Reconstruction of penetrable obstacles in acoustics

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Abstract

We develop a reconstruction algorithm to determine penetrable obstacles in a region in the plane from acoustic measurements made on the boundary. This algorithm uses complex geometrical optics solutions to the Helmholtz equation with polynomial-type phase functions. We have also tested the algorithm with simulated data.

1 Introduction

Let D be an unknown obstacle with an unknown index of refraction subset of a larger domain Ω with an homogeneous index of refraction. Assume that D is penetrable. We send an acoustic wave from the boundary of Ω. Suppose that we are given all possible Cauchy data or the Dirichlet-to-Neumann measured on $\partial\Omega$. The inverse problem we consider in this paper is to determine the shape of D using the boundary measurements.

In this paper, we consider this problem in the plane, that is, we assume $D \Subset \Omega \subset \mathbb{R}^2$. For simplicity, we suppose that both D and Ω have C^2 boundaries. Let $\gamma_D \in C^2(\overline{D})$ satisfy $\gamma_D \geq c_\gamma$ for some positive constant c_γ

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and denote $\widetilde{\gamma} := 1 + \gamma_D \chi_D$, where χ_D denotes the characteristic function of D. Let $k > 0$ and consider the steady state acoustic wave equation in Ω with Dirichlet condition

$$\begin{cases} \nabla \cdot (\widetilde{\gamma} \nabla v) + k^2 v = 0 \text{ in } \Omega, \\ v = f \text{ on } \partial\Omega. \end{cases} \tag{1.1}$$

In the case that $\widetilde{\gamma} = 1$, the problem (1.1) is the boundary value problem for the Helmholtz equation

$$\begin{cases} \Delta v_0 + k^2 v_0 = 0 \text{ in } \Omega, \\ v_0 = f \text{ on } \partial\Omega. \end{cases} \tag{1.2}$$

Throughout the paper, we assume that k^2 is not a Dirichlet eigenvalue of the operator $-\nabla \cdot (\widetilde{\gamma} \nabla \bullet)$ and $-\Delta$ in Ω. It is known that for any $f \in H^{1/2}(\partial\Omega)$, there exists a unique solution v to (1.1). Thus, we can define the Dirichlet-to-Neumann map $\Lambda_D : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ for (1.1) by

$$\Lambda_D f := \left. \frac{\partial v}{\partial \nu} \right|_{\partial\Omega} \text{ for } f \in H^{1/2}(\partial\Omega).$$

The inverse problem consists of determining D from Λ_D. The domain D can also be treated as an inclusion embedded in Ω. The aim of this work is to give a reconstruction algorithm for this problem. Note that the information on the medium parameter γ_D inside D is not known a priori.

The main tool in our reconstruction method is the complex geometrical optics (CGO) solutions with polynomial-type phase functions for the Helmholtz equation. This type of CGO solutions has been introduced in [17] for general second order elliptic equations or systems having the Laplacian as the principal part, which includes the Helmholtz equation. However, in order to obtain more explicit forms in the lower orders of the CGO solutions, we will not adopt the approach in [17]. Instead, we will take advantage of the transformation between the harmonic functions and the solutions to the Helmholtz equation in two dimensions found by Vekua [19] (also see [8]) to construct the needed CGO solutions.

Having obtained the CGO solutions with polynomial phases for the Helmholtz equation, we apply them to determine the shape of D by Λ_D. CGO solutions have been found to be useful in detecting some geometrical information of D in several inverse problems. For the inclusion problem in the static case, i.e., $k = 0$, there are several articles, and some of them include numerical results, dealing with either the conductivity equation (the first equation of (1.1) with $k = 0$) or the isotropic elasticity [1], [5], [6], [4], [16], [17], and [18]. This type of method was called the enclosure method by Ikehata. We refer to his survey paper [7] for some of the early developments.

For the reconstruction of penetrable obstacles or inclusions in acoustics by the enclosure type method, we mention the work [9] by Ikehata. In this paper he considers the reconstruction of a penetrable *polygon* having homogeneous medium different from the background one by a single pair of Cauchy data in two dimensions. Using CGO solutions with linear phases, he showed that one can reconstruct the convex hull of the polygonal obstacle using a single measurement. In our paper, we consider a general penetrable obstacle and assume the medium inside of the penetrable obstacle is an unknown inhomogeneous function. Using CGO solutions with polynomial-type phases, we are able to reconstruct more information on the shape of the penetrable obstacle from the Dirichlet-to-Neumann map. Especially, in theory, we can completely reconstruct certain class of penetrable objects such as star-shaped obstacles. For other related results, we would like to mention that Nakamura and Yoshida [15] used CGO solutions with limiting Carleman weights introduced in [11] to reconstruct some non-convex sound-hard obstacles from the Dirichlet-to-Neumann map. The level set of the limiting Carleman weights are circles (in two dimensions) or spheres (in three dimensions). Also, we mention that the uniqueness of determining a penetrable obstacle by the scattering amplitude at a fixed energy was proven by Isakov [10] and Kirsch, Kress [12].

Unlike the static case, for the enclosure type method in the Helmholtz equation, we need to analyze the effect coming from the term k^2 due to the loss of positivity in the equation. More precisely, we have to be able to bound the L^2 norm of $w := v - v_0$ in Ω in terms of v_0 in D (see (3.16)) and on ∂D (see (3.19)). The estimate (3.16) is easier. Our main focus is on (3.19). In the impenetrable case, this can be achieved using elliptic regularity with smooth coefficient and the Sobolev embedding theorem (see [15]). However, in the penetrable case, the coefficient is merely L^∞. We do not have enough smoothness on the solution to apply the Sobolev embedding theorem. The main technical part of this work is to establish the needed estimate.

In section 2 we construct the CGO solutions to the Helmholtz equation with polynomial phase functions and their properties. In section 3 we establish some identities and the estimates we need. In section 4 we prove our main result on the determination of D from Λ_D under a "curvature assumption" on ∂D on the intersection of the level sets of the real part of the phases of the CGO with ∂D (see Theorem 4.1). In section 5 we show that the curvature assumption is satisfied for a large class of CGO solutions. In section 7 we state the reconstruction algorithm. Finally in section 8 we test this algorithm with simulated data.

2 CGO solutions

In this section we want to construct CGO solutions with polynomial phases for the Helmholtz equation. We do this by combining the idea in [17] and the transform introduced by Vekua (see (13.9) on page 58 in [19]). Let us first introduce $\eta(x) := c_*\big((x_1 - x_{*,1}) + i(x_2 - x_{*,2})\big)^N$ as the phase function, where $c_* \in \mathbb{C}$ satisfies $|c_*| = 1$, N is a positive integer, and $x_* = (x_{*,1}, x_{*,2}) \in \mathbb{R}^2 \setminus \overline{\Omega}$. Without loss of generality we may assume that $x_* = 0$ using an appropriate translation. We put $\eta_{\mathrm{R}}(x) := \mathrm{Re}\,\eta(x)$. Note that

$$\eta_{\mathrm{R}}(x) = r^N \cos N(\theta - \theta_*) \text{ for } x = r(\cos\theta,\, \sin\theta) \in \mathbb{R}^2.$$

We now define an open cone

$$\Gamma := \left\{ r(\cos\theta,\, \sin\theta) : |\theta - \theta_*| < \frac{\pi}{2N} \right\}$$

with an opening angle π/N (see Figure 1). It is clear that $\eta_{\mathrm{R}}(x) > 0$ for all $x \in \Gamma$.

Given any $h > 0$, $\check{V}_h(x) := \exp\big(\eta(x)/h\big)$ is a harmonic function. Following Vekua [19], we define a map T_k on any harmonic function $\check{V}(x)$ by

$$\begin{aligned} T_k\check{V}(x) &:= \check{V}(x) - \int_0^1 \check{V}(tx)\,\frac{\partial}{\partial t}\Big\{ J_0\big(k|x|\sqrt{1-t}\big)\Big\}dt \\ &= \check{V}(x) - k|x| \int_0^1 \check{V}\big((1-s^2)x\big)\, J_1\big(k|x|s\big)\, ds \end{aligned}$$

where J_m is the Bessel function of the first kind of order m. We now set $V_h^\sharp(x) := T_k\check{V}_h(x)$. Then $V_h^\sharp(x)$ satisfies the Helmholtz equation $\Delta V_h^\sharp + k^2 V_h^\sharp = 0$ in $\mathbb{R}^2$. The function $V_h^\sharp$ is a CGO solution to the Helmholtz equation in Γ by the following lemma.

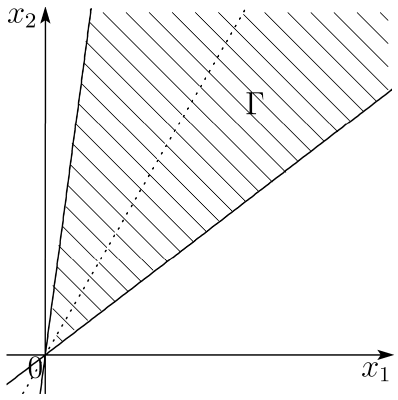

Figure 1: The definition of Γ

Lemma 2.1. *We can write*

$$V_h^\sharp(x) = \exp\left(\frac{\eta(x)}{h}\right)\bigl(1 + R_0(x)\bigr) \ \text{in } \Gamma, \tag{2.1}$$

where $R_0(x) = R_0(x; h)$ *satisfies*

$$|R_0(x)| \le h\,\frac{k^2|x|^2}{4\eta_{\mathrm{R}}(x)}, \qquad \left|\frac{\partial R_0}{\partial x_j}(x)\right| \le \frac{Nk^2|x|^{N+1}}{4\eta_{\mathrm{R}}(x)} + h\,\frac{k^2|x_j|}{2\eta_{\mathrm{R}}(x)} \quad \text{in } \Gamma.$$

Proof. Let $x \in \Gamma$. We note that

$$\begin{aligned} |R_0(x)| &\le k|x| \int_0^1 \exp\left(\frac{1}{h}\,\mathrm{Re}\Bigl(\eta\bigl((1-s^2)x\bigr) - \eta(x)\Bigr)\right) \bigl|J_1\bigl(k|x|s\bigr)\bigr|\,ds \\ &\le \frac{k^2|x|^2}{2} \int_0^1 \exp\left(\frac{1}{h}\,\mathrm{Re}\Bigl(\eta\bigl((1-s^2)x\bigr) - \eta(x)\Bigr)\right) s\,ds \end{aligned}$$

since

$$R_0(x) = -k|x| \int_0^1 \exp\left(\frac{1}{h}\Bigl(\eta\bigl((1-s^2)x\bigr) - \eta(x)\Bigr)\right) J_1\bigl(k|x|s\bigr)\,ds \tag{2.2}$$

and $|J_1(t)| \le t/2$ for $t \ge 0$. On the other hand, we can see that

$$\begin{aligned} \mathrm{Re}\Bigl(\eta\bigl((1-s^2)x\bigr) - \eta(x)\Bigr) &= \mathrm{Re}\Bigl(-\eta(x)\bigl(1-(1-s^2)^N\bigr)\Bigr) \\ &= -\eta_{\mathrm{R}}(x)\,\bigl(1-(1-s^2)^N\bigr) \le -\eta_{\mathrm{R}}(x)\,s^2 \end{aligned} \tag{2.3}$$

for any $0 < s < 1$ using the formula

$$1 - s^2 - (1-s^2)^N = (1-s^2)\big(1-(1-s^2)^{N-1}\big) \geq 0 \text{ for any } s \in (0,1).$$

