APPROXIMATE QUANTUM AND ACOUSTIC CLOAKING

ALLAN GREENLEAF, YAROSLAV KURYLEV,
MATTI LASSAS AND GUNTHER UHLMANN

ABSTRACT. For any $E \geq 0$, we construct a sequence of bounded potentials V_n^E, $n \in \mathbb{N}$, supported in an annular region $B_{out} \setminus B_{inn} \subset \mathbb{R}^3$, which act as approximate cloaks for solutions of Schrödinger's equation at energy E: For any potential $V_0 \in L^\infty(B_{inn})$ such that E is not a Neumann eigenvalue of $-\Delta + V_0$ in B_{inn}, the scattering amplitudes $a_{V_0 + V_n^E}(E, \theta, \omega) \to 0$ as $n \to \infty$. The V_n^E thus not only form a family of approximately transparent potentials, but also function as approximate invisibility cloaks in quantum mechanics. On the other hand, for E close to interior eigenvalues, resonances develop and there exist *almost trapped states* concentrated in B_{inn}. We derive the V_n^E from singular, anisotropic transformation optics-based cloaks by a de-anisotropization procedure, which we call *isotropic transformation optics*. This technique uses truncation, inverse homogenization and spectral theory to produce nonsingular, isotropic approximate cloaks. As an intermediate step, we also obtain approximate cloaking for a general class of equations including the acoustic equation.

1. INTRODUCTION

A fundamental problem is to describe the scattering of waves by a potential, as governed by the time-independent Schrödinger equation at energy $E \geq 0$,

$$(1) \qquad (-\Delta + V(x))\psi(x) = E\psi(x), \quad x \in \mathbb{R}^d,$$
$$\psi(x) = \exp(iE^{1/2}x \cdot \theta) + \psi_{sc}(x),$$

where $\theta \in \mathbb{S}^{d-1}$ and $\psi_{sc}(x)$ satisfies the Sommerfeld radiation condition. We restrict ourselves in this paper to compactly supported potentials V, so

$$\psi_{sc}(x) = C_d E^{\frac{d-3}{4}} \frac{a_V(E, \frac{x}{|x|}, \theta)}{|x|^{\frac{d-1}{2}}} \exp(iE^{1/2}|x|) + o\left(|x|^{-\frac{d-1}{2}}\right), \quad \text{as } |x| \to \infty.$$

The function $a_V(E, \omega, \theta)$ is the *scattering amplitude* at energy E of the potential V. The associated inverse scattering problem consists of trying to determine V from the scattering amplitude, or measurements of waves at the boundary of some region Ω containing the support of V.

Recently, Zhang, *et al.*, [65] described quantum mechanical cloaking at any fixed energy E. Their construction starts with a homogeneous, isotropic mass tensor and a potential $V \equiv 0$, and subjects this pair to a singular change of variables. This "blowing up a point" transformation had been used in [28, 29] to produce conductivities that hide objects from detection by electrostatic measurements[1], and was subsequently used to describe the same phenomenon for electromagnetic waves [54]; one now refers to a specification of material parameters having this effect as a *cloak*. The cloaking Schrödinger equation in [65], which has an anisotropic, singular mass tensor, is equivalent with the Helmholtz equation (at frequency $\omega = \sqrt{E}$) for an associated singular Riemannian metric, and thus covered by cloaking for the Helmholtz equation in 3D, as we analyzed in [20, Sec. 3]. Similarly, cloaks for acoustics in 3D have been described in [10, 15][2]; again, these are in fact direct consequences of cloaking for the Helmholtz equation in 3D, *cf.* [24].

Thus, for ideal 3D cloaking in each of scalar optics, quantum mechanics and acoustics, one knows from [20] that any finite energy distributional solution decouples into a sum of a wave on the exterior of the cloak, unaffected (in terms of scattering or boundary measurements) by the cloak, and a wave within the cloaked region satisfying the Neumann boundary condition at the cloaking surface. Hence, if E is not a Neumann eigenvalue, then the wave must vanish within the cloaked region and cloaking works as advertised. On the other hand, if E is an eigenvalue, the cloaked region supports interior resonances, or *trapped states*. This is an unphysical situation, since the Dirichlet problem on Ω no longer has unique solutions, and this can be considered as a failure of cloaking, *per se*. However, what emerges from this failure in the setting of approximate cloaking described below is, we believe, quite useful.

In this paper, we construct, for each energy E, a family $\{V_n^E\}_{n=1}^\infty$ of potentials, supported in an annulus $B_{out} \setminus B_{inn} \subset \mathbb{R}^3$, which are not only almost transparent, in that the scattering amplitudes $a_{V_n^E}(E, \cdot, \cdot) \to 0$ as $n \to \infty$, but also act as *approximate cloaks* for potentials supported

---

[1]The 2D version has the same electrostatic cloaking property [36].

[2]See [14] for the 2D case, and [46, 52] for more regarding elastic and acoustic cloaking.

in the inner ball B_{inn}: for any potential $V_0 \in L^\infty(B_{inn})$ for which E is not an eigenvalue of $-\Delta + V_0$ in B_{inn}, the scattering amplitudes $a_{V_0+V_n^E}(E, \cdot, \cdot) \to 0$ as $n \to \infty$, as well. There are also approximate versions of the interior resonances supported by the ideal cloak: there exist energies close to the Neumann eigenvalues of the cloaked region B_{inn}, near which there are waves largely concentrated in B_{inn}, which we call *almost trapped states*. Furthermore, the quality of the approximate cloaking degrades as we move towards these energies, with waves being able to penetrate B_{inn}, and the presence of this region and V_0 detectable by scattering or boundary measurements. In addition, we are able to include a magnetic potential in the Schrödinger equation; this allows one to switch between the approximate cloak and almost trapped state alternatives by application of a suitable homogeneous magnetic field. We have given physical applications of this elsewhere [25, 26].

To put these results in a mathematical context, recall that in several dimensions numerous uniqueness results are available for the inverse scattering and boundary value problems, subject to some regularity assumptions, in terms of the scattering data at a fixed energy E and the closely related Dirichlet-to-Neumann (DN) operator. Physically, these correspond to far- and near-field measurements, resp. In fact, for compactly supported potentials the scattering data is equivalent to the DN operator measured on the boundary of a domain containing the support of the potential [6]. In dimensions $d \geq 3$, Sylvester and Uhlmann [62] showed that a smooth potential is determined by the associated DN map, and this was extended to L^∞ in [49]. Lavine and Nachman [39] and Chanillo [8] generalized this further, showing that, if a potential $V \in L^{\frac{d}{2}}(\mathbb{R}^d)$ or has small norm in the Fefferman-Phong space F_p, $p > \frac{d-1}{2}$, resp., then the DN map determines V. For $d = 2$, Nachman [48] proved that uniqueness holds if V has the special form $V = -\Delta(\sqrt{\gamma})/\sqrt{\gamma}$ arising from the gauge transformation of the conductivity equation to the Schrödinger equation, with $\gamma \in W^{2,p}$, $p > 1$, while Sun and Uhlmann [61] showed that uniqueness holds for a generic class of potentials. Very recently, Bukhgeim[7] has shown that uniqueness holds for all potentials in $L_{comp}^p(\mathbb{R}^2)$, $p > 2$.

On the other hand, for each positive energy E, Regge [57] constructed (noncompactly supported) potentials which are transparent, *i.e.*, for which the scattering amplitude $a_V(E, \theta, \omega) \equiv a_0(E, \theta, \omega) = 0$. See also [50, 59, 30], where the last gives a construction of transparent potentials in the Schwartz class $\mathcal{S}(\mathbb{R}^2)$. In the physics literature, Hendi, Henn and

Leonhardt [31] have described central, *i.e.*, radial, potentials which are transparent on the level of the ray geometry.

We construct the families $\{V_n^E\}$ of approximately cloaking potentials by means of a result of independent interest. The use of changes of variables to produce novel optical effects on waves or to facilitate computations has been considered in the physics literature, (see[3], *e.g.*, Dolin [17] or more recently Ward and Pendry [63]), and is now generally referred to as *transformation optics* (TO). However, to produce cloaking and other extreme effects, nonsingular changes of variables are insufficient. The recently proposed plans for cloaking are based on singular transformations and consist of medium parameters which are both *anisotropic* and *singular*[4], whether for the conductivity (electrostatics) [28, 29]; index of refraction (Helmholtz) [41, 20, 35]; permittivity and permeability (Maxwell) [54, 20]; mass density (acoustic) [14, 10, 15, 24]; or effective mass (Schrödinger)[65]. Physical realization of such designs is now potentially feasible due to the rapidly developing area of *metamaterials*, but the singularity and extreme anisotropy make characterizing and fabricating the materials to implement such designs an enormous obstacle to manufacturing invisibility devices, These same remarks are valid for other TO designs, such as [42, 9, 44, 21, 23, 55], some of which are singular.

We propose here a general method, which we refer to by the oxymoronic *isotropic transformation optics*, for dealing with both the anisotropy and singularity of TO material parameters. We describe this in detail in the context of cloaking, but it should be applicable to a wider range of TO designs. We in fact derive the quantum mechanical approximate cloaks from approximate cloaks for a general class of equations that includes the acoustic equation. Using ideas from Nguetseng [51], Allaire [1], Cherkaev [13] and elsewhere, we show how to find cloaking material parameters that are at once both isotropic and nonsingular, at the price of replacing perfect (ideal) cloaking with *approximate* cloaking (of arbitrary accuracy). This method, starting with transformation optics-based designs and constructing approximations to them, first by *nonsingular*, but still anisotropic, material parameters, and then by nonsingular *isotropic* parameters, appears to be a very flexible tool for creating physically realistic theoretical designs, easier to implement than the ideal ones due to the relatively tame nature of the materials

---

[3]We thank A. Kildishev for this reference.

[4]By *singular*, we mean that at least one of the eigenvalues goes to zero or infinity at some points.

needed, yet (up to an arbitrarily small error) essentially capturing the desired effects on wave propagation.

We review the ideal electrostatic cloak of [28], and extend this in Sec. 2.1 to a class of equations which will allow us to deal with both general acoustic and magnetic Schrödinger equations. These equations have coefficients which, borrowing the terminology from acoustics, we refer to as mass density and bulk modulus and which are singular at the cloaking surface Σ, *i.e.*, the interface between the cloaked and uncloaked regions. The desingularization process begins in Sec. 2.2, by truncating the mass density away from Σ, on the outer side of which the ideal cloaking parameters are singular. (Similar truncations have been considered before in the context of cylindrical or 2D cloaking, *cf.* [58, 22, 11, 36].) We show in Sec. 2.3 that the Dirichlet forms for the ideal cloaks can be well-approximated (in the sense of Γ-convergence) by these truncations. We then desingularize the bulk modulus in Sec. 2.4.

So far, the approximately cloaking mass densities are still anisotropic. As is well known in effective medium theory, homogenization of isotropic material parameters may lead to anisotropic ones [45]; in Secs. 3 and 4, we use this phenomenon in reverse, showing that the Dirichlet forms obtained in Sec. 2 can be well-approximated by those for certain nonsingular *isotropic* conductivities, which thus provide approximate cloaks for the general class of acoustic-like equations. These then allow us to obtain in Thm. 5.1 families of approximate quantum cloaks. In Sec. 6 we study failure of cloaking near exceptional energies, mirroring the failure of the existence of unique solutions for the ideal cloak at Neumann eigenvalues of the cloaked region; see the Remark at the end of the section. Further physical applications, including a new type of ion trap, can be found in [25, 26]. Finally, numerical simulations are presented in Sec. 7.

