The Derivative Formula for Kubota-Leopoldt p-adic L-functions at Trivial Zeros

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Introduction

Suppose that ψ is an even Dirichlet character and that p is an odd prime. The Kubota-Leopoldt p-adic L-function $L_p(s,\psi)$ is an analytic function of a p-adic variable characterized by the interpolation property

$$L_p(1-n,\psi) \;=\; \big(1-\psi_n(p)p^{n-1}\big)L(1-n,\psi_n)$$

for all integers $n \geq 1$. Here $\psi_n = \psi\omega^{-n}$, where ω is the Dirichlet character of conductor p satisfying $\omega(a) \equiv a \pmod{p\mathbf{Z}_p}$ for all integers a.

In particular, we have $L_p(0,\psi) \;=\; \big(1-\psi_1(p)\big)L(0,\psi_1)$. Thus, $L_p(s,\psi)$ vanishes at $s=0$ when $\psi_1(p)=1$. This talk will mostly be about the derivative $L_p'(0,\psi)$ in that case.

$L_p(s, \psi)$ as a function on a family of Galois representations

In terms of Galois representations, one can think of $L(1-n, \psi_n)$ in the above interpolation property as $L(0, V_{n-1})$, where V_{n-1} is the 1-dimensional vector space over $\mathbf{Q}_p$ on which $G_{\mathbf{Q}} = \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ acts by

$$\psi_n \chi^{n-1} = \psi \omega^{-n} \chi^{n-1} = \psi \omega^{-1} \omega^{1-n} \chi^{n-1} = \psi_1 (\chi \omega^{-1})^{n-1} = \psi_1 \kappa^{n-1} \ ,$$

where $\chi : G_{\mathbf{Q}} \to \mathbf{Z}_p^{\times}$ is defined by the action of $G_{\mathbf{Q}}$ on the group μ_{p^∞} of p-power roots of unity and $\kappa = \chi \omega^{-1}$. One defines $L(z, V_{n-1})$ by an Euler product as usual.

Notice that κ is a homomorphism from $G_{\mathbf{Q}}$ to $1 + p\mathbf{Z}_p$. Thus, it makes sense to write V_{-s} for $s \in \mathbf{Z}_p$, the 1-dimensional space on which G_Q acts by $\psi_1 \kappa^{-s}$. One can then regard $L_p(s, \psi)$ as a function of the family of Galois representations V_{-s}. They all have the same residual representation as ψ_1. Furthermore, notice that $G_{\mathbf{Q}}$ acts on V_0 by ψ_1.

The map κ

The homomorphism κ factors through $\mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q})$, where $\mathbf{Q}_\infty$ denotes the cyclotomic $\mathbf{Z}_p$-extension of $\mathbf{Q}$, a subfield of $\mathbf{Q}(\mu_{p^\infty})$. Let $\Gamma = \mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q})$. Thus, $\Gamma \cong 1 + p\mathbf{Z}_p \cong \mathbf{Z}_p$.

If K is a finite extension of $\mathbf{Q}$ and $p \nmid [K : \mathbf{Q}]$, then $K_\infty = K\mathbf{Q}_\infty$ is a Galois extension of K and $\mathrm{Gal}(K_\infty/K)$ is canonically isomorphic to Γ. We will regard κ as the corresponding homomorphism

$$G_K \longrightarrow \mathrm{Gal}(K_\infty/K) \longrightarrow \Gamma = \mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q}) \longrightarrow 1 + p\mathbf{Z}_p \quad .$$

Then $\{ \kappa^s \mid s \in \mathbf{Z}_p \}$ is a subset of $\mathrm{Hom}_{cont}(G_K, \overline{\mathbf{Q}}_p^\times)$

A p-adic analogue of a theorem of Lerch

Let d be the conductor of ψ_1. Assume that $p \nmid d$. Bruce Ferrero and I proved the following formula in 1977:

$$L_p'(0,\psi) \;=\; \sum_{c=1}^{d} \psi_1(c) log_p\big(\Gamma_p(c/d)\big) \;+\; L_p(0,\psi) log_p(d) \quad .$$

Here $\Gamma_p(x)$ is Morita's p-adic Gamma function and log_p is the p-adic log function (defined on $1+p\mathbf{Z}_p$ and extended to $\mathbf{Z}_p^\times$). The interpolation property for $\Gamma_p(x)$ is

$$\Gamma_p(n) \;=\; (-1)^n \prod_{\substack{a=1 \\ p \nmid a}}^{n-1} a$$

This extends to a continuous function for $x \in \mathbf{Z}_p$.