Hence, since $\eta_{\mathrm{R}}(x) > 0$ in Γ, we have

$$\begin{aligned}
|R_0(x)| &\leq \frac{k^2|x|^2}{2}\int_0^1 \exp\left(\frac{1}{h}\,\mathrm{Re}\Big(\eta\big((1-s^2)x\big)-\eta(x)\Big)\right) s\,ds \\
&\leq \frac{k^2|x|^2}{2}\int_0^1 \exp\left(-\frac{\eta_{\mathrm{R}}(x)\,s^2}{h}\right) s\,ds < h\,\frac{k^2|x|^2}{4\eta_{\mathrm{R}}(x)}.
\end{aligned}$$

In a similar fashion we can obtain the estimate for $\partial R_0/\partial x_j$ since we have

$$\begin{aligned}
\nabla R_0(x) = -\frac{k|x|}{h}\int_0^1 &\Big((1-s^2)(\nabla\eta)\big((1-s^2)x\big) - \nabla\eta(x)\Big) \\
&\times \exp\left(\frac{1}{h}\Big(\eta\big((1-s^2)x\big)-\eta(x)\Big)\right) J_1\big(k|x|s\big)\,ds \\
- k^2 x &\int_0^1 \exp\left(\frac{1}{h}\Big(\eta\big((1-s^2)x\big)-\eta(x)\Big)\right) J_0\big(k|x|s\big)\,s\,ds
\end{aligned}$$

and $|J_0(t)| \leq 1$ for any $t \geq 0$. □

From the above lemma, we conclude that $V_h^\sharp$ is a CGO solution to the Helmholtz equation in $\Gamma \cap \Omega$. We now extend it to the whole domain Ω by using an appropriate cut-off. Let $l_s := \{x \in \Gamma : \eta_{\mathrm{R}}(x) = 1/s\}$ for $s > 0$. This is the level curve of η_{R} (see Figure 2). For $\varepsilon > 0$ small enough and $t^\sharp > 0$ large enough, we define the function $\phi_t \in C^\infty(\mathbb{R}^2)$ by

$$\phi_t(x) = \begin{cases} 1 & \text{for } x \in \overline{\bigcup_{0<s<t+\varepsilon/2} l_s},\ t \in [0,t^\sharp], \\ 0 & \text{for } x \in \mathbb{R}^2 \setminus \bigcup_{0<s<t+\varepsilon} l_s,\ t \in [0,t^\sharp] \end{cases}$$

(see Figure 3) and

$$|\partial_x^\alpha \phi_t(x)| \leq C_\phi \quad \text{for } |\alpha| \leq 2,\ x \in \Omega,\ t \in [0,t^\sharp]$$

for some positive constant C_ϕ depending only on Ω, N, $t^\sharp$ and ε. Next we

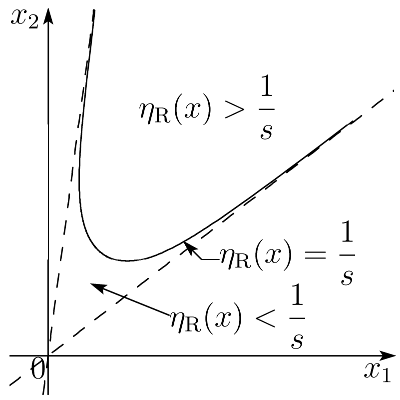

Figure 2: A level curve of η_{R}.

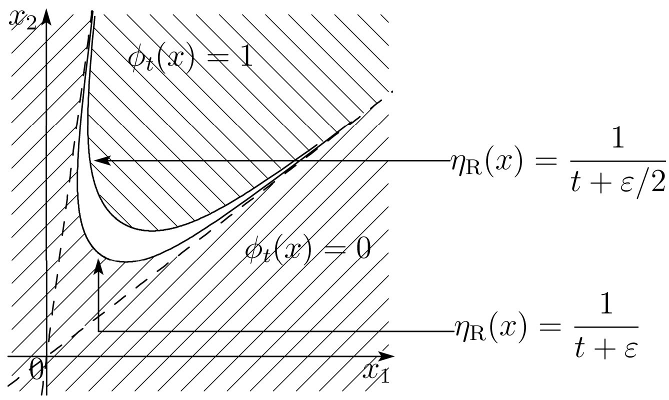

Figure 3: The definition of ϕ_t

define the function $V_{t,h}$ by

$$V_{t,h}(x) := \phi_t(x) \exp\left(-\frac{1}{ht}\right) V_h^\sharp(x) \text{ for } x \in \overline{\Omega}.$$

Then we know by Lemma 2.1 that the dominant parts of $V_{t,h}$ and its derivatives are as follows:

$$V_{t,h}(x) = \begin{cases} 0 \text{ for } x \in \Omega \setminus \bigcup_{0<s<t+\varepsilon} l_s, \\ \exp\left(\frac{1}{h}\left(-\frac{1}{t} + \eta(x)\right)\right) \left(\phi_t(x) + S_0(x)h\right) \\ \hfill \text{for } x \in \Omega \cap \bigcup_{0<s<t+\varepsilon} l_s, \end{cases} \tag{2.4}$$

$$\nabla V_{t,h}(x) = \begin{cases} 0 \text{ for } x \in \Omega \setminus \bigcup_{0<s<t+\varepsilon} l_s, \\ \frac{1}{h} \exp\left(\frac{1}{h}\left(-\frac{1}{t} + \eta(x)\right)\right) \left(\phi_t(x)\nabla\eta(x) + \boldsymbol{S}(x)h\right) \\ \hfill \text{for } x \in \Omega \cap \bigcup_{0<s<t+\varepsilon} l_s \end{cases} \tag{2.5}$$

for $t \in (0, t^\sharp]$ and $h \in (0, 1]$, where $S_0(x) = S_0(x; t, h)$ and $\boldsymbol{S}(x) = \boldsymbol{S}(x; t, h)$ satisfy

$$|S_0(x)|,\ |\boldsymbol{S}(x)| \le C_{\mathrm{V}} \text{ for any } x \in \Omega \cap \bigcup_{0<s<t+\varepsilon} l_s,\ t \in (0, t^\sharp],\ h \in (0, 1]$$

with a positive constant C_{V} depending only on Ω, N, $t^\sharp$, ε and k. Unfortunately, the function $V_{t,h}$ does not satisfy the Helmholtz equation in Ω. However, if we let $v_{0,t,h}$ be the solution to the Helmholtz equation in Ω with boundary value $f_{t,h} := V_{t,h}|_{\partial\Omega}$, then the error between $V_{t,h}$ and $v_{0,t,h}$ is exponentially small as shown in the following lemma.

Lemma 2.2. *There exist constants $C_0, C_0' > 0$ and $a > 0$ such that*

$$\|v_{0,t,h} - V_{t,h}\|_{H^2(\Omega)} \le \frac{C_0'}{h} e^{-a_t/h} \le C_0 e^{-a/h}$$

for any $h \in (0, 1]$, where the constants C_0 and C_0' depend only on Ω, k, N, $t^\sharp$ and ε; the constant a depends only on $t^\sharp$ and ε; and we set $a_t := 1/t - 1/(t + \varepsilon/2)$.

This lemma can be proved in the same way as Lemma 4.1 in [17]. So we omit the details here.

For our inverse problem, the difference between the two Dirichlet-to-Neumann maps Λ_D and $\Lambda_\emptyset$ plays a crucial role. We define the functional $E(t,h)$ by

$$E(t,h) := \int_{\partial\Omega} (\Lambda_D - \Lambda_\emptyset) f_{t,h} \overline{f}_{t,h} \, d\sigma. \tag{2.6}$$

Roughly speaking, for a fixed large N, we can show that if $D \cap \left(\bigcup_{0<s<t} l_s\right) = \emptyset$ then $E(t,h) \to 0$ as $h \to +0$; if $D \cap \left(\bigcup_{0<s<t} l_s\right) \neq \emptyset$ then $E(t,h) \to +\infty$ as $h \to +0$. We will state our main result more precisely in Theorem 4.1.

3 Some identities and estimates

In this section, we derive some identities and estimates for solutions to some Dirichlet problems which are needed later. We denote $C > 0$ a general constant in this section. The constant C depends only on Ω, D, γ_D and k. When a constant depends on other data, we will denote the dependence by writing as a subscript, for example C_q the dependence of the constant on q. For a fixed $f \in H^{1/2}(\partial\Omega)$, let v_0 and v be the solutions to the Dirichlet problems (1.2) and (1.1), respectively. As before, we put $w = v - v_0$. Note that w satisfies the Dirichlet problem

$$\begin{cases} \nabla \cdot (\widetilde{\gamma} \nabla w) + k^2 w = -\nabla \cdot ((\widetilde{\gamma} - 1) \nabla v_0) \ \text{ in } \Omega, \\ w = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.1}$$

We first show an estimate for w.

Lemma 3.1. *For any $2 < q \leq 4$, we have*

$$\|w\|_{L^\infty(\Omega)} \leq C_q \|\nabla v_0\|_{L^q(D)}. \tag{3.2}$$

Proof. Recall that k^2 is not a Dirichlet eigenvalue of $-\nabla \cdot (\widetilde{\gamma} \nabla \bullet)$ in Ω. From the well-posedness of the boundary value problem (3.1) (see Corollary 8.7 in [3], for example), we have

$$\|w\|_{H^1(\Omega)} \leq C \left\| (\widetilde{\gamma} - 1) \nabla v_0 \right\|_{L^2(\Omega)} \leq C \left\| \nabla v_0 \right\|_{L^2(D)}.$$

On the other hand, since $W := w$ satisfies the Dirichlet problem

$$\begin{cases} \nabla \cdot (\widetilde{\gamma} \nabla W) = -k^2 w - \nabla \cdot ((\widetilde{\gamma} - 1) \nabla v_0) \ \text{ in } \Omega, \\ W = 0 \text{ on } \partial\Omega, \end{cases}$$

we have the estimate

$$\begin{aligned} \|w\|_{L^\infty(\Omega)} &\leq C_q \left(\|k^2 w\|_{L^{q/2}(\Omega)} + \|(\widetilde{\gamma} - 1)\nabla v_0\|_{L^q(\Omega)} \right) \\ &\leq C_q \left(\|w\|_{L^2(\Omega)} + \|\nabla v_0\|_{L^q(D)} \right) \end{aligned}$$

by Theorem 8.16 in [3] and Hölder's inequality. Hence we get that

$$\begin{aligned} \|w\|_{L^\infty(\Omega)} &\leq C_q \left(\|w\|_{L^2(\Omega)} + \|\nabla v_0\|_{L^q(D)} \right) \leq C_q \left(\|w\|_{H^1(\Omega)} + \|\nabla v_0\|_{L^q(D)} \right) \\ &\leq C_q \left(\|\nabla v_0\|_{L^2(D)} + \|\nabla v_0\|_{L^q(D)} \right) \leq C_q \|\nabla v_0\|_{L^q(D)} \end{aligned}$$

by using Hölder's inequality again. □

We next prove some useful identities.

