The scattering relation and the Dirichlet-to-Neumann map

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Abstract. We describe a connection between the scattering relation, the Hilbert transform and the geodesic X-ray transform for non-trapping Riemannian manifolds with strictly convex boundary. This connection was used to solve the boundary rigidity problem in two dimensions for simple manifolds [**PU1**]. The key point in this development is the proof that the scattering relation determines the Dirichlet-to-Neumann map for the Laplace-Beltrami operator. We give another application of the above mentioned connection: an inversion procedure to reconstruct the conformal factor of a metric from its boundary distance function for two dimensional simple manifolds [**PU2**]. We also use this connection to give a characterization of the range of the geodesic X-ray transform in terms of the scattering relation for non-trapping manifolds with strictly convex boundary. and inversion formulas, on two dimensional simple manifolds, for the geodesic X-ray transform acting on scalar functions and vector fields for metrics with constant curvature and Fredholm type inversion formulas in the general case [**PU3**].

1. Travel Time Tomography and Boundary Rigidity

The question of determining the sound speed or index of refraction of a medium by measuring the first arrival times of waves arose in geophysics in an attempt to determine the substructure of the Earth by measuring at the surface of the Earth the travel times of seismic waves. An early success of this inverse method was the estimate by Herglotz [**H**] and Wiechert and Zoeppritz [**WZ**] of the diameter of the Earth and the location of the mantle, crust and core. The assumption used in those papers is that the index of refraction (speed of waves) depends only on the radius. A more realistic model is to assume that the it depends on position. The inverse kinematic problem can be formulated mathematically as determining a Riemannian metric on a bounded domain (the Earth) given by $ds^2 = \frac{1}{c^2(x)}dx^2$, where c is a positive function, from the length of geodesics (travel times) joining points in the boundary.

More recently it has been realized, by measuring the travel times of seismic waves, that the inner core of the Earth might exhibit anisotropic behavior, that is the speed of waves depends also on direction there with the fast direction parallel

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to the Earth's spin axis. In [**Cr**] it is given this explanation for the different time residuals Given the complications presented by modeling the Earth as an anisotropic elastic medium we consider a simpler model of anisotropy, namely that the wave speed is given by a symmetric, positive definite matrix $g = (g_{ij})(x)$, that is, a Riemannian metric in mathematical terms. The problem is to determine the metric from the lengths of geodesics joining points in the boundary (the surface of the Earth in the motivating example). It is useful to consider a more general and geometric formulation of the problem.

Let (M, g) be a compact Riemannian manifold with boundary ∂M. Let $d_g(x, y)$ denote the geodesic distance between x and y, two points in the boundary. This is defined as the infimum of the length of all sufficiently smooth curves joining the two points. The function d_g measures the first arrival time of waves joining points of the boundary. The inverse problem we discuss in this section is whether we can determine the Riemannian metric g knowing $d_g(x, y)$ for any $x \in \partial M$, $y \in \partial M$. This problem also arose in rigidity questions in Riemannian geometry [**Mi**], [**Cro1**], [**Gr**].

The metric g cannot be determined from this information alone. We have $d_{\psi^* g} = d_g$ for any diffeomorphism $\psi : M \to M$ that leaves the boundary pointwise fixed, i.e., $\psi|_{\partial M} = Id$, where Id denotes the identity map and $\psi^* g$ is the pull-back of the metric g. The natural question is whether this is the only obstruction to unique identifiability of the metric. It is easy to see that this is not the case. Namely one can construct a metric g and find a point x_0 in M so that $d_g(x_0, \partial M) > \sup_{x,y\in\partial M} d_g(x, y)$. For such a metric, d_g is independent of a change of g in a neighborhood of x_0. The hemisphere of the round sphere is another example.

Therefore it is necessary to impose some a-priori restrictions on the metric. One such restriction is to assume that the Riemannian manifold (M, g) is **simple**, i.e., M is simply-connected, any geodesic has no conjugate points and ∂M is strictly convex. ∂M is strictly convex if the second fundamental form of the boundary is positive definite in every boundary point.

R. Michel conjectured in [**Mi**] that simple manifolds are boundary distance rigid, that is d_g determines g uniquely up to an isometry which is the identity on the boundary. This is known for simple subspaces of Euclidean space (see [**Gr**]), simple subspaces of an open hemisphere in two dimensions (see [**Mi1**]), simple subspaces of symmetric spaces of constant negative curvature [BCG], simple two dimensional spaces of negative curvature (see [**Cro2**] or [**O**]). We remark that simplicity of a compact manifold with boundary can be determined from the boundary distance function.

Michel's conjecture was proven in generality in [**PU1**] in two dimensions.