$L'_p(0, \psi_1)$ when $\psi_1(p) = 1$

At precisely the same time that Ferrero and I proved the above formula, Gross and Koblitz proved a formula relating certain products of the $\Gamma_p(c/d)$'s to Gaussian sums for $\mathbf{F}_{p^f}$, where f is the order of $p + d\mathbf{Z}$ in $(\mathbf{Z}/d\mathbf{Z})^\times$. If $\psi_1(p) = 1$, then those products show up in the above formula for $L'_p(0, \psi_1)$, which is then a linear combination of p-adic logs of algebraic numbers. As a consequence, one can prove that $L'_p(0, \psi_1) \neq 0$ by using a theorem from transcendental number theory (the Baker-Brumer theorem).

In the above, one extends log_p to a homomorphism $log_p : \mathbf{Q}_p^\times \to \mathbf{Z}_p$ by taking $log_p(p) = 0$. The kernel of log_p is $\mu_{p-1} p^{\mathbf{Z}}$.

A special case

Suppose ψ_1 has order 2. Let F be the corresponding imaginary quadratic field. Since $\psi_1(p) = 1$, we have $p^h = \pi\overline{\pi}$ where $\pi \in \mathcal{O}_F$ and $h = h_F$, the class number of F. Then the formula becomes

$$L_p'(0, \psi_1) \;=\; \frac{4}{|\mathcal{O}_F^\times|} \cdot log_p(\overline{\pi}) \;=\; \mathcal{L}(\psi_1) \cdot L(0, \psi_1)$$

where the "$\mathcal{L}$-invariant" $\mathcal{L}(\psi_1)$ is defined by

$$\mathcal{L}(\psi_1) \;=\; \frac{log_p(\frac{\pi}{\overline{\pi}})}{ord_p(\frac{\pi}{\overline{\pi}})}$$

The nonvanishing of $L_p'(0, \psi_1)$ becomes clear in this case.

New proofs.

Another quite different proof of the above derivative formula has been given in a recent paper by Dasgupta, Darmon, and Pollack. The proof works even for the p-adic L-functions over totally real number fields constructed by Deligne and Ribet.

In the rest of this talk, we describe a new proof of the formula for $L_p'(0, \psi_1)$ when $\psi_1(p) = 1$ (due to Benjamin Lundell, Shaowei Zhang, and myself). In place of the Gross-Koblitz formula, it uses properties of a certain p-adic L-function of two-variables, including the so-called Main Conjecture for that function (proved by Karl Rubin).

We begin by briefly outlining a proof of a derivative formula for another p-adic L-function using a two-variable approach.

$L_p(s, E)$, where E is an elliptic curve defined over $\mathbf{Q}$

For an elliptic curve $E/\mathbf{Q}$ with good, ordinary or multiplicative reduction at p, a p-adic L-function $L_p(s, E)$ can be defined.

Mazur & Swinnerton-Dyer (1974),

Mazur, Tate, & Teitelbaum (1985).

Just as for the Kubota-Leopoldt p-adic L-function, the interpolation property for $L_p(s, E)$ sometimes forces that function to have a zero. This happens when E has split, multiplicative reduction at p. In that case, one always has $L_p(1, E) = 0$.

The formula for $L'_p(1, E)$, when E has split multiplicative reduction at p

The formula proposed by Mazur, Tate, and Teitelbaum is

$$L'_p(1, E) \; = \; \mathcal{L}(E) \cdot \frac{L(1, E)}{\Omega_E} \quad ,$$

where

$$\mathcal{L}(E) = \frac{log_p(q_E)}{ord_p(q_E)}$$

and $q_E \in \mathbf{Q}_p^\times$ is defined by

$$j_E \; = \; \frac{1}{q_E} \; + \; 744 \; + \; 196884 q_E \; + \;$$

It is the "Tate period" for E.

The nonvanishing of $\mathcal{L}(E)$

It was proved by K. Barré-Sirieix, G. Diaz, F. Gramain, and G. Philibert that q_E is transcendental.

Therefore, $\mathcal{L}(E) \neq 0$.

A proof of the formula, briefly and inaccurately sketched

We briefly outline the proof by Glenn Stevens and myself for the formula.