Lemma 3.2. *We have*

$$\int_{\partial\Omega} (\Lambda_D - \Lambda_\emptyset) f \, \overline{f} \, d\sigma = \operatorname{Re} \int_\Omega (\widetilde{\gamma} - 1) \nabla v \cdot \nabla \overline{v_0} \, dx. \tag{3.3}$$

Proof. It is clear that

$$\begin{aligned} \int_{\partial\Omega} \frac{\partial v}{\partial \nu} \, \overline{\varphi} \, d\sigma &= \int_{\partial\Omega} \widetilde{\gamma} \, \frac{\partial v}{\partial \nu} \, \overline{\varphi} \, d\sigma = \int_\Omega \nabla \cdot \left(\widetilde{\gamma} \, \overline{\varphi} \, \nabla v \right) dx \\ &= \int_\Omega \nabla \cdot \left(\widetilde{\gamma} \nabla v \right) \overline{\varphi} \, dx + \int_\Omega \widetilde{\gamma} \, \nabla v \cdot \nabla \overline{\varphi} \, dx \\ &= -k^2 \int_\Omega v \, \overline{\varphi} \, dx + \int_\Omega \widetilde{\gamma} \, \nabla v \cdot \nabla \overline{\varphi} \, dx \end{aligned}$$

for any $\varphi \in H^1(\Omega)$. Since $v = v_0 = f$ on $\partial\Omega$, the left-hand side of the identity above has the same value whether we take $\varphi = v$ or $\varphi = v_0$ and it is equal to $\int_{\partial\Omega} \Lambda_D f \, \overline{f} \, d\sigma$. Thus we have

$$\begin{aligned} \int_{\partial\Omega} \Lambda_D f \, \overline{f} \, d\sigma &= -k^2 \int_\Omega v \, \overline{v_0} \, dx + \int_\Omega \widetilde{\gamma} \, \nabla v \cdot \nabla \overline{v_0} \, dx \\ &= -k^2 \int_\Omega |v|^2 \, dx + \int_\Omega \widetilde{\gamma} \, |\nabla v|^2 \, dx \in \mathbb{R}. \end{aligned}$$

The right-hand side of the identity above is real. Hence, by taking the real part, we obtain

$$\int_{\partial\Omega} \Lambda_D f\,\overline{f}\,d\sigma = -k^2\,\mathrm{Re}\int_{\Omega} v\,\overline{v_0}\,dx + \mathrm{Re}\int_{\Omega} \widetilde{\gamma}\,\nabla v\cdot\nabla\overline{v_0}\,dx. \tag{3.4}$$

In the same way, we show that

$$\int_{\partial\Omega} \Lambda_{\emptyset} f\,\overline{f}\,d\sigma = -k^2\,\mathrm{Re}\int_{\Omega} v\,\overline{v_0}\,dx + \mathrm{Re}\int_{\Omega} \nabla v\cdot\nabla\overline{v_0}\,dx \tag{3.5}$$

since

$$\begin{aligned}\int_{\partial\Omega} \frac{\partial v_0}{\partial\nu}\,\overline{\varphi}\,d\sigma &= \int_{\Omega} \Delta v_0\,\overline{\varphi}\,dx + \int_{\Omega} \nabla v_0\cdot\nabla\overline{\varphi}\,dx \\ &= -k^2\int_{\Omega} v_0\,\overline{\varphi}\,dx + \int_{\Omega} \nabla v_0\cdot\nabla\overline{\varphi}\,dx\end{aligned}$$

for any $\varphi\in H^1(\Omega)$. Now (3.3) follows easily from (3.4) and (3.5). □

The estimates (3.8) and (3.9) in the following lemma play an essential role in our reconstruction algorithm.

Lemma 3.3.

$$\begin{aligned}&\int_{\partial\Omega} (\Lambda_D-\Lambda_{\emptyset}) f\,\overline{f}\,d\sigma \\ &= -\int_{\Omega} \widetilde{\gamma}|\nabla w|^2\,dx + k^2\int_{\Omega}|w|^2\,dx + \int_{\Omega}\big(\widetilde{\gamma}-1\big)|\nabla v_0|^2\,dx,\end{aligned} \tag{3.6}$$

$$\begin{aligned}&\int_{\partial\Omega} (\Lambda_D-\Lambda_{\emptyset}) f\,\overline{f}\,d\sigma \\ &= \int_{\Omega} |\nabla w|^2\,dx - k^2\int_{\Omega}|w|^2\,dx + \int_{\Omega}\big(\widetilde{\gamma}-1\big)|\nabla v|^2\,dx.\end{aligned} \tag{3.7}$$

In particular, we have

$$\int_{\partial\Omega} (\Lambda_D-\Lambda_{\emptyset}) f\,\overline{f}\,d\sigma \le k^2\int_{\Omega}|w|^2\,dx + \int_{D}\gamma_D\,|\nabla v_0|^2\,dx, \tag{3.8}$$

$$\int_{\partial\Omega} (\Lambda_D-\Lambda_{\emptyset}) f\,\overline{f}\,d\sigma \ge \int_{D}\frac{\gamma_D}{1+\gamma_D}|\nabla v_0|^2\,dx - k^2\int_{\Omega}|w|^2\,dx. \tag{3.9}$$

Proof. By multiplying the identity

$$0 = \nabla \cdot (\widetilde{\gamma} \nabla w) + \nabla \cdot ((\widetilde{\gamma} - 1) \nabla v_0) + k^2 w$$

by $\overline{w}$ and integrating the result over Ω, we get

$$\begin{aligned}
0 &= \int_\Omega \nabla \cdot \big(\widetilde{\gamma} \nabla w\big)\, \overline{w}\, dx + \int_\Omega \nabla \cdot \big((\widetilde{\gamma} - 1)\nabla v_0\big)\, \overline{w}\, dx + \int_\Omega k^2 w\, \overline{w}\, dx \\
&= -\int_\Omega \widetilde{\gamma}\, |\nabla w|^2\, dx + \int_{\partial\Omega} \widetilde{\gamma}\, \frac{\partial w}{\partial \nu}\, \overline{w}\, d\sigma \\
&\quad - \int_\Omega (\widetilde{\gamma} - 1) \nabla v_0 \cdot \nabla \overline{w}\, dx + \int_{\partial\Omega} (\widetilde{\gamma} - 1)\, \frac{\partial v_0}{\partial \nu}\, \overline{w}\, d\sigma + k^2 \int_\Omega |w|^2\, dx \\
&= -\int_\Omega \widetilde{\gamma}\, |\nabla w|^2\, dx - \int_\Omega (\widetilde{\gamma} - 1) \nabla v_0 \cdot \nabla \overline{w}\, dx + k^2 \int_\Omega |w|^2\, dx \\
&= -\int_\Omega \widetilde{\gamma}\, |\nabla w|^2\, dx - \int_\Omega (\widetilde{\gamma} - 1) \nabla v_0 \cdot \nabla \overline{v}\, dx \\
&\qquad + \int_\Omega (\widetilde{\gamma} - 1) |\nabla v_0|^2\, dx + k^2 \int_\Omega |w|^2\, dx.
\end{aligned}$$

Taking the real part of this identity and substituting the identity (3.3) immediately leads to (3.6). The identity (3.8) is an easy consequence of (3.6).

Similarly, by multiplying the identity

$$0 = \nabla \cdot ((\widetilde{\gamma} - 1)\, \nabla v) + \Delta w + k^2 w$$

by $\overline{w}$ and integrating the result over Ω, we obtain

$$\begin{aligned}
0 &= \int_\Omega \nabla \cdot \big((\widetilde{\gamma} - 1)\nabla v\big)\, \overline{w}\, dx + \int_\Omega \Delta w\, \overline{w}\, dx + \int_\Omega k^2 w\, \overline{w}\, dx \\
&= -\int_\Omega (\widetilde{\gamma} - 1) \nabla v \cdot \nabla \overline{w}\, dx - \int_\Omega |\nabla w|^2\, dx + \int_\Omega k^2 |w|^2\, dx \\
&= -\int_\Omega (\widetilde{\gamma} - 1) |\nabla v|^2\, dx + \int_\Omega (\widetilde{\gamma} - 1) \nabla v \cdot \nabla \overline{v_0}\, dx \\
&\qquad - \int_\Omega |\nabla w|^2\, dx + k^2 \int_\Omega |w|^2\, dx,
\end{aligned}$$

which implies (3.7). Finally, (3.9) follows from (3.7) and the formula

$$\begin{aligned}
|\nabla w|^2 + \big(\widetilde{\gamma} - 1\big)|\nabla v|^2 &= \widetilde{\gamma}\, |\nabla v|^2 - 2\, \mathrm{Re}\, \nabla v \cdot \nabla \overline{v_0} + |\nabla v_0|^2 \\
&= \widetilde{\gamma} \left| \nabla v - \frac{1}{\widetilde{\gamma}} \nabla v_0 \right|^2 + \left(1 - \frac{1}{\widetilde{\gamma}} \right) |\nabla v_0|^2.
\end{aligned}$$

□

In view of (3.8) and (3.9), we need to estimate $\|w\|_{L^2(\Omega)}$. To begin with, we consider the boundary value problem

$$\begin{cases} \nabla \cdot (\widetilde{\gamma} \nabla p) + k^2 p = \overline{w} \text{ in } \Omega, \\ p = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.10}$$

Note that there exists a unique solution $p \in H^1(\Omega)$ to (3.10). We can derive the following estimates for p.

Lemma 3.4. *Let p be the solution to* (3.10)*, then*

$$\|p\|_{H^1(\Omega)} \le C\|w\|_{L^2(\Omega)}, \tag{3.11}$$

$$\|p\|_{L^\infty(\Omega)} \le C\|w\|_{L^2(\Omega)}. \tag{3.12}$$

Furthermore, for any $2 < q \le 4$ and any $0 < \alpha < 1$, we have

$$\|p\|_{C^\alpha(\overline{\Omega})} \le C_{q,\alpha}\Big(\|w\|_{L^2(\Omega)} + \|\nabla v_0\|_{L^q(D)}\Big). \tag{3.13}$$

Proof. The estimate (3.11) follows directly from the well-posedness of the boundary value problem (3.10). On the other hand, we have

$$\|p\|_{L^\infty(\Omega)} \le C\|{-k^2 p} + \overline{w}\|_{L^2(\Omega)} \tag{3.14}$$

by Theorem 8.16 in [3] since $P := p$ satisfies the Dirichlet problem

$$\begin{cases} \nabla \cdot (\widetilde{\gamma} \nabla P) = -k^2 p + \overline{w} \text{ in } \Omega, \\ P = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.15}$$

Combining (3.14) and (3.11), we obtain (3.12). Finally, by virtue of Corollary 7.3 in [13], we have

$$\|p\|_{C^\alpha(\overline{\Omega})} \le C_\alpha\|{-k^2 p} + \overline{w}\|_{L^\infty(\Omega)}$$

Then (3.13) follows from by this estimate, (3.12) and (3.2). □

We now prove the first upper bound on the L^2 norm of w.

Lemma 3.5. *We have*

$$\int_\Omega |w|^2\, dx \le C \int_D |\nabla v_0|^2\, dx. \tag{3.16}$$

Proof. By the first equation of (3.10), we see that

$$\begin{aligned}\int_\Omega |w|^2\,dx &= \int_\Omega w\Big(\nabla\cdot\big(\widetilde{\gamma}\nabla p\big)+k^2 p\Big)\,dx\\ &= -\int_\Omega \nabla w\cdot\widetilde{\gamma}\nabla p\,dx + k^2\int_\Omega w\,p\,dx\\ &= \int_\Omega\Big(\nabla\cdot\big(\widetilde{\gamma}\nabla w\big)+k^2 w\Big)p\,dx = -\int_\Omega \nabla\cdot\Big(\big(\widetilde{\gamma}-1\big)\nabla v_0\Big)p\,dx\\ &= \int_\Omega\big(\widetilde{\gamma}-1\big)\nabla v_0\cdot\nabla p\,dx = \int_D \gamma_D\nabla v_0\cdot\nabla p\,dx. \qquad (3.17)\end{aligned}$$

Hence we get (3.16) by the Cauchy-Schwarz inequality and (3.11). □

From (3.8), (3.9) and (3.16), we immediately obtain

Corollary 3.6.

$$\left|\int_{\partial\Omega}(\Lambda_D-\Lambda_\emptyset)f\,\overline{f}\,d\sigma\right| \le C\int_D|\nabla v_0|^2\,dx. \qquad (3.18)$$

Now we prove another bound on the L^2 norm of w. We first define

$$I_{x_0,\alpha} := \int_{\partial D}\left|\frac{\partial v_0}{\partial\nu}(x)\right||x-x_0|^\alpha\,d\sigma(x)$$

for any $x_0\in\Omega$ and $0<\alpha<1$.