THEOREM 1.1. *Let $(M, g_i), i = 1, 2$ be two dimensional simple compact Riemannian manifolds with boundary. Assume*

$$d_{g_1}(x, y) = d_{g_2}(x, y) \quad \forall (x, y) \in \partial M \times \partial M.$$

Then there exists a diffeomorphism $\psi : M \to M$, $\psi|_{\partial M} = Id$, so that

$$g_2 = \psi^* g_1.$$

In the case that both g_1 and g_2 are conformal to the Euclidean metric e (i.e., $(g_k)_{ij} = \alpha_k \delta_{ij}$, $k = 1, 2$ with δ_{ij} the Kronecker symbol), as mentioned earlier, the problem we are considering here is known in seismology as the inverse kinematic problem. In this case, it has been proven by Mukhometov in two dimensions [**Mu1**] that if $(M, g_i), i = 1, 2$ is simple and $d_{g_1} = d_{g_2}$, then $g_1 = g_2$.

More generally the same method of proof shows that if $(M, g_i), i = 1, 2$, are simple compact Riemannian manifolds with boundary and they are in the same conformal class then the metrics are determined by the boundary distance function. More precisely we have:

THEOREM 1.2. *Let $(M, g_i), i = 1, 2$ be simple compact Riemannian manifolds with boundary. Assume $g_1 = \rho g_2$ for a positive, smooth function $\rho, \rho|_{\partial M} = 1$ and $d_{g_1} = d_{g_2}$ then $g_1 = g_2$.*

This result and a stability estimate were proven in [**Mu1**]. We remark that in this case the diffeomorphism ψ that is present in the general case must be the identity if the metrics are conformal to each other. For related results and generalizations see [**Be**], [**BG**], [**Cro1**], [**GN**], [**MR**].

As a consequence of the method of proof of Theorem 1.1 we derived in [**PU2**] a reconstruction formula for the conformal factor in the two dimensional case. More precisely we have:

THEOREM 1.3. *Let (M, g) be a simple two dimensional manifold and ρ be a smooth positive function on M so that $(M, \rho g)$ is also simple. Then we develop a reconstruction procedure to recover ρ from $d_{\rho g}(x, y), x, y \in \partial M$.*

The proof of Theorem 1.1 involves a connection between the scattering relation and the Dirichlet-to-Neumann map (DN) associated to the Laplace-Beltrami operator. In section 2 we define the scattering relation which quantizes the scattering operator. In Section 3 we discuss the geodesic X-ray transform and the connection between the scattering relation the Hilbert transform and the geodesic X-ray transform (see Theorem 3.3 and Theorem 3.4). In Section 4 we discuss the main step of the proof of Theorem 1.1 which consists in showing that, under the assumptions of the theorem, we can determine the Dirichlet-to-Neumann map if we know the scattering relation. We also sketch the proof of Theorem 1.3. In Section 5 we use the connection indicated in section 3, to give a characterization of the range of the geodesic X-ray transform in terms of the scattering relation and we give Fredholm type inversion formulas for the geodesic X-ray transform acting on scalar functions and vector fields.

2. The scattering relation

Suppose we have a Riemannian metric in Euclidean space which is the Euclidean metric outside a compact set. The inverse scattering problem for metrics is to determine the Riemannian metric by measuring the scattering operator (see [**Gu**]). A similar obstruction to the boundary rigidity problem occurs in this case with the diffeomorphism ψ equal to the identity outside a compact set. It was proven in [**Gu**] that from the wave front set of the scattering operator, one can determine, under some non-trapping assumptions on the metric, the **scattering relation** on the boundary of a large ball. This uses high frequency information of the scattering operator. In the semiclassical setting Alexandrova has shown for a large class of operators that the scattering operator associated to potential and metric perturbations of the Euclidean Laplacian is a semiclassical Fourier integral operator quantized by the scattering relation [**A1**], [**A2**]. The scattering relation maps the point and direction of a geodesic entering the manifold to the point and direction of exit of the geodesic. We proceed to define in more detail the scattering relation and its relation with the boundary distance function.

Let ν denote the unit-inner normal to ∂M. We denote by $\Omega\,(M) \to M$ the unit-sphere bundle over M:

$$\Omega(M) = \bigcup_{x\in M} \Omega_x, \quad \Omega_x = \{\xi \in T_x(M) : |\xi|_g = 1\}.$$

$\Omega(M)$ is a (2 dim $M-1$)-dimensional compact manifold with boundary, which can be written as the union $\partial\Omega\,(M) = \partial_+\Omega\,(M) \cup \partial_-\Omega\,(M)$

$$\partial_\pm\Omega\,(M) = \{(x,\xi) \in \partial\Omega\,(M)\,,\ \pm(\nu\,(x)\,,\xi) \geq 0\ \}.$$

