On p-adic Artin L-functions II

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1 Introduction

Let p be a prime. Iwasawa's famous conjecture relating Kubota-Leopoldt p-adic L-functions to the structure of certain Galois groups has been proven by Mazur and Wiles in [10]. Wiles later proved a far-reaching generalization involving p-adic L-functions for Hecke characters of finite order for a totally real number field in [14]. As we discussed in [5], an analogue of Iwasawa's conjecture for p-adic Artin L-functions can then be deduced. The formulation again involves certain Galois groups. However, one can reformulate this result in terms of Selmer groups for the Artin representations. There are several advantages to such a reformulation. First of all, it fits perfectly into the much broader framework described in [6] which relates the p-adic L-function for a motive to the corresponding Selmer group. The crucial assumption in [6] that the motive be ordinary at p (or at least potentially ordinary) is satisfied by an Artin motive and all of its Tate twists.

A second advantage of a reformulation involving Selmer groups is that the issue of how to define the μ-invariant becomes resolved in a natural and transparent way. Thirdly, the arguments in [5] can be simplified. In particular, there is no need for singling out the class of Artin representations which are called type S in [5]. The purpose of this paper is to explain these advantages.

Suppose that F is a totally real number field. Consider an Artin representation

$$\rho : \ G_F \longrightarrow \mathrm{Aut}_{\mathcal{E}}(V) \ ,$$

where G_F is the absolute Galois group of F and V is a finite dimensional vector space over a finite extension $\mathcal{E}$ of $\mathbf{Q}_p$. We will assume that ρ is totally even. This means that ρ factors through $\Delta = \mathrm{Gal}(K/F)$, where K is a finite extension of F which is also totally real. Let $\mathcal{O}$ be the ring of integers of $\mathcal{E}$. Let T be an $\mathcal{O}$-lattice in V which is G_F-invariant. Furthermore, let $D = V/T$, a discrete $\mathcal{O}$-module.

Let F_∞ denote the cyclotomic $\mathbf{Z}_p$-extension of F. The Selmer group associated to D over F_∞ is defined by

$$\mathrm{Sel}_D(F_\infty) \;=\; \ker\Big(H^1(F_\infty, D) \longrightarrow \prod_{\eta \nmid p} \; H^1(F_{\infty,\eta}, D)\Big) \;\; .$$

Here η runs over all the primes of F_∞ except for the finitely many primes lying over p. The archimedean primes are included in the product, although this is only important when $p = 2$. One defines the field $F_{\infty,\eta}$ to be the union of the η-adic completions of the finite extensions of F contained in F_∞.

To relate the above definition to the way Selmer groups are defined in [6], note that if ρ is a totally even Artin representation of G_F over $\mathbf{C}$, then the Artin L-function $L(s,\rho)$ does not have a critical value at $s = 1$ in the sense of Deligne. However, its value at $s = 1 - n$ is critical in that sense when n is even and positive. One can write

$$L(1-n,\rho) = L(1,\rho(n))$$

where $\rho(n)$ is the n-th Tate twist. The underlying representation space for $\rho(n)$ over $\mathcal{E}$ is $V(n) = V \otimes \chi_F^n$, where $\chi_F : G_F \to \mathbf{Z}_p^\times$ is the p-power cyclotomic character. In the notation of [6], we have $F^+V(n) = V(n)$ when $n \geq 1$. (This is so for all the primes above p.) Let $T(n) = T \otimes \chi^n$ and $D(n) = V(n)/T(n)$. The corresponding Selmer group $\mathrm{Sel}_{D(n)}(F_\infty)$, as it is defined in [6], is just as above, but with $D(n)$ replacing D. Let $d = [F(\mu_q) : F]$, where $q = p$ when p is odd and $q = 4$ when $p = 2$. If we take $n \equiv 0 \pmod{d}$, then $D(n) \cong D$ for the action of G_{F_∞}. Thus, the two Selmer groups are then the same, although the action of $\mathrm{Gal}(F_\infty/F)$ on those groups is somewhat different. (See remark 2.11.)

Since $\mathrm{Sel}_D(F_\infty)$ is a discrete $\mathcal{O}$-module and $\Gamma_F = \mathrm{Gal}(F_\infty/F)$ acts naturally and continuously on it, we can regard $\mathrm{Sel}_D(F_\infty)$ as a discrete $\Lambda_{(\mathcal{O},F)}$-module, where $\Lambda_{(\mathcal{O},F)} = \mathcal{O}[[\Gamma_F]]$. It is not difficult to show that the Pontryagin dual $X_D(F_\infty)$ of $\mathrm{Sel}_D(F_\infty)$ is a finitely generated, torsion $\Lambda_{(\mathcal{O},F)}$-module. (See proposition 2.1.) We denote the characteristic ideal of that $\Lambda_{(\mathcal{O},F)}$-module by I_ρ. It is a principal ideal in the ring $\Lambda_{(\mathcal{O},F)}$. As the notation suggests, this ideal depends only on ρ, and not on the choice of the Galois-invariant $\mathcal{O}$-lattice T, as we show in proposition 2.4.

Another discrete $\Lambda_{(\mathcal{O},F)}$-module to be considered is $H^0(F_\infty, D)$. Its Pontryagin dual $Y_D(F_\infty)$ is clearly a finitely-generated $\mathcal{O}$-module and hence a torsion $\Lambda_{(\mathcal{O},F)}$-module. Let J_ρ denote the characteristic ideal of $Y_D(F_\infty)$. This ideal is nontrivial if and only if ρ has at least one irreducible constituent which factors through Γ_F.

The p-adic L-function associated to ρ will be denoted by $L_p(s,\rho)$. It is characterized by a certain interpolation property. In case ρ is 1-dimensional, this functions have been constructed by Deligne and Ribet in [1], by Cassou-Noguès in [3], and by Barsky in [2]. One can then define $L_p(s,\rho)$ if ρ has arbitrary dimension by using a classical theorem of Brauer from group theory.

One can associate to $L_p(s,\rho)$ a certain element θ_ρ in the fraction field of $\Lambda_{(\mathcal{O},F)}$. For an odd prime p, the Main Conjecture is the assertion that the fractional ideals $\Lambda_{(\mathcal{O},F)}\theta_\rho$ and $I_\rho J_\rho^{-1}$ are the same. This is proved in section 4 as a consequence of theorems of Wiles proved in [14]. For $p=2$, there is an extra power of 2 in the formulation, but this case appears to still be open.

2 Basic results concerning the Selmer group

We will prove several useful propositions. We continue to make the same assumptions as in the introduction. In particular, F is a totally real number field and ρ is a totally even Artin representation of G_F defined over a field $\mathcal{E}$, a finite extension of $\mathbf{Q}_p$. The ring of integers in $\mathcal{E}$ is denoted by $\mathcal{O}$.

We will use the traditional terminology for modules over a topological ring Λ. If S is a discrete Λ-module, and X is its Pontryagin dual, then we say that S is a cofinitely-generated Λ-module if X is finitely-generated. If X is a torsion Λ-module, we say that S is cotorsion.

Suppose that V, T, and $D=V/T$ are as in the introduction. Let $d=\dim_{\mathcal{E}}(V)$. As an $\mathcal{O}$-module, we have $D\cong(\mathcal{E}/\mathcal{O})^d$. The Selmer group $\mathrm{Sel}_D(F_\infty)$ is a discrete $\Lambda_{(\mathcal{O},F)}$-module.

Proposition 2.1. *The $\Lambda_{(\mathcal{O},F)}$-module $S_D(F_\infty)$ is cofinitely-generated and cotorsion.*

Proof. Suppose that ρ factor through $\mathrm{Gal}(K/F)$, where K is a totally real, finite Galois extension of F. Let $\Delta=\mathrm{Gal}(K_\infty/F_\infty)$ and let M_∞ be the maximal abelian pro-p extension of K_∞ which is unramified at the primes of K_∞ not lying over p (including the archimedean primes). One can consider $X(K_\infty)=\mathrm{Gal}(M_\infty/K_\infty)$ as a module over $\Lambda_{(\mathbf{Z}_p,K)}=\mathbf{Z}_p[[\Gamma_K]]$. A well-known theorem of Iwasawa asserts that $X(K_\infty)$ is finitely-generated and torsion as a $\Lambda_{(\mathbf{Z}_p,K)}$-module. The fact that it is torsion is equivalent to the fact that the so-called weak Leopoldt conjecture is valid for K_∞/K.