Lemma 3.7. *For any $x_0\in\Omega$, $0<\alpha<1$ and $2<q\le 4$, we have*

$$\int_\Omega |w|^2\,dx \le C_{q,\alpha}\Big(I_{x_0,\alpha}^2 + I_{x_0,\alpha}\|\nabla v_0\|_{L^q(D)} + \|v_0\|_{L^2(D)}^2\Big). \qquad (3.19)$$

Proof. By (3.17), we get

$$\begin{aligned}&\int_\Omega |w|^2\,dx = \int_D \gamma_D\nabla v_0\cdot\nabla p\,dx\\ &= -\int_D \nabla\cdot\big(\gamma_D\nabla v_0\big)\,p\,dx + \int_{\partial D}\gamma_D\,\frac{\partial v_0}{\partial\nu}\,p\,d\sigma\\ &= -\int_D \nabla\gamma_D\cdot\nabla v_0\,p\,dx - \int_D\gamma_D\,\Delta v_0\,p\,dx + \int_{\partial D}\gamma_D\,\frac{\partial v_0}{\partial\nu}\,p\,d\sigma\end{aligned}$$

$$= -\int_D \nabla\gamma_D \cdot \nabla v_0 \, p \, dx + k^2 \int_D \gamma_D \, v_0 \, p \, dx + \int_{\partial D} \gamma_D \, \frac{\partial v_0}{\partial \nu} \, p \, d\sigma. \tag{3.20}$$

Hence it is enough to estimate each term of the right-hand side of (3.20).

We first show that we can estimate the first term as

$$\left| \int_D \nabla\gamma_D \cdot \nabla v_0 \, p \, dx \right| \le C \|v_0\|_{L^2(D)} \|w\|_{L^2(\Omega)}. \tag{3.21}$$

Note that

$$\begin{aligned} \left| \int_D \nabla\gamma_D \cdot \nabla v_0 \, p \, dx \right| &= \left| -\int_D v_0 \, \nabla \cdot (p \nabla\gamma_D) \, dx + \int_{\partial D} v_0 \, \frac{\partial \gamma_D}{\partial \nu} \, p \, d\sigma \right| \\ &\le C \Big(\|v_0\|_{L^2(D)} \|p\|_{H^1(D)} + \|v_0\|_{H^{-1/2}(\partial D)} \|p\|_{H^{1/2}(\partial D)} \Big) \\ &\le C \Big(\|v_0\|_{L^2(D)} + \|v_0\|_{H^{-1/2}(\partial D)} \Big) \|p\|_{H^1(D)}. \end{aligned}$$

Therefore, using (3.11), we can obtain (3.21) if we show that

$$\|v_0\|_{H^{-1/2}(\partial D)} \le C \|v_0\|_{L^2(D)}. \tag{3.22}$$

To derive (3.22), we remark that

$$\|\psi\|_{H^{-1/2}(\partial D)} \le C \|\psi\|_{H_\Delta(D)}$$

holds for $\psi \in \mathcal{D}'(D)$ satisfying $\psi \in L^2(D)$ and $\Delta\psi \in L^2(D)$, where $\|\psi\|_{H_\Delta(D)}$ is defined by $\|\psi\|_{H_\Delta(D)}^2 := \|\psi\|_{L^2(D)}^2 + \|\Delta\psi\|_{L^2(D)}^2$. (see, for example, Lemma 1.1 in [2]). Using this fact, we immediately obtain that

$$\|v_0\|_{H^{-1/2}(\partial D)} \le C \Big(\|v_0\|_{L^2(D)} + \|\Delta v_0\|_{L^2(D)} \Big) = C \Big(\|v_0\|_{L^2(D)} + \|k^2 v_0\|_{L^2(D)} \Big),$$

which is (3.22). To estimate the second term, we simply use (3.11) and obtain

$$\left| \int_D \gamma_D \, v_0 \, p \, dx \right| \le C \|v_0\|_{L^2(D)} \|p\|_{L^2(D)} \le C \|v_0\|_{L^2(D)} \|w\|_{L^2(\Omega)}. \tag{3.23}$$

We now estimate the last term of the right-hand side of (3.20). We have

$$\begin{aligned} &\left| \int_{\partial D} \gamma_D \, \frac{\partial v_0}{\partial \nu} \, p \, d\sigma \right| \\ &\le \left| \int_{\partial D} \gamma_D(x) \, \frac{\partial v_0}{\partial \nu}(x) \, \big(p(x) - p(x_0)\big) \, d\sigma(x) \right| + \left| p(x_0) \int_{\partial D} \gamma_D \, \frac{\partial v_0}{\partial \nu} \, d\sigma \right| \\ &\le C_{q,\alpha} \Big(\|w\|_{L^2(\Omega)} + \|\nabla v_0\|_{L^q(D)} \Big) I_{x_0,\alpha} + C \|w\|_{L^2(\Omega)} \left| \int_{\partial D} \gamma_D \, \frac{\partial v_0}{\partial \nu} \, d\sigma \right| \end{aligned}$$

by (3.13) and (3.12). On the other hand, using (3.22) and Hölder's inequality, we can estimate

$$\begin{aligned}\left|\int_{\partial D}\gamma_D\,\frac{\partial v_0}{\partial\nu}\,d\sigma\right| &= \left|\int_{\partial D}\frac{\partial\gamma_D}{\partial\nu}\,v_0\,d\sigma+\int_D\gamma_D\,\Delta v_0\,dx-\int_D\Delta\gamma_D\,v_0\,dx\right|\\ &= \left|\int_{\partial D}\frac{\partial\gamma_D}{\partial\nu}\,v_0\,d\sigma-k^2\int_D\gamma_D\,v_0\,dx-\int_D\Delta\gamma_D\,v_0\,dx\right|\\ &\le C\Big(\|v_0\|_{H^{-1/2}(\partial D)}+\|v_0\|_{L^1(D)}\Big)\le C\|v_0\|_{L^2(D)}.\end{aligned}$$

Consequently, we get

$$\begin{aligned}&\left|\int_{\partial D}\gamma_D\,\frac{\partial v_0}{\partial\nu}\,p\,d\sigma\right|\\ &\le C_{q,\alpha}\Big(\|w\|_{L^2(\Omega)}+\|\nabla v_0\|_{L^q(D)}\Big)I_{x_0,\alpha}+C\|w\|_{L^2(\Omega)}\|v_0\|_{L^2(D)}.\end{aligned}\tag{3.24}$$

Combining (3.20), (3.21), (3.23) and (3.24), we then have

$$\begin{aligned}\int_\Omega|w|^2\,dx &= -\int_D\nabla\gamma_D\cdot\nabla v_0\,p\,dx+k^2\int_D\gamma_D\,v_0\,p\,dx+\int_{\partial D}\gamma_D\,\frac{\partial v_0}{\partial\nu}\,p\,d\sigma\\ &\le C\|v_0\|_{L^2(D)}\|w\|_{L^2(\Omega)}+C_{q,\alpha}\Big(\|w\|_{L^2(\Omega)}+\|\nabla v_0\|_{L^q(D)}\Big)I_{x_0,\alpha}\\ &\le \widetilde{\varepsilon}\|w\|^2_{L^2(\Omega)}+\frac{C}{\widetilde{\varepsilon}}\|v_0\|^2_{L^2(D)}\\ &\quad+\widetilde{\varepsilon}\|w\|^2_{L^2(\Omega)}+\frac{C_{q,\alpha}}{\widetilde{\varepsilon}}I^2_{x_0,\alpha}+C_{q,\alpha}\|\nabla v_0\|_{L^q(D)}I_{x_0,\alpha}.\end{aligned}$$

The lemma follows by taking $\widetilde{\varepsilon}>0$ small enough. □

4 The main theorem and its proof

In this section, we prove our main theorem, Theorem 4.1, which is the key to our reconstruction method. Here we denote the general constants by $C, c > 0$. The constants C and c depend only on Ω, D, γ_D, k, N, c_*, $t^\sharp$ and ε. As in Section 3, when a constant depends on other data, we denote the dependence by subscript.

To begin, substituting $v_0 = v_{0,t,h}$ and $f = f_{t,h}$ $(= v_{0,t,h}|_{\partial\Omega})$ to (3.18) yields

$$|E(t,h)|\le C\int_D|\nabla v_{0,t,h}|^2\,dx.\tag{4.1}$$

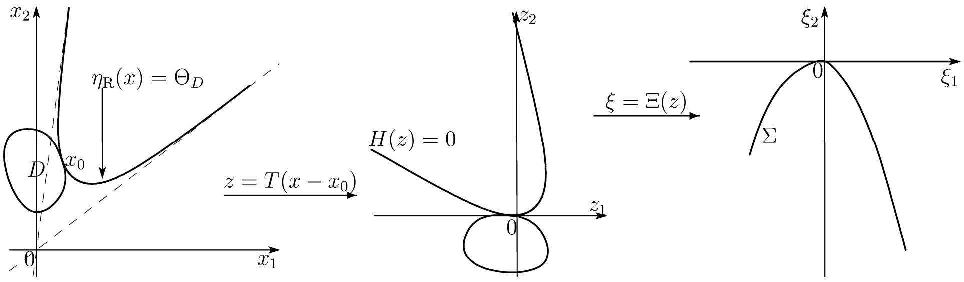

Figure 4: The coordinates transformation in defining a relative curvature.

On the other hand, applying estimate (3.19) to (3.9) for $f = f_{t,h}$, we have

$$E(t,h) \geq c \int_D |\nabla v_{0,t,h}|^2 \, dx - C_{q,\alpha}\Big(I_{t,h,x_0,\alpha}^2 + I_{t,h,x_0,\alpha} \|\nabla v_{0,t,h}\|_{L^q(D)} + \|v_{0,t,h}\|_{L^2(D)}^2\Big) \tag{4.2}$$

for $x_0 \in \Omega$, $2 < q \leq 4$ and $0 < \alpha < 1$, where

$$I_{t,h,x_0,\alpha} := \int_{\partial D} \left| \frac{\partial v_{0,t,h}}{\partial \nu}(x) \right| |x - x_0|^\alpha \, d\sigma(x).$$

Next we introduce a notion of relative curvature with respect to the level curve of η_R. Assume $D \cap \Gamma \neq \emptyset$ and put $\Theta_D := \sup_{x \in D \cap \Gamma} \eta_{\mathrm{R}}(x)$. Let $x_0 \in \{x \in \Gamma : \eta_{\mathrm{R}}(x) = \Theta_D\} \cap \partial D$. By simple translation and rotation T, i.e. the change of variables $z = T(x - x_0)$, we can take the unit outer normal vector of $T(D - x_0)$ at $z = 0$ as the vector $(0,1)$. We now put $H(z) := \eta_{\mathrm{R}}(T^{-1}z + x_0) - \Theta_D$, and apply the coordinates transformation $(\xi_1, \xi_2) = \Xi(z) := (z_1, H(z))$ to a neighborhood of $z = 0$. The transformation Ξ maps $\eta_{\mathrm{R}}(x) = \Theta_D$ near x_0 to the line $\xi_2 = 0$, and D near x_0 to a subdomain of the half space $\{\xi \in \mathbb{R}^2 : \xi_2 \leq 0\}$. Let Σ be the image of ∂D near x_0 by this transformation (see Figure 4). We then call the curvature of Σ at $\xi = 0$ the *relative* curvature to $\eta_{\mathrm{R}}(x) = \Theta_D$ of ∂D at x_0. We now can state our main theorem.