The manifold of inner vectors $\partial_+\Omega\,(M)$ and outer vectors $\partial_-\Omega\,(M)$ intersect at the set of tangent vectors

$$\partial_0\Omega\,(M) = \{(x,\xi) \in \partial\Omega\,(M)\,, \quad (\nu\,(x)\,,\xi) = 0\ \}.$$

Let (M,g) be an n-dimensional compact manifold with boundary. We say that (M,g) is **non-trapping** if each maximal geodesic is finite. Let (M,g) be non-trapping and the boundary ∂M is strictly convex. Denote by $\tau(x,\xi)$ the length of the geodesic $\gamma(x,\xi,t), t \geq 0$, starting at the point x in the direction $\xi \in \Omega_x$. This function is smooth on $\Omega(M) \setminus \partial_0\Omega(M)$. The function $\tau^0 = \tau|_{\partial\Omega(M)}$ is equal zero on $\partial_-\Omega(M)$ and is smooth on $\partial_+\Omega(M)$. Its odd part with respect to ξ

$$\tau^0_-(x,\xi) = \frac{1}{2}\left(\tau^0(x,\xi) - \tau^0\,(x,-\xi)\right)$$

is a smooth function.

Definition 2.1. *Let (M,g) be non-trapping with strictly convex boundary. The scattering relation $\alpha : \partial\Omega\,(M) \to \partial\Omega\,(M)$ is defined by*

$$\alpha(x,\xi) = (\gamma(x,\xi,2\tau^0_-(x,\xi)), \dot\gamma(x,\xi,2\tau^0_-(x,\xi))).$$

The scattering relation is a diffeomorphism $\partial\Omega\,(M) \to \partial\Omega\,(M)$. Notice that $\alpha|_{\partial_+\Omega(M)} : \partial_+\Omega\,(M) \to \partial_-\Omega\,(M)\,,\ \alpha|_{\partial_-\Omega(M)} : \partial_-\Omega\,(M) \to \partial_+\Omega\,(M)$ are diffeomorphisms as well. Obviously, α is an involution, $\alpha^2 = id$ and $\partial_0\Omega\,(M)$ is the hypersurface of its fixed points, $\alpha(x,\xi) = (x,\xi),\ (x,\xi) \in \partial_0\Omega\,(M)$.

A natural inverse problem is whether the scattering relation determines the metric g up to an isometry which is the identity on the boundary. This information takes into account all the travel times not just the first arrivals.

In the case that (M,g) is a simple manifold, and we know the metric at the boundary (and this is determined if d_g is known, see [**Mi**], knowing the scattering relation is equivalent to knowing the boundary distance function ([**Mi**]) so that we concentrate on studying the scattering relation.

We introduce the operators of even and odd continuation with respect to α:

$$A_\pm w(x,\xi) = w(x,\xi), \qquad (x,\xi) \in \partial_+\Omega\,(M)\,,$$

$$A_\pm w(x,\xi) = \pm\,(\alpha^* w)\,(x,\xi), \ \ (x,\xi) \in \partial_-\Omega\,(M)\,.$$

The scattering relation preserves the measure $|(\xi,\nu)|d\Sigma$, ($d\Sigma$ is the measure of the boundary $\partial\Omega(M)$ induced by the metric g) and therefore the operators

$A_\pm : L^2_\mu(\partial_+\Omega(M)) \to L^2_{|\mu|}(\partial\Omega(M))$ are bounded, where $L^2_{|\mu|}(\partial\Omega(M))$ is the real Hilbert space with scalar product

$$(u,v)_{L^2_{|\mu|}(\partial\Omega(M))} = \int\limits_{\partial\Omega(M)} |\mu|\, uv d\Sigma, \quad \mu = (\xi,\nu).$$

and $L^2_\mu(\partial\Omega_+(M))$ is the real Hilbert space with scalar product

$$(u,v)_{L^2_\mu(\partial\Omega_+(M))} = \int\limits_{\partial\Omega_+(M)} \mu uv d\Sigma.$$

The adjoint of $A_\pm$ is a bounded operator $A^*_\pm : L^2_{|\mu|}(\partial\Omega(M)) \to L^2_\mu(\partial_+\Omega(M))$ given by

$$A^*_\pm u = (u \pm u\circ\alpha)|_{\partial_+\Omega(M)}.$$

3. The geodesic X-ray transform

The X-ray transform integrates a function along lines. Radon found in 1917 an inversion formula in two dimensions to determine a function knowing the X-ray transform. This formula is non-local in the sense that in order to find the function at a point x one needs to know the integral of the function along lines far from the point. Radon's inversion formula has been implemented numerically using the filtered backprojection algorithm which is used today in CT scans. Another important transform in medical imaging and other applications is the Doppler transform which integrates a vector field along lines. The motivation is ultrasound Doppler tomography. It is known that blood flow is irregular and faster around tumor tissue than in normal tissue and Doppler tomography attempts to reconstruct the blood flow pattern. Mathematically the problem is to what extend a vector field is determined from its integral along lines.