We have $H^0(K_\infty,D)=D$. Also, $H^1(\Delta,D)$ is finite. Hence the restriction map

$$H^1(F_\infty,D)\longrightarrow H^1(K_\infty,D)^\Delta \tag{1}$$

has finite kernel. We can identify Γ_K with $\mathrm{Gal}(F_\infty/K\cap F_\infty)$, a subgroup of Γ_F. The map (1) is Γ_K-equivariant. Now $H^1(K_\infty,D)=\mathrm{Hom}(\mathrm{Gal}(K_\infty^{ab}/K_\infty),D)$, where K_∞^{ab} is the maximal abelian extension of K_∞. It is clear that the image of $\mathrm{Sel}_D(F_\infty)$ under the map (1) is contained in $\mathrm{Hom}(X(K_\infty), D)$, which is a cofinitely-generated, cotorsion $\Lambda_{(\mathbf{Z}_p,K)}$-module according to Iwasawa's theorem. Since (1) has finite kernel, it follows that $\mathrm{Sel}_D(F_\infty)$ is cofinitely-generated and cotorsion as a $\Lambda_{(\mathbf{Z}_p,K)}$-module, and therefore as a $\Lambda_{(\mathcal{O},F)}$-module. ■

Remark 2.2. With the notation of the above proof, the cokernel of the map (1) is also finite. This follows from the fact that $H^2(\Delta, D)$ is finite. Assume that the order of $\mathrm{im}(\rho)$ is not divisible by p. We can then assume that $p \nmid |\Delta|$. In particular, $K \cap F_\infty = F$. Hence the map $\Gamma_K \to \Gamma_F$ is an isomorphism. We then have $\mathrm{Gal}(K_\infty/F) \cong \Delta \times \Gamma_F$. Furthermore, (1) is an isomorphism. The induced map

$$\mathrm{Sel}_D(F_\infty) \longrightarrow \mathrm{Hom}_\Delta(X(K_\infty), D) \tag{2}$$

is also easily verified to be an isomorphism. In addition to assuming $p \nmid |\Delta|$, assume that ρ is absolutely irreducible. Let e_ρ be the idempotent for ρ in $\mathcal{O}[\Delta]$. Then

$$\mathrm{Hom}_\Delta(X(K_\infty),\ D) \ \cong\ \mathrm{Hom}_{\mathcal{O}[\Delta]}\big(X(K_\infty) \otimes_{\mathbf{Z}_p} \mathcal{O},\ D\big) \ \cong\ \mathrm{Hom}_{\mathcal{O}[\Delta]}\big(e_\rho X(K_\infty) \otimes_{\mathbf{Z}_p} \mathcal{O},\ D\big) \ .$$

Thus, the $\Lambda_{(\mathcal{O},F)}$-modules $X_D(F_\infty)$ and $e_\rho X(K_\infty) \otimes_{\mathbf{Z}_p} \mathcal{O}$ are closely related. In fact, the characteristic ideal of the second module is I_ρ^d.

Remark 2.3. Suppose that X is any $\Lambda_{(\mathcal{O},F)}$-module and that I is the characteristic ideal of X. Let π be a generator of the maximal ideal of $\mathcal{O}$. The μ-invariant of X will be denoted by $\mu_F(X)$. It is the integer μ characterized by $I \subseteq \pi^\mu \Lambda_{(\mathcal{O},F)}$, $\ I \nsubseteq \pi^{\mu+1}\Lambda_{(\mathcal{O},F)}$. A conjecture of Iwasawa (at least for odd primes p) asserts that the μ-invariant of the $\Lambda_{(\mathcal{O},K)}$-module X_{K_∞} should vanish. This should be true even for $p = 2$. If this is so, then the proof of proposition 2.1 would show that $\mu_F\big(X_D(F_\infty)\big) = 0$.

It will be useful to have an alternative definition of $\mathrm{Sel}_D(F_\infty)$. Let Σ be a finite set of primes of F containing the archimedean primes, the primes lying over p, and the ramified primes for ρ. For each $v \in \Sigma$, define

$$\mathcal{H}_v(F_\infty, D) = \varinjlim_n \bigoplus_{\nu | v} H^1(F_{n,\nu}, D)$$

where, for each n, ν runs over the primes of F_n lying over v. The maps defining the direct limit are induced by the local restriction maps. If v is a finite prime, then

$$\mathcal{H}_v^1(F_\infty, D) \ = \ \bigoplus_{\nu | v} H^1(F_{\infty,\nu}, D)$$

where ν runs over the finite set of primes of F_∞ lying over v. The p-cohomological dimension of $G_{F_{\infty,\nu}}$ is 1, and so $H^2(F_{\infty,\nu}, D[\pi]) = 0$. It follows that $H^1(F_{\infty,\nu}, D)$ is $\mathcal{O}$-divisible. Now assume that $v \nmid p$. Then, according to proposition 2 in [6], the $\mathcal{O}$-corank of $H^1(F_{\infty,\nu}, D)$ is finite. It therefore follows that the Pontryagin dual of $\mathcal{H}^1_v(F_\infty, D)$ is a torsion-free $\mathcal{O}$-module of finite rank. Hence it is a free $\mathcal{O}$-module. It is also a $\Lambda_{(\mathcal{O},F)}$-module. Since its $\mathcal{O}$-rank is finite, it must be a torsion $\Lambda_{(\mathcal{O},F)}$-module whose μ-invariant vanishes.

If p is odd and v is an archimedean prime, then $\mathcal{H}^1_v(F_\infty, D) = 0$. However, if $p = 2$, then $\mathcal{H}^1_v(F_\infty, D)$ is nontrivial. More precisely, since all the F_n's are totally real, and ρ is totally even, we have

$$H^1(F_{n,\nu}, D) = H^1(\mathbf{R}, D) = D[2] \cong (\mathcal{O}/2\mathcal{O})^d$$

if ν is archimedean. One sees that $\mathcal{H}^1_v(F_\infty, D)$ is a direct limit of modules isomorphic to $(\mathcal{O}/2\mathcal{O})[\mathrm{Gal}(F_n/F)]^d$. The Pontryagin dual of $\mathcal{H}^1_v(F_\infty, D)$ is isomorphic to $\big(\Lambda_{(\mathcal{O},F)}/2\Lambda_{(\mathcal{O},F)}\big)^d$ as a $\Lambda_{(\mathcal{O},F)}$-module. It is a torsion $\Lambda_{(\mathcal{O},F)}$-module, but its μ-invariant is positive and is determined by $d = \dim_{\mathcal{E}}(V)$.

One can similarly define $\mathcal{H}^2_v(F_\infty, D)$ as a direct limit by replacing the H^1's by H^2's. However, for any finite prime v, the p=cohomological dimension of $G_{F_{\infty,\nu}}$ is 1 and so $\mathcal{H}^2_v(F_\infty, D)$ vanishes. This is also true if $v|\infty$ because $H^2(\mathbf{R}, D) = 0$.

The following definition is equivalent to the one given in the introduction:

$$\mathrm{Sel}_D(F_\infty) = \ker\Big(H^1(F_\Sigma/F_\infty, D) \longrightarrow \prod_{v\in\Sigma, v\nmid p} \mathcal{H}^1_v(F_\infty, D)\Big). \tag{3}$$

To verify the equivalence, we first point out that if v is a non-archimedean prime of F and $v \nmid p$, and if ν is a prime of F_∞ lying over v, then $F_{\infty,\nu}$ is the unramified $\mathbf{Z}_p$-extension of F_v. It follows that the restriction map $H^1(F_{\infty,\nu}, D) \to H^1(F_v^{unr}, D)$ is injective. Here F_v^{unr} is the maximal unramified extension of F_v and contains $F_{\infty,\nu}$. Therefore, requiring a cocycle class to be trivial in $H^1(F_{\infty,\nu}, D)$ is equivalent to requiring it to be trivial in $H^1(F_\nu^{unr}, D)$.

Suppose now that ϕ is a 1-cocycle for G_{F_∞} with values in D. Note that we have $H^1(F_\Sigma, D) = \mathrm{Hom}(G_{F_\Sigma}, D)$. Also, G_{F_Σ} is generated topologically by the inertia subgroups I_η of G_F for all primes η of F_Σ lying over some $v \notin \Sigma$. Thus, the class $[\phi]$ in $H^1(F_\infty, D)$ has a trivial restriction to all those I_η's if and only if $[\phi]$ is in

$$\ker\big(H^1(F_\infty, D) \to H^1(F_\Sigma, D)\big) \;=\; \mathrm{im}\big(H^1(F_\Sigma/F_\infty, D) \to H^1(F_\infty, D)\big) \;\; .$$

Thus, the cocycle classes in $H^1(F_\Sigma/F_\infty, D)$ can be identified under the inflation map with the cocyle classes in $H^1(F_\infty, D)$ which are unramified at all primes of F_∞ not lying over primes in Σ. The equivalence of (3) and the earlier definition follows.