Theorem 4.1. *Assume $D \cap \Gamma \neq \emptyset$. Suppose that $\{x \in \Gamma : \eta_{\mathrm{R}}(x) = \Theta_D\} \cap \partial D$ consists only of one point x_0 and the relative curvature to $\eta_{\mathrm{R}}(x) = \Theta_D$ of ∂D at x_0 is not zero. Then there exist positive constants C_1, c_1 and h_1 such that for any $0 < t \leq t^\sharp$ and $0 < h \leq h_1$ the following holds:*

(I) *if* $1/t > \Theta_D$ *then*

$$|E(t,h)| \le \begin{cases} \dfrac{C_1}{h^2} \exp\left(\dfrac{2}{h}\Big(-\dfrac{1}{t} + \dfrac{1}{t+\varepsilon/2}\Big)\right) & \text{if } \Theta_D \le \dfrac{1}{t+\varepsilon/2}, \\ \dfrac{C_1}{h^2} \exp\left(\dfrac{2}{h}\Big(-\dfrac{1}{t} + \Theta_D\Big)\right) & \text{if } \dfrac{1}{t+\varepsilon/2} < \Theta_D < \dfrac{1}{t}. \end{cases}$$

(II) *if* $1/t \le \Theta_D$ *then*

$$E(t,h) \ge c_1 \exp\left(\frac{2}{h}\Big(-\frac{1}{t} + \Theta_D\Big)\right) h^{-1/2}.$$

Remark 4.2. If $D \cap \Gamma = \emptyset$ then we can prove

$$|E(t,h)| \le \frac{C_1}{h^2} \exp\left(\frac{2}{h}\Big(-\frac{1}{t} + \frac{1}{t+\varepsilon/2}\Big)\right)$$

in the same way as the proof of Theorem 4.1 (I) since $\nabla V_{t,h} \equiv 0$ in D by (2.5).

Remark 4.3. In the main theorem, Theorem 4.1, we impose some restriction on the curvature of ∂D at x_0. However, in Section 5, we will show that the curvature assumption is always satisfied as long as N is large enough.

Proof of Theorem 4.1. (I) By estimate (4.1), Lemma 2.2, and formula (2.5), it is easy to see that

$$\begin{aligned} |E(t,h)| &\le C\|\nabla V_{t,h}\|^2_{L^2(D)} + \frac{C}{h^2} e^{-2a_t/h} \\ &\le \frac{C}{h^2} \exp\left(\frac{2}{h}\Big(-\frac{1}{t} + \Theta_D\Big)\right) + \frac{C}{h^2} e^{-2a_t/h} \quad \text{for } 0 < h \le 1. \end{aligned}$$

Thus the estimates of $E(t,h)$ in (I) is obvious.

(II) In view of (4.2), it suffices to estimate $\int_D |\nabla v_{0,t,h}|^2\, dx$ from below and other remaining terms in the right side of (4.2) from above. Using Lemma 2.2 and (2.5), we can get that

$$\begin{aligned} &\|\nabla v_{0,t,h}\|^2_{L^2(D)} \\ &\ge \frac{1}{2}\|\nabla V_{t,h}\|^2_{L^2(D)} - \|\nabla(V_{t,h} - v_{0,t,h})\|^2_{L^2(D)} \\ &\ge \frac{1}{2}\left\| \frac{1}{h} \exp\left(\frac{1}{h}\Big(-\frac{1}{t} + \eta(x)\Big)\right) \big(\phi_t(x)\nabla\eta(x) + \boldsymbol{S}(x)h\big) \right\|^2_{L^2(D')} - C_0^2 e^{-2a/h} \end{aligned}$$

$$
\begin{aligned}
&\geq \frac{1}{4}\left\|\frac{1}{h}\exp\left(\frac{1}{h}\Big(-\frac{1}{t}+\eta(x)\Big)\right)\phi_t(x)\nabla\eta(x)\right\|_{L^2(D')}^2 \\
&\quad -\frac{1}{2}\left\|\exp\left(\frac{1}{h}\Big(-\frac{1}{t}+\eta(x)\Big)\right)\boldsymbol{S}(x)\right\|_{L^2(D')}^2 - C_0^2 e^{-2a/h} \\
&\geq c\,\frac{1}{h^2}\left\|\exp\left(\frac{1}{h}\Big(-\frac{1}{t}+\eta_{\mathrm{R}}(x)\Big)\right)\right\|_{L^2(D'')}^2 \\
&\quad - C\left\|\exp\left(\frac{1}{h}\Big(-\frac{1}{t}+\eta_{\mathrm{R}}(x)\Big)\right)\right\|_{L^2(D')}^2 - C_0^2 e^{-2a/h},
\end{aligned}
\tag{4.3}
$$

where we set $D' := D \cap \overline{\bigcup_{0<s<t+\varepsilon} l_s}$ and $D'' := D \cap \overline{\bigcup_{0<s<t+\varepsilon/2} l_s}$. On the other hand, by (2.4), (2.5), Lemma 2.2 and the Sobolev embedding theorem (for two dimensions), we have

$$
I_{t,h,x_0,\alpha} \leq C\,\frac{1}{h}\int_{\partial D}\exp\left(\frac{1}{h}\Big(-\frac{1}{t}+\eta_{\mathrm{R}}(x)\Big)\right)|x-x_0|^{\alpha}\,d\sigma(x) + Ce^{-a/h}, \tag{4.4}
$$

$$
\|\nabla v_{0,t,h}\|_{L^q(D)} \leq C\,\frac{1}{h}\left\|\exp\left(\frac{1}{h}\Big(-\frac{1}{t}+\eta_{\mathrm{R}}(x)\Big)\right)\right\|_{L^q(D')} + C_q e^{-a/h}, \tag{4.5}
$$

$$
\|v_{0,t,h}\|_{L^2(D)} \leq C\left\|\exp\left(\frac{1}{h}\Big(-\frac{1}{t}+\eta_{\mathrm{R}}(x)\Big)\right)\right\|_{L^2(D')} + C_0 e^{-a/h}. \tag{4.6}
$$

Therefore, our task now is to estimate

$$
\left\|\exp\left(\frac{1}{h}\Big(-\frac{1}{t}+\eta_{\mathrm{R}}(x)\Big)\right)\right\|_{L^2(D'')}^2 \tag{4.7}
$$

from below and

$$
\left\|\exp\left(\frac{1}{h}\Big(-\frac{1}{t}+\eta_{\mathrm{R}}(x)\Big)\right)\right\|_{L^q(D')}, \tag{4.8}
$$

$$
\int_{\partial D}\exp\left(\frac{1}{h}\Big(-\frac{1}{t}+\eta_{\mathrm{R}}(x)\Big)\right)|x-x_0|^{\alpha}\,d\sigma(x) \tag{4.9}
$$

from above, where the index q in (4.8) is $q = 2$ (for (4.3) and (4.6)) or $2 < q \leq 4$ (for (4.5)).

We first look at (4.7). By translation and rotation with the orthogonal matrix T, we can assume that $x_0 = 0$ and the unit outer normal vector of

∂D at $x_0 = 0$ is $(0,1)$. Then we can see that

$$(4.7) = \iint_{T(D''-x_0)} \exp\left(\frac{2}{h}\Big(H(z) - \frac{1}{t} + \Theta_D\Big)\right) dz$$
$$= \exp\left(\frac{2}{h}\Big(-\frac{1}{t} + \Theta_D\Big)\right) \iint_{T(D''-x_0)} e^{2H(z)/h}\, dz$$

where we have used $H(z) = \eta_{\mathrm{R}}(T^{-1}z + x_0) - \Theta_D$. We now make the change of variables $(\xi_1, \xi_2) = \Xi(z) := (z_1, H(z))$. Notice that there exists a neighborhood U_0 of $z = 0$ such that the map $\xi = \Xi(z)$ is injective from U_0 to $\Xi(U_0)$ since we have $\det(\partial\Xi/\partial z)(0) = N|x_0|^{N-1} \neq 0$. In particular, there exist positive constants $a_\sharp$ and $a^\sharp$ such that

$$a_\sharp \leq \det\frac{\partial \Xi}{\partial z} \leq a^\sharp \text{ in } U_0, \quad \text{i.e.} \quad \frac{1}{a^\sharp} \leq \det\frac{\partial z}{\partial \xi} \leq \frac{1}{a_\sharp} \text{ in } \Xi(U_0).$$

Consequently, we have

$$\iint_{T(D''-x_0)} e^{2H(z)/h}\, dz \geq \iint_{T(D''-x_0)\cap U_0} e^{2H(z)/h}\, dz \geq \frac{1}{a^\sharp} \iint_{\widetilde{U}_0} e^{2\xi_2/h}\, d\xi,$$

where $\widetilde{U}_0 := \Xi(T(D''-x_0)\cap U_0)$. We now parameterize the boundary $\partial\widetilde{U}_0$ near 0. We remark that the boundary $\partial\widetilde{U}_0$ near 0 is the image of ∂D near x_0 under the coordinates transform given above. Therefore we can parameterize the boundary $\partial\widetilde{U}_0$ near 0 by $\xi_2 = l(\xi_1)$ (we may choose a smaller neighborhood U_0 if needed), and express $\widetilde{U}_0$ near 0 as $\xi_2 \leq l(\xi_1)$. Moreover, by the assumption on the curvature of ∂D, there exist positive constants $K_\sharp$ and $K^\sharp$ such that

$$K_\sharp \xi_1^2 \leq -l(\xi_1) \leq K^\sharp \xi_1^2 \text{ for } (\xi_1, l(\xi_1)) \in \Xi(U_0).$$

Then for $\delta_1 > 0$ small enough (see Figure 5), we can estimate

$$\iint_{\widetilde{U}_0} e^{2\xi_2/h}\, d\xi \geq \iint_{\substack{|\xi_1|\leq\delta_1 \\ -K^\sharp\delta_1^2 \leq \xi_2 \leq l(\xi_1)}} e^{2\xi_2/h}\, d\xi_1 d\xi_2$$
$$\geq \iint_{\substack{|\xi_1|\leq\delta_1 \\ -K^\sharp\delta_1^2 \leq \xi_2 \leq -K^\sharp\xi_1^2}} e^{2\xi_2/h}\, d\xi_1 d\xi_2$$
$$= \frac{2}{\sqrt{K^\sharp}} \int_{-K^\sharp\delta_1^2}^{0} e^{2\xi_2/h} \sqrt{-\xi_2}\, d\xi_2$$
$$= \frac{h^{3/2}}{(2K^\sharp)^{1/2}} \int_0^{2K^\sharp\delta_1^2/h} e^{-\tau}\tau^{1/2}\, d\tau \geq c\, h^{3/2}$$

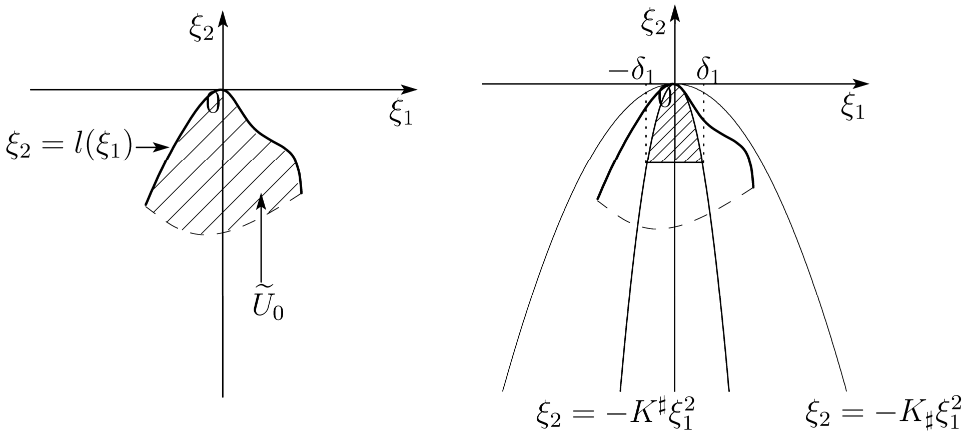

Figure 5: The choice of $\delta_1 > 0$.

for any $0 < h \ll 1$. Summing up, we obtain

$$\left\| \exp\left(\frac{1}{h}\Big(-\frac{1}{t} + \eta_{\mathrm{R}}(x)\Big)\right) \right\|^2_{L^2(D'')} \geq c\, h^{3/2} \exp\left(\frac{2}{h}\Big(-\frac{1}{t} + \Theta_D\Big)\right) \tag{4.10}$$

for any $0 < h \ll 1$.