2. Approximating cloaking material parameters by nonsingular anisotropic parameters

We will denote by $B(a, R)$ the ball of radius R centered at a in $\mathbb{R}^3$, sometimes denoted simply $B(R)$ when centered at the origin, O. Let

$M_1 = \overline{B}(O_1, 3) \subset \mathbb{R}^3$ and $M_2 = \overline{B}(O_2, 1)$, with O_1, O_2 being two copies of O, considered as disjoint compact manifolds with boundary; set $M = M_1 \cup M_2$. Also, let $\Omega = \overline{B}(3) \subset \mathbb{R}^3$, and $F^1 : M_1 \setminus \{O_1\} \rightarrow \Omega \setminus \overline{B}(1) \subset \mathbb{R}^3$ be the map

$$(2) \qquad\qquad F^1(x) = x, \quad \text{for } 2 < |x| \le 3,$$

$$F^1(x) = \left(\frac{|x|}{2} + 1 \right) \frac{x}{|x|}, \quad \text{for } 0 < |x| \le 2.$$

Define also $F^2 : M_2 \rightarrow \overline{B}(1)$ as the identity map,

$$(3) \qquad\qquad\qquad F^2(x) = x.$$

Together, these form a surjective map $F = (F^1, F^2)$ from the *cloaking manifold* (or virtual space) $M \setminus \{O_1\}$ to the *cloaking device* (or physical space) Ω. By a conductivity we mean a measurable map with values in the symmetric non-negative $\mathbb{R}^{3 \times 3}$ matrices. Let $\gamma_0 = 1$ be the constant isotropic conductivity on $\mathbb{R}^3$ and define the conductivity σ_1 on Ω as

$$(4) \qquad \sigma_1 = F^1_* \gamma_0, \quad x \in \Omega \setminus \overline{B}(1), \qquad \sigma_1 = 2\, F^2_* \gamma_0, \quad x \in B(1),$$

which has a singularity on the *cloaking surface* $\Sigma := \partial B(1)$, both in that one of the eigenvalues (corresponding to the radial direction) tends to 0 as $r \searrow 1$ and that there is a jump discontinuity across the sphere Σ. This conductivity σ_1 is, up to the radius of Ω and the factor 2 in the second of formulae (4), used here for technical reasons, the one introduced in [28, 29] and shown to be indistinguishable from γ_0, *vis-a-vis* electrostatic boundary measurements at $\partial \Omega$. In fact, $\sigma_1|_{B(1)}$ can be replaced by *any* smooth, non-degenerate anisotropic conductivity tensor and its values will be undetectable at $\partial \Omega$. The same construction of $\sigma_1|_{\Omega \setminus B(1)}$, applied instead to the electric permittivity ϵ and magnetic permeability μ in Maxwell's equations, was proposed in [54] (see also [41]) to *cloak* the region $B(1)$ from observation by electromagnetic waves at positive frequency; we thus refer to σ_1 as a *cloaking* conductivity, and following the physics literature, we will refer to (Ω, σ_1) as the *ideal* cloak.

This gives rise to the Dirichlet problem for the singular conductivity equation,

$$(5) \qquad\qquad \nabla \cdot \sigma_1 \nabla u = 0, \quad \text{in } \Omega, \quad u|_{\partial \Omega} = h,$$

and to the corresponding singular conductivity operator $\mathcal{A}$,

$$(6) \qquad\qquad\qquad \mathcal{A}u := -g^{-1/2} \nabla \cdot \sigma_1 \nabla u,$$

that we consider with Dirichlet boundary condition $u|_{\partial\Omega} = 0$. Here, we use the singular Riemannian metric $(g_{jk})_{j,k=1}^3$ associated to the conductivity σ_1, namely,

$$(7) \qquad g^{1/2}g^{jk} = \sigma_1^{jk}, \quad g = |\det[g^{ij}]|^{-1} = |\det[\sigma_1^{ij}]|^2,$$

cf. [28, 29]. We denote by g both the metric and the corresponding scalar function, the meaning being clear from the context.

A rigorous definition of the meaning of (5) and the operator $\mathcal{A}$ is given in the following sections. In particular, the operator $\mathcal{A}$ is self-adjoint on $L_g^2(\Omega)$, the weighted L^2-space defined using the weight $g^{1/2}$. For a general weight $w(x) \geq 0$, we denote by $L^2(\Omega, wdx) = L^2(wdx)$ the weighted space,

$$L^2(\Omega, wdx) = \{u : \Omega \to \mathbb{C} \text{ measurable}, \|u\|_{L^2(wdx)}^2 := \int_\Omega |u|^2 \, w \, dx < \infty\}.$$

For simplicity, we denote $L_g^2 = L_g^2(\Omega) := L^2(\Omega, g^{\frac{1}{2}}dx)$, the natural L^2 space for the metric g, and the norm in this space by $\|\cdot\|_g$. Note that

$$(8) \qquad L^2(\Omega) \subset L_g^2(\Omega), \quad \|u\|_g \leq \sqrt{8}\,\|u\|.$$

We also use the Sobolev spaces

$$H_g^1(\Omega) = \{u \in L_g^2(\Omega) : u|_{\Omega\backslash\Sigma} \in H_{loc}^1(\Omega\backslash\Sigma), \int_{\Omega\backslash\Sigma} \sigma_1^{jk}\partial_j u \overline{\partial_k u} \, dx < \infty\},$$

$$H_{0,g}^1(\Omega) = \{v \in H_g^1(\Omega) : v|_{\partial\Omega} = 0\}.$$

Here and below, we use Einstein summation convention, summing over indices j and k appearing both as sub- and super-indices. Observe that $H^1(\Omega) \subset H_g^1(\Omega)$ and

$$(9) \qquad \|u\|_{H_g^1} = (\|u\|_{L_g^2}^2 + \int_\Omega \sigma_1^{jk}\partial_j u \overline{\partial_k u} \, dx)^{\frac{1}{2}} \leq \sqrt{8}\,\|u\|_{H^1}.$$

Throughout, we also use the following standard terminology: when considering convergence of sequences $\{x_n\}_{n=1}^\infty$ in a Hilbert space H, we say that x_n converges strongly to x in H, if $\|x_n - x\|_H \to 0$, as $n \to \infty$, while x_n converges weakly to x, if $(x_n - x, y)_H \to 0$, as $n \to \infty$, for any $y \in H$.

2.1. **Ideal cloaking for more general equations.** In this paper we treat equations more general than (5), which are important for physical applications, cf. [25, 26]. To this end, consider a family of equations that simultaneously includes both the magnetic Schrödinger equation and the acoustic equation; for simplicity, we will refer to equations of

this type as *acoustic*. Let q and $b = (b_1, b_2, b_3)$ be scalar- and vector-valued functions, which eventually will represent the electric and magnetic potentials; we will assume that

$$b \in L^\infty(\Omega; \mathbb{R}^3), \quad q \in L^\infty(\Omega; \mathbb{R}), \quad q \geq 0.$$

We note that the condition $q \geq 0$ is merely a convenience, since for general $q \in L^\infty(\Omega; \mathbb{R})$ we can always add a constant to achieve positivity, and of course this just shifts the spectrum.

To deal rigorously with the elliptic boundary value problem

$$(10) \quad -g^{-1/2}(\nabla + ib) \cdot \sigma_1(\nabla + ib)u + qu - \lambda u = f, \quad u|_{\partial\Omega} = h,$$

which has singular coefficients, we consider the corresponding quadratic form. In the following, we use the notation

$$\nabla^b = \nabla + ib(x).$$

Lemma 2.1. *The quadratic form*

$$(11) \qquad a_1[u, u] = \int_{\Omega \setminus \Sigma} \sigma_1 \nabla^b u \cdot \overline{\nabla^b u} dx + \int_{\Omega \setminus \Sigma} qg^{1/2}|u|^2 \, dx,$$

defined in the domain $\mathcal{D}(a_1) = H^1_{0,g}(\Omega)$ *is closed. Moreover, the embedding* $\mathcal{D}(a_1) \hookrightarrow L^2_g(\Omega)$ *is compact, and there is* $C_b > 1$ *so that*

$$(12) \qquad C_b^{-1}\|u\|^2_{H^1_{0,g}(\Omega)} \leq \|u\|^2_g + a_1[u, u] \leq C_b\|u\|^2_{H^1_{0,g}(\Omega)}.$$

Proof. To prove the assertion, we need to show two facts. First, we need to prove that $a_1[u, u] < \infty$ for $u \in H^1_{0,g}(\Omega)$. Second, we have to prove the closedness of the form a_1 on $H^1_{0.g}(\Omega)$.

We start with the fact [20] that the map F_* is unitary from $L^2(M_1) \oplus L^2(M_2)$ to $L^2_g(\Omega)$ and from $H^1_0(M_1) \oplus H^1(M_2)$ to $H^1_{0,g}(\Omega)$, respectively. For $v = (v_1, v_2)$ and $u = F_*v$, we have

$$(13) \qquad (v_1, v_2) \in L^2(M_1) \oplus L^2(M_2) \quad \text{iff } u \in L^2_g(\Omega);$$
$$\qquad (v_1, v_2) \in H^1_0(M_1) \oplus H^1(M_2) \quad \text{iff } u \in H^1_{0,g}(\Omega).$$

Hence, as $H^1(M_j)$, $j = 1, 2$ is compactly embedded into $L^2(M_j)$, we see that the space $H^1_g(\Omega)$ is compactly embedded into $L^2_g(\Omega)$.

Using definition (4) and the transformation rule for $1-$forms, we see that

$$(14) \qquad a_1[u,u] = \alpha[v,v] = \alpha^1[v_1,v_1] + \alpha^2[v_2,v_2];$$

$$\alpha^1[v_1,v_1] = \int_{M_1} |\nabla v_1 + i\beta_1\, v_1|^2 dx + \int_{M_1} \kappa_1\, |v_1|^2 dx,$$

$$\alpha^2[v_2,v_2] = 2\int_{M_2} |\nabla v_2 + i\beta_2\, v_2|^2 dx + 8\int_{M_2} \kappa_2\, |v_2|^2 dx.$$

Here the $1-$forms $\beta_1 = \beta|_{M_1}$, $\beta_2 = \beta|_{M_2}$, and functions $\kappa_1 = \kappa|_{M_1}$, $\kappa_2 = \kappa|_{M_2}$, are given by $b = F_*\beta$, $q = F_*\kappa$, that is,

$$b_j(x) = \frac{\partial F^k}{\partial y^j}(y)\beta_k(y), \quad q(x) = \kappa(y), \quad x = F(y) \in \Omega \setminus \Sigma.$$

It follows from (2), (3) that β_2, κ_1 and κ_2 are bounded, but β_1 has a singularity at $x = 0$, of the order $1/|x|$, and

$$(15) \qquad\qquad |\beta_1(x)| \leq \frac{C\|b\|_{L^\infty}}{|x|}.$$

Let us consider $\alpha^2[v_2, v_2]$ as an unbounded non-negative quadratic form in $L^2(M_2)$, with domain $\mathcal{D}(\alpha^2) = H^1(M_2)$. Then α^2 is closed. The quadratic form $\alpha^1[v_1, v_1]$ requires further analysis. We consider $\alpha^1[v_1, v_1]$ as an unbounded non-negative quadratic form in $L^2(M_1)$ having the domain

$$(16) \quad \mathcal{D}(\alpha^1) = \{v_1 \in L^2(M_1): \nabla v_1 + i\beta_1 v_1 \in L^2(M_1),\ v_1|_{\partial M_1} = 0\}.$$

Note that the condition $\nabla v_1 + i\beta_1 v_1 \in L^2(M_1)$ implies that the trace $v_1|_{\partial M_1}$ is well-defined. By [40, 60], $C_0^\infty(M_1)$ is dense on $\mathcal{D}(\alpha^1)$ and α^1 on $\mathcal{D}(\alpha^1)$ is a closed, non-negative quadratic form.