The paper of Mazur, Tate, and Teitelbaum constructs p-adic L-functions $L_p(s, f)$ for modular forms f of arbitrary weight. The function $L_p(s, E)$ is $L_p(s, f_E)$, where f_E is the modular form of weight 2 corresponding to E.

By Hida Theory, there is a Hida family of modular forms f_k, where $k \geq 2$, such that f_k is of weight k and $f_2 = f_E$.

The main ingredient in our proof: There is a two-variable p-adic L-function $L_p(s, k)$ (constructed by Kitagawa-Mazur) such that, when k is an integer ≥ 2, we have

$$L_p(s, k) = c_k L_p(s, f_k)$$

for some constants c_k with $c_2 = 1$.

Properties of $L_p(s, k)$

1. Assuming that $L(z, E)$ has an even order zero at $z = 1$, $L_p(s, E)$ has an odd order zero at $s = 1$ and so does $L_p(s, f_k)$ at $s = \frac{k}{2}$ when $k \geq 2$. Thus, $L_p(\frac{k}{2}, k) = 0$ for all $k \in \mathbf{Z}_p$.

2. $L_p(s, 2) = L_p(s, E)$

3. $L_p(1, k) = \left(1 - \alpha_p(k)^{-1}\right) L_p^*(k)$ for $k \in \mathbf{Z}_p$, where $\alpha_p(k)$ and $L_p^*(k)$ are analytic functions for $k \in \mathbf{Z}_p$. Furthermore,

$$\alpha_p(2) = 1, \qquad \textit{and} \qquad L_p^*(2) = \frac{L(1, E)}{\Omega_E} \quad .$$

Computation of $L_p'(1, E)$

The properties on the previous slide imply that

$$L_p'(1, E) = -2\alpha_p'(2) L_p^*(2) = -2\alpha_p'(2) \frac{L(1, E)}{\Omega_E} \quad .$$

Thus, one must prove that $\alpha_p'(2) = -\frac{1}{2}\mathcal{L}(E)$. This is proved by a Galois cohomology argument . It involves the Galois representation attached to the Hida family. The Tate period enters the argument since the extension class associated with the exact sequence

$$0 \longrightarrow \mu_{p^\infty} \longrightarrow E[p^\infty] \longrightarrow \mathbf{Q}_p/\mathbf{Z}_p \longrightarrow 0$$

is given by the Kummer cocycles defined by p-power roots of q_E.

The two-variable $\mathfrak{p}$-adic L-function of Katz. Its domain of definition.

Suppose that K is an imaginary quadratic field and that p splits in K. There are two prime ideals $\mathfrak{p}$ and $\overline{\mathfrak{p}}$ lying over p. The map $\kappa : G_K \to 1 + p\mathbf{Z}_p$ was defined before. It factors through $\mathrm{Gal}(K_\infty/K)$, where K_∞ is the cyclotomic $\mathbf{Z}_p$-extension of K.

Let L_∞ denote the unique $\mathbf{Z}_p$-extension of K in which $\overline{\mathfrak{p}}$ is unramified. The prime $\mathfrak{p}$ is ramified in L_∞/K. We choose λ so that it factors through $\mathrm{Gal}(L_\infty/K)$ and defines an isomorphism

$$\mathrm{Gal}(L_\infty/K) \longrightarrow 1 + p\mathbf{Z}_p \quad .$$

We can make the choice of λ unique by requiring that it be the Galois representation corresponding to a Grossencharacter for K of type A_0 with infinity type $(1, 0)$.

$\mathrm{Hom}_{cont}(G_K, \overline{\mathbf{Q}}_p^\times)$ contains $\{ \kappa^s \lambda^k \mid (s, k) \in \mathbf{Z}_p \times \mathbf{Z}_p \}$

The two-variable $\mathfrak{p}$-adic L-function of Katz

Let $\psi_1 = \psi\omega^{-1}$ be as before. We assume from here on that $\psi_1(p) = 1$. Let F be the cyclic extension of $\mathbf{Q}$ cut out by ψ_1. Thus, p splits completely in $F/\mathbf{Q}$.

Choose any imaginary quadratic field K in which p splits completely and such that $K \cap F = \mathbf{Q}$. Let $\varphi = \psi_1\big|_{G_K}$.