Next, we estimate (4.8). It is enough to estimate the integral on some neighborhood of x_0. Indeed, when U_{x_0} is a neighborhood of x_0, there exists $\delta_2 > 0$ such that $\overline{D'} \setminus U_{x_0} \subset \{x \in \Gamma : \eta_{\mathrm{R}}(x) \leq \Theta_D - \delta_2\}$. Then we obtain

$$\begin{aligned} &\left\| \exp\left(\frac{1}{h}\Big(-\frac{1}{t} + \eta_{\mathrm{R}}(x)\Big)\right) \right\|_{L^q(D' \setminus U_{x_0})} \\ &\leq \exp\left(\frac{1}{h}\Big(-\frac{1}{t} + \Theta_D\Big)\right) \left\| e^{-\delta_2/h} \right\|_{L^q(D' \setminus U_{x_0})} \\ &\leq C \exp\left(\frac{1}{h}\Big(-\frac{1}{t} + \Theta_D - \delta_2\Big)\right). \end{aligned} \tag{4.11}$$

Here we use the same notations as in estimating (4.7) (Denote $U_0 = T(U_{x_0} -$

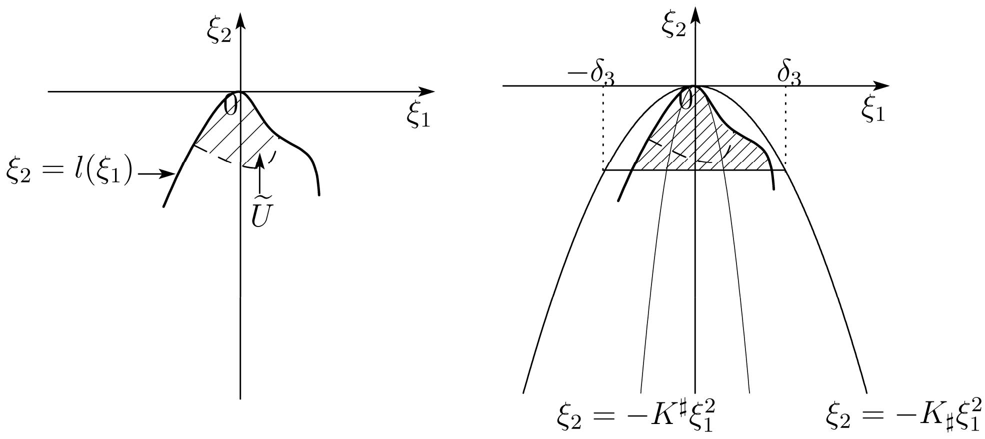

Figure 6: The choice of $\delta_3 > 0$.

$x_0)$). Using the similar arguments as above, we can derive

$$\begin{aligned}
&\left\| \exp\left(\frac{1}{h}\Big(-\frac{1}{t}+\eta_{\mathrm{R}}(x)\Big)\right)\right\|^q_{L^q(D'\cap U_{x_0})} \\
&= \exp\left(\frac{q}{h}\Big(-\frac{1}{t}+\Theta_D\Big)\right)\iint_{T\left((D'\cap U_{x_0})-x_0\right)} e^{qH(z)/h}\,dz \\
&\le \frac{1}{a_\sharp}\exp\left(\frac{q}{h}\Big(-\frac{1}{t}+\Theta_D\Big)\right)\iint_{\widetilde{U}} e^{q\xi_2/h}\,d\xi \qquad (4.12)
\end{aligned}$$

where $\widetilde{U} := \Xi\left(T\left((D'\cap U_{x_0})-x_0\right)\right)$. Now we can take $\delta_3 > 0$ such that $\widetilde{U} \subset \{\xi\in\mathbb{R}^2 : |\xi_1|\le\delta_3,\ -K_\sharp\delta_3^2\le\xi_2\le l(\xi_1)\}$ by choosing the neighborhood U_{x_0} of x_0 small enough (see Figure 6). Then we have

$$\begin{aligned}
\iint_{\widetilde{U}} e^{q\xi_2/h}\,d\xi &\le \iint_{\substack{|\xi_1|\le\delta_3 \\ -K_\sharp\delta_3^2\le\xi_2\le l(\xi_1)}} e^{q\xi_2/h}\,d\xi \\
&\le \iint_{\substack{|\xi_1|\le\delta_3 \\ -K_\sharp\delta_3^2\le\xi_2\le -K_\sharp\xi_1^2}} e^{q\xi_2/h}\,d\xi \le C_q h^{3/2} \qquad (4.13)
\end{aligned}$$

for any $0 < h \ll 1$. Combining (4.11), (4.12), (4.13) yields

$$\left\| \exp\left(\frac{1}{h}\Big(-\frac{1}{t}+\eta_{\mathrm{R}}(x)\Big)\right)\right\|_{L^q(D')} \le C_q h^{3/(2q)}\exp\left(\frac{1}{h}\Big(-\frac{1}{t}+\Theta_D\Big)\right) \qquad (4.14)$$

for any $0 < h \ll 1$.

Lastly, we turn to (4.9). As before, it suffices to estimate the integral on some neighborhood of x_0. Thus we compute

$$\begin{aligned}
&\int_{\partial D\cap U_{x_0}} \exp\left(\frac{1}{h}\Big(-\frac{1}{t}+\eta_{\mathrm{R}}(x)\Big)\right)|x-x_0|^\alpha\, d\sigma(x) \\
&= \exp\left(\frac{1}{h}\Big(-\frac{1}{t}+\Theta_D\Big)\right)\int_{T\left((\partial D\cap U_{x_0})-x_0\right)} e^{H(z)/h}\,|z|^\alpha\, d\sigma(z).
\end{aligned}$$

Then by choosing a sufficiently small neighborhood U_{x_0} of x_0, we have

$$\begin{aligned}
&\int_{T\left((\partial D\cap U_{x_0})-x_0\right)} e^{H(z)/h}\,|z|^\alpha\, d\sigma(z) \\
&\le C'\int_{-\delta_4}^{\delta_4} e^{-K_\sharp z_1^2/h}\,|z_1|^\alpha\, dz_1 \\
&= h^{(\alpha+1)/2}\,\frac{C'}{K_\sharp^{(\alpha+1)/2}}\int_0^{K_\sharp\delta_4^2/h} e^{-\tau}\tau^{(\alpha-1)/2}\, d\tau \le C_\alpha h^{(\alpha+1)/2}
\end{aligned}$$

for any $0<h\ll 1$ since

$$K_\sharp z_1^2 \le -H(z) \le K^\sharp z_1^2 \text{ for } z\in T(\partial D - x_0) \text{ close to } 0.$$

Therefore we obtain

$$\begin{aligned}
&\int_{\partial D} \exp\left(\frac{1}{h}\Big(-\frac{1}{t}+\eta_{\mathrm{R}}(x)\Big)\right)|x-x_0|^\alpha\, d\sigma(x) \\
&\qquad\qquad \le C_\alpha h^{(\alpha+1)/2}\exp\left(\frac{1}{h}\Big(-\frac{1}{t}+\Theta_D\Big)\right) \qquad (4.15)
\end{aligned}$$

for any $0<h\ll 1$.

Now by (4.2), (4.3), (4.4), (4.5), (4.6), (4.10), (4.14) and (4.15), we conclude that

$$\begin{aligned}
E(t,h) \ge &\left(ch^{-\frac{1}{2}} - C_{q,\alpha}\left(h^{\alpha-1}+h^{\frac{\alpha}{2}+\frac{3}{2q}-\frac{3}{2}}+h^{\frac{3}{2}}\right)\right) \\
&\qquad\times\exp\left(\frac{2}{h}\Big(-\frac{1}{t}+\Theta_D\Big)\right) \qquad (4.16)
\end{aligned}$$

for any $0<h\ll 1$. It is easy to see that we can choose $0<\alpha_0<1$ and $2<q_0\le 4$ such that

$$\beta := \max\left\{\frac{\alpha_0}{2}+\frac{3}{2q_0}-\frac{3}{2},\ \alpha_0-1\right\} > -\frac{1}{2}$$

Then (4.16) implies

$$E(t,h) \geq \left(ch^{-1/2} - C_{q_0,\alpha_0}h^{\beta}\right)\exp\left(\frac{2}{h}\Big(-\frac{1}{t}+\Theta_D\Big)\right)$$
$$\geq ch^{-1/2}\exp\left(\frac{2}{h}\Big(-\frac{1}{t}+\Theta_D\Big)\right)$$

for any $0 < h \ll 1$. □

5 Remarks on the curvature assumption

In this section we would like to show that the curvature condition assumed in Theorem 4.1 always holds provided the degree N of η is sufficiently large. To this end, in order to indicate the dependence on c_*, θ_*, and N, we write $\eta(x; c_*, N) = \eta(x) = c_*(x_1 + ix_2)^N$ and similarly write $\eta_{\mathrm{R}}(x) = \eta_{\mathrm{R}}(x; c_*, N)$. Also, we denote

$$\Gamma(N, \theta_*) := \left\{ r(\cos\theta,\, \sin\theta) : |\theta - \theta_*| < \frac{\pi}{2N} \right\}.$$

Recall that $\eta_{\mathrm{R}}(x) = r^N \cos N(\theta - \theta_*)$ for $x = r(\cos\theta,\, \sin\theta) \in \mathbb{R}^2$ and $\eta_{\mathrm{R}}(x) > 0$ when $x \in \Gamma(N, \theta_*)$.

Lemma 5.1. *Let $c_* \in \mathbb{C}$ satisfy $|c_*| = 1$, N be a positive integer and $t_0 > 0$. Assume that x_0 is on the level curve $\eta_{\mathrm{R}}(x; c_*, N) = 1/t_0$. Then there exist $c'_* \in \mathbb{C}$ with $|c'_*| = 1$, $\theta'_* \in \mathbb{R}$, a positive integer $N' > N$ and a positive number t' such that $\eta_{\mathrm{R}}(x; c'_*, N') = r^{N'} \cos N'(\theta - \theta'_*)$ and the following holds:*

(i) *The point x_0 is on the level curve $\eta_{\mathrm{R}}(x; c'_*, N') = 1/t'$.*

(ii) *If $\eta_{\mathrm{R}}(x; c'_*, N') \geq 1/t'$ and $x \in \Gamma(N', \theta'_*) \setminus \{x_0\}$ then $\eta_{\mathrm{R}}(x; c_*, N) > 1/t_0$.*

Proof. Without loss of generality, we can assume that $c_* = 1$. Let $x_0 = r_0(\cos\theta_0,\, \sin\theta_0)$. By the assumption, we have $r_0^N \cos N\theta_0 = 1/t_0$ and $|\theta_0| < \pi/(2N)$. Now we choose a positive integer $N' > N$ such that $|(N' - N)\theta_0 - 2k'\pi| < \pi/(2N)$ for some $k' \in \mathbb{Z}$. We put $\alpha' := -(N' - N)\theta_0$, $\theta'_* := -\alpha'/N'$,

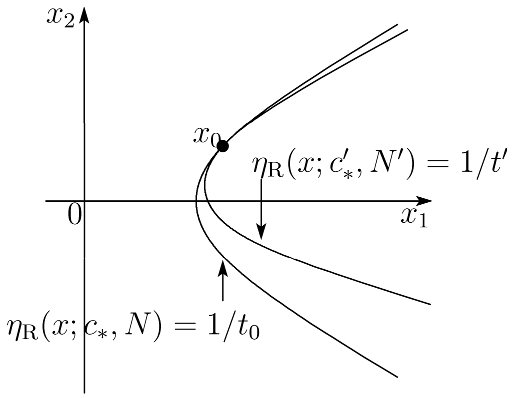

Figure 7: Intersection of two level curves.