By Hardy's inequality [37], it follows from (15) that

$$(17) \qquad\qquad \|\beta_1 v_1\|_{L^2(M_1)} \leq C\|b\|_{L^\infty} \|v_1\|_{H^1(M_1)},$$

so that

$$(18) \qquad\qquad H_0^1(M_1) \subset \mathcal{D}(\alpha^1) \quad \text{is dense.}$$

Let $v_1 \in \mathcal{D}(\alpha^1)$. Then, by [40] $|v_1| \in H_0^1(M_1)$. Using (17) again, we obtain

$$\|\beta_1 v_1\|_{L^2(M_1)} = \|\,\beta_1|v_1|\,\|_{L^2(M_1)} < \infty.$$

Thus, by definition (16), this yields that $\nabla v_1 \in L^2(M_1)$, and hence $v_1 \in H_0^1(M_1)$. This shows that $\mathcal{D}(\alpha^1) \subset H_0^1(M_1)$ which, together with (18), implies that $\mathcal{D}(\alpha^1) = H_0^1(M_1)$. Thus, the domain of the closed

form α is $\mathcal{D}(\alpha) = H_0^1(M_1) \oplus H^1(M_2)$. Using the transformation rule
(14) for $u = F_* v$, and (13), we conclude that $a_1[u, u] < \infty$ for all $u \in H_{0,g}^1(\Omega)$ and that the quadratic form a_1 with domain $\mathcal{D}(a_1) = H_{0,g}^1(\Omega)$
is closed. The inequality (12) follows from this by the open mapping
theorem. $\square$

By the theory of quadratic forms [33], the closed quadratic form $a_1[u, u]$
defines an associated non-negative self-adjoint operator $\mathcal{A}_1$ in $L_g^2(\Omega)$,
having domain

(19) $\mathcal{D}(\mathcal{A}_1) := \{u \in H_{0,g}^1(\Omega) \ : \ \text{there is } f \in L_g^2(\Omega) \text{ such that}$
$$a_1[u, v] = (f, v)_g \text{ for } v \in H_{0,g}^1(\Omega)\},$$

that is, for $u \in \mathcal{D}(A_1)$ we have, for all $v \in H_{0,g}^1(\Omega)$,

$$(20) \quad \int_{\Omega \backslash \Sigma} \sigma_1 \nabla^b u \cdot \overline{\nabla^b v} dx + \int_{\Omega \backslash \Sigma} q g^{1/2} u \overline{v} \, dx = \int_{\Omega \backslash \Sigma} g^{1/2} f \overline{v} \, dx$$

with $\mathcal{A}_1 u = f \in L_g^2(\Omega)$. By Lemma 2.1, the spectrum of $\mathcal{A}_1$ consists of
discrete eigenvalues with finite dimensional eigenspaces.

Observe that, formally integrating by parts in (20), we come to (10)
with $h = 0$. To better understand the nature of the operator $\mathcal{A}_1$, we
give also an alternative definition in the case when $b \in C(\Omega; \mathbb{R}^3)$. In
this case, we use the fact that the map $D_\sigma^j : \phi \mapsto \sigma_1^{jk}(\partial_k \phi + i b_k \phi)$,
defined initially for $\phi \in C_0^\infty(\Omega)$, has a bounded extension

$$D_\sigma^j : H_{0,g}^1(\Omega) \to \mathcal{M}(\Omega; \mathbb{R}^3),$$

where $\mathcal{M}(\Omega; \mathbb{R})$ denotes the space of Borel measures on Ω, *cf.* [20,
Lem. 3.2]. For $b \in C(\Omega; \mathbb{R}^3)$, an equivalent definition of the operator
$\mathcal{A}_1$ is then

$$(21) \quad \mathcal{A}_1(u) := -g^{-1/2}(\partial_j + i b_j) D_\sigma^j u + q u,$$
$$\mathcal{D}(\mathcal{A}_1) = \{u \in H_{0,g}^1(\Omega) : \ (\partial_j + i b_j) D_\sigma^j u \in L^2(\Omega, g^{-1/2} dx)\}.$$

2.2. **Approximate cloaking by truncation.** For any $1 < R < 2$,
consider the nonsingular truncations of (10),

$$(22) \qquad -\nabla^b \cdot \sigma_R \nabla^b u + q g^{1/2} u - \lambda g^{1/2} u \ = \ 0 \quad \text{in } \Omega,$$
$$u|_{\partial \Omega} = h,$$

where σ_R are measurable anisotropic conductivities in Ω satisfying

(23) $\sigma_R|_{\Omega \backslash B(0, 5/2)} = \gamma_0, \quad \sigma_R|_{B(0,1)} = 2\gamma_0,$
$$\lim_{R \searrow 1} \sigma_R(x) = \sigma(x), \quad c_1(R-1)\gamma_0 \le \sigma_R(x) \le c_2 \gamma_0, \quad \text{for } x \in \Omega,$$
$$\sigma_{R_2}(x) \le \sigma_{R_1}(x), \quad \text{for } R_2 \ge R_1,$$

for some $c_1 \leq 1/2$, $c_2 \geq 2$. For instance, we can choose

$$(24) \qquad \sigma_R(x) = \begin{cases} F_*\gamma_0, & x \in \Omega \setminus \overline{B}(R), \\ 2\gamma_0, & x \in B(R). \end{cases}$$

We note that, by smoothing the conductivities (24), it is possible to construct conductivities $\sigma_R \in C^2(\overline{\Omega}; \mathbb{R}^2)$, which we use in Sec. 5, satisfying (23).

We denote the solution of (22) by $u = u_R^h$. Note that, for $b = 0$, $q = 0$, equation (22) is an *acoustic* equation with *mass density* σ_R and *bulk modulus* $g^{1/2}$ with $k = \sqrt{\lambda}$; in the quantum mechanical setting, σ_R corresponds to the effective mass and $(q - \lambda)g^{1/2}$ to the potential. By abuse of notation, even for $b \neq 0$ and $q \neq 0$, we will refer to σ_R as the mass density.

Observe that, for each $R > 1$, the mass density σ_R is nonsingular, *i.e.*, is bounded from above and below with, however, the lower bound going to 0 as $R \searrow 1$. Moreover, for any $x \in \mathbb{R}^3$, the symmetric matrix valued function $R \mapsto \sigma_R(x)$ is increasing [5] as a function of R, and therefore decreases as $R \searrow 1$. Nonsingular regularizations or truncations of singular ideal cloaks have previously been considered in [58, 22, 36].

To motivate the treatment here, consider for $R > 1$ the Dirichlet-to-Neumann (DN) map $\Lambda_R^\lambda : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ that maps

$$(25) \qquad \Lambda_R^\lambda : u|_{\partial\Omega} \mapsto (\partial_\nu u + i\nu \cdot bu)|_{\partial\Omega},$$

where u solves (22). The DN map corresponds to the Dirichlet-to-Neumann quadratic form, which by abuse of notation is denoted also by Λ_R^λ,

$$(26) \qquad \Lambda_R^\lambda[h] = \int_{\partial\Omega} (\Lambda_R^\lambda h)(x)\overline{h(x)} \, dS(x),$$

where we denote $h = u|_{\partial\Omega} \in H^{1/2}(\partial\Omega)$; for $\lambda \leq 0$, the Dirichlet-to-Neumann form may be also represented as

$$(27) \qquad \Lambda_R^\lambda[h] = \inf \left(a_R[u, u] - \lambda\|u\|_g^2 \right),$$

where infimum is taken over all $u \in H^1(\Omega)$ with $u|_{\partial\Omega} = h$. However, to treat general λ, and the general class of equations (22), we will use the definition (31) below.

---

[5] Note that, due to the behavior of the eigenvalues of σ_1, this simple but important monotonicity property fails in the 2D case, which we are not treating.

Returning to equation (22), note that for $\lambda < 0$ and $R > 1$ the solution can be obtained from the minimization problem for the quadratic functional associated to the sesquilinear functional

$$(28) \quad a_R[u,v] = \int_\Omega \sigma_R(x)\nabla^b u(x) \cdot \overline{\nabla^b v(x)} \, dx + \int_\Omega q(x)g^{1/2}(x)u(x)\overline{v(x)} \, dx$$

Moreover, we have

$$u_R^h = \operatorname{argmin}\left(a_R[u,u] - \lambda \|u\|_g^2 \right),$$

where minimization is taken over $u \in H^1(\Omega)$ such that $u|_{\partial\Omega} = h$. Observe that the DN form (27) is related to a_R defined by (28).

Thus, the solution u_R^h of the acoustic equation (22) with the Dirichlet boundary value h is the minimizer of $a_R[u,u] - \lambda \|u\|_g^2$.

Now consider the quadratic form $a_R[u,u]$ restricted to $u \in H_0^1(\Omega)$.

Lemma 2.2. *For $R > 1$, the sesquilinear form (28) with domain $\mathcal{D}(a_R) = H_0^1(\Omega)$, is closed and positive definite on $L_g^2(\Omega)$.*

Proof. Clearly a_R is non-negative. The fact that a_R is closed on $\mathcal{D}_R = H_0^1(\Omega)$ follows from the same considerations as those in Lemma 2.1. Next we show that $\nabla^b u = 0$ iff $u = 0$ which, due to the compactness of the embedding $H_0^1(\Omega) \hookrightarrow L_g^2$, implies the positive-definiteness of a_R. Assume, on the contrary, that there is a non-zero $\psi \in H_0^1(\Omega)$ such that $\nabla^b \psi = 0$ in Ω. Continue b and ψ by 0 to $\mathbb{R}^3 \setminus \Omega$. and also extend σ_R to $\mathbb{R}^3 \setminus \Omega$ as $\gamma_0 = 1$ in $\mathbb{R}^3 \setminus B(3)$. Then $\psi \in H^1(\mathbb{R}^3)$ and $\nabla^b \psi = 0$ in $\mathbb{R}^3$, and hence $\nabla^b \cdot \nabla^b \psi = 0$ in $\mathbb{R}^3 \setminus \{O\}$. Using unique continuation for elliptic equations with non-smooth first order terms, see [4, 34], we deduce that $\psi = 0$ in $\mathbb{R}^3 \setminus \{O\}$. $\square$

As σ_R is bounded from below, Lemma 2.2 implies that there is $c_R > 0$ such that

$$(29) \qquad \int_\Omega \sigma_R \nabla^b u \cdot \overline{\nabla^b u} \, dx \geq c_R \|u\|_{L^2(\Omega)}^2, \quad u \in H_0^1(\Omega).$$

Similarly to (19), (20), the sesquilinear form a_R defines a self-adjoint operator $\mathcal{A}_R$ in L_g^2, and, analogously to (21), we have

$$(30) \qquad \mathcal{A}_R(u) := -g^{-1/2}\nabla^b \cdot (\sigma_R \nabla^b u) + qu,$$
$$\mathcal{D}(\mathcal{A}_R) = \{u \in H_0^1(\Omega): \ \nabla^b \cdot (\sigma_R \nabla^b u) \in L^2(\Omega, g^{-1/2}dx)\}.$$

Using the operator $\mathcal{A}_R$, $R > 1$ we see that, for $\lambda \notin \text{spec}(\mathcal{A}_R)$,

$$(31) \qquad \Lambda_R^\lambda h = (\partial_\nu + i\nu \cdot b)\left(u_0 + v_R^\lambda\right)|_{\partial\Omega};$$

$$v_R^\lambda := \mathcal{R}_R(\lambda)\left(\nabla^b \cdot (\sigma_R \nabla^b u_0) - qu_0 + \lambda u_0\right).$$

Here $u_0 \in H^1(\Omega)$ with $\text{supp}(u_0) \subset \Omega \setminus \overline{B(O,2)}$ satisfies $u_0|_{\partial\Omega} = h$, while $\mathcal{R}_R(\lambda)$ is the resolvent,

$$(32) \qquad \mathcal{R}_R(\lambda) = (\mathcal{A}_R - \lambda I)^{-1} : H^{-1}(\Omega) \to L^2(\Omega).$$

We will use the notation (32) for $R = 1$ as well as $R > 1$, in which case $\mathcal{R}_1(\lambda) : H_g^{-1}(\Omega) \to L_g^2(\Omega)$. Note that the right-hand side in (31) should be understood in the sense of the pairing $H^{-1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$,

$$\int_{\partial\Omega} ((\partial_\nu + i\nu \cdot b)(u_0 + v_R^\lambda))\overline{\psi}\, dS = a_R[u_0 + v_R^\lambda,\, \psi] - \lambda(u_0 + v_R^\lambda, \psi)_{L_g^2},$$

for all $\psi \in H^1(\Omega)$, and we again consider a_R on the whole of $H^1(\Omega)$.

Next, consider the DN map in the ideal case $R = 1$. Following [20], we say that u is a *finite energy solution* of the boundary value problem (10) with $h \in H^{1/2}(\partial\Omega)$ and $f \in H_g^{-1}(\Omega)$, if $u \in H_g^1(\Omega)$ and

$$(33) \quad a_1[u - u_0, \phi] = -\int_\Omega (\sigma_1 \nabla^b u_0 \cdot \overline{\nabla^b \phi} + (qg^{1/2}u_0 - \lambda g^{1/2}u - f)\overline{\phi}\, dx,$$

for every $\phi \in H_{0,g}^1(\Omega)$. Here $H_g^{-1}(\Omega)$ is the dual space to $H_{0,g}^1(\Omega)$.