It will also be useful to point out that the global-to-local map in (3) is surjective. This follows proposition 2.1 in [7]. It is only proved there for $F = \mathbf{Q}$ and odd p, but the argument works if F is totally real and for any p. One assumption is that $\mathrm{Sel}_D(F_\infty)$ is $\Lambda_{\mathcal{O},F}$-cotorsion, which is satisfied by proposition 2.1 above. The other assumption is that $H^0(F_\infty, D^*)$ is finite. Here $D^* = \mathrm{Hom}(T, \mu_{p^\infty})$ and the finiteness is clear since F_∞ is totally real and $H^0(\mathbf{R}, \mu_{p^\infty})$ is finite (and even trivial if p is odd).

Let T' be another G_F-invariant $\mathcal{O}$-lattice in V. Let $D' = V/T'$. We consider $X_D(F_\infty)$ and $X_{D'}(F_\infty)$ as $\Lambda_{(\mathcal{O},F)}$-modules.

Proposition 2.4. *The $\Lambda_{(\mathcal{O},F)}$-modules $X_D(F_\infty)$ and $X_{D'}(F_\infty)$ have the same characteristic ideal.*

Proof. As above, let π be a generator of the maximal ideal of $\mathcal{O}$. Scaling by a power of π, we may assume that $T \subseteq T'$. We then have a G_F-equivariant map $\varphi : D \to D'$ with finite kernel. Such a map φ is called a G_F-isogeny. It is surjective. Let $\Phi = \ker(\varphi)$. Then $\Phi \subseteq D[\pi^t]$ for some $t \geq 0$. There is also a G_F-isogeny $\psi : D' \to D$ such that $\psi \circ \varphi$ is the map $D \to D$ given by multiplication by π^t.

The map φ induces a map from $\mathrm{Sel}_D(F_\infty)$ to $\mathrm{Sel}_{D'}(F_\infty)$ whose kernel is killed by π^t. Similarly, ψ induces such a map from $\mathrm{Sel}_{D'}(F_\infty)$ to $\mathrm{Sel}_D(F_\infty)$ and the compositum is multiplication by π^t. It follows that the characteristic ideals I and I' of $X_D(F_\infty)$ and $X_{D'}(F_\infty)$, respectively, are related as follows: $I' = \pi^s I$ for some $s \in \mathbf{Z}$. Thus, the proposition is equivalent to showing that the μ-invariants for the two modules are equal.

Assume first that p is odd. In the definition (3), $\mathcal{H}_v^1(F_\infty, D) = 0$ when $v|\infty$ and $\mathcal{H}_v^1(F_\infty, D)$ has finite $\mathcal{O}$-corank when $v \in \Sigma, v \nmid p$. Thus, the μ-invariants for the Pontryagin duals of $\mathrm{Sel}_D(F_\infty)$ and $H^1(F_\Sigma/F, D)$ are equal. The same statement is true for $\mathrm{Sel}_{D'}(F_\infty)$ and $H^1(F_\Sigma/F, D')$. And so it suffices to prove that the Pontryagin duals of $H^1(F_\Sigma/F, D)$ and $H^1(F_\Sigma/F, D')$ have the same μ-invariants. This is sufficient even for $p = 2$. This follows because the map (3) and the corresponding global-to-local map for D' are both surjective. Furthermore, for any archimedean prime v, the μ-invariants of $\mathcal{H}_v^1(F_\infty, D)$ and $\mathcal{H}_v^1(F_\infty, D')$ are equal.

Using the notation in the proof of proposition 2.1, we have an exact sequence

$$H^0(F_\Sigma/F_\infty, D') \longrightarrow H^1(F_\Sigma/F_\infty, \Phi) \longrightarrow H^1(F_\Sigma/F_\infty, D) \longrightarrow H^1(F_\Sigma/F_\infty, D')$$
$$\longrightarrow H^2(F_\Sigma/F_\infty, \Phi) \longrightarrow H^2(F_\Sigma/F_\infty, D) \quad .$$

Now the μ-invariant of $H^0(F_\Sigma/F_\infty, D')$ certainly vanishes. Also, $H^2(F_\Sigma/F_\infty, D) = 0$. One can verify this for odd p by using propositions 3 and 4 in [6]. For $H^2(F_\Sigma/F, D)$ is $\Lambda_{(\mathcal{O},F)}$-cotorsion by proposition 3, and $\Lambda_{(\mathcal{O},F)}$-cofree by proposition 4. For $p = 2$, $H^2(F_\Sigma/F, D)$ is still $\Lambda_{(\mathcal{O},F)}$-cotorsion. The analogue of proposition 4 is that

$$\ker\bigl(H^2(F_\Sigma/F_\infty, D) \to \prod_{v|\infty} \mathcal{H}_v^2(F_\infty, D)\bigr)$$

is $\Lambda_{(\mathcal{O},F)}$-cofree, and hence must vanish. However, since $\mathcal{H}_v^2(F_\infty, D)$ vanishes, it follows that $H^2(\mathbf{R}, D) = 0$. Consequently, we indeed have $H^2(F_\Sigma/F_\infty, D) = 0$.

To complete the proof, we must show that $H^1(F_\Sigma/F_\infty, \Phi)$ and $H^2(F_\Sigma/F_\infty, \Phi)$ have the same μ-invariants as $\Lambda_{(\mathcal{O},F)}$-modules. In the statement of the proposition, we can reduce to the case where Φ is killed by π. Let $\widetilde{\Lambda}_{(\mathcal{O},F)} = \Lambda_{(\mathcal{O},F)}/\pi\Lambda_{(\mathcal{O},F)}$. It then suffices to show that

$$\mathrm{corank}_{\widetilde{\Lambda}_{(\mathcal{O},F)}}\bigl(H^1(F_\Sigma/F_\infty, \Phi)\bigr) = \mathrm{corank}_{\widetilde{\Lambda}_{(\mathcal{O},F)}}\bigl(H^2(F_\Sigma/F_\infty, \Phi)\bigr) \quad .$$

Here Φ is a representation space for G_F over $\mathcal{O}/(\pi)$ and is totally even. The Euler-Poincaré characteristic over F_∞, which is the alternating sum of the $\widetilde{\Lambda}_{(\mathcal{O},F)}$-coranks of $H^i(F_\Sigma/F_\infty, \Phi)$ for $0 \leq i \leq 2$, is 0. The above equality follows. ■

Remark 2.5. As mentioned in remark 2.3, $\mu\bigl(X_D(F_\infty)\bigr)$ should always vanish. The above proof would then show that $X_D(F_\infty)$ and $X_{D'}(F_\infty)$ are pseudo-isomorphic as $\Lambda_{(\mathcal{O},F)}$-modules. This is also true if $\mu\bigl(X_D(F_\infty)\bigr) = 1$. In contrast, for non-Artin motives, the μ-invariant of the Pontryagin dual of a Selmer group can be nonzero and can change under isogeny. This phenomenon was first pointed out by Mazur in [9]. The exact change in the μ-invariant under an isogeny is studied in [13] and [11]. In fact, proposition 2.4 is just a special case of the main theorem in [11] when p is an odd prime.

Remark 2.6. If $\varphi : D \to D'$ is a G_F-isogeny and $\ker(\phi) = D[\mathfrak{m}^t]$ for some $t \geq 0$, then $D \cong D'$ as G_F-modules. This follows because $\mathfrak{m}$ is a principal ideal. Any other G_F-isogeny will be called nontrivial. Such G_F-isogenies φ exist if and only if $T/\pi T$ is reducible as a G_F-representation space over the residue field $\mathcal{O}/(\pi)$. Now, if ρ is irreducible over $\mathcal{E}$ and $\mathrm{im}(\rho)$ has order prime to p, then it is well-known that $T/\pi T$ is also irreducible. In contrast, if $\mathrm{im}(\rho)$ has order divisible by p, then $T/\pi T$ may be reducible even if ρ is irreducible.