In this paper we consider the case of integrating functions and vector fields along geodesics of a Riemannian metric. This arises in geophysics since the ray paths are no longer straight lines. We obtain inversion formulas for the constant curvature case and Fredholm type formulas in general which are *non-local*. We define next the geodesic X-ray transform for any compact Riemannian manifold (M,g) with boundary of any dimension.

We embed (M,g) into a compact Riemannian manifold (S,g) with no boundary. Let φ_t be the geodesic flow on $\Omega(S)$ and $\mathcal{H} = \frac{d}{dt}\varphi_t|_{t=0}$ be the geodesic vector field. Let u^f be the solution of the boundary value problem

$$\mathcal{H}u = -f, \quad u|_{\partial_-\Omega(M)} = 0,$$

which can be written as

$$u^f(x,\xi) = \int\limits_0^{\tau(x,\xi)} f(\varphi_t(x,\xi))dt, \quad (x,\xi)\in\Omega(M).$$

In particular

$$\mathcal{H}\tau = -1.$$

The trace

$$If = u^f|_{\partial_+\Omega(M)}$$

is called **the geodesic X-ray transform** of the function f. If the manifold (M,g) is non-trapping (that is every geodesic has finite length) and has a strictly convex boundary the operator $I : C^\infty(\Omega(M)) \to C^\infty(\partial_+\Omega(M))$.

Clearly a function f is not determined by its geodesic X-ray transform alone, since it depends on more variables than If. We consider the geodesic X-ray transform acting on symmetric tensor fields.

We denote by $f_m(x,\xi)$ an homogeneous polynomial of degree m with respect to ξ, induced by the symmetric tensor field f on (M,g) of m degree :

$$f_m(x,\xi) = f_{i_1 \dots i_m}(x)\, \xi^{i_1} \dots \xi^{i_m}.$$

The operator I_m, defined by

$$I_m f = I f_m$$

is called **the geodesic X-ray transform** of the symmetric tensor field. If the manifold (M,g) is non-trapping and the boundary ∂M is strictly convex $I_m : C^\infty(M, S_m(M)) \to C^\infty(\partial_+\Omega(M))$, where $S_m(M)$ denotes the bundle of symmetric tensors over (M,g). It is known that any symmetric smooth enough tensor field f may be decomposed in a potential and solenoidal part [**Sh**]:

$$f = dp + h, \quad p|_{\partial M} = 0, \quad \delta h = 0,$$

where δ notes the divergence and $d = \sigma\nabla$ is the symmetric part of covariant derivative. It is easy to see that the geodesic X-ray transform of the potential part dp is zero. We denote by $C^\infty_{sol}(M, S_m(M))$ the space of smooth solenoidal symmetric tensor fields.

We will consider in this paper only the case of the geodesic X-ray transform acting on functions independent of ξ and the geodesic X-ray transform acting on vector fields which, following the notation above, are denoted by I_0 and I_1 respectively. It is known that I_0 is injective on simple manifolds [**Mu1**] and that I_1 is injective acting on solenoidal vector fields on simple manifolds [**An**]. We mention also that the transform I_2 arises in the linearization of the boundary rigidity problem (see [**Sh**]). We define $\psi : \Omega(M) \to \partial_-\Omega(M)$ by

$$\psi(x,\xi) = \varphi_{-\tau(x,-\xi)}(x,\xi), (x,\xi) \in \Omega(M).$$

So, φ is a retract which maps vector (x,ξ) along geodesic $\gamma(x,\xi,t)$ in back direction into incoming vector. The solution of the boundary value problem for the transport equation

$$\mathcal{H}u = 0, \quad u|_{\partial_+\Omega(M)} = w$$

can be written in the form

$$u = w_\psi = w \circ \psi.$$

The adjoint of the operator I_m is the bounded operator $I_m^* : L^2_\mu(\partial_+\Omega(M)) \to L^2(M, S_m(M))$ which is given by

$$(I_m^* w)^{i_1 \cdots i_m}(x) = \int\limits_{\Omega_x} w_\psi(x,\xi)\xi^{i_1} \dots \xi^{i_m} d\Omega_x.$$

The Hilbert space $L^2(M, S_m(M))$ may be considered as subspace of $L^2(\Omega(M))$ of homogeneous polynomials with respect to ξ of m degree. The field $I_m^* w$ is solenoidal in the sense of the theory of distributions. Notice, that adjoint of the bounded operator $I : L^2(\Omega(M)) \to L^2_\mu(\partial_+\Omega(M))$ is given by

$$I^* w = w_\psi.$$

We also remark that by the fundamental theorem of calculus we have

$$I\mathcal{H}f = (f \circ \alpha - f)|_{\partial_+\Omega(M)} = -A_-^* f^0, f^0 = f|_{\partial\Omega(M)}. \tag{1}$$