On the other hand,

$$(34) \qquad -(\nabla + i\beta_1) \cdot (\nabla + i\beta_1)v_1 + \kappa_1 v_1 - \lambda v_1 = \widetilde{f}_1, \quad \text{in } M_1,$$

$$v_1|_{\partial M_1} = h,$$

$$(35) \qquad -\frac{1}{4}(\nabla + i\beta_2) \cdot (\nabla + i\beta_2)v_2 + \kappa_2 v_2 - \lambda v_2 = \widetilde{f}_2, \quad \text{in } M_2,$$

$$(\partial_\nu + i\nu \cdot \beta_2)v_2|_{\partial M_2} = 0,$$

are satisfied in the *weak sense* if

$$(36) \quad \alpha[v - v_0, \psi] = -\sum_{j=1}^2 \int_{M_j} (c_j \nabla^\beta v_0 \cdot \overline{\nabla^\beta \psi} + (\kappa_j u_0 - \lambda v - \widetilde{f}_j)\overline{\psi})\, dx,$$

for all $\psi \in H_0^1(M_1) \oplus H^1(M_2)$, where $c_1 = 1$, $c_2 = \frac{1}{4}$. Here $v_0 \in H^1(M_1) \oplus H^1(M_2)$ is supported in $M_1 \setminus \overline{B(2)}$ and satisfies $v_0|_{\partial M_1} = h$.

Lemma 2.3. *Let $h \in H^{1/2}(\partial\Omega)$ and $f \in H_g^{-1}(\Omega)$. A function $u \in H_g^1(\Omega)$ is a finite energy solution in the sense (33) of the boundary value problem (10) if and only if $v = (v_1, v_2) \in H^1(M_1) \oplus H^1(M_2)$, $u = F_* v$ satisfies equations (34) and (35) in the weak sense (36) with $f = F_*(\widetilde{f}_1, \widetilde{f}_2)$.*

In particular, for any $\lambda \in \mathbb{R}$ the Cauchy data, on $\partial\Omega$, of solutions to (22) satisfy

$$(37) \quad \{(u|_{\partial\Omega}, (\partial_\nu + i\nu \cdot b)u|_{\partial\Omega}) : -\nabla^b \cdot \sigma_1 \nabla^b u + q g^{1/2} u = g^{1/2} \lambda u \text{ in } \Omega\}$$
$$= \{(v_1|_{\partial\Omega}, (\partial_\nu + i\nu \cdot \beta_1)v_1|_{\partial\Omega}) : -\nabla^{\beta_1} \cdot \nabla^{\beta_1} v_1 + \kappa_1 v_1 = \lambda v_1 \text{ in } \Omega\}$$
$$\subset H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$$

Proof. By Lemma 2.1 and formulae (13) and (14) we see that $u \in \mathcal{D}(a_1)$ if and only if $u = F_* v$, $v = (v_1, v_2) \in \mathcal{D}(\alpha)$, and

$$(38) \quad a_1[\widetilde{\psi}, \widetilde{\phi}] = \alpha[\psi, \phi] \quad \text{for } \phi, \psi \in \mathcal{D}(\alpha), \widetilde{\phi} = F_*\phi, \widetilde{\psi} = F_*\psi.$$

Let $v_0 \in H^1(M_1) \oplus H^1(M_2)$ be supported in $M_1 \setminus \overline{B(2)}$ and satisfy $v_0|_{\partial M_1} = h$, and $u_0 \in H^1(\Omega)$ be such that $u_0 = F_* v_0$. Using formula (38) with $\widetilde{\psi} = u - u_0$ and $\psi = v - v_0$, we see that $u \in H^1_g(\Omega)$ is a finite energy solution if and only if $v = (v_1, v_2) \in H^1(M_1) \oplus H^1(M_2)$, $u = F_* v$ satisfies (36) for all $\phi \in H^1_0(M_1) \oplus H^1(M_2)$, that is, v satisfies equations (34) and (35) in the weak sense. $\qquad\square$

Assume next that $\lambda \notin \mathrm{spec}(\mathcal{A}_1)$. Then the solution to equations (10), in the sense of definition (36), may be found in terms of the resolvent $\mathcal{R}_1(\lambda)$ of $\mathcal{A}_1$, *cf.* (32). Indeed, comparing (19), (20) with equation (33), we see that its solution u has the form

$$(39) \quad u = u_0 + \mathcal{R}_1(\lambda)\left(f + \nabla^b \cdot \sigma_1 \nabla^b u_0 - q u_0 + \lambda u_0\right),$$
$$u_0|_{\partial\Omega} = h, \ \mathrm{supp}(u_0) \subset \Omega \setminus \overline{B(2)},$$

at least when $f \in L^2_g(\Omega)$ and $h \in H^{3/2}(\partial\Omega)$ so that $u_0 \in H^2(\Omega)$. Since $\mathcal{D}(a_1) = \mathcal{D}(\mathcal{A}_1^{1/2}) = H^1_{0,g}(\Omega)$, we see that $H^{-1}_g(\Omega) = \mathcal{D}(\mathcal{A}_1^{-1/2})$. Therefore, the operator $\mathcal{R}_1(\lambda)$ can be extended by continuity to a bounded operator from $H^{-1}_g(\Omega)$ onto $H^1_{0,g}(\Omega)$. This makes it possible to generalize (39) for all $u_0 \in H^1_g(\Omega)$, i.e. $h \in H^{1/2}(\partial\Omega)$, and $f \in H^{-1}_g(\Omega)$. Observe that the right-hand side in (37) is related to the unbounded selfadjoint operator A_{out} in $L^2(\Omega)$, $\mathcal{D}(A_{out}) \subset H^1_0(\Omega)$, associated with the form α_1, see (14) where we use $\Omega = M_1$. More precisely, this operator is the unbounded selfadjoint operator in $L^2(\Omega)$ given by

$$(40) \quad A_{out} = -\nabla^{\beta_1} \cdot \nabla^{\beta_1} + \kappa_1,$$
$$\mathcal{D}(A_{out}) = \{v_1 \in H^1(\Omega) : \nabla^{\beta_1} \cdot \nabla^{\beta_1} v_1 \in L^2(\Omega), \ v_1|_{\partial\Omega} = 0\};$$

Moreover, when $b \in C^1(\overline{\Omega}; \mathbb{R}^3)$, the selfajdoint operator associated to the form α_2 on $B(1) = M_2$ is the operator

$$(41) \qquad A_{in} = -\frac{1}{4} \nabla^{\beta_2} \cdot \nabla^{\beta_2} + \kappa_2,$$

$$\mathcal{D}(A_{in}) = \{v_2 \in H^2(B(1)): \ \partial_\nu v|_{\partial B(1)} = 0\}.$$

When $\lambda \notin \mathrm{spec}(A_{out})$, the set (37) coincides with the graph of the DN-map

$$(42) \qquad \Lambda_{out}^\lambda : v_1|_{\partial\Omega} \mapsto (\partial_\nu + i\nu \cdot \beta_1)v_1|_{\partial\Omega},$$

where v_1 solves equation (34) with $\widetilde{f}_1 = 0$.

Note that if $b = 0$, $q = 0$, then $\lambda = 0$ is an eigenvalue of $\mathcal{A}_1$ with the corresponding eigenfunctions of the form (*cf.* [29, 20])

$$(43) \qquad u(x) = \begin{cases} 0, & \text{for } x \in \Omega \setminus \overline{B}(1), \\ c_0 & \text{for } \quad x \in B(1), \end{cases} \quad c_0 \neq 0.$$

However, as follows from Lemma 2.3 even, in this case the Cauchy data on $\partial\Omega$ of solutions of the equation (10) with $\lambda = 0$ coincide with the Cauchy data of the solutions of $\Delta v_1 = 0$ on $\partial M_1 = \partial\Omega$.

2.3. $\Gamma-$**convergence and spectral convergence.**
In this section we establish $\Gamma-$convergence and spectral convergence results for a_R as $R \searrow 1$. To that end, following the comment in the previous section, observe that since $\sigma_{R_1} \geq \sigma_{R_2}$ for $R_1 \geq R_2$, one thus has

$$(44) \qquad a_{R_1}[v, v] \geq a_{R_2}[v, v], \ v \in H^1(\Omega), \quad \text{if } R_1 \geq R_2.$$

This implies that $R \mapsto a_R[v, v]$ is decreasing as $R \searrow 1$. For $R > 1$, consider non-linear (quadratic) functionals $a_R : L_g^2(\Omega) \mapsto \mathbb{R}^+ := \mathbb{R} \cup \{+\infty\}$,

$$(45) \qquad a_R(v) = \begin{cases} a_R[v, v] & \text{when } v \in H_0^1(\Omega), \\ \infty & \text{otherwise.} \end{cases}$$

For the ideal cloak, *i.e.*, $R = 1$, define

$$(46) \qquad a_1(v) = \begin{cases} a_1[v, v] & \text{when } v \in H_{0,g}^1(\Omega), \\ \infty & \text{otherwise.} \end{cases}$$

We will make extensive use of De Giorgi's $\Gamma-$convergence, see, *e.g.*, [3, 16].

Definition 2.4. *Let $\{J_R : 1 \leq R \leq 2\}$ be a family of functionals on a Hilbert space $\mathcal{H}$. We say that the J_R Γ-converge to J_1, or $J_1 = \Gamma-\lim_{R \searrow 1} J_R$ on $\mathcal{H}$, if*

(i) for every $v \in \mathcal{H}$, and all sequences v_R converging to v in $\mathcal{H}$ as $R \searrow 1$, $J_1(v) \leq \liminf_{R \searrow 1} J_R(v_R)$; and

(ii) for every $w \in \mathcal{H}$ there exists a sequence w_R converging to w in $\mathcal{H}$ such that $J_1(w) \geq \limsup_{R \searrow 1} J_R(w_R)$.

We then have the following result:

Lemma 2.5. *The functionals a_R Γ-converge to a_1 as $R \searrow 1$,*

$$(47) \qquad\qquad \Gamma-\lim_{R \searrow 1} a_R = a_1, \quad on\ L_g^2(\Omega).$$

Proof. By Lemma 2.1, $H_{0,g}^1(\Omega)$ is a Hilbert space, when endowed with the norm $(\|u\|_{L_g^2}^2 + a_1(u))^{1/2}$. Since the functionals a_{σ_R} are pointwise decreasing as $R \searrow 1$, it follows from [16, Prop 5.7] that the functionals a_R $\Gamma-$converge on L_g^2 to the

$$(48) \qquad\qquad \widetilde{a}_1 = \mathrm{sc}^- G,$$

that is, the lower semicontinuous envelope of G (see [16, Def. 3.1]), where $G : L_g^2 \to \mathbb{R}^+ := \mathbb{R} \cup \{+\infty\}$ is defined by

$$(49)\ G(v) = \begin{cases} \int_\Omega (\sigma_1 \nabla^b u \cdot \overline{\nabla^b u}\, dx + qg^{1/2}|u|^2)\, dx, & \text{for } u \in H_0^1(\Omega), \\ \infty, & \text{otherwise.} \end{cases}$$

By [16, Prop 11.10], the function $\widetilde{a}_1 : L_g^2(\Omega) \to \mathbb{R} \cup \{\infty\}$ is a quadratic form. Moreover, by [16, Prop 12.16], its domain

$$\mathcal{D}(\widetilde{a}_1) = \{u \in L_g^2(\Omega) : \widetilde{a}_1(u) < \infty\},$$

endowed with the norm $(\|u\|_g^2 + \widetilde{a}_1(u))^{1/2}$, is a Hilbert space.

Now $H_0^1(\Omega)$ is contained in both $\mathcal{D}(\widetilde{a}_1)$ and $\mathcal{D}(a_1) = H_{0,g}^1(\Omega)$, and the norms of these Hilbert spaces coincide on $H_0^1(\Omega)$. Moreover, by the proof of [20, Lem. 3.3], $H_0^1(\Omega)$ is dense in $\mathcal{D}(a_1)$. Thus $\mathcal{D}(a_1) \subset \mathcal{D}(\widetilde{a}_1)$.

On the other hand, as $G(v) \geq a_1(v)$ for all $v \in L_g^2(\Omega)$ and a_1 is lower semicontinuous (see [16, Prop. 2.16]), it follows that the lower semicontinuous envelope $\widetilde{a}_1$ of G also satisfies $\widetilde{a}_1(v) \geq a_1(v)$. Hence $\mathcal{D}(\widetilde{a}_1) \subset \mathcal{D}(a_1)$. Thus, $\mathcal{D}(\widetilde{a}_1) = \mathcal{D}(a_1)$. $\qquad\square$

Let us next consider the resolvent $\mathcal{R}_R(\lambda)$ for $\lambda < 0$. The following result then holds:

Lemma 2.6. *For any $\lambda < 0$, the resolvents $\mathcal{R}_R(\lambda)$, $R > 1$, strongly converge on $L_g^2(\Omega)$ to $\mathcal{R}_1(\lambda)$, i.e., for any $f \in L_g^2(\Omega)$,*

$$(50) \qquad\qquad \lim_{R \searrow 1} \mathcal{R}_R(\lambda)f = \mathcal{R}_1(\lambda)f$$

strongly in $L_g^2(\Omega)$ and weakly in $H_{0,g}^1(\Omega)$.

