Comment on "Scattering Theory Derivation of a 3D Acoustic Cloaking Shell"

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In a recent Letter, Cummer et al. [1] give a description of material parameters for acoustic wave propagation giving rise to a 3D spherical cloak, and verify the cloaking phenomenon on the level of scattering coefficients. A similar configuration has been given in [2]. In this Comment, we show that these theoretical constructions follow directly from earlier work [3, Sec. 3] on full wave analysis of cloaking for the Helmholtz equation with respect to Riemannian metrics. Furthermore, the analysis there covers the case of acoustically radiating objects being enclosed in the cloaked region.

For a Riemannian metric $g = (g_{ij})$ in n-dimensional space, the Helmholtz equation with source term is

$$\frac{1}{\sqrt{|g|}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(\sqrt{|g|}\, g^{ij} \frac{\partial u}{\partial x_j} \right) + k^2 u = f, \qquad (1)$$

where $|g| = \det(g_{ij})$ and $(g^{ij}) = g^{-1} = (g_{ij})^{-1}$. The equation of acoustics, in the notation of [1], is

$$\nabla \cdot (\overline{\overline{\rho}}^{-1} \nabla p) + \frac{\omega^2}{\lambda} p = 0. \qquad (2)$$

The connection between (1) and (2) is given by

$$k = \omega, \quad \overline{\overline{\rho}}^{-1} = (\sqrt{|g|}\, g^{ij}), \quad \sqrt{|g|} = \lambda^{-1}. \qquad (3)$$

In [3], we showed that the degenerate cloaking metrics g for electrostatics constructed in [4], giving the same boundary measurements as the Euclidian metric $g_0 = (\delta_{ij})$, also cloak with respect to solutions of the Helmholtz equation at any nonzero frequency k and with any source f. An example in 3D, with respect to spherical coordinates (r, θ, ϕ), is

$$g^{-1} = \begin{pmatrix} 2(r-1)^2 \sin\theta & 0 & 0 \\ 0 & 2\sin\theta & 0 \\ 0 & 0 & 2(\sin\theta)^{-1} \end{pmatrix} \qquad (4)$$

on $\{1 < r \le 2\}$, with the cloaked region being the ball $\{0 \le r < 1\}$. Note that $\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\phi}$ are not normalized to have length 1; otherwise, (4) agrees with [1, (24-25)] and [2, (8)]. This g is the image of g_0 under the singular transformation $(r, \theta, \phi) = F(r', \theta', \phi')$ defined by $r = 1 + \frac{r'}{2}$, $\theta = \theta'$, $\phi = \phi'$, $0 < r' \le 2$, which blows up the point $r' = 0$ to the cloaking surface $\Sigma = \{r = 1\}$. The same transformation was later used in [5] and gives rise to the cloaking structure that is referred to in [3] as the *single coating*. It was shown in [3, Thm.1] that if the cloaked region is given any nondegenerate metric, then finite energy waves u that satisfy the Helmholtz equation (1) on $\{r < 2\}$ in the sense of distributions have the same set of Cauchy data at $r = 2$, i.e., the same acoustic boundary measurements, as do the solutions for the Helmholtz equation for g_0 with source term $f \circ F$. (This also holds for Maxwell's equations, explaining the "mirage effect" of [6].) The part of f supported within the cloaked region is undetectable at $r = 2$. Furthermore, on the boundary Σ of the cloaked region, the normal derivative of u must vanish, so that within Σ the acoustic waves propagate as if Σ were lined with a sound–hard surface.

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