The two-variable $\mathfrak{p}$-adic L-function $L_{\mathfrak{p}}(\cdot)$ is defined on the following domain: $\mathrm{Hom}_{cont}\big(G_K, \overline{\mathbf{Q}}_p^\times\big)$. We will consider the restriction of that function to

$$\{\ \varphi\kappa^s\lambda^k \ \big|\ (s,k) \in \mathbf{Z}_p \times \mathbf{Z}_p\ \} \quad .$$

Or, one can regard $L_{\mathfrak{p}}(\cdot)$ as a function on the family $\mathrm{Ind}_{G_K}^{G_{\mathbf{Q}}}(\varphi\kappa^s\lambda^k)$ of 2-dimensional Galois representations.

Properties of $L_{\mathfrak{p}}(\varphi\kappa^s\lambda^k) = L_{\mathfrak{p}}(s,k)$

1. Interpolation property : For $(s,k) \in \mathbf{Z} \times \mathbf{Z}$ satisfying $1 \leq s \leq k$. For fixed $k \in \mathbf{Z}$, $k \geq 1$,

$L_{\mathfrak{p}}(s,k) = c_{k+1} \cdot (\textit{the } p-\textit{adic } L-\textit{function for a CM form of weight } k+1)$

with precise constants c_{k+1}.

2. Gross Factorization Theorem: For the line $k = 0$. Let $\varepsilon =$ the quadratic character corresponding to K. We have

$$L_{\mathfrak{p}}(0,s) = L_{\mathfrak{p}}(\varphi\kappa^s) = L_p(s,\psi)L_p(1-s,\varepsilon\psi_1^{-1})$$

So $L_{\mathfrak{p}}(0,0) = 0$ and $\left.\dfrac{dL_{\mathfrak{p}}(s,0)}{ds}\right|_{s=0} = L_p'(0,\psi)L_p(1,\varepsilon\psi_1^{-1})$.

More properties of $L_{\mathfrak{p}}(s,k)$

3. For the line $s=0$. Katz's Kronecker Limit Formula:

$$\left.\frac{dL_{\mathfrak{p}}(0,k)}{dk}\right|_{k=0} = L(0,\psi_1)L_p(1,\varepsilon\psi_1^{-1})$$

Thus, the ratio $\left(\left.\frac{dL_{\mathfrak{p}}(s,0)}{ds}\right|_{s=0}\right)\Big/\left(\left.\frac{dL_{\mathfrak{p}}(0,k)}{dk}\right|_{k=0}\right)$ is equal $L_p'(0,\psi)\Big/L(0,\psi_1)$.

This should be $\mathcal{L}(\psi_1)$.

1. The direction where $L_{\mathfrak{p}}(s,k)$ has a double zero

The linear term in the power series expansion for $L_{\mathfrak{p}}(s,k)$ is $as+bk$, where

$$a = \left.\frac{dL_{\mathfrak{p}}(s,0)}{ds}\right|_{s=0}, \qquad b = \left.\frac{dL_{\mathfrak{p}}(0,k)}{dk}\right|_{k=0}$$

One should have $a/b = \mathcal{L}(\psi_1)$.

We will now assume (for simplicity) that ψ_1 has order dividing $p-1$.

2. The direction where $L_{\mathfrak{p}}(s,k)$ has a double zero

This direction involves $\mathcal{L}(\psi_1)$. Let D_∞ be a $\mathbf{Z}_p$-extension of K. Then

$$K \subset D_\infty \subset K_\infty L_\infty$$

Then $\mathrm{Gal}(K_\infty L_\infty / D_\infty)$ is isomorphic to $\mathbf{Z}_p$. Suppose δ is a topological generator.

Then $\kappa^s \lambda^k$ factors through $\mathrm{Gal}(D_\infty/K)$ when $\kappa^s\lambda^k(\delta) = 1$. The set

$$\{ (s,k) \mid \kappa^s \lambda^k(\delta) = 1\}$$

is the line $as + bk = 0$, where $a = log_p\big(\kappa(\delta)\big)$, $b = log_p\big(\lambda(\delta)\big)$.