Proof. The quadratic forms $a_R(u) - \lambda\|u\|_g^2$, $R \geq 1$ in the Hilbert space $L_g^2(\Omega)$ are associated to the unbounded selfadjoint operators $\mathcal{A}_R - \lambda I$. Thus we can use [16, Thm. 13.6] to show that the resolvents $(\mathcal{A}_R - \lambda I)^{-1}$ satisfy (50) in $L_g^2(\Omega)$. Indeed, to show the strong convergence (50) in $L_g^2(\Omega)$ it is sufficient to prove the following three properties, (51), (52) and (53):

(51) $a_R : L_g^2(\Omega) \to \mathbb{R} \cup \{\infty\}$ are lower semicontinuous;

(52) $\Gamma{-}\lim\limits_{R \searrow 1} a_R = a_1 \quad$ on $L_g^2(\Omega)$;

(53) $a_1(u) - \lambda\|u\|_g^2 \leq \liminf\limits_{R \searrow 1} \left(a_R(u_R) - \lambda\|u_R\|_g^2\right)$ as $u_R \rightharpoonup u$ in $L_g^2(\Omega)$,

where $\rightharpoonup$ denotes weak convergence in $L_g^2(\Omega)$.

Clearly, the quadratic forms $a_R(u) - \lambda\|u\|_g^2$ are lower semicontinuous on $L_g^2(\Omega)$, see proof of Lemma 2.5. By Lemma 2.5, the quadratic forms a_R Γ−converge to a_1. Thus (51) and (52) are valid. To see (53), it suffices to consider the case when $u_R \rightharpoonup u$ in L_g^2 and

$$\liminf_{R \searrow 1} \left(a_R(u_R) - \lambda\|u_R\|_g^2\right) < \infty.$$

Next we consider a sequence $R^k \searrow 1$ such that

$$\liminf_{R \searrow 1} \left(a_R(u_R) - \lambda\|u_R\|_g^2\right) = \lim_{k \to \infty} \left(a^k(u^k) - \lambda\|u^k\|_g^2\right),$$

where $a^k = a_{R^k}$ and $u^k = u_{R^k} \in H_0^1(\Omega)$. Since u^k converges weakly in L_g^2, the norms $\|u^k\|_g$ are uniformly bounded. Since also

(54) $$a_1(u^k) \leq a^k(u^k),$$

we see that the sequence u^k is uniformly bounded in $H_g^1(\Omega)$. Let us now choose a subsequence of u^k (still denoted by u^k) which converges weakly to u in $H_{0,g}^1(\Omega)$. As the embedding $H_g^1 \hookrightarrow L_g^2$ is compact, this sequence converges strongly in $L_g^2(\Omega)$. Using the weak convergence in $H_{0,g}^1(\Omega)$, we see that

(55) $a_1(u) - \lambda\|u\|_g^2 = \lim\limits_{k \to \infty} \int_\Omega \left(\sigma_1 \nabla^b u^k \cdot \overline{\nabla^b u} + g^{1/2} q u^k \overline{u} - \lambda g^{1/2} u^k \overline{u}\right) dx.$

By the Cauchy-Schwarz inequality,

$$\left| \int_\Omega \left(\sigma_1 \nabla^b u^k \cdot \overline{\nabla^b u} + g^{1/2} q u^k \overline{u} - \lambda g^{1/2} u^k \overline{u} \right) dx \right|$$

$$\leq \left(\int_\Omega \left(\sigma_1 \nabla^b u^k \cdot \overline{\nabla^b u^k} + g^{1/2} q |u^k|^2 - \lambda g^{1/2} |u^k|^2 \right) dx \right)^{\frac{1}{2}} \cdot$$

$$\cdot \left(\int_\Omega \left(\sigma_1 \nabla^b u \cdot \overline{\nabla^b u} + g^{1/2} q |u|^2 - \lambda g^{1/2} |u|^2 \right) dx \right)^{\frac{1}{2}} .$$

Using (54) and (55), this implies that

$$a_1(u) - \lambda \|u\|_g^2 \leq \liminf_{k\to\infty} \left(a_1(u^k) - \lambda \|u^k\|_g^2 \right)^{1/2} \cdot \left(a_1(u) - \lambda \|u\|_g^2 \right)^{1/2}$$

$$\leq \liminf_{k\to\infty} \left(a_k(u^k) - \lambda \|u^k\|_g^2 \right)^{1/2} \cdot \left(a_1(u) - \lambda \|u\|_g^2 \right)^{1/2} .$$

The desired inequality (53) follows immediately, proving (50) in $L_g^2(\Omega)$.

Finally, to prove the (50) holds weakly in $H_{0,g}^1(\Omega)$, observe that, since $\sigma_R \geq \sigma_1$,

$$(56) \quad \int_\Omega (\sigma_R \nabla^b v \cdot \overline{\nabla^b v} + g^{1/2} q |v|^2) \, dx \geq \int_\Omega (\sigma_1 \nabla^b v \cdot \overline{\nabla^b v} + g^{1/2} q |v|^2) \, dx$$

for $v \in H_0^1(\Omega)$. On the other hand, denoting $u_R = \mathcal{R}_R(\lambda) f$ and using $\lambda \leq 0$, we have

$$\int_\Omega (\sigma_R \nabla^b u_R \cdot \overline{\nabla^b u_R} + g^{1/2} q |u_R|^2) \, dx \leq \|f\|_g \|u_R\|_g.$$

The above two inequalities, together with the strong convergence (50) in $L_g^2(\Omega)$, show that the $\|u_R\|_{H_g^1}$ are uniformly bounded. Thus, if weak convergence (50) in $H_{0,g}^1(\Omega)$ is not valid, there is a $v \neq u$ and a subsequence $R_n \searrow 1$ such that, $u_{R_n} \rightharpoonup v$ in H_g^1. Then $u_{R_n} \to v$ in $L_g^2(\Omega)$, which is in contradiction with the strong convergence (50) in $L_g^2(\Omega)$. Thus (50) holds weakly in $H_{0,g}^1(\Omega)$. $\square$

In some applications, *e.g.*, dealing with scattering of plane waves $e^{i<k,x>}$, $k \in \mathbb{R}^3$, by the cloaking device, the main interest concerns not values $\lambda < 0$, but rather $\lambda = |k|^2 > 0$. To analyze this case, let us first consider the behavior of the spectra, spec($\mathcal{A}_R$), as $R \searrow 1$.

Lemma 2.7. *Let K be a compact set with $K \cap spec(\mathcal{A}_1) = \emptyset$. Then, for R sufficiently close to 1, $K \cap spec(\mathcal{A}_R) = \emptyset$.*

Proof. It suffices to consider $K = [a, b] \subset \mathbb{R}$. Assume, to the contrary, the existence of a sequence $R_n \searrow 1$, $\mu_n \in [a, b]$, and functions $u_n \in$

$H^1_0 \subset H^1_{0,g}$, $\|u_n\|_g = 1$, such that

(57) $$\mathcal{A}_{R_n} u_n = \mu_n u_n.$$

Then,

$$\int_\Omega \left(\sigma_{R_n} \nabla^b u_n \cdot \overline{\nabla^b u_n} + g^{1/2} q |u_n|^2 \right) dx = \mu_n \int_\Omega g^{1/2} |u_n|^2 \, dx = \mu_n \le b.$$

Therefore, as $\sigma_R \ge \sigma_1$ and $q \ge 0$, this implies $\|u_n\|^2_{H^1_g} \le b + 1$. Thus there exists a subsequence of u_n and μ_n, which we relabel as the original sequence, $u \in H^1_{0,g}$ and $\mu \in [a, b]$, such that

$$u_n \rightharpoonup u \quad \text{weakly in } H^1_g; \quad u_n \to u \quad \text{strongly in } L^2_g; \quad \text{and} \ \mu_n \to \mu$$

as $n \to \infty$. Thus, in particular, $\|u\|_g = 1$.

Taking, *e.g.*, $\lambda = -1$ in Lemma 2.6, we know that $\mathcal{R}_R(-1) \to \mathcal{R}_1(-1)$ as $R \searrow 1$ in the strong operator topology on L^2_g. Consider $\mathcal{R}_1(-1)u$; then

$$\mathcal{R}_1(-1)u = \lim_{n\to\infty} \mathcal{R}_{R_n}(-1)u = \lim_{n\to\infty} \mathcal{R}_{R_n}(-1)u_n, \quad \text{in } L^2_g(\Omega)$$

where, in the last step we have used fact that $\mathcal{A}$ are non-negative operators yielding the estimate $\|\mathcal{R}_{R_n}(-1)\|_{L^2_g \to L^2_g} \le 1$.

However, taking into account (57),

$$\mathcal{R}_{R_n}(-1)u_n = \frac{1}{1+\mu_n} u_n \to \frac{1}{1+\mu} u \quad \text{in } L^2_g(\Omega) \text{ as } n \to \infty.$$

Thus, $\mathcal{R}_1(-1)u = (1+\mu)^{-1}u$ with $\|u\|_{L^2_g} = 1$, implying that $\mu \in \text{spec}(\mathcal{A}_1)$ with u being an associated eigenfunction. This contradiction proves the statement. $\qquad\square$

Lemma 2.8. *Let $K \subset \mathbb{C}$ be compact subset such that $K \cap spec(\mathcal{A}_1) = \emptyset$. Then, for any $f \in L^2_g(\Omega)$ and $\lambda \in K$,*

(58) $$\lim_{R\searrow 1} \mathcal{R}_R(\lambda)f = \mathcal{R}(\lambda)f$$

strongly in $H^1_{0,g}(\Omega)$, and the convergence is uniform for $\lambda \in K$.

Proof. Let $\delta > 0$ satisfies

$$K_\delta \cap \text{spec}(\mathcal{A}_1) = \emptyset, \quad K_\delta = \{z \in \mathbb{C} : \text{dist}(z, K) \le \delta\}.$$

It then follows from Lemma 2.7 that, for R sufficiently close to 1, $K_{\delta/2} \cap \text{spec}(\mathcal{A}_R) = \emptyset$. As all $\mathcal{A}_R$, $R \ge 1$ are self-adjoint in $L^2_g(\Omega)$, we see that, for R sufficiently close to 1,

$$\|\mathcal{R}_R(\lambda)\|_{L^2_g \to L^2_g} \le \frac{2}{\delta}, \quad \text{when } \lambda \in K.$$

This implies that $\mathbb{C} \setminus \mathrm{spec}(\mathcal{A}_1) = \Delta_b$, where Δ_b denotes the *region of boundedness* for the family of operators $\mathcal{A}_R$, $R \geq 1$, *i.e.*, the set of those $\lambda \in \mathbb{C}$ for which the norms $\|(\mathcal{A}_R - \lambda)^{-1}\|_{L_g^2 \to L_g^2}$ are bounded by some constant $C_\lambda > 0$ for all $R > 1$, see [33, Sec. VIII.1.1]. As $\mathrm{spec}(\mathcal{A}_1)$ is countable, Δ_b is connected. On the other hand, by Lemma 2.6, $\mathbb{R}_- \subset \Delta_s$, where Δ_s is the *region of strong convergence* for the above family, *i.e.*, the set of $\lambda \in \mathbb{C}$ such that for $f \in L_g^2(\Omega)$

(59) $$\lim_{R \to 1} \mathcal{R}_R(\lambda)f = \mathcal{R}_1(\lambda)f \quad \text{in } L_g^2(\Omega),$$

Therefore, by [33, Thm. VIII.1.2],

(60) $$\Delta_s = \Delta_b = \mathbb{C} \setminus \mathrm{spec}(\mathcal{A}_1).$$

By [20, Lemma 3], $H_0^1(\Omega)$ is dense in $H_{0,g}^1(\Omega) = \mathcal{D}(a_1)$. Thus $H_0^1(\Omega)$ is a core of the quadratic form $a_1[\cdot, \cdot]$. Now $H_0^1(\Omega) = \mathcal{D}(a_R)$, for $R > 1$ and a_R are monotonically increasing with $R \geq 1$ on $H_0^1(\Omega)$. Thus it follows from [33, Th. VIII.3.6], that

(61) $$\lim_{R \searrow 1} a_1(\mathcal{R}_R(\lambda)f - \mathcal{R}_1(\lambda)f) = 0,$$

uniformly for $\lambda \in K$ where K is an arbitrary compact subset of Δ_s. By Lemma 2.1, the desired convergence (58) now follows from (59)–(61). $\square$

Let $\mu \notin \mathrm{spec}(\mathcal{A}_1)$. It follows from Lemma 2.7 that $\mu \notin \mathrm{spec}(\mathcal{A}_R)$ for $R > 1$ sufficiently close to 1. For $\mu \notin \mathrm{spec}(\mathcal{A}_R)$, we denote by $N_R(\mu)$ the subspace of $L_g^2(\Omega)$ spanned by the eigenfunctions of $\mathcal{A}_R$ with eigenvalues $\lambda_j < \mu$. We also denote by P_R^μ, the orthoprojectors in $L_g^2(\Omega)$ onto $N_R(\mu)$. By [33, Th. III.6.17], these (Riesz) projectors P_R have the representation

(62) $$P_R^\mu u = \frac{1}{2\pi i} \int_\Gamma (\mathcal{A}_R - z)^{-1} u \, dz,$$

where the contour $\Gamma \subset \mathbb{C}$ surrounds all the eigenvalues λ_j of $\mathcal{A}_R$ satisfying $\lambda_j < \mu$ and only those.

