriate integral equation for this problem is:

$$-\mu(\hat{Q}) + \frac{1}{\pi}\int_S \mu(Q)\left[\frac{\partial}{\partial \mathbf{n}_Q}\left(\log|Q-\hat{Q}|\right)+1\right] ds(Q) = 2f(\hat{Q}), \quad \hat{Q}\in S,$$

where $\mu(Q)$ is the jump in the solution u at the point Q. The solution at any point P outside the ellipse D is then given by:

$$u(P) = \frac{1}{2\pi}\int_S \mu(Q)\left[\frac{\partial}{\partial \mathbf{n}_Q}\left(\log|Q-P|\right)+1\right] ds(Q).$$

Test your code by taking the exact solution to be

$$u(x,y) = 1 + 2\left(\frac{x-x_0}{(x-x_0)^2+(y-y_0)^2}\right)$$

for some point (x_0, y_0) inside the ellipse. Once you have computed the density function $\mu(Q)$, use the integral representation to evaluate $u(P)$ at several points outside the ellipse. Again determine how the error decreases with the mesh spacing h between the boundary points and how the accuracy is affected by the eccentricity of the ellipse.

Use the integral representation for $u(P)$ to evaluate the solution at points near the boundary of the ellipse and explain what you find.