3. The direction where $L_{\mathfrak{p}}(s,k)$ has a double zero

Recall that ψ_1 is an odd character of $\mathrm{Gal}(F/\mathbf{Q})$ and that p splits completely in $F/\mathbf{Q}$. There is a $\mathbf{Z}_p$-extension F_∞ of F which is Galois over $\mathbf{Q}$ and such that $\mathrm{Gal}(F/\mathbf{Q})$ acts on $\mathrm{Gal}(F_\infty/F)$ by the character ψ_1. Completing at a prime v above p, we have $F_v = \mathbf{Q}_p$ and $F_{\infty,v}$ is a $\mathbf{Z}_p$-extension of $\mathbf{Q}_p$.

Any $\mathbf{Z}_p$-extension of $\mathbf{Q}_p$ is determined by its universal norm subgroup which is of the form $\mu_{p-1}\langle q\rangle$, where $ord_p(q) \neq 0$ (except for the unramified $\mathbf{Z}_p$-extension of $\mathbf{Q}_p$). Excluding the unramified $\mathbf{Z}_p$-extension, a $\mathbf{Z}_p$-extension is determined by $\dfrac{log_p(q)}{ord_p(q)}$.

In the special case where ψ_1 has order 2, the universal norm subgroup for $F_{\infty,v}$ contains $\pi/\overline{\pi}$. (Recall that $\pi \in \mathcal{O}_F$ and $\pi\overline{\pi} = p^h$.) In general, one applies an idempotent to some p-unit π in F.

4. The direction where $L_{\mathfrak{p}}(s,k)$ has a double zero

There is one $\mathbf{Z}_p$-extension D_∞ of K such that

$$D_{\infty,\bar{\mathfrak{p}}} = F_{\infty,v}$$

One associates a Selmer group to the representation $\varphi = \psi_1\big|_{G_K}$ over any $\mathbf{Z}_p$-extension D_∞ of K and also over the $\mathbf{Z}_p^2$-extension $K_\infty L_\infty$ of K. The latter Selmer group has a characteristic ideal generated (essentially) by $L_{\mathfrak{p}}(s,k)$. This is a special case of the "Main Conjecture" formulated by Yager and proved by Rubin.

5. The direction where $L_{\mathfrak{p}}(s,k)$ has a double zero

For any $\mathbf{Z}_p$-extension D_∞/K, we denote the Selmer group for φ by $Sel_\varphi(D_\infty)$. There is a natural action of $\mathrm{Gal}(D_\infty/K)$ on that object.

Let I denote the augmentation ideal in $\mathbf{Z}_p[[\mathrm{Gal}(D_\infty/K)]]$. When D_∞ is any $\mathbf{Z}_p$-extension of K, then $Sel_\varphi(D_\infty)[I]$ has $\mathbf{Z}_p$-corank 1. Usually, $Sel_\varphi(D_\infty)[I^2]$ also has $\mathbf{Z}_p$-corank 1. The one exception is when D_∞ is chosen as above. Then $Sel_\varphi(D_\infty)[I^2]$ has $\mathbf{Z}_p$-corank 2.

The local condition at $\bar{\mathfrak{p}}$ is that cocycle classes be unramified. For the exceptional $\mathbf{Z}_p$-extension D_∞ (and none of the others $\mathbf{Z}_p$-extensions of K), the elements of $Sel_\varphi(D_\infty)[I]$ are actually locally trivial at $\bar{\mathfrak{p}}$, and not just locally unramified. This is what allows one to show that $Sel_\varphi(D_\infty)[I^2]$ has $\mathbf{Z}_p$-corank 2.

The corresponding line $as + bk = 0$ is the direction where $L_{\mathfrak{p}}(s,k)$ has a double zero.

6. The direction where $L_{\mathfrak{p}}(s,k)$ has a double zero

One can restrict $\kappa^s\lambda^k$ to the local Galois group $G_{K_{\bar{\mathfrak{p}}}}$. One wants this to factor through $D_{\infty,\bar{\mathfrak{p}}}/\mathbf{Q}_p$. By local class field theory, if q is any universal norm for $D_{\infty,\bar{\mathfrak{p}}}/\mathbf{Q}_p$, then one wants

$$\kappa^s\lambda^k\big(Rec(q)\big) = 1$$

This suffices to determine the line $as + bk = 0$.

In the special case where ψ_1 has order 2, one can take $q = \pi/\overline{\pi}$. One finds that

$$a/b \;=\; \frac{log_p(\frac{\pi}{\overline{\pi}})}{ord_p(\frac{\pi}{\overline{\pi}})} \;=\; \mathcal{L}(\psi_1)$$

Thank you!