Proposition 2.7. *Suppose that ρ_1 and ρ_2 are totally even Artin representations of G_F. Let $\rho = \rho_1 \oplus \rho_2$. Then $I_\rho = I_{\rho_1} I_{\rho_2}$ and $J_\rho = J_{\rho_1} J_{\rho_2}$.*

Proof. We assume that $\mathcal{E}$ is a field of definition for ρ_1, ρ_2, and ρ. Let V_1 and V_2 be the underlying representations spaces for ρ_1 and ρ_2 over $\mathcal{E}$, and let $V = V_1 \oplus V_2$. Let T_1 and T_2 be Galois invariant $\mathcal{O}$-lattices in V_1 and V_2. Let $T = T_1 \oplus T_2$. Then $D = D_1 \oplus D_2$. With this choice of $\mathcal{O}$-lattices, it is clear that $\mathrm{Sel}_D(F_\infty) \cong \mathrm{Sel}_{D_1}(F_\infty) \oplus \mathrm{Sel}_{D_2}(F_\infty)$. The first equality in the proposition follows. We also have $H^0(F_\infty, D) \cong H^0(F_\infty, D_1) \oplus H^0(F_\infty, D_2)$, giving the second equality. ■

Suppose that F' is a totally real, finite extension of F and that ρ' is a totally even Artin representation of $G_{F'}$. Thus, ρ' factors through $\mathrm{Gal}(K'/F')$, where K' is totally real. We can then define $\rho = \mathrm{Ind}_{G_{F'}}^{G_F}(\rho')$. Then ρ is an Artin representation of G_F and factors through $\mathrm{Gal}(K/F)$, where K is the Galois closure of K' over F. Note that K is totally real, and hence ρ is also totally even. Furthermore, there is an injective homomorphism $\Gamma_{F'} \to \Gamma_F$. Therefore, we can regard $\Lambda_{(\mathcal{O},F')}$ as a subring of $\Lambda_{(\mathcal{O},F)}$. One sees easily that $\Lambda_{(\mathcal{O},F)}$ is a finite integral extension of $\Lambda_{(\mathcal{O},F')}$ and that the degree is $[\Gamma_F : \Gamma_{F'}] = [F' \cap F_\infty : F]$. The characteristic ideals I and J are essentially unchanged by induction. To be precise, we have

Proposition 2.8. *With the above notation, we have $I_\rho = I_{\rho'}\Lambda_{(\mathcal{O},F)}$ and $J_\rho = J_{\rho'}\Lambda_{(\mathcal{O},F)}$.*

To simplify notation in the following proof, we will write just write $\mathrm{Ind}_{F'}^F(\rho')$ in place of $\mathrm{Ind}_{G_{F'}}^{G_F}(\rho')$. We will also write ρ_∞ and ρ'_∞ for the restrictions of ρ and ρ' to F_∞ and F'_∞, respectively.

Proof. We consider separately the two cases where $F' \cap F_\infty = F$ and $F' \subset F_\infty$. That will suffice because if $E = F' \cap F_\infty$, then $E_\infty = F_\infty$ and $\mathrm{Ind}_E^F\big(\mathrm{Ind}_{F'}^E(\rho')\big) = \rho$.

Suppose first that $F' \cap F_\infty = F$. In this case, we can identify $\Gamma_{F'}$ with Γ_F and hence $\Lambda_{(\mathcal{O},F')}$ with $\Lambda_{(\mathcal{O},F)}$. For brevity, let $G = G_F, G' = G_{F'}$, and $N = G_{F_\infty}$, a normal supgroup of G. Then $N \cap G' = G_{F'_\infty}$. Furthermore, suppose that K is a finite, totally real Galois extension of F which contains F' and such that ρ' factors through $\mathrm{Gal}(K/F')$. Then ρ factors through $\mathrm{Gal}(K/F)$. Furthermore, since $NG' = G$, we have $[G : G'] = [N : N \cap G']$, and it follows that

$$\rho|_N = \mathrm{Ind}_{G'}^G(\rho')\big|_N = \mathrm{Ind}_{N\cap G'}^N(\rho'|_N) \quad .$$

Consequently, $\rho_\infty \cong \mathrm{Ind}_{F'_\infty}^{F_\infty}(\rho'_\infty)$.

Choose the Galois invariant $\mathcal{O}$-lattices for ρ and ρ' so that $\mathrm{Ind}_{G'}^G(D') = D$. Here we can replace G and G' by N and $N \cap G'$. Then $H^0(F_\infty, D) \cong H^0(F'_\infty D')$, and the isomorphism is equivariant for the action of $\Gamma_F = \Gamma_{F'}$. It follows that $J_\rho = J_{\rho'}$.

Note that $K_\infty = KF_\infty$ contains F'_∞. Let M_∞ be defined exactly as in the proof of proposition 2.1. Then, since the inertia subgroups of G_{K_∞} for all primes $\eta \nmid p$ generate that group topologically, we have injective maps

$$(4) \qquad \mathrm{Sel}_D(F_\infty) \to H^1(M_\infty/F_\infty, D), \qquad \mathrm{Sel}_{D'}(F'_\infty) \to H^1(M_\infty/F'_\infty, D')$$

The cokernels of these maps are finite. To see this, suppose that v is a nonarchimedean prime of F not dividing p. The inertia subgroup of $\mathrm{Gal}(K_\infty/F_\infty)$ for any $\eta|v$ is finite. Consequently, the inertia subgroup of $\mathrm{Gal}(M_\infty/F_\infty)$ for any prime of M_∞ over η will be finite. Now η lies over some prime ν of F_∞. One need only pick one such inertia subgroup for each prime ν of F_∞ lying over v. Now if A is any finite subgroup of $\mathrm{Gal}(M_\infty/F_\infty)$, then $H^1(A, D)$ is finite. It follows that the cokernel of the first map in (4) is indeed finite. Similarly, this is also true for the second map.

Let $U = G_{M_\infty}$. Then we can identify $\mathrm{Ind}_{G'/U}^{G/U}(\rho')$ with ρ, viewed as a representation of $G/U = \mathrm{Gal}(M_\infty/F)$, and also of the subgroup $\mathrm{Gal}(M_\infty/F_\infty)$. According to Shapiro's Lemma, we then have a canonical isomorphism

$$H^1(M_\infty/F_\infty, D) \longrightarrow H^1(M_\infty/F'_\infty, D') \tag{5}$$

The map is Γ_F-equivariant and so the isomorphism is as discrete $\Lambda_{(\mathcal{O},F)}$-modules. Their Pontryagin duals are isomorphic as $\Lambda_{(\mathcal{O},F)}$-modules. It follows that the Pontryagin duals of $\mathrm{Sel}_D(F_\infty)$ and $\mathrm{Sel}_{D'}(F'_\infty)$ are pseudo-isomorphic and therefore that $I_\rho = I_{\rho'}$, as stated. We remark in passing that, with a little more care, one can verify that (5) actually defines an isomorphism of $\mathrm{Sel}_D(F_\infty)$ to $\mathrm{Sel}_{D'}(F'_\infty)$.

Suppose now that $F' \subset F_\infty$. Then $F'_\infty = F_\infty$ and $\Gamma_{F'}$ is a subgroup of Γ_F of finite index t. We use the previous notation, but now we have $N \subset G' \subset G$. Thus, ρ_∞ and ρ'_∞ are the restrictions of ρ and ρ' to N, respectively. In this case, ρ_∞ is a direct sum of representations obtained from ρ'_∞ by composing with certain automorphisms of N. The automorphism are just the restriction of certain inner automorphisms of G, namely the inner automorphisms defined by some set of coset representatives $g_1, ..., g_t$ for G' in G. We take g_1 to be the identity. Denote these representations of G' by $\rho'_1, ..., \rho'_t$. (They are not necessarily distinct.) Define the corresponding discrete G'-modules $D'_1, ..., D'_t$ obtained from D' by composing with the above specified automorphisms of N. We have $D'_1 = D'$. For each i, $1 \le i \le t$, conjugating by g_i also defines an isomorphism of $H^1(N, D')$ to $H^1(N, D'_i)$. Since Γ_F is commutative, this isomorphism is $\Gamma_{F'}$-equivariant. Also, the isomorphism induces an isomorphism of $\mathrm{Sel}_{D'}(F_\infty)$ to $\mathrm{Sel}_{D'_i}(F_\infty)$. Thus, the $\mathrm{Sel}_{D'_i}(F_\infty)$'s are all isomorphic to $\mathrm{Sel}_{D'}(F_\infty)$ as $\Lambda_{(\mathcal{O},F')}$-modules.

As a G_{F_∞}-module, $D \cong \oplus_{1 \le i \le t} D'_i$. Furthermore, $D'_i = g_i(D')$. Thus,

$$\mathrm{Sel}_D(F_\infty) \cong \underset{1 \le i \le t}{\oplus} \mathrm{Sel}_{D'_i}(F_\infty)$$

as $\Gamma_{F'}$-modules and the action of Γ_F permutes the summands in the corresponding way. The same thing is true for the Pontryagin duals. It follows that

$$X_D(F_\infty) \;\cong\; X_{D'}(F_\infty) \otimes_{\Lambda_{(\mathcal{O},F)'}} \Lambda_{(\mathcal{O},F)}$$

and therefore $I_\rho = I_{\rho'}\Lambda_{(\mathcal{O},F)}$ as stated. ■

Propositions 2.7 and 2.8 show that it is enough to consider I_τ and J_τ where τ is a totally even, absolutely irreducible Artin representation of $G_{\mathbf{Q}}$. For if ρ is a totally even Artin representation over F, then $\mathrm{Ind}_F^{\mathbf{Q}}(\rho)$ is a direct sum of absolutely irreducible Artin representations which must also be totally even. The field $\mathcal{E}$ must be chosen to be sufficiently large. Both ρ and all of the absolutely irreducible constituents of $\mathrm{Ind}_F^{\mathbf{Q}}(\rho)$ must be realizable over $\mathcal{E}$. The next remark shows that the choice of $\mathcal{E}$ is otherwise not too significant.