The space $C_\alpha^\infty(\partial_+\Omega(M))$ is defined by

$$C_\alpha^\infty(\partial_+\Omega(M)) = \{w \in C^\infty(\partial_+\Omega(M)) : w_\psi \in C^\infty(\Omega(M))\}.$$

In [**PU1**] the following characterization of the space of smooth solutions of the transport equation was given

LEMMA 3.1.

$$C_\alpha^\infty(\partial_+\Omega(M)) = \{w \in C^\infty(\partial_+\Omega(M)) : A_+ w \in C^\infty(\partial\Omega(M))\}.$$

In the scalar case the following result holds on the solvability of $I_m^*, m = 0, 1$ [**PU1**].

THEOREM 3.2. *Let (M, g) be a simple, compact Riemannian manifold with boundary. Then the operator $I_0^* : C_\alpha^\infty(\partial_+\Omega(M)) \to C^\infty(M)$ is onto.*

The analog result for vector fields was proven in [**Pe**].

THEOREM 3.3. *Let (M, g) be a simple, compact Riemannian manifold with boundary. Then for any field $v \in C_{sol}^\infty(M, T(M))$ there exists a function $w \in C_\alpha^\infty(\partial_+\Omega(M))$ and harmonic function $h \in C^\infty(M)$, such that*

$$v = I_1^* w + \nabla h.$$

Now we define the Hilbert transform:

$$Hu(x,\xi) = \frac{1}{2\pi}\int_{\Omega_x} \frac{1 + (\xi,\eta)}{(\xi_\perp,\eta)} u(x,\eta)\, d\Omega_x(\eta), \quad \xi \in \Omega_x,$$

where the integral is understood as a principle value integral. Here $\perp$ means a 90^o degree rotation. In coordinates $(\xi_\perp)_i = \varepsilon_{ij}\xi^j$, where

$$\varepsilon = \sqrt{\det g}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The Hilbert transform H transforms even (respectively odd) functions with respect to ξ to even (respectively odd) ones. If H_+ (respectively H_-) is the even (respectively odd) part of the operator H:

$$H_+u(x,\xi) = \frac{1}{2\pi}\int_{\Omega_x} \frac{(\xi,\eta)}{(\xi_\perp,\eta)} u(x,\eta)\, d\Omega_x(\eta),$$

$$Hu_-(x,\xi) = \frac{1}{2\pi}\int_{\Omega_x} \frac{1}{(\xi_\perp,\eta)} u(x,\eta)\, d\Omega_x(\eta)$$

and u_+, u_- are the even and odd parts of the function u, then $H_+u = Hu_+, H_-u = Hu_-$.

We introduce the notation $\mathcal{H}_\perp = (\xi_\perp, \nabla) = -(\xi, \nabla_\perp)$, where $\nabla_\perp = \varepsilon\nabla$ and ∇ is the covariant derivative with respect to the metric g. The following commutator formula for the geodesic vector field and the Hilbert transform, is a crucial ingredient in the proofs of the main theorems surveyed in this paper (see [**PU1**]).

THEOREM 3.4. *Let (M, g) be a two dimensional Riemannian manifold. For any smooth function u on $\Omega(M)$ we have the identity*

$$[H, \mathcal{H}]u = \mathcal{H}_\perp u_0 + (\mathcal{H}_\perp u)_0$$

where

$$u_0(x) = \frac{1}{2\pi}\int_{\Omega_x} u(x,\xi)d\Omega_x$$

is the average value.

We define

$$P_- = A_-^* H_- A_+, \quad P_+ = A_-^* H_+ A_+.$$

Under the same assumptions of Theorem 3.3 The following factorizations holds:

THEOREM 3.5.

$$P_- = -\frac{1}{2\pi} I\delta_\perp I_1^*, \quad P_+ = -\frac{1}{2\pi} I_1 \nabla_\perp I_0^*. \tag{2}$$

4. The scattering relation and the Dirichlet-to-Neumann Map

Let (M, g) be a compact Riemannian manifold with boundary. The Laplace-Beltrami operator associated to the metric g is given in local coordinates by

$$\Delta_g u = \frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\sqrt{\det g}\, g^{ij}\frac{\partial u}{\partial x_j}\right)$$

where (g^{ij}) is the inverse of the metric g. Let us consider the Dirichlet problem

$$\Delta_g u = 0 \text{ on } M, \quad u\Big|_{\partial M} = f.$$

We define the DN map in this case by

$$\Lambda_g(f) = (\nu, \nabla u|_{\partial M})$$

The inverse problem is to recover g from Λ_g.

A similar obstruction to the one indicated in Section 1 holds for this problem. Namely

$$\Lambda_{\psi^* g} = \Lambda_g$$

where ψ is a C^∞ diffeomorphism of M which is the identity on the boundary.

In addition in the two dimensional case the Laplace-Beltrami operator is conformally invariant. More precisely

$$\Delta_{\beta g} = \frac{1}{\beta}\Delta_g$$

for any function β, $\beta \neq 0$. Therefore we have that for $n = 2$

$$\Lambda_{\beta(\psi^* g)} = \Lambda_g$$

for any non-zero β satisfying $\beta|_{\partial M} = 1$.

Therefore the best that one can do in two dimensions is to show that we can determine the conformal class of the metric g up to an isometry which is the identity on the boundary That this is the case is a result proven in [**LeU**] for simple metrics and for general connected two dimensional Riemannian manifolds with boundary in [**LaU**].

More precisely we have:

THEOREM 4.1. *Let (M, g) be a connected, compact Riemannian surface with boundary. Then $(\Lambda_g, \partial M)$ determines uniquely the conformal class of (M, g).*

As it was shown in [LaU] it is enough to measure the DN map in an open subset of the boundary.

The connection in two dimensions between the DN map and the scattering relation is given by

THEOREM 4.2. *Let $(M, g_i), i = 1, 2$, be compact, simple two dimensional Riemannian manifolds with boundary. Assume that $\alpha_{g_1} = \alpha_{g_2}$. Then $\Lambda_{g_1} = \Lambda_{g_2}$.*

The proof of Theorem 1.1 is reduced then to the proof of Theorem 4.2. In fact from Theorem 4.2 and Theorem 4.1 we get that we can determine the conformal class of the metric up to an isometry which is the identity on the boundary. Now by Theorem 1.2 we have that the conformal factor must be one proving that the metrics are isometric via a diffeomorphism which is the identity at the boundary. In other words $d_{g_1} = d_{g_2}$ implies that $\alpha_{g_1} = \alpha_{g_2}$. By Theorem 4.2 $\Lambda_{g_1} = \Lambda_{g_2}$. By Theorem 4.1, there exists a diffeomorphism $\psi : M \longrightarrow M$, $\psi|_{\partial M} =$ Identity and a function $\beta \neq 0, \beta|_{\partial M} =$ identity such that $g_1 = \beta\psi^* g_2$. By Mukhometov's theorem $\beta = 1$ showing that $g_1 = \psi^* g_2$ proving Theorem 1.1.

Before starting the proof of Theorem 4.2 we recall that Michel [**Mi1**] has proven that for two dimensional manifolds Riemannian manifolds with strictly convex boundary one can determine from the boundary distance function, up to the natural obstruction, all the derivatives of the metric at the boundary. This result was generalized to any dimensions in [**LSU**].

The proof of Theorem 4.2 consists in showing that from the scattering relation we can determine the traces at the boundary of conjugate harmonic functions, which is equivalent information to knowing the DN map associated to the Laplace-Beltrami operator.

Sketch of the proof of Theorem 4.2

Let (h, h_*) be a pair of conjugate harmonic functions on M,

$$\nabla h = \nabla_\perp h_*, \quad \nabla h_* = -\nabla_\perp h.$$

Notice, that $\delta\nabla = \triangle$ is the Laplace-Beltrami operator and $\delta\nabla_\perp = 0$. Let $I_0^* w = h$. Since $I_1 \mathcal{H}_\perp h = I_1 \mathcal{H} h_* = -A_-^* h_*^0$, where $h_*^0 = h_*|_{\partial M}$, we obtain from the second identity (3.2)

$$2\pi A_-^* H_+ A_+ w = -A_-^* h_*^0. \tag{1}$$

The following theorem gives the key to obtain the DN map from the scattering relation.

THEOREM 4.3. *Let M be a 2-dimensional simple manifold. Let $w \in C_\alpha^\infty(\partial_+\Omega(M))$ and h_* the harmonic continuation of function h_*^0. Then the equation (1) holds iff the functions $h = I_0^* w$ and h_* are conjugate harmonic functions.*

In summary we have the following procedure to obtain the DN map from the scattering relation. For a given smooth function h_*^0 on ∂M we find a solution $w \in C_\alpha^\infty(\partial_+\Omega(M))$ of the equation (1). Then the functions $h^0 = 2\pi(A_+ w)_0$ (notice, that $2\pi(A_+ w)_0 = I_0^* w|_{\partial M}$) and h_*^0 are the traces of conjugate harmonic functions. It is easy to see that this gives the DN map.

We now sketch the proof of Theorem 1.3. We first start with the a general simple manifold.

Let (M, g) be a a simple two-dimensional compact Riemannian manifold with boundary. As usual the scalar product in $L^2(M)$ is defined by

$$(u,v) = \int_M uv\sqrt{\det g}dx.$$

We have that

$$\int_M fI_0^*w\sqrt{\det g}dx = (I_0 f, w) \tag{2}$$

where in the right hand side the inner product is in $L^2_\mu(\partial_+\Omega(M))$. A conformal Killing vector field X satisfies the equation

$$\frac{1}{2}(\nabla_i X_j + \nabla_j X_i) = g_{ij}\frac{\delta X}{2} \tag{3}$$

where ∇ denotes the covariant derivative and δ is the divergence. For such vector field we have

$$\mathcal{H}_g(X,\xi) = \frac{1}{2}\delta X. \tag{4}$$

We remark that the right hand side of (4.4) does not depend on ξ. We also note that the local 1-parameter flow generated by the vector field X consists of local conformal isometries of the metric g.

From (3.1) we have that the geodesic X-ray transform $I_0(\delta X)$ is given by

$$I_0(\delta X) = -2A_-^*(X^0,\xi), \text{ where } X^0 = X|_{\partial M}. \tag{5}$$

Then putting $f = \delta X$ in (4.2) with X solution of (4.3) we get for any given $w \in C_\alpha^\infty(\partial_+\Omega(M))$:

(6)

$$\int_M (X, \nabla I_0^* w)\sqrt{\det g}dx = -2\int\limits_{\partial\Omega_+(M)} \mu w A_-^*(X^0,\xi)d\Sigma + 2\pi\int_{\partial M}(X^0,\nu)(A_+w)_0 d\Gamma,$$

where $d\Gamma$ is the measure of the boundary ∂M, induced by the metric g.

Now we specialize to the case of M a bounded domain of Euclidean space with smooth boundary. We provide M with the Riemannian metric g with lengths given by given by

$$ds^2 = \frac{1}{c^2(x)}dx^2$$

where $c(x)$ is a smooth and positive function on M. In other words the metric g is conformal to the Euclidean metric, $g_{ij} = \delta_{ij}/c^2$. The function $c(x)$ models the sound speed (index of refraction) of the medium M. We denote by $\rho = 1/c^2$.

We will develop a (linear) method of reconstruction of the sound speed from d_g and then prove Theorem 1.3. In this case the vector field X as in (4.3) is a Cauchy-Riemann vector field, more exactly its contravariant components satisfy the Cauchy-Riemann equations

$$\frac{\partial X^1}{\partial x^1} = \frac{\partial X^2}{\partial x^2}, \quad \frac{\partial X^1}{\partial x^2} + \frac{\partial X^2}{\partial x^1} = 0.$$

Let h be a harmonic function. (In our case $\Delta_g = c^2\Delta_e$ and consequently $\Delta_e h = 0$). We know that we can find find a solution $w \in C_\alpha^\infty(\partial_+\Omega(M))$ of equation (1), where h_*^0 is the trace of the conjugate harmonic function. Then $h = I_0^* w$. Thus we can calculate the integral in the left hand side of (4.6) for a holomorphic vector field X and harmonic function h. We denote by

$$S_{X,h}[\rho] = \int_M (X, \nabla h)\rho(x)dx$$

(since $\sqrt{\det g} = \rho$) which is known if we know the scattering relation.

The problem of finding ρ is then reduced to finding enough holomorphic vector fields u and harmonic functions h so that the product $(X, \nabla h)$ is dense in an appropriate space. This is similar to a question considered by Calderón for the linearized inverse conductivity problem at a constant conductivity [**Ca**]. See [**U1**] for further developments.

We choose

$$X^1 = \zeta_2 e^{<x,\zeta>}, \quad X^2 = \zeta_1 e^{<x,\zeta>}, \quad h = e^{<x,\sigma>} \tag{7}$$

with complex vector $\zeta, \sigma \in C^2; \zeta\cdot\zeta = \sigma\cdot\sigma = 0$ with $\sigma \neq -\overline{\zeta}$. Here $<,>$ denotes the standard Euclidean inner product. We remark that we can write for $\zeta \in C^2; \zeta{\cdot}\zeta = 0$, in the form

$$\zeta = \eta + ik, \text{ with } \eta, k \in \mathbb{R}^2 \text{ satisfying } |k| = |\eta|, < k, \eta >= 0.$$

Substituting (4.7) in (4.6) we obtain:

$$S_{X,h}[\rho] = (\zeta_2\sigma_1 + \zeta_1\sigma_2)\int_M \rho(x)e^{<x,\zeta+\sigma>}dx.$$

Therefore we get

$$\frac{S_{X,h}[\rho]}{(\zeta_2\sigma_1 + \zeta_1\sigma_2)} = \int_M \rho(x)e^{<x,\zeta+\sigma>}dx.$$

Now by taking the limit

$$lim_{\sigma\to-\overline{\zeta}} \quad \frac{S_{X,h}[\rho]}{(\zeta_2\sigma_1 + \zeta_1\sigma_2)} = \int_M \rho(x)e^{2i<x,k>}dx$$

we recover the Fourier transform of ρ.

Thus we have given a recovery procedure to obtain the conformal factor ρ from the scattering relation.

Now let us consider the general case in Theorem 1.3. We can find a conformal diffeomorphism $\phi : (\widetilde{M}, g) \longrightarrow (D, e)$ where D is the unit disk and e denotes the Euclidean metric. Therefore in the argument above we replace $x^i, i = 1, 2$ by $\phi(x^i), i = 1, 2$ and we proceed in a completely analogous fashion. This concludes the sketch of proof of Theorem 1.3.

5. Range and inversion of the geodesic X-ray transform

Let $T(M)$ be the tangent bundle of M. We denote by δ the divergence operator $\delta : C^\infty(M, TM)) \to C^\infty(M)$. In local coordinates this is given by $\delta u = g^{kj}\nabla_k u_j$ using Einstein's summation convention.

We define the operator $\delta_\perp : C^\infty(M, T(M)) \to C^\infty(M)$ by

$$\delta_\perp u = -\delta u_\perp.$$

Then

$$\delta_\perp\nabla_\perp f = \delta\nabla f = \Delta f, \;\; \delta_\perp\nabla f = -\delta\nabla_\perp f = 0.$$

We now give the characterization of the range of I_0 and I_1 in terms of the scattering relation only. We have that these are the projections of the operators P_-, P_+ respectively. For the details see [**PU2**].

THEOREM 5.1. *Let (M, g) be simple two dimensional compact Riemannian manifold with boundary. Then*
i) The maps

$$\begin{aligned} \delta_\perp I_1^* &: C_\alpha^\infty(\partial_+\Omega(M)) \to C^\infty(M), \\ \nabla_\perp I_0^* &: C_\alpha^\infty(\partial_+\Omega(M)) \to C_{sol}^\infty(M, T(M)) \end{aligned}$$

are onto.
ii). A function $u \in C^\infty(\partial_+\Omega(M))$ belong to Range I_0 iff $u = P_- w$, $w \in C_\alpha^\infty(\partial_+\Omega(M))$.
iii). A function $u \in C^\infty(\partial_+\Omega(M))$ belong to Range I_1 iff $u = P_+ w$, $w \in C_\alpha^\infty(\partial_+\Omega(M))$.

PROPOSITION 5.1. *The operator $W : C_0^\infty(M) \to C^\infty(M)$, defined by*

$$Wf = (\mathcal{H}_\perp u^f)_0$$

can be extended to a smoothing operator $W : L^2(M) \to C^\infty(M)$.

We remark that in the case of constant Gaussian curvature $W = 0$ and this does not depend on whether the metric has conjugate points so that the inversion formulas of Theorem 5.2 hold for all two dimensional manifolds with boundary with constant curvature.

The inversion formulas are (see [**PU2**])

THEOREM 5.2. *Let (M, g) be a two-dimensional simple manifold. Then we have*

$$\begin{aligned} f + W^2 f &= \frac{1}{2\pi}\delta_\perp I_1^* w, \quad w = \frac{1}{2}\alpha^* H(I_0 f)^-|_{\partial_+\Omega(M)}, \;\; f \in L^2(M), \\ h + (W^*)^2 h &= \frac{1}{2\pi} I_0^* w, \quad w = \frac{1}{2}\alpha^* H(I_1 \mathcal{H}_\perp h)^+|_{\partial_+\Omega(M)}, \; h \in H_0^1(M), \end{aligned}$$

where $W, W^* : L^2(M) \to C^\infty(M)$. In the case of a manifold of constant curvature $W = 0, \quad W^* = 0$.

6. Final Remarks

The Hilbert transform for 2-dimensional Riemannian manifolds is the map that relates the restrictions on the boundary of conjugate harmonic functions. In this sense the Hilbert transform, up to a constant, is just the Dirichlet-to-Neumann (DN) map. In Section 4 we fixed a point x and started with the microlocal Hilbert transform on the circle Ω_x in the tangent space with Euclidean metric and we ended with the global Hilbert transform (the DN map).

The scattering relation and the boundary distance function are determined by the singularities of the DN map associated to the wave equation for the Laplace-Beltrami operator, the so-called hyperbolic (or dynamic) Dirichlet-to-Neumann map [**U2**]. We have found, in two dimensions, a connection between the scattering relation and the *elliptic* Dirichlet-to-Neumann map which led to a solution of the boundary rigidity problem in two dimensions. Is there a similar connection in higher dimensions?

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