$c'_* := \exp(i\alpha')$ and $t' := 1/\eta_R(x_0; c'_*, N')$. Note that

$$\eta_R(x; 1, N) > \frac{1}{t_0} \text{ is equivalent to } r > r_0 \left(\frac{\cos N\theta_0}{\cos N\theta}\right)^{1/N} \quad \text{for } x \in \Gamma(N, 0),$$

$$\eta_R(x; c'_*, N') > \frac{1}{t'} \text{ is equivalent to } r > r_0 \left(\frac{\cos(N'\theta_0 + \alpha')}{\cos(N'\theta + \alpha')}\right)^{1/N'} \quad \text{for } x \in \Gamma(N', \theta'_*)$$

and $\Gamma(N', \theta'_*) \subset \Gamma(N, 0)$. Thus it is enough to show that

$$\left(\frac{\cos N\theta_0}{\cos N\theta}\right)^{1/N} < \left(\frac{\cos(N'\theta_0 + \alpha')}{\cos(N'\theta + \alpha')}\right)^{1/N'} \quad \text{for } |\theta - \theta'_*| < \frac{\pi}{2N'},\ \theta \neq \theta_0. \quad (5.1)$$

We remark that (5.1) is equivalent to

$$\psi(\theta) < \psi(\theta_0) \text{ for } |\theta - \theta'_*| < \frac{\pi}{2N'},\ \theta \neq \theta_0, \quad (5.2)$$

where

$$\psi(\theta) := \frac{(\cos(N'\theta + \alpha'))^{1/N'}}{(\cos N\theta)^{1/N}}.$$

It is straightforward to check that

$$\psi'(\theta) = -(\cos(N'\theta + \alpha'))^{1/N'-1}(\cos N\theta)^{-1/N-1} \sin(N' - N)(\theta - \theta_0).$$

Therefore, ψ is monotone increasing on the interval $[\theta'_* - \pi/(2N'), \theta_0]$ and monotone decreasing on the interval $[\theta_0, \theta'_* + \pi/(2N')]$. Hence (5.2) holds and so does (5.1). □

Denote $\Theta_D(c_*, N, \Gamma) := \sup_{x\in D\cap\Gamma} \eta_{\mathrm{R}}(x; c_*, N)$. We next show that we can always assume that the relative curvature is not zero by taking N sufficiently large.

Lemma 5.2. *Let $c_* \in \mathbb{C}$ satisfy $|c_*| = 1$, N be a positive integer. Assume that θ_* satisfies $\eta_{\mathrm{R}}(x; c_*, N) = r^N \cos N(\theta - \theta_*)$. Let $x_0 \in \{x \in \Gamma(N, \theta_*) : \eta_{\mathrm{R}}(x; c_*, N) = \Theta_D(c_*, N, \Gamma(N, \theta_*))\} \cap \partial D$. Then there exist $c'_* \in \mathbb{C}$ with $|c'_*| = 1$ and a positive integer N' such that the relative curvature to $\eta_{\mathrm{R}}(x; c'_*, N') = \Theta_D(c'_*, N', \Gamma(N', \theta'_*))$ of ∂D at x_0 is negative, where θ'_* satisfies $\eta_{\mathrm{R}}(x; c'_*, N') = r^{N'} \cos N'(\theta - \theta'_*)$.*

Proof. We first calculate the relative curvature explicitly. As before, in the new coordinates $z = T(x - x_0)$, x_0 moves to the origin and the unit outer normal of ∂D at x_0 is transformed to $(0, 1)$. We parameterize the boundary $T(\partial D - x_0)$ near the origin by $z_2 = m(z_1)$. Note that $m(0) = m'(0) = 0$. On the other hand, let $H(z) := \eta_{\mathrm{R}}(T^{-1}z + x_0) - \Theta_D$ and use the coordinates transformation $\xi = \Xi(z) := (z_1, H(z))$ to a small neighborhood of $z = 0$, then we can parameterize the boundary $\Xi(T(\partial D - x_0))$ by $\xi_2 = l(\xi_1)$. Obviously, we have $l(z_1) = H(z_1, m(z_1))$. Recall that

$$\frac{\partial H}{\partial z_1}(0) = 0, \quad \frac{\partial H}{\partial z_2}(0) = N|x_0|^{N-1}, \quad \frac{\partial^2 H}{\partial z_1^2}(0) = -N(N-1)|x_0|^{-2}\eta_{\mathrm{R}}(x_0).$$

So we have $l'(0) = 0$ and

$$\begin{aligned} l''(0) &= \frac{\partial^2 H}{\partial z_1^2}(0) + 2m'(0)\frac{\partial^2 H}{\partial z_2 \partial z_1}(0) + m''(0)\frac{\partial H}{\partial z_2}(0) + m'(0)^2\frac{\partial^2 H}{\partial z_2^2}(0) \\ &= -N(N-1)|x_0|^{-2}\eta_{\mathrm{R}}(x_0) + m''(0)N|x_0|^{N-1}. \end{aligned}$$

In other words, the relative curvature to $\eta_{\mathrm{R}}(x; c_*, N) = \Theta_D(c_*, N, \Gamma(N, \theta_*))$ of ∂D at x_0 is

$$-N(N-1)|x_0|^{-2}\eta_{\mathrm{R}}(x_0; c_*, N) + m''(0)N|x_0|^{N-1}. \tag{5.3}$$

Let $t_0 := 1/\Theta_D(c_*, N, \Gamma(N, \theta_*))$. As above, we can take $c_* = 1$ without loss of generality. Now we want to compare the relative curvature to $\eta_{\mathrm{R}}(x; 1, N) =$

$1/t_0$ with that to $\eta_{\rm R}(x; c'_*, N') = 1/t'$ where θ'_*, c'_*, N' and t' were given in Lemma 5.1. It should be noted that $1/t' = \Theta_D(c'_*, N', \Gamma(N', \theta'_*))$ and $x_0 \in \{x \in \Gamma(N', \theta'_*) : \eta_{\rm R}(x; c'_*, N') = \Theta_D(c'_*, N', \Gamma(N', \theta'_*))\} \cap \partial D$. From (5.3) the relative curvature to $\eta_{\rm R}(x; 1, N) = 1/t_0$ of ∂D at $x_0 = r_0(\cos\theta_0, \sin\theta_0)$ is

$$\begin{aligned} &- N(N-1)|x_0|^{-2}\eta_{\rm R}(x_0; 1, N) + m''(0)N|x_0|^{N-1} \\ &\qquad = -N(N-1)r_0^{N-2}\cos N\theta_0 + m''(0)Nr_0^{N-1}. \end{aligned} \tag{5.4}$$

In view of the definition of x_0, we see that (5.4) is non-positive. So it is negative, we need not do anything. Thus we assume that (5.4) is zero. Then by (5.3) the relative curvature to $\eta_{\rm R}(x; c'_*, N') = 1/t'$ of ∂D at x_0 is

$$\begin{aligned} &- N'(N'-1)|x_0|^{-2}\eta_{\rm R}(x_0; c'_*, N') + m''(0)N'|x_0|^{N'-1} \\ &= -N'(N'-1)r_0^{N'-2}\cos N\theta_0 + m''(0)N'r_0^{N'-1} \\ &= -N'(N'-N)r_0^{N'-2}\cos N\theta_0 < 0. \end{aligned}$$

The proof is completed. □

6 Reconstruction algorithm

In view of Theorem 4.1, we are able to reconstruct some part of ∂D using boundary measurements on $\partial\Omega$ by looking into the asymptotic behavior of $E(t, h)$ for various t's. More precisely, let

$$t_D := \sup\left\{t \in (0, t^\sharp) : \lim_{h\to 0} E(t, h) = 0\right\}.$$

It should be noted that $E(t, h)$ depends on, besides h and t, Ω, k, D, γ_D (see (1.1)), c_*, x_*, N (appear in the phase function $\eta(x)$), ε, $t^\sharp$ (appear in the cut-off function $\phi_t(x)$). Thus t_D depends on Ω, k, D, γ_D , c_*, x_*, N, ε and $t^\sharp$. If $t_D = t^\sharp$, then $D \cap \bigcup_{0<s<t^\sharp} l_s = \emptyset$. On the other hand, if $t_D < t^\sharp$, then there exists a point $x_D \in l_{t_D} \cap \partial D$.

By taking N arbitrarily large (the opening angle of $\Gamma(N, \theta_*)$ becomes arbitrarily small), we can reconstruct even more information of ∂D. A point x_0 on ∂D is said to be detectable if there exists a semi-straight line L starting from x_0 such that L does not intersect ∂D except x_0. For example, if D is star-shaped, every point of ∂D is detectable. We can prove the following corollary similarly to Corollary 5.4 in [17].

Corollary 6.1. *Every detectable point of ∂D can be reconstructed from Λ_D.*

Now we state our reconstruction algorithm.
Step 1. Pick a point $x_* \in \mathbb{R}^2 \setminus \overline{\Omega}$. Given $N \in \mathbb{N}$ and $c_* \in \mathbb{C}$ satisfying $|c_*| = 1$. Choose the cone Γ which intersects Ω.
Step 2. Fix $\varepsilon > 0$ small enough and $t^\sharp > 0$ large enough. Take $t \in (0, t^*)$ such that $\left(\bigcup_{0<s<t} l_s\right) \cap \Omega \neq \emptyset$. Construct $V_{t,h}$ and determine the Dirichlet data $f_{t,h} := V_{t,h}|_{\partial\Omega}$.
Step 3. Compute $E(t,h) := \int_{\mathrm{supp}\, f_{t,h}} (\Lambda_D - \Lambda_\emptyset) f_{t,h} \overline{f}_{t,h}\, d\sigma$.
Step 4. If $E(t,h)$ is arbitrarily small as h tends to zero, then increase t and repeat Steps 2 and 3; if $E(t,h)$ is arbitrarily large as h tends to zero, then decrease t and repeat Steps 2 and 3.
Step 5. Repeat Step 4 in order to get a good approximation of ∂D in Γ.
Step 6. Move the cone Γ around x_* by taking a different c_*. Repeat Steps 2–5.
Step 7. Choose a larger N, new c_* and new cone Γ. Repeat Steps 2–6.
Step 8. Pick a different x_* and repeat Steps 1–7.

7 Numerical results

In this section, we demonstrate some numerical results of our method with synthetic data. The numerical code used here is modified from that in [18]. We refer to [18] for more detailed description of the program. We consider a rectangle domain

$$\Omega = \{(x_1, x_2) : -1 < x_1 < 1,\ -1.01 < x_2 < -0.01\}.$$

We will apply Dirichlet $f_{t,h} = v_{0,t,h}|_{\partial\Omega}$ on $\partial\Omega$. We also want to point out that the Dirichlet condition $f_{t,h}$ is localized in $\Gamma \cap \partial\Omega$. In the numerical computation, we use $f_{t,h}$ localized on $\{(x_1, -0.01) : -1 < x < 1\}$ (top boundary), $\{(x_1, -1.01) : -1 < x < 1\}$ (bottom boundary) by choosing vertex points x_* on $\{x_2 = -1.02\}$ and $\{x_2 = 0\}$, respectively. In other words, we probe the region from the top and bottom boundaries of the domain. For simplicity, we take $N = 4$ and $k = 5$. To improve the effectiveness of our numerical method, we will use an approximation of $f_{t,h}$. To obtain such approximate Dirichlet data, we shall study the asymptotic expansion of the CGO solution $V_h^\sharp$. We defer the derivation of the asymptotic expansion of $V_h^\sharp$ to Appendix. We test our method on seven cases without and with noise. The results are shown in Figure 8.