Remark 2.9. Suppose that $\mathcal{E}'$ and $\mathcal{E}$ are finite extensions of $\mathbf{Q}_p$ with rings of integers $\mathcal{O}'$ and $\mathcal{O}$, respectively. Assume that $\mathcal{E}' \subseteq \mathcal{E}$. Let $\Gamma = \mathbf{Z}_p$. Let $\Lambda' = \mathcal{O}'[[\Gamma]]$ and $\Lambda = \mathcal{O}[[\Gamma]]$, and let $\mathcal{L}'$ and $\mathcal{L}$ be their fraction fields. Thus, Λ is the integral closure of Λ' in $\mathcal{L}$. Since Λ' is integrally closed in $\mathcal{L}'$, one has $\Lambda' = \Lambda \cap \mathcal{L}$. It follows that if I' is a principal ideal in Λ', and $I = I'\Lambda$, then one can recover I' from I by $I' = I \cap \Lambda'$.

In particular, suppose that ρ' is a totally even Artin representation of G_F over $\mathcal{E}'$, and D' is the corresponding discrete Galois module. Extending scalars to $\mathcal{E}$, one obtains an Artin representation ρ over $\mathcal{E}$, and one can take $D = D' \otimes_{\mathcal{O}'} \mathcal{O}$ as the corresponding discrete Galois module. One sees easily that $\mathrm{Sel}_D(F_\infty) \cong \mathrm{Sel}_{D'}(F_\infty) \otimes_{\mathcal{O}'} \mathcal{O}$. The characteristic ideals are related by $I_\rho = I_{\rho'}\Lambda_{(\mathcal{O},F)}$. Hence $I_{\rho'} = I_\rho \cap \Lambda_{(\mathcal{O}',F)}$. Similar statements hold for the ideals $J_{\rho'}$ and J_ρ.

Our final result in this section concerns the effect of twisting the Galois representation ρ. Suppose that ξ is a 1-dimensional Artin representation of G_F which factors through Γ_F. We must choose $\mathcal{E}$ sufficiently large so that ξ has values in $\mathcal{E}$. Thus, $\xi : \Gamma_F \to \mathcal{O}^\times$ is a continuous homomorphism. We denote the twist $\rho \otimes \xi$ simply by by $\rho\xi$. The corresponding discrete G_F-module is $D \otimes \xi = D \otimes_{\mathcal{O}} \mathcal{O}(\xi)$, where $\mathcal{O}(\xi)$ is the free $\mathcal{O}$-module of rank 1 on which G_F acts by ξ. We use a similar notation below for other discrete and compact $\mathcal{O}$-modules. For brevity, we denote $D \otimes \xi$ by D_ξ. Of course, as $\mathcal{O}$-modules and G_{F_∞}-modules, we can identify D_ξ and D. The actions of Γ_F on the corresponding Galois cohomology groups are related by

$$H^1(F_\Sigma/F_\infty, D_\xi) \ \cong H^1(F_\Sigma/F_\infty, D) \otimes \xi$$

and therefore we have the following Γ_F-equivariant isomorphism of discrete $\mathcal{O}$-modules:

$$\mathrm{Sel}_{D_\xi}(F_\infty) \ \cong \ \mathrm{Sel}_D(F_\infty) \otimes \xi \quad . \tag{6}$$

Both of these $\mathcal{O}$-modules are $\Lambda_{(\mathcal{O},F)}$-modules and the isomorphism is a $\Lambda_{(\mathcal{O},F)}$-module isomorphism.

It follows from (6) that $X_{D_\xi}(F_\infty) \cong X_D(F_\infty) \otimes \xi^{-1}$ as $\Lambda_{(\mathcal{O},F)}$-modules. Furthermore, noting that

$$H^0(F_\Sigma/F_\infty, D_\xi) = H^0(F_\Sigma/F_\infty, D) \otimes \xi \quad ,$$

it follows that $Y_{D_\xi}(F_\infty) \cong Y_D(F_\infty) \otimes \xi^{-1}$ as $\Lambda_{(\mathcal{O},F)}$-modules. These isomorphisms give a simple relationship between I_{ρ_ξ} and I_ρ and between J_{ρ_ξ} and J_ρ, as we now discuss.

In general, suppose that Γ is a commutative pro-p group. Let $\Lambda_{\mathcal{O}} = \mathcal{O}[[\Gamma]]$. We have the natural inclusion map $\varepsilon : \Gamma \to \Lambda^\times$. Suppose that $\xi : \Gamma_F \to \mathcal{O}^\times$ is a continuous homomorphism. Since $\mathcal{O}^\times \subset \Lambda_{\mathcal{O}}^\times$, we obtain a continuous homomorphism $\xi\varepsilon : \Gamma \to \Lambda_{\mathcal{O}}^\times$. This is the map $\gamma \mapsto \xi(\gamma)\gamma$ for all $\gamma \in \Gamma$. We can then extend $\xi\varepsilon$ to a continuous $\mathcal{O}$-algebra homomorphism $tw_\xi : \Lambda_{\mathcal{O}} \to \Lambda_{\mathcal{O}}$. This is an automorphism of $\Lambda_{\mathcal{O}}$. The inverse map is $tw_{\xi^{-1}}$.

Now suppose that $\Gamma \cong \mathbf{Z}_p$ and that $\xi : \Gamma \to \mathcal{O}^\times$ is a continuous homomorphism. Let X be a finitely-generated, torsion $\Lambda_{\mathcal{O}}$-module. For any $\lambda \in \Lambda_{\mathcal{O}}$, $X[\lambda]$ denotes $\{x \in X \mid \lambda x = 0\ \}$. The characteristic ideal I_X is determined by the invariants $\mu(X)$ and $\mathrm{rank}_{\mathcal{O}}\big(X[\theta^t]\big)$, where θ varies over the irreducible elements in $\Lambda_{\mathcal{O}}$ and $t \ge 1$. We write $X_{\xi^{-1}}$ for $X \otimes \xi^{-1}$. It is easily seen that $\mu(X) = \mu(X_{\xi^{-1}})$. We regard X and $X_{\xi^{-1}}$ as the same $\mathcal{O}$-modules, but with different $\mathcal{O}$-linear actions of Γ. If $\gamma \in \Gamma$, and $x \in X = X_{\xi^{-1}}$, we denote the first action by $\gamma \cdot x$ and the second by $\gamma \cdot\cdot x$. Thus, $\gamma \cdot\cdot x = \xi(\gamma)^{-1}\gamma \cdot x$. That is, we have $(\xi(\gamma)\gamma) \cdot\cdot x = \gamma \cdot x$ for all $\gamma \in \Gamma, x \in X$. It follows that $tw_\xi(\theta) \cdot\cdot x = \theta \cdot x$ for all $\theta \in \Lambda_{\mathcal{O}}$ and $x \in X$. In particular, for any irreducible element $\theta \in \Lambda_{\mathcal{O}}$ and $t \ge 1$, we have

$$X_{\xi^{-1}}[tw_\xi(\theta^t)] \ = \ X[\theta^t] \ \ .$$

Consequently, we have the following result.

Proposition 2.10. *Let ξ be a character of finite order of Γ_F. Then $I_{\rho\xi} = tw_\xi(I_\rho)$ and $J_{\rho\xi} = tw_\xi(J_\rho)$.*

Remark 2.11. In the introduction, we mentioned the Selmer group $\mathrm{Sel}_{D(n)}(F_\infty)$ associated to the Tate twist $D(n)$ when $n \ge 2$ and $n \equiv 0 \pmod{p-1}$. Note that $\chi_F^n = \kappa_F^n$ factors through Γ_F. The above discussion shows how the actions of Γ_F on $\mathrm{Sel}_D(F_\infty)$ and $\mathrm{Sel}_{D(n)}(F_\infty)$ differ. In fact, if θ generates the characteristic ideal of $X_D(F_\infty)$, then $tw_{\kappa_F^n}(\theta)$ generates the characteristic ideal of $X_{D(n)}(F_\infty)$.

3 The definition of p-adic Artin L-functions.

If F is totally real and $\rho : G_F \to GL_d(\mathbf{C})$ is a totally even Artin representation, then the corresponding Artin L-function $L(z,\rho)$ is a meromorphic function on $\mathbf{C}$. We will let $L^*(z,\rho)$ denote the function given by the same Euler product as $L(z,\rho)$, but with the Euler factors for the primes of F lying above p omitted. A theorem of Siegel implies that $L(1-n,\rho) \in \mathbf{Q}(\psi)$ for all integers $n \ge 1$. Here ψ is the character of ρ and $\mathbf{Q}(\psi)$ is the field generated by its values. Furthermore, $L(1-n,\rho) \neq 0$ when n is even. For our purpose, we will consider the values $L^*(1-n,\rho)$. They are also algebraic numbers and are nonzero for even $n \geq 2$.