Lemma 2.9. *Let $\mu \notin spec(\mathcal{A}_1)$. For R sufficiently close to 1*

(63) $$dim(N_R(\mu)) = dim(N_1(\mu)).$$

Moreover,

(64) $$\lim_{R \searrow 1} \|P_R^\mu - P_1^\mu\|_{L_g^2 \to L_g^2} = 0.$$

Proof. Recall that $\mathcal{D}(a_R) = H_0^1(\Omega)$ are independent of $R > 1$, $H_0^1(\Omega) \subset \mathcal{D}(a_1) = H_{0,g}^1(\Omega)$, and $a_R[u,u]$ are decreasing, as $R \searrow 1$, for all $u \in \mathcal{D}(a_R)$. Thus the identity (63) follows directly by [33, Th. VIII.3.15].

Using representation (62) and Lemma 2.8, we see that

$$\lim_{R \searrow 1} P_R^\mu = P_1^\mu \quad \text{strongly in } L_g^2(\Omega).$$

As P_R^μ and P_1^μ are orthoprojectors, this and (63) yields (64) by [33] (see Lemmas VIII.1.23 and VIII.1.24). □

We remark that in the course of this paper we need a number of results concerning convergence of orthoprojectors that appear similar to (64), *e.g.*, (78), (114) and (133), but these are for different operators or with respect to different operator norms and require separate proofs.

2.4. **Approximating the singular bulk modulus $g^{1/2}$ by nonsingular densities.** Above, in the operator $\mathcal{A}_R = -g^{-1/2}\nabla^b \cdot \sigma_R \nabla^b + q$ there appears the determinant g of the metric (also denoted g!), which vanishes at the cloaking surface Σ. We now consider how to approximate the scalar function g by functions g_m that are bounded from below with positive constants. To this end, we introduce the functions

$$(65) \qquad g_m(x) = \max\left(g(x), 1/m\right), \quad m \in \mathbb{Z}_+.$$

Then $L^2(\Omega, g_m^{1/2}dx) \subset L_g^2(\Omega)$ and

$$(66) \qquad \|f\|_g \leq \|f\|_{L^2(\Omega, g_m^{1/2}dx)}, \quad \text{for } f \in L^2(\Omega, g_m^{1/2}dx).$$

The multiplication map $f \mapsto g^{1/2}f$ is unitary from $L_g^2 = L^2(\Omega, g^{\frac{1}{2}}dx)$ onto $L^2(\Omega, g^{-\frac{1}{2}}dx)$. Note that $L^2(\Omega, g^{-\frac{1}{2}}dx) \subset L^2(\Omega) \subset L_g^2(\Omega)$. Next we will consider operators $g^{1/2}\mathcal{A}_R u$. For $f \in L_g^2(\Omega)$, we have

$$(67) \qquad (\mathcal{A}_R - \lambda)u = f,$$

where both sides are in $L_g^2(\Omega)$, if and only if u is a solution to the acoustic equation with mass density tensor σ_R, bulk modulus $g^{1/2}$, and potential $qg^{1/2}$),

$$(68) \qquad (g^{1/2}\mathcal{A}_R - \lambda g^{1/2})u = F,$$

where $F = g^{1/2}f \in L^2(\Omega, g^{-\frac{1}{2}}dx)$. By the above considerations, we have for $f \in L_g^2(\Omega)$

$$(69) \qquad (\mathcal{A}_R - \lambda)^{-1}f = (g^{1/2}\mathcal{A}_R - \lambda g^{1/2})^{-1}(g^{1/2}f).$$

Later in this section we keep $R > 1$ fixed. Define an unbounded self-adjoint operator $\mathcal{B}_R$ in $L^2(\Omega)$, having the same differential expression as the operator $g^{1/2}\mathcal{A}_R$, but with different domain,

$$(70) \qquad \mathcal{B}_R u := -\nabla^b \cdot \left(\sigma_R \nabla^b u\right) + g^{1/2} q u,$$

$$\mathcal{D}(\mathcal{B}_R) = \{u \in H_0^1(\Omega) : \nabla^b \cdot \left(\sigma_R \nabla^b u\right) \in L^2(\Omega)\}.$$

Since $\mathcal{D}(\mathcal{A}_R) \subset H_0^1(\Omega)$, see (30), $\mathcal{B}_R$ is an extension of $g^{1/2}\mathcal{A}_R$ and, in particular, $\nabla^b \cdot (\sigma_R \nabla^b u) \in L^2(\Omega, g^{-\frac{1}{2}}dx)$ for $u \in \mathcal{D}(\mathcal{A}_R)$; however, $\nabla^b \cdot (\sigma_R \nabla^b u) \in L^2(\Omega)$ for $u \in \mathcal{D}(\mathcal{B}_R)$. Note that by (69),

$$(71) \quad (\mathcal{A}_R - \lambda)^{-1} f = (\mathcal{B}_R - \lambda g^{1/2})^{-1} (g^{1/2} f) \quad \text{for } f \in L_g^2(\Omega),$$

where $\lambda \notin \text{spec}(\mathcal{A}_R)$. We will use this formula extensively later for $f \in L^2(\Omega) \subset L_g^2(\Omega)$.

Lemma 2.10. *The operator $\mathcal{B}_R - \lambda g^{1/2}$ has a bounded inverse if and only if $\lambda \notin spec(\mathcal{A}_R)$.*

Proof. For $\lambda < 0$ the operator $\mathcal{B}_R - \lambda g^{1/2}$ is strictly positive and, since $\mathcal{D}(\mathcal{B}_R) \subset H_0^1(\Omega)$, has a compact resolvent. Therefore, the operator $(\mathcal{B}_R - \lambda g^{1/2})^{-1}$ exists for $\lambda < 0$ and is bounded in $L^2(\Omega)$. Since the multiplication, $u \mapsto g^{1/2}u$, is bounded in $L^2(\Omega)$, by the analytic Fredholm theory [56] the operator-valued function

$$\lambda \mapsto (\mathcal{B}_R - \lambda g^{1/2})^{-1} = \left[I - (\lambda + 1)(\mathcal{B}_R + g^{1/2})^{-1} g^{1/2}\right]^{-1} (\mathcal{B}_R + g^{1/2})^{-1}$$

is a meromorphic operator-valued function of $\lambda \in \mathbb{C}$. Therefore, if the inverse $(\mathcal{B}_R - \lambda g^{1/2})^{-1}$ does not exist for a given $\lambda \geq 0$, then there is $v \in H_0^1(\Omega)$ such that

$$(\mathcal{B}_R - \lambda g^{1/2})v = 0.$$

In this case $\mathcal{B}_R v = \lambda g^{1/2} v \in L^2(\Omega, g^{-\frac{1}{2}}dx)$ and we see that $v \in \mathcal{D}(\mathcal{A}_R)$, i.e., $\lambda \in \text{spec}(\mathcal{A}_R)$. On the other hand, if $\lambda \in \text{spec}(\mathcal{A}_R)$, i.e.

$$-g^{-1/2}\nabla^b \cdot \left(\sigma_R \nabla^b u\right) + qu = \lambda u,$$

then $\nabla^b \cdot \left(\sigma_R \nabla^b u\right) = qg^{1/2}u - \lambda g^{1/2}u \in L^2(\Omega)$, i.e., $u \in \mathcal{D}(\mathcal{B}_R)$, so that $\mathcal{B}_R - \lambda g^{1/2}$ does not have a bounded inverse. $\qquad \square$

Next we consider the uniform convergence of resolvents. To this end we introduce operators $\mathcal{B}_{R,m}$, $m \in \mathbb{Z}_+$, in $L^2(\Omega)$, of the form

$$(72) \quad \mathcal{B}_{R,m} u := -\nabla^b \cdot \left(\sigma_R \nabla^b u\right) + q g_m^{1/2} u,$$

$$\mathcal{D}(\mathcal{B}_{R,m}) = \{u \in H_0^1(\Omega) : \nabla^b \cdot \left(\sigma_R \nabla^b u\right) \in L^2(\Omega)\} = \mathcal{D}(\mathcal{B}_R).$$

The operator $\mathcal{B}_{R,m}$ is associated with the operator $\mathcal{A}_{R,m}$ in the same way that $\mathcal{A}_R$ is associated with $\mathcal{B}_R$, where the operator $\mathcal{A}_{R,m}$ is the self-adjoint operator in $L^2(\Omega, g_m^{1/2} dx)$ defined by

$$(73) \qquad \mathcal{A}_{R,m} u := -g_m^{-1/2} \nabla^b \cdot \left(\sigma_R \nabla^b u \right) + qu,$$
$$\mathcal{D}(\mathcal{A}_{R,m}) = \{ u \in H_0^1(\Omega) : \nabla^b \cdot \left(\sigma_R \nabla^b u \right) \in L^2(\Omega) \}.$$

Note that $\mathcal{D}(\mathcal{A}_{R,m}) = \mathcal{D}(\mathcal{B}_{R,m})$.

Lemma 2.11. *Let $R > 1$ and $K \subset \mathbb{C}$ be compact and such that $K \cap \operatorname{spec}(\mathcal{A}_R) = \emptyset$. Then there is an $m_R \in \mathbb{Z}_+$ such that $K \cap \operatorname{spec}(\mathcal{A}_{R,m}) = \emptyset$ for $m > m_R$, and*

$$(74) \qquad \lim_{m \to \infty} \| (\mathcal{A}_{R,m} - \lambda)^{-1} - (\mathcal{A}_R - \lambda)^{-1} \|_{L^2(\Omega) \to H_0^1(\Omega)} = 0,$$

uniformly for $\lambda \in K$.

Proof. By the assumptions on K, the inverse $(\mathcal{B}_R - \lambda g^{1/2})^{-1}$ exists and is a continuous function of $\lambda \in K$ with respect to the $L^2(\Omega)$-operator norm topology. Let

$$d = \max_{\lambda \in K} \| (\mathcal{B}_R - \lambda g^{1/2})^{-1} \|_{L^2(\Omega) \to L^2(\Omega)} < \infty.$$

Denote $V_{R,m}(\lambda) = (\lambda - q)(g_m^{1/2} - g^{1/2}) \in L^\infty(\Omega)$, so that

$$(75) \qquad \lim_{m \to \infty} \| V_{R,m}(\lambda) \|_{L^\infty(\Omega)} = 0$$

uniformly for $\lambda \in K$. Thus there is $m(K) > 0$ such that $\| V_{R,m}(\lambda) \|_{L^\infty} \le (2d)^{-1}$ for $m > m(K)$, $\lambda \in K$. Therefore, $(\mathcal{B}_{R,m} - \lambda g_m^{1/2})^{-1}$ exists for $\lambda \in K$ and is given by

$$(76) \quad \left(\mathcal{B}_{R,m} - \lambda g_m^{1/2} \right)^{-1} = \left(\mathcal{B}_R - \lambda g^{1/2} \right)^{-1} \left[I + V_{R,m}(\lambda)(\mathcal{B}_R - \lambda g^{1/2})^{-1} \right]^{-1}$$

where the right hand side can be written as a Neumann series. This also shows that there is an m_R such that $K \cap \operatorname{spec}(\mathcal{A}_{R,m}) = \emptyset$ for $m > m_R$.