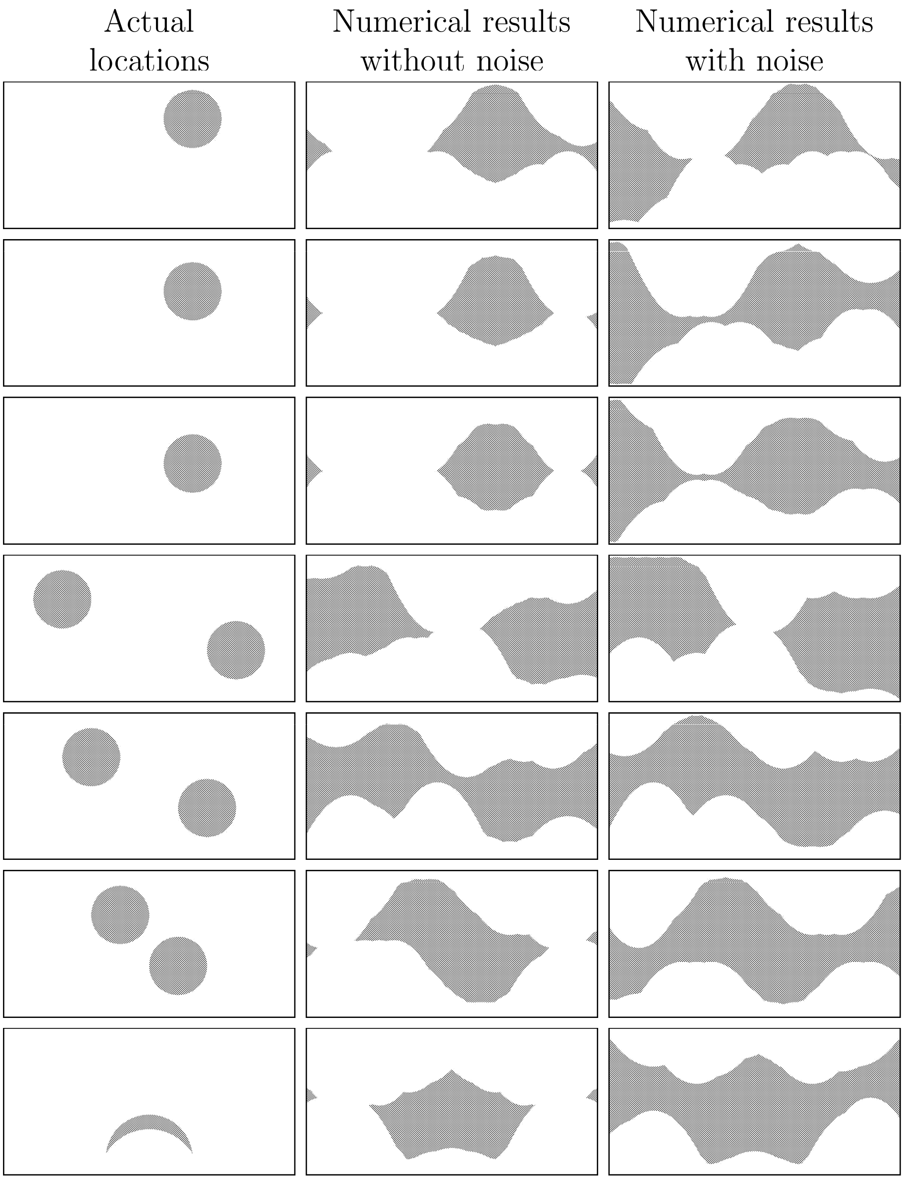

Figure 8: The first column represents the actual location of the inclusion. The second column is the numerical reconstructions with noiseless simulated data. The third column is the numerical reconstructions with noisy data. All white areas are inclusion-free regions.

8 Conclusions

In this work, we present an enclosure type reconstruction method for identifying penetrable obstacles in acoustics in two dimensions. Our main tool is the CGO solutions with polynomial phases for the Helmholtz equation. We construct these types of solutions from the harmonic functions via a transform introduced by Vekua. Doing so, we have a better description on the lower order terms of the CGO solutions, which is useful in numerical computations. Our theory shows that we are able to reconstruct precise geometrical information of some penetrable objects by the boundary measurements or the Dirichlet-to-Neumann map. To prove the main theorem, it requires a delicate analysis on solutions to the elliptic equation with discontinuous coefficients.

We also provide some numerical results based on our method. Since this inverse problem is notoriously ill-posed, some numerical results in Figure 8 are sensitive to noise. On other hand, it has been formally justified in [14] that the stability deteriorates when the object is further away from the boundary. Some of the figures in Figure 8 clearly demonstrate this phenomenon. For example, the figures in row seven show that the lower part of the inclusion is better resolved than the top part. Here we can see that the lower part of the inclusion is closer to the boundary. Due to ill-posedness, there are still many challenging issues on the numerical computation for inverse problems.

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Appendix. Asymptotic expansion of the CGO solutions

In this appendix, we prove an asymptotic expansion of the CGO solution $V_h^\sharp$. We adopt the notations of Section 1.

Lemma A.1. *We can write*

$$V_h^\sharp(x) = \exp\left(\frac{\eta(x)}{h}\right)\left(1 - \frac{k^2|x|^2}{4N\eta(x)}\,h + O(h^2)\right) \ \text{in } \Gamma\cap\Omega \text{ as } h\to +0.$$

More precisely, we have

$$V_h^\sharp(x) = \exp\left(\frac{\eta(x)}{h}\right)\left(1 - \frac{k^2|x|^2}{4N\eta(x)}\,h + R_1(x)\,h\right) \ \text{in } \Gamma,$$

where $R_1(x) = R_1(x;h)$ satisfies

$$\begin{aligned}|R_1(x)| \le{} & \frac{k^2}{4N|x|^{N-2}}\exp\left(-\frac{N\eta_{\mathrm{R}}(x)}{h}\right) \\ & + \frac{k^2N(N-1)|x|^{N+2}}{4\eta_{\mathrm{R}}(x)^3}\,h + \frac{k^4|x|^4}{32\eta_{\mathrm{R}}(x)^2}\,h \ \text{in } \Gamma. \qquad \text{(A.1)}\end{aligned}$$

Proof. By (2.1) and (2.2), it suffices to show that

$$R_0(x) = -\frac{k^2|x|^2}{4N\eta(x)}\,h + R_1(x)\,h \text{ in } \Gamma \tag{A.2}$$

and estimate (A.1). Using (2.2) and the property

$$J_1(\tau) = \frac{\tau}{2} + K_1(\tau), \text{ where } |K_1(\tau)| \le \frac{1}{16}\tau^3 \text{ for } \tau > 0,$$

we have

$$\begin{aligned} R_0(x) = &-\frac{k^2|x|^2}{2}\int_0^1 \exp\left(\frac{1}{h}\Big(\eta\big((1-s^2)x\big)-\eta(x)\Big)\right) s\,ds \\ &- k|x|\int_0^1 \exp\left(\frac{1}{h}\Big(\eta\big((1-s^2)x\big)-\eta(x)\Big)\right) K_1\big(k|x|s\big)\,ds. \end{aligned} \tag{A.3}$$

From (2.3), we can estimate the second term of the right-hand side of (A.3) by

$$\begin{aligned} &\left|-k|x|\int_0^1 \exp\left(\frac{1}{h}\Big(\eta\big((1-s^2)x\big)-\eta(x)\Big)\right) K_1\big(k|x|s\big)\,ds\right| \\ &\le \frac{k^4|x|^4}{16}\int_0^1 \exp\left(\frac{1}{h}\,\mathrm{Re}\Big(\eta\big((1-s^2)x\big)-\eta(x)\Big)\right) s^3\,ds \\ &\le \frac{k^4|x|^4}{16}\int_0^1 \exp\left(-\frac{1}{h}\,\eta_{\mathrm{R}}(x)\,s^2\right) s^3\,ds = \frac{k^4|x|^4}{32}\frac{h^2}{\eta_{\mathrm{R}}(x)^2}\int_0^{\eta_{\mathrm{R}}(x)/h} e^{-\tau}\tau\,d\tau \\ &= \frac{k^4|x|^4}{32}\frac{h^2}{\eta_{\mathrm{R}}(x)^2}\left\{1-\left(\frac{\eta_{\mathrm{R}}(x)}{h}+1\right)\exp\left(-\frac{\eta_{\mathrm{R}}(x)}{h}\right)\right\} < \frac{k^4|x|^4}{32\eta_{\mathrm{R}}(x)^2}\,h^2. \end{aligned}$$

On the other hand, setting $s^2 = \tau h$, the integral of the first term on the right-hand side of (A.3) can be written as

$$\int_0^1 \exp\left(\frac{1}{h}\Big(\eta\big((1-s^2)x\big)-\eta(x)\Big)\right) s\,ds = \frac{h}{2}\int_0^{1/h}\Phi(h;\tau,x)\,d\tau,$$

where

$$\begin{aligned} \Phi(h;\tau,x) &:= \exp\left(-\frac{1}{h}\Big(1-(1-\tau h)^N\Big)\eta(x)\right) \\ &= \exp\left(-N\tau\eta(x)\int_0^1(1-\tau h t)^{N-1}\,dt\right). \end{aligned}$$

Now we compute

$$\int_0^{1/h} \Phi(h;\tau,x)\,d\tau = \int_0^{1/h} \left(\Phi(0;\tau,x) + \int_0^h \frac{\partial}{\partial s}\Phi(s;\tau,x)\,ds\right) d\tau$$

$$= \int_0^{1/h} \exp(-N\tau\eta(x))\,d\tau$$

$$+ N(N-1)\,\eta(x) \int_0^{1/h}\int_0^h \tau^2 \int_0^1 (1-t\tau s)^{N-2}\,t\,dt\,\Phi(s;\tau,x)\,ds\,d\tau$$

$$= \frac{1}{N\eta(x)} - \frac{\exp(-N\eta(x)/h)}{N\eta(x)}$$

$$+ N(N-1)\,\eta(x) \int_0^{1/h}\int_0^h \tau^2 \int_0^1 (1-t\tau s)^{N-2}\,t\,dt\,\Phi(s;\tau,x)\,ds\,d\tau$$

and

$$\left|\int_0^{1/h}\int_0^h \tau^2 \int_0^1 (1-t\tau s)^{N-2}\,t\,dt\,\Phi(s;\tau,x)\,ds\,d\tau\right|$$

$$\leq \frac{1}{2}\int_0^{1/h}\int_0^h \tau^2 \exp\left(-\frac{1}{s}\Big(1-(1-\tau s)^N\Big)\eta_{\mathrm{R}}(x)\right) ds\,d\tau$$

$$\leq \frac{1}{2}\int_0^{1/h}\int_0^h \tau^2 \exp\big(-\tau\,\eta_{\mathrm{R}}(x)\big)\,ds\,d\tau \quad \text{(by (2.3))}$$

$$= \frac{h}{2\eta_{\mathrm{R}}(x)^3}\left[2-\exp\left(-\frac{\eta_{\mathrm{R}}(x)}{h}\right)\left(\frac{\eta_{\mathrm{R}}(x)^2}{h^2}+\frac{2\eta_{\mathrm{R}}(x)}{h}+2\right)\right] < \frac{1}{\eta_{\mathrm{R}}(x)^3}\,h.$$

Putting all estimates together immediately yields this lemma. □

We remark that we can derive more elaborate asymptotic expansion of the CGO solutions in a similar way. For example, the asymptotic expansion of $V_h^\sharp$ up to h^2 is

$$V_h^\sharp(x) = \exp\left(\frac{\eta(x)}{h}\right)\left[1 - \frac{k^2|x|^2}{4N\eta(x)}\,h + \frac{1}{N^2\eta(x)^2}\left\{-\frac{(N-1)k^2|x|^2}{4} + \frac{k^4|x|^4}{32}\right\} h^2 + O(h^3)\right].$$