The above L-values behave well under conjugacy in the following sense. If $g \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, then $\psi' = g \circ \psi$ is the character of another totally even Artin representation ρ' of G_F. We then have $L(1-n,\rho') = g\big(L(1-n,\rho)\big)$ for all $n \geq 1$. The Euler factors for primes above p behave similarly, and so we have the same conjugacy properties for the values $L^*(1-n,\rho)$. Therefore, if we arbitrarily choose embeddings of $\overline{\mathbf{Q}}$ into $\mathbf{C}$ and into $\overline{\mathbf{Q}}_p$, then the complex algebraic numbers $L^*(1-n,\rho)$ and the values of ψ can all be regarded as elements of $\overline{\mathbf{Q}}_p$. The character ψ is then the character of an Artin representation $\rho : G_F \to GL_d(\overline{\mathbf{Q}}_p)$. Of course, ψ determines ρ up to equivalence. The values $L^*(1-n,\rho)$ are also determined by the $\overline{\mathbf{Q}}_p$-valued character ψ, and do not depend on the choice of embeddings. In fact, if ψ has values in a finite extension $\mathcal{E}$ of $\mathbf{Q}_p$, then the values $L^*(1-n,\rho)$ for $n \geq 1$ are also in $\mathcal{E}$, and are nonzero when n is even.

These L-values also behave well under induction. Suppose that F' is a finite, totally real extension of F. Suppose that $\rho = \mathrm{Ind}_{F'}^{F}(\rho')$, where ρ' is a totally even Artin representation of $G_{F'}$. We have $L(z,\rho) = L(z,\rho')$. The same identity is true if we delete the Euler factors for primes above p. Thus,

$$L^*(1-n,\rho) = L^*(1-n,\rho')$$

for all $n \geq 1$.

The p-adic L-function $L_p(s,\rho)$ satisfies the following interpolation property:

$$L_p(1-n,\rho) \;=\; L^*(1-n,\rho)$$

for all $n \equiv 0 \pmod{[F(\mu_p):F]}$ if p is odd, or all $n \equiv 0 \pmod 2$ if $p = 2$. It is a meromorphic function defined on a certain disc $\mathcal{D}$ in $\overline{\mathbf{Q}}_p$. The existence of such a function was proved by Deligne and Ribet when ρ is of dimension 1. In this case, it is holomorphic on $\mathcal{D}$, except possibly at $s = 1$.

Suppose now that ρ factors through $\Delta = \mathrm{Gal}(K/F)$, where K is a finite, totally real, Galois extension of F. The existence of $L_p(s,\rho)$ then follows if ρ is induced from a 1-dimensional representation ρ' of a subgroup Δ' of Δ. Then ρ is a so-called *monomial* representation. If $\Delta' = \mathrm{Gal}(K/F')$, then $\rho = \mathrm{Ind}_{F'}^{F}(\rho')$ and we have $L_p(s,\rho) = L_p(s,\rho')$. Thus, $L_p(s,\rho)$ is again holomorphic on $\mathcal{D}$, except possibly at $s = 1$.

In general, a theorem of Brauer states that there exist monomial representations $\rho_1, ..., \rho_s$ $\sigma_1, ..., \sigma_t$ of Δ, where $s, t \geq 0$, such that

$$\rho \oplus \Big(\bigoplus_{j=1}^{t} \sigma_j \Big) \cong \bigoplus_{i=1}^{s} \rho_i \tag{7}$$

and so we can define $L_p(s,\rho)$ as the quotient $\prod_{i=1}^{s} L_p(s,\rho_i) \big/ \prod_{j=1}^{t} L_p(s,\sigma_j)$. The above interpolation property is indeed satisfied by this function.

Let $\Gamma = \Gamma_{\mathbf{Q}} = \mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q})$. There is a canonical isomorphism $\kappa : \Gamma \to 1 + q\mathbf{Z}_p$, where $q = p$ for odd p and $q = 4$ for $p = 2$. It is defined as the composite map

$$\mathrm{Gal}(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q}) \xrightarrow{\chi} \mathbf{Z}_p^\times \longrightarrow 1 + q\mathbf{Z}_p$$

which indeed factors through Γ. Here χ is the p-power cyclotomic character. The second map is just the projection map for the decomposition $\mathbf{Z}_p^\times = W \times (1 + q\mathbf{Z}_p)$, where W is the group of roots of unity in $\mathbf{Q}_p$. For any $s \in \mathbf{Z}_p$, one can define $\kappa^s : \Gamma \to 1 + q\mathbf{Z}_p$, which is a continuous group homomorphism. It extends to a continuous $\mathcal{O}$-algebra homomorphism $\Lambda_{(\mathcal{O},\mathbf{Q})} \to \mathcal{O}$ which we also denote by κ^s. Furthermore, for any F, the restriction map $\Gamma_F \to \Gamma$ defines an injective homomorphism $\Lambda_{(\mathcal{O},F)} \to \Lambda_{(\mathcal{O},\mathbf{Q})}$. We identify $\Lambda_{(\mathcal{O},F)}$ with its image and define κ_F^s to be the restriction of κ^s to that subring.

Let $\mathcal{L}_{(\mathcal{O},F)}$ denote the fraction field of $\Lambda_{(\mathcal{O},F)}$. Suppose that $\theta \in \mathcal{L}_{(\mathcal{O},F)}$. Write $\theta = \alpha\beta^{-1}$, where $\alpha, \beta \in \Lambda_{(\mathcal{O},F)}$ and $\beta \neq 0$. The Weierstrass preparation theorem implies that $\kappa_F^s(\beta) \neq 0$ for all but finitely many $s \in \mathbf{Z}_p$. Thus, excluding a finite set of values of s, one can make the definition $\kappa_F^s(\theta) = \kappa_F^s(\alpha)\kappa_F^s(\beta)^{-1}$. Furthermore, one has the following property:

If $\theta_1,\ \theta_2 \in \mathcal{L}_{(\mathcal{O},F)}$ and $\kappa_F^s(\theta_1) = \kappa_F^s(\theta_2)$ for infinitely many $s \in \mathbf{Z}_p$, then $\theta_1 = \theta_2$.

One verifies this by writing $\theta_1 = \alpha_1\beta_1^{-1},\ \theta_2 = \alpha_2\beta_2^{-1}$, and applying the Weierstrass preparation theorem to $\alpha_1\beta_2 - \alpha_2\beta_1$.

One can associate to $L_p(s,\rho)$ a nonzero element θ_ρ of $\mathcal{L}_{(\mathcal{O},F)}$. It is characterized as follows:

$$L_p(1-s,\rho) \;=\; \kappa_F^s(\theta_\rho) \qquad \text{for all but finitely many } s \in \mathbf{Z}_p. \tag{8}$$

for all but finitely many $s \in \mathbf{Z}_p$. If ρ is 1-dimensional, then Deligne and Ribet's construction of $L_p(1-s,\rho)$ proves the existence of such a θ_ρ. Furthermore, they show that

$$J_\rho\theta_\rho \;\subseteq\; 2^{[F:\mathbf{Q}]}\Lambda_{(\mathcal{O},F)} \quad ,$$

where J_ρ is the ideal in $\Lambda_{(\mathcal{O},F)}$ defined in the introduction. Note that $J_\rho = \Lambda_{(\mathcal{O},F)}$ unless ρ factors through Γ_F. If ρ is monomial, one can use proposition 2.8 to prove that θ_ρ exists. Then θ_ρ has the integrality property

$$J_\rho\theta_\rho \;\subseteq\; 2^{[F:\mathbf{Q}]\deg(\rho)}\Lambda_{(\mathcal{O},F)} \tag{9}$$

since if ρ is induced from a 1-dimensional Artin representation ρ' of $G_{F'}$, where F' is a finite extension of F, then $[F':\mathbf{Q}] = [F:\mathbf{Q}]\deg(\rho)$.

If ρ has arbitrary dimension, then the existence of θ_ρ satisfying (8) follows from (7). One assumes at first that $\mathcal{E}$ is large enough so that all of the monomial representations ρ_i and σ_j are realizable over $\mathcal{E}$. The θ_{ρ_i}'s and θ_{σ_j}'s are nonzero elements in the fraction fields of various subrings of $\Lambda_{(\mathcal{O},F)}$. One can then define

$$\theta_\rho = \prod_{i=1}^{s} \theta_{\rho_i} \Big/ \prod_{j=1}^{t} \theta_{\sigma_j} \tag{10}$$

With this definition, we can only say that θ_ρ is an element in the fraction field of $\Lambda_{(\mathcal{O},F)}$.

The behavior of the values $L^*(1-n,\rho)$ under conjugacy implies a similar behavior for the elements θ_ρ. To be precise, suppose that $\gamma \in G_{\mathbf{Q}_p}$. Let $\mathcal{O}' = \gamma(\mathcal{O})$. Let $\rho' = \gamma \circ \rho$. Note that γ induces a continuous isomorphism from $\Lambda_{(\mathcal{O},F)}$ to $\Lambda_{(\mathcal{O}',F)}$. This isomorphism extends to an isomorphism of the fraction fields, which we also denote by γ. We then have $\theta_{\rho'} = \gamma\big(\theta_\rho\big)$.

Concerning the choice of $\mathcal{O}$, the above conjugacy property and a straightforward Galois theory argument show that one can even take $\mathcal{O}$ to be the extension of $\mathbf{Z}_p$ generated by the values of the character of ρ. In the next section, we will see that the integrality property (9) still holds when p is odd.

The above remarks give us the following properties of the θ_ρ's which are parallel to the assertions in propositions 2.7 and 2.8.

Proposition 3.1. *With the same notation as in proposition 2.7, we have* $\theta_\rho = \theta_{\rho_1}\theta_{\rho_2}$.

Proposition 3.2. *With the same notation as in proposition 2.8, we have* $\theta_\rho = \theta_{\rho'}$.

We will also need the analogue of proposition 2.10. It relies on another property of the p-adic L-functions constructed by Deligne and Ribet. The interpolation property for θ_ρ stated before can be expressed as follows:

$$\kappa_F^n(\theta_\rho) \;=\; L^*(1-n,\rho) \;=\; L^*(1,\rho\kappa_F^n)$$

for all $n \geq 2$ satisfying $n \equiv 0 \pmod{p-1}$ if p is odd (or $n \equiv 0 \pmod{2}$ if $p=2$). The underlying $\mathcal{E}$ representation space for $\rho\kappa_F^n$ is the Tate twist $V(n)$. However, if ξ is a character of Γ_F of finite order, and $\mathcal{O}$ contains the values of ξ, then Deligne and Ribet also show that

$$\kappa_F^n\xi(\theta_\rho) \;=\; L^*(1-n,\rho\xi) \;=\; L^*(1,\rho\xi\kappa_F^n) \;=\; L^*(1,\rho\kappa_F^n\xi).$$

Furthermore, one has $\kappa_F^n(\theta_{\rho\xi}) = L^*(1,\rho\xi\kappa_F^n)$. Thus, we have $\kappa_F^n\xi(\theta_\rho) = \kappa_F^n(\theta_{\rho\xi})$ for the above values of n.

Suppose that $\varphi : \Gamma_F \to \mathcal{O}^\times$ is any continuous homomorphism. Let ξ be as above. Then both φ and $\varphi\xi : \Gamma_F \to \mathcal{O}^\times$ extend to continuous $\mathcal{O}$-algebra homomorphisms φ and $\varphi\xi$ from $\Lambda_{(\mathcal{O},F)}$ to $\mathcal{O}$. We also have the continuous $\mathcal{O}$-algebra homomorphism $\varphi \circ tw_\xi : \Lambda_{(\mathcal{O},F)} \to \mathcal{O}$. Such $\mathcal{O}$-algebra homomorphisms are determined uniquely by their restrictions to Γ_F. Note that

$$(\varphi \circ tw_\xi)(\gamma) \ = \ \varphi(\xi(\gamma)\gamma) = (\varphi\xi)(\gamma).$$

Therefore, we also have $(\varphi \circ tw_\xi)(\theta) = (\varphi\xi)(\theta)$ for all $\theta \in \Lambda_{(\mathcal{O},F)}$. Applying this to $\varphi = \kappa_F^n$, where $n \geq 2$ and $n \equiv 0 \pmod{p-1}$ (or $n \equiv 2 \pmod 2$ if $p = 2$), we obtain

$$\kappa_F^n\big(tw_\xi(\theta_\rho)\big) \ = \ (\kappa_F^n \circ tw_\xi)(\theta_\rho) \ = \ \kappa_F^n(\theta_{\rho\xi})$$

for all such n and therefore it follows that $tw_\xi(\theta_\rho) = \theta_{\rho\xi}$.

Proposition 3.3. *If ξ is a 1-dimension Artin representations of G_F which factors through Γ_F, then $\theta_{\rho\xi} = tw_\xi(\theta_\rho)$.*

4 Relationship of Selmer groups to p-adic L-functions

We can state the relationship quite succinctly in terms of the notation of the preceding sections. We refer to this statement as the Iwasawa Main Conjecture for ρ. As before, we assume that ρ is realizable over a finite extension $\mathcal{E}$ of $\mathbf{Q}_p$ with ring of integers $\mathcal{O}$. Let $m(\rho) = [F : \mathbf{Q}]\dim(\rho)$, which is just the degree of the representation $\mathrm{Ind}_F^{\mathbf{Q}}(\rho)$.

IMC(ρ). *Suppose that F is a totally real number field and that ρ is a totally even Artin representation of G_F. Then $I_\rho = J_\rho \theta_\rho 2^{-m(\rho)}$.*

Note that I_ρ is an ideal in $\Lambda_{(\mathcal{O},F)}$ by definition, but the assertion that $J_\rho\theta_\rho 2^{-m(\rho)}$ is an ideal in that ring, and not just a *"fractional ideal"*, is not at all clear from the definitions.

It is interesting to note that when $p = 2$, one can omit the extra power of 2 appearing in the formulation of **IMC**(ρ) by merely omitting the local conditions at the archimedean primes in the definition of the Selmer group. One obtains a larger Selmer group $\mathrm{Sel}_D^{\#}(F_\infty)$ and the characteristic ideal of the Pontryagin dual of $\mathrm{Sel}_D^{\#}(F_\infty)\big/\mathrm{Sel}_D(F_\infty)$ is precisely $2^{m(\rho)}\Lambda_{(\mathcal{O},F)}$. One sees this by using the surjectivity of the map (3) together with the structure of $\mathcal{H}_v^1(F_\infty, D)$ for archimedean v as described in section 2. If $I_\rho^{\#}$ denotes the characteristic ideal of the Pontryagin dual of $\mathrm{Sel}_D^{\#}(F_\infty)$, then **IMC**$(\rho)$ is equivalent to the assertion that $I_\rho^{\#} = J_\rho\theta_\rho$.

We also consider a weaker form of $\mathbf{IMC}(\rho)$. It amounts to the above equality up to multiplication by a power of a uniformizing parameter π of $\mathcal{O}$. We will denote the ring $\Lambda_{(\mathcal{O},F)}[\frac{1}{\pi}]$ by $\Lambda^*_{(\mathcal{O},F)}$. It is a subring of the fraction field $\mathcal{L}_{(\mathcal{O},F)}$ of $\Lambda_{(\mathcal{O},F)}$.

$\mathbf{IMC}(\rho)^*$. *We have* $I_\rho \Lambda^*_{(\mathcal{O},F)} = J_\rho \theta_\rho \Lambda^*_{(\mathcal{O},F)}$.

The main purpose of this section is to point out that the results proved by Wiles in [14] are sufficient to prove $\mathbf{IMC}(\rho)$ for all $p \geq 3$. Let

$$A(\rho) \;=\; I_\rho^{-1} J_\rho \theta_\rho 2^{-m(\rho)}$$

which is a principal fractional ideal of $\Lambda_{(\mathcal{O},F)}$, i.e., a nonzero free $\Lambda_{(\mathcal{O},F)}$-submodule of the fraction field $\mathcal{L}_{(\mathcal{O},F)}$. Such fractional ideals form a group.

If I is a nonzero principal ideal of $\Lambda_{(\mathcal{O},F)}$, and θ is a generator of I, then we will refer to $\mu(\Lambda_{(\mathcal{O},F)}/I)$ as the μ-invariant associated to I, or to θ. We denote it by μ_I. For a principal fractional ideal $I = I_1 I_2^{-1}$, we define the associated μ-invariant by $\mu_I = \mu_{I_1} - \mu_{I_2}$. A simple direct argument, or the fact that $\Lambda_{(\mathcal{O},F)}$ is a UFD, shows that μ_I is well-defined.

Suppose that $\rho = \rho_1 \oplus \rho_2$, where ρ_1 and ρ_2 are totally even Artin representations of G_F. It is clear that $m(\rho) = m(\rho_1) + m(\rho_2)$. It therefore follows from propositions 2.7 and 3.1 that

$$A(\rho) \;=\; A(\rho_1)A(\rho_2) \quad . \tag{11}$$

Suppose that F' is a finite, totally real extension of F, that ρ' is a totally even Artin representation of $G_{F'}$, and that $\rho = \mathrm{Ind}_{F'}^{F}(\rho')$. It is clear that $m(\rho) = m(\rho')$, and so propositions 2.8 and 3.2 imply that

$$A(\rho) \;=\; A(\rho') \quad . \tag{12}$$

The conjecture $\mathbf{IMC}(\rho)$ asserts that $A(\rho) = \Lambda_{(\mathcal{O},F)}$. The conjecture $\mathbf{IMC}(\rho)^*$ asserts that $A(\rho)$ is generated by a power of π. Suppose that ρ factors through $\Delta = \mathrm{Gal}(K/F)$, where K is totally real. It is clear from (11) and (12) that if one proves $\mathbf{IMC}(\rho')$ (respectively, $\mathbf{IMC}(\rho')^*$) for all 1-dimensional representation ρ' of all subgroups Δ' of Δ, then $\mathbf{IMC}(\rho)$ (respectively, $\mathbf{IMC}(\rho)^*$) follows.

Now suppose that ξ is a 1-dimensional Artin representation which factors through Γ_F. Obviously, $m(\rho\xi) = m(\rho)$. It therefore follows from propositions 2.10 and 3.3 that

$$A(\rho\xi) \;=\; tw_\xi\big(A(\rho)\big) \tag{13}$$

When p is an odd prime, Wiles proves $\mathbf{IMC}(\rho)^*$ for a certain class of totally even, 1-dimensional Artin representations ρ over any totally real number field F, namely the representations which factor through $\mathrm{Gal}(K/F)$ where $K \cap F_\infty = F$. (These are the representations of type S.) This result is Theorem 1.3 in [14]. Therefore, $A(\rho)$ is generated by a power of π for all those ρ's. However, one sees easily that if ρ is 1-dimensional, but not of type S, then there exists a 1-dimensional Artin representation ξ factoring through Γ_F such that $\rho\xi$ is of type S. Since $tw_\xi(\pi) = \pi$, it follows that if $A(\rho\xi) = (\pi^t)$, then $A(\rho) = (\pi^t)$. Therefore, $\mathbf{IMC}(\rho)^*$ is established when p is an odd prime, F is any totally real number field, and ρ is any 1-dimensional, even Artin representation. It then follows from (11) and (12) that $\mathbf{IMC}(\rho)^*$ holds for an arbitrary totally even Artin representation of G_F when p is odd. For $p = 2$, Wiles does prove partial results, but not enough to prove $\mathbf{IMC}(\rho)^*$

Now consider $\mathbf{IMC}(\rho)$. Wiles proves this assertion when p is odd and ρ is 1-dimensional and has order prime to p. It follows from Theorem 1.3 and 1.4 in [14]. The above remarks and the lemma below allow one to establish $\mathbf{IMC}(\rho)$ for all ρ.

An alternative way to deal with the μ-invariants when ρ is 1-dimensional, but of order divisible by p, is described in [4], pages 9 and 10. It is in terms of Galois groups instead of Selmer groups. We should also add that theorem 1.4 in [14] involves odd 1-dimensional representations instead of even. However, the equivalence of that theorem with what we need here is a consequence of the so-called *"Reflection Principle"*. It is also a consequence of theorem 2 in [6].

Lemma 4.1. *Suppose that G is a finite group and that ψ is the character of a representation of G over $\overline{\mathbf{Q}}_p$. Assume that the values of ψ are in $\mathbf{Q}_p^{unr}$. Then, there exists subgroups H_i of G and a 1-dimensional character ψ_i of each H_i, where $1 \le i \le t$ for some $t \ge 1$, such that*

1. $m\psi = \sum_{i=1}^t \ m_i \mathrm{Ind}_{H_i}^G(\psi_i)$, *where* $m, m_1, ..., m_t \in \mathbf{Z}$, $m \ge 1$
2. *Each* ψ_i *has order prime to* p

Proof. Brauer's theorem asserts that $\psi = \sum_{i=1}^s \ a_i \mathrm{Ind}_{K_i}^G(\varphi_i)$, where the a_i's are integers, the K_i's are subgroups of G, and, for each i, φ_i is a 1-dimensional character of K_i. If $\sigma \in \mathrm{G}_{\mathbf{Q}_p^{unr}}$, then σ fixes the values of ψ and one also has $\psi = \sum_{i=1}^t \ n_i \mathrm{Ind}_{K_i}^G(\sigma \circ \varphi_i)$. The extension of $\mathbf{Q}_p^{unr}$ generated by the values of all the φ_i's is finite. If m is its degree, then, by taking the trace, one obtains

$$m\psi = \sum_{i=1}^s \ b_i \mathrm{Ind}_{K_i}^G(\tau_i)$$

where τ_i is the sum of the conjugates of φ_i over $\mathbf{Q}_p^{unr}$ and the b_i's are integers. Since induction is transitive, it is enough to prove the lemma for each of the characters τ_i of K_i. The values of τ_i are in $\mathbf{Q}_p^{unr}$.

Let K be a subgroup of G and let φ be a 1-dimensional character of K. Then $\varphi = \alpha\beta$, where α has order prime to p and β has p-power order. Let p^k be the order of β. We can assume that $k \geq 1$. The values of α are in $\mathbf{Q}_p^{unr}$. Let τ_φ and τ_β denote the sums of the conjugates of φ and of β over $\mathbf{Q}_p^{unr}$, respectively. Note that $\tau_\varphi = \alpha\tau_\beta$.

Let $J = \ker(\beta)$. Thus, β factors through K/J, which is cyclic of order p^k. The conjugates of β over $\mathbf{Q}_p^{unr}$ are the characters of K/J of order p^k. Let J_1 be the subgroup of K such that $J \subset J_1 \subseteq K$ and J_1/J is cyclic of order p. Let $\epsilon_{_J}$ and $\epsilon_{_{J_1}}$ denote the trivial characters of J and J_1, respectively. Then $\mathrm{Ind}_J^K(\epsilon_{_J})$ is the sum of all the irreducible characters of K/J and $\mathrm{Ind}_{J_1}^K(\epsilon_{_{J_1}})$ is the sum of the irreducible characters factoring through K/J_1. Hence the difference is τ_β. Thus, $\tau_\beta = \mathrm{Ind}_J^K(\epsilon_{_J}) - \mathrm{Ind}_{J_1}^K(\epsilon_{_{J_1}})$. It follows that

$$\tau_\varphi = \mathrm{Ind}_J^K(\alpha_{_J}) - \mathrm{Ind}_{J_1}^K(\alpha_{_{J_1}}),$$

where $\alpha_{_J}$ and $\alpha_{_{J_1}}$ denote the restrictions of α to J and to J_1, respectively. Their orders are prime to p. ■

Assume now that p is odd and that ρ is an arbitrary totally real Artin representation over F. We assume that ρ is realizable over a finite extension $\mathcal{E}$ of $\mathbf{Q}_p$. Assume further that $\mathcal{E}$ is Galois over $\mathbf{Q}_p$. Since $\mathbf{IMC}(\rho)^*$ is established, it suffices to just consider the μ-invariants to prove $\mathbf{IMC}(\rho)$. Now if ρ and ρ' are conjugate over $\mathbf{Q}_p$ under the action of $\mathrm{Gal}(\mathcal{E}/\mathbf{Q}_p)$, then one sees easily that I_ρ and $I_{\rho'}$ are conjugate under the natural action of $\mathrm{Gal}(\mathcal{E}/\mathbf{Q}_p)$ on $\Lambda_{(\mathcal{O},F)}$. In particular, the μ-invariants associated to those ideals are equal. The μ-invariants associated to J_ρ and $J_{\rho'}$ are 0. In addition, θ_ρ and $\theta_{\rho'}$ are conjugate too, and so the μ-invariants associated to those elements of $\mathcal{L}_{(\mathcal{O},F)}$ are equal. It follows that the μ-invariants associated to $A(\rho)$ and $A(\rho')$ are equal.

Let $\tilde{\rho} = \oplus_\sigma \sigma$, where σ runs over the conjugates of ρ over $\mathbf{Q}_p$. Then the character of $\tilde{\rho}$ has values in $\mathbf{Q}_p$, and hence in $\mathbf{Q}_p^{unr}$. Furthermore, the above remarks show that it suffices to prove that the μ-invariant for $A(\tilde{\rho})$ vanishes. The μ-invariant for $A(\rho)$ will then also vanish. Lemma 4.1, the behavior of $A(\cdot)$ under induction and direct sums, and Theorem 1.4 in [14] imply that the μ-invariant for $A(\tilde{\rho})$ is indeed zero.

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