For any $\lambda' \in K$, $(\mathcal{B}_R - \lambda' g^{1/2})^{-1}$ is a bounded operator from $L^2(\Omega)$ to $H_0^1(\Omega)$, and for if $|\lambda - \mu| < (2d)^{-1}$ we have

$$\left(\mathcal{B}_R - \lambda g^{1/2} \right)^{-1} = \left(\mathcal{B}_R - \lambda' g^{1/2} \right)^{-1} \left[I + (\lambda' - \lambda) g^{1/2} (\mathcal{B}_R - \lambda' g^{1/2})^{-1} \right]^{-1}.$$

Using this we see that the norm of $(\mathcal{B}_R - \lambda g^{1/2})^{-1} : L^2(\Omega) \to H_0^1(\Omega)$ is uniformly bounded in $\lambda \in K$. Using formulae (75) and (76), we see that

$$(77) \quad \lim_{m \to \infty} \| \left(\mathcal{B}_{R,m} - \lambda g_m^{1/2} \right)^{-1} - \left(\mathcal{B}_R - \lambda g^{1/2} \right)^{-1} \|_{L^2(\Omega) \to H_0^1(\Omega)} = 0$$

uniformly for $\lambda \in K$, and that the norms of operators $(\mathcal{B}_{R,m} - \lambda g_m^{1/2})^{-1}$:
$L^2(\Omega) \to H_0^1(\Omega)$ are uniformly bounded for $\lambda \in K$. This proves

$$\lim_{m \to \infty} \| \left(\mathcal{B}_{R,m} - \lambda g_m^{1/2} \right)^{-1} - \left(\mathcal{B}_R - \lambda g^{1/2} \right)^{-1} \|_{L^2(\Omega) \to H_0^1(\Omega)} = 0.$$

Additionally, the multiplication operators $g_m^{1/2}$, $g^{1/2}$ are bounded on $L^2(\Omega)$, uniformly in m, and

$$\|g_m^{1/2} - g^{1/2}\|_{L^2 \to L^2} \le m^{-1/2} \to 0, \quad \text{as } m \to \infty.$$

Together with equation (77) and the boundedness of $(\mathcal{B}_{R,m} - \lambda g_m^{1/2})^{-1}$ as operators from $L^2(\Omega)$ to $H_0^1(\Omega)$, this implies that

$$\lim_{m \to \infty} \left\| \left(\mathcal{B}_{R,m} - \lambda g_m^{1/2} \right)^{-1} g_m^{1/2} - \left(\mathcal{B}_R - \lambda g^{1/2} \right)^{-1} g^{1/2} \right\|_{L^2(\Omega) \to H_0^1(\Omega)} = 0.$$

This in turn implies equation (74), due to formula (71) and the relations between $\mathcal{B}_{R,m}$, $\mathcal{B}_R$ and $\mathcal{A}_{R,m}$, $\mathcal{A}_R$, which follow from their definitions (30), (70-72), and (73). $\qquad\qquad\square$

Let $R > 1$, $\mu \notin \mathrm{spec}(\mathcal{A}_R)$. It follows from Lemma 2.11 that $\mu \notin \mathrm{spec}(\mathcal{A}_{R,m})$ for m sufficiently large. Denote by $N_{R,m}(\mu)$ the subspace of $L^2(\Omega, g_m^{1/2}dx)$ spanned by the eigenfunctions of $\mathcal{A}_{R,m}$ with eigenvalues in $(-\infty, \mu)$, cf. the definition of $N_R(\mu)$. Also denote by $P_{R,m}^\mu$ the orthogonal eigenprojectors onto $N_{R,m}(\mu)$ in $L^2(\Omega, g_m^{1/2}dx)$.

Clearly, since, for $m \in \mathbb{Z}_+$, $L^2(\Omega, g_m^{1/2}dx) = L^2(\Omega)$ as sets, we can consider $P_{R,m}^\mu$ as projectors, although not orthogonal, on $L^2(\Omega)$. Recall that P_R^μ is an orthoprojector in $L_g^2(\Omega)$ onto $N_R(\mu) \subset H_0^1(\Omega)$. Restricting it to $L^2(\Omega)$, we obtain a projector, which we still call P_R^μ, on $L^2(\Omega)$. Again, P_R^μ is not an orthoprojector on $L^2(\Omega)$. However, we can compare these projectors, as well as spaces $N_{R,m}(\mu)$, $N_R(\mu)$.

Corollary 2.12. *Let $R > 1$ and $\mu \notin spec(\mathcal{A}_R)$. The Riesz projectors $P_{R,m}^\mu$ corresponding to the operators $\mathcal{A}_{R,m}$ satisfy*

$$(78) \qquad \lim_{m \to \infty} \|P_{R,m}^\mu - P_R^\mu\|_{L^2(\Omega) \to H_0^1(\Omega)} = 0.$$

Moreover, for m sufficiently large,

$$(79) \qquad dim(N_{R,m}(\mu)) = dim(N_R(\mu)).$$

Proof. Let $\Gamma \subset \mathbb{C}$ be a contour surrounding only λ_0 from $\mathrm{spec}(\mathcal{A}_R)$. When m is large enough, the Riesz projectors $P_{R,m}^\mu$ have a representation analogous to (62), obtained by replacing $\mathcal{A}_R$ by $\mathcal{A}_{R,m}$ and using

the contour Γ. Thus

$$\lim_{m\to\infty} \|P_{R,m}^{\mu} - P_R^{\mu}\|_{L^2(\Omega)\to H_0^1(\Omega)} \leq$$

$$\lim_{m\to\infty} \frac{1}{2\pi} \int_{\Gamma} \|(\mathcal{A}_{R,m} - z)^{-1} - (\mathcal{A}_R - z)^{-1}\|_{L^2(\Omega)\to H_0^1(\Omega)} \, dl(z),$$

where dl is the arclength measure on Γ. Taking into account (74), this formula implies (78).

Using equation (78), we see that there exists an m_0 such that for $m \geq m_0$ we have $\|P_{R,m}^{\mu} - P_R^{\mu}\|_{L^2(\Omega)\to L^2(\Omega)} < 1$. Using, *e.g.*, [33, Cor. IV.2.6], we see that this proves (79). $\square$

3. Approximating anisotropic by isotropic mass densities

We now show, using techniques from homogenization theory, *cf.* [1, 13, 16], that we can approximate arbitrarily closely, on the level of the operators, the nonsingular anisotropic approximate mass densities σ_R, for any fixed $R > 1$, by a family of nonsingular *isotropic* mass densities $\sigma_{R,\varepsilon}$, $\varepsilon > 0$, which will thus also function as approximate cloaks. This can be considered as the reverse of the traditional homogenization theory.

3.1. **Inverse homogenization with magnetic potential.** Observe that, as all the approximate cloaks σ_R, $R > 1$, are rotationally invariant, it is natural to use spherical coordinates. Namely, we will use either the Euclidian coordinates $x = (x^1, x^2, x^3)$, or the spherical coordinates $s = s(x) = (r(x), \theta(x), \phi(x))$. Note that we use the same notation, x, for a point inside Ω and its Euclidian coordinates, $x = (x^1, x^2, x^3)$. Which meaning is intended will be always clear from the context. We denote by $X : (x^1, x^2, x^3) \mapsto (r, \theta, \phi)$ the corresponding coordinate transformation. To exploit the rotational invariance, we will employ in the homogenization process, *cf.* [1, 13], cells which are *cubes* in these spherical coordinates.

To approximate the anisotropic mass densities in spherical coordinates in the ball $\Omega = B(0, 3)$, let us consider isotropic mass densities of the form

$$(80) \qquad \sigma_\varepsilon(x) = \sigma(x, r(x)/\varepsilon), \quad \sigma(x, r') = h(x, r')I \in \mathbb{R}^{3\times 3}.$$

Here $h(x, r')$ is a scalar valued function, to be chosen later, that is periodic in r' with period 1 and is bounded from above and below, *i.e.*,

$$(81) \qquad h(x, r' + 1) = h(x, r'), \quad 0 < c_1 \leq h(x, r') \leq c_2.$$

We will consider densities for which $h(x, r')$ is independent of r' for x with $r(x) < 1$ and $5/2 < r(x) < 3$, that is,

(82) $h(x, r') = h(x)$ if $|x| < 1$ or $5/2 < |x| < 3$.

We make this assumption since later we will use the isotropic mass densities to approximate the non-singular anisotropic mass densities σ_R, $R > 1$ that are isotropic for $r(x) < 1$ and $5/2 < r(x) < 3$.

Let (r, θ, ϕ) and (r', θ', ϕ') be spherical coordinates corresponding to two different scales. Then, in these coordinates,

$$(X_*(\sigma_\varepsilon))(s) = \sigma^*(s, s/\varepsilon),$$

where

$$\sigma^*(s, r') = h(s, r')\det(DX(x))^{-1}DX(x)DX^t(x)|_{X(x)=s}$$

$$= h(s, r') \begin{pmatrix} r^2 \sin\theta & 0 & 0 \\ 0 & \sin\theta & 0 \\ 0 & 0 & \frac{1}{\sin\theta} \end{pmatrix}.$$

Here and later we denote by σ, sometimes with various indices, various mass tensors (or matrices) in the Euclidian coordinates, while σ^* always stands for their representation in the spherical coordinates.

In the following, the material on homogenization is a quite straightforward generalization of known results [1, 13]. However, as we need to introduce changes due both to the presence of a magnetic potential and the use of spherical coordinates, for completeness we give details of the arguments.

In the small-scale coordinates $t = (r', \phi', \theta')$, we denote by $e^1 = (1, 0, 0)$, $e^2 = (0, 1, 0)$, and $e^3 = (0, 0, 1)$ the vectors corresponding to the differential forms dr', $d\theta'$ and $d\phi'$, respectively. Let $W^j(s, t)$, $j = 1, 2, 3$, be the solutions of

(83) $\nabla_t \cdot \sigma^*(s, t)(\nabla_t W^j(s, t) + e^j) = 0, \quad t = (r', \theta', \phi') \in \mathbb{R}^3,$

that are 1-periodic functions in r', θ' and ϕ' variables (noting that the periodicity in r', θ' and ϕ' has no relation to periodicity in the "large-scale" spherical coordinates θ, ϕ), and satisfy, $\int_{[0,1]^3} W^j(s, t)dt = 0$, for all s, where $dt = dr'd\theta'd\phi'$. Since σ^* is independent of θ', ϕ', the above conditions imply that $W^j = 0$ for $j = 2, 3$. As for W^1, it satisfies

$$\frac{\partial}{\partial r'}\left(h(s, r')\frac{\partial W^1}{\partial r'}\right) = -\frac{\partial h(s, r')}{\partial r'},$$

with W^1 being 1-periodic with respect to (θ', ϕ'). These imply that W^1 is independent of (θ', ϕ') with

$$\frac{\partial W^1}{\partial r'} = -1 + \frac{C_0}{h(s, r')}.$$

To find the constant C_0 we use the periodicity of W^1, now with respect to r', to get that C_0 is given by the harmonic means h^{harm} of h,

$$(84) \qquad C_0 = h^{harm}(s) := \left(\int_0^1 \frac{dr'}{h(s, r')} \right)^{-1}.$$

Define the corrector matrices [1] as

$$(85) \qquad P_j^k(s, t) = \frac{\partial}{\partial t^j} W^k(s, t) + \delta_j^k.$$

Then the homogenized mass density in the spherical coordinates, σ_{hom}^*, is given by

$$(86) \qquad (\sigma_{hom}^*)^{jk}(s) = \sum_{p=1}^3 \int_{[0,1]^3} (\sigma^*)^{jp}(s, t) P_p^k(s, t) \, dt.$$

We note that, applying integration by parts and using definition (85) and equations (83), formula (86) can be written also in a more symmetric form

$$(\sigma_{hom}^*)^{jk}(s) = \sum_{p,q=1}^3 \int_{[0,1]^3} (\sigma^*)^{pq}(s, t) P_p^j(s, t) P_q^k(s, t) \, dt.$$

Using the above formulae for the W^i, it follows from (86) that

$$\sigma_{hom}^*(s) = \begin{pmatrix} h^{harm}(s) r^2 \sin(\theta) & 0 & 0 \\ 0 & h^a(s) \sin(\theta) & 0 \\ 0 & 0 & h^a(s) \sin^{-1}(\theta) \end{pmatrix}.$$

Here $h^a(s)$ denotes the arithmetic means of h in the second variable,

$$h^a(s) = \int_{[0,1]} h(s, r') \, dr'.$$

Returning to the Euclidian coordinates, one sees that the conductivity, $\sigma_{hom}(x) = X_* \sigma_{hom}^*(x)$, has the form

$$(87) \qquad \sigma_{hom}(x) = \omega_1(x)\Pi(x) + \omega_2(x)(I - \Pi(x)),$$

with

$$(88) \qquad \omega_1(x) = h^{harm}(x), \quad \omega_2(x) = h^a(x)$$

and $\Pi(x): \mathbb{R}^3 \to \mathbb{R}^3$ being the projection to the radial direction,

$$\Pi(x)\, v = \left(v \cdot \frac{x}{|x|} \right) \frac{x}{|x|},$$

represented by the matrix $(|x|^{-2} x^j x^k)_{j,k=1}^3$.

Next, we analyze the Dirichlet problems for elliptic equations

(89) $- \nabla^b \cdot \sigma_\varepsilon \nabla^b u_\varepsilon + Q u_\varepsilon = f, \quad u_\varepsilon|_{\partial\Omega} = h.$

Here, $b(x) = (b_1(x), b_2(x), b_3(x))$ is the magnetic potential and $Q(x)$ is a scalar function, with $Q, b_j \in L^\infty(\Omega; \mathbb{R})$, and σ_ε are isotropic mass densities bounded from above and below by positive constants independent of ε. Moreover, $f \in H^{-1}(\Omega)$, and $h \in H^{1/2}(\partial\Omega)$. Our goal is to show that the solutions u_ε convergence to the solution of the equation

(90) $- \nabla^b \cdot \sigma_{hom} \nabla^b u + Q u = f, \quad u|_{\partial\Omega} = h.$

By adapting the technique of Allaire [1], we can prove the following result.

Proposition 3.1. *Let σ_ε, $\varepsilon > 0$ be mass densities in Ω satisfying (80), (81), and (82), σ_{hom} be the mass density defined by (87), $Q \in L^\infty(\Omega)$, $Q(x) \geq 0$, and $b = (b_1(x), b_2(x), b_3(x))$ be a vector field, $b \in L^\infty(\Omega; \mathbb{R}^3)$. Then the solutions u_ε of equations (89) and solution u of equation (90) satisfy*

(91) $\displaystyle\lim_{\varepsilon \to 0} u_\varepsilon = u \quad \text{weakly in } H^1(\Omega).$

Proof. Let $E_h \in H^1(\Omega)$, $\operatorname{supp}(E_h) \subset \{5/2 \leq |x| \leq 3\}$ be an extension of h, *i.e.*, $E_h|_{\partial\Omega} = h$. Writing $u_\varepsilon = E_h + v_\varepsilon$ and $u = E_h + v$, we see that the functions v_ε and v satisfy

(92) $-\nabla^b \cdot \sigma_\varepsilon \nabla^b v_\varepsilon + Q v_\varepsilon = \widetilde{f}, \quad \text{on } \Omega, \quad v_\varepsilon|_{\partial\Omega} = 0,$

(93) $-\nabla^b \cdot \sigma_{hom} \nabla^b v + Q v = \widetilde{f} \quad \text{on } \Omega, \quad v|_{\partial\Omega} = 0.$

Here

(94) $\widetilde{f} = f - \nabla^b \cdot \sigma_\varepsilon \nabla^b E_h + Q E_h \in H^{-1}(\Omega),$

is independent of ε. For proving (91) it is enough to show that v_ε converges to v weakly in $H_0^1(\Omega)$.

Let us recall that by Lemma 2.2, a_R is (strictly) positive definite. Since $Q(x) \geq 0$, we see that there exists a $c_0 > 0$ such that

$$\int_\Omega \left(|\nabla^b u(x)|^2 + Q(x)|u(x)|^2 \right) dx \geq c_0 \|u\|_{H_0^1(\Omega)}^2 \quad \text{for } u \in H_0^1(\Omega).$$

As the mass densities σ_ε are uniformly bounded from above and below by positive constants, it follows from the proof of Lemma 2.2 that there is also $c_1 > 0$ such that

$$\int_\Omega \left(\nabla^b u \cdot \sigma_\varepsilon \overline{\nabla^b u} + Q(x)|u|^2\right) dx \geq c_1 \|u\|^2_{H^1_0(\Omega)} \quad \text{for } u \in H^1_0(\Omega).$$

Thus, using the Lax-Milgram lemma we see that the solutions $v_\varepsilon(x)$ of equations (92) satisfy

$$(95) \qquad \qquad \|v_\varepsilon\|_{H^1_0(\Omega)} \leq c\|\widetilde{f}\|_{H^{-1}(\Omega)},$$

where $c > 0$ is independent of $\varepsilon > 0$. Therefore, the solutions $v_\varepsilon(x)$ are uniformly bounded in $H^1_0(\Omega)$. Thus, for an arbitrary sequence $\varepsilon_n \to 0$, the corresponding v_{ε_n} have a subsequence that converges weakly to some function $w(x)$ in $H^1_0(\Omega)$. Let us show that w coincides with the solution v of equation (93).

To this end we consider convergence in a finer, two-scale, sense on local coordinate neighborhoods. Let $U \subset \mathbb{S}^2$ be an open set on which we can define, in a regular manner, spherical coordinates. For example, by choosing two antipodal points as the South and North poles and connecting those by a meridian γ, we can take U so that $\overline{U} \subset\subset \mathbb{S}^2 \setminus \gamma$ and define polar coordinates on U. Let $\Omega' = \{r\omega : \omega \in U, r \in (r_1, r_2)\} \subset \Omega$ with some $r_2 > r_1 > 0$. Clearly, $X : \Omega' \to \mathbb{R}^3$ defines the spherical coordinates, $x \mapsto (r, \theta, \phi)$, with domain $W := X(\Omega') = (r_1, r_2) \times U$.

Rewrite now the equations (92) and (93) on W in these spherical coordinates and multiply the equations so obtained by $\det(DX(X^{-1}(s)))$. The resulting equations are

$$(96) \quad -(\nabla_s + ib^*(s)) \cdot \sigma^*(s, \frac{s}{\varepsilon})(\nabla_s + ib^*(s))v_\varepsilon(s) + Q^*(s)v_\varepsilon(s) = \widehat{f}(s),$$

$$(97) \quad -(\nabla_s + ib^*(s)) \cdot \sigma^*_{hom}(s)(\nabla_s + ib^*(s))v(s) + Q^*(s)v(s) = \widehat{f}(s),$$

where $s \in W$. Here

$$\begin{aligned} Q^*(s) &= \det(DX(X^{-1}(s)))Q(X^{-1}(s)) \in L^\infty(W), \\ b_j^*(s) &= \partial_j X^k(X^{-1}(s))b_k(X^{-1}(s)) \in L^\infty(W; \mathbb{R}^3), \end{aligned}$$

correspond to the electric potential Q and magnetic potential b in the spherical coordinates

and $\widehat{f}(s) = \det(DX(X^{-1}(s)))\widetilde{f}(X^{-1}(s)) \in H^{-1}(W)$. To simplify notations, we continue to denote the functions v and v_ε in the spherical coordinates by $v(s) = ((X^{-1})^*v)(s)$, $v_\varepsilon(s) = ((X^{-1})^*v_\varepsilon)(s)$. Finally, ∇_s is the vector field $(\partial_r, \partial_\theta, \partial_\phi)$.

In the following, even though in the forthcoming applications the mass densities $\sigma_\varepsilon^*(x, r(x)/\varepsilon)$ will depend only on the small-scale variable $r' = r(x)/\varepsilon$, we consider the general case when the mass densities depend on all small-scale coordinates $t = (r', \theta', \phi')$. Let $T = [0,1]^3$ and $C_\#^m(T)$ denote those $C^m(T)$ functions that can be continued as $\mathbb{Z}^3$-periodic functions in $\mathbb{R}^3$ which are in $C^m(\mathbb{R}^3)$. By definition, a family $v_\varepsilon(s)$, $v_\varepsilon \in L^2(W)$ is said to *two-scale converge*, as $\varepsilon \to 0$, to a function $v_0(s,t) \in L^2(W \times T)$ if, for all test functions $\psi(s,t)$ in $C_0^\infty(W; C_\#^\infty(T))$, we have

$$(98) \qquad \lim_{\varepsilon \to 0} \int_W v_\varepsilon(s) \psi(s, \frac{s}{\varepsilon}) \, ds = \int_W \int_T v_0(s,t) \psi(s,t) \, ds dt.$$

By [1], the two-scale convergence of v_ε implies the weak convergence of $v_\varepsilon(s)$ in $L^2(W)$ to the function $w(s) = \int_T v_0(s,t) \, dt$, so that the two-scale convergence gives finer information on the convergence than the weak convergence. For example, functions of the form $u(s, s/\varepsilon)$ two-scale converge to $u(s,t)$.

By [1], every bounded family $v_\varepsilon(s) \in L^2(W)$ contains a two-scale converging sequence. Moreover, if $v_\varepsilon(s)$ is a bounded family in $H^1(W)$ that converges in $L^2(W)$ to $w(s)$ as $\varepsilon \to 0$, then $v_\varepsilon(s)$ also two-scale converges to $w(s)$ and there is a function $w_1(s,t) \in L^2(W \times T)$, so that $\nabla_s v_\varepsilon(s)$ two-scale converge to $\nabla_s w(s) + \nabla_t w_1(s,t)$. For example, if functions $v_\varepsilon(s)$ have the form $v_\varepsilon(s) = u_0(s) + \varepsilon u_1(s, s/\varepsilon)$, then $\nabla_s v_\varepsilon(s)$ two-scale converge to $u(s,t) = \nabla u_0(s) + \nabla_t u_1(s,t)$.

As noted above, the solutions $v_\varepsilon(x)$ of equations (92) are uniformly bounded in $H_0^1(\Omega)$. Consider the restrictions of these functions on Ω' and rewrite them in the spherical coordinates on W. Then any sequence of $v_\varepsilon(s)$ has a subsequence $v_{\varepsilon_j}(s)$, $j \in \mathbb{Z}_+$, $\varepsilon_j \to 0$ as $j \to \infty$, that weakly converges, in $H^1(W)$, to some function $w(s)$. By [1] $v_{\varepsilon_j}(s)$ also two-scale converge to $w(s)$, as $\varepsilon \to 0$, and there is a function $w_1(s,t) \in L^2(W \times T)$ so that $\nabla v_{\varepsilon_j}(s)$ two-scale converge to $\nabla_s w(s) + \nabla_t w_1(s,t)$.

Let us now multiply both sides of equation (96) by a test function $\phi(s) + \varepsilon \phi_1(s, s/\varepsilon)$, where $\phi(s) \in C_0^\infty(W)$ and $\phi_1(s,t) \in C_0^\infty(W; C_\#^\infty(T))$, and integrate over W. Using integration by parts, we obtain

$$(99) \quad \int_W \sigma^*(s, \frac{s}{\varepsilon})[(\nabla_s + ib^*(s))v_\varepsilon(s)] \cdot \overline{(\eta_1(s, \frac{s}{\varepsilon}) + \varepsilon \eta_2(s, \frac{s}{\varepsilon}))} \, ds +$$

$$+ \int_W Q^*(s) v_\varepsilon(s) \overline{[\phi(s) + \varepsilon \phi_1(s, \frac{s}{\varepsilon})]} \, ds = \int_{\Omega'} \widehat{f}(s) \overline{[\phi(s) + \varepsilon \phi_1(s, \frac{s}{\varepsilon})]} \, ds,$$

where

$$\eta_1(s,t) = \nabla_s\phi(s) + ib^*(s)\phi(s) + \nabla_t\phi_1(s,t),$$
$$\eta_2(s,t) = \nabla_s\phi_1(s,t) + ib^*(s)\phi_1(s,t).$$

Substitute $\varepsilon = \varepsilon_j$ in (99) and let $j \to \infty$. First, we observe that since $\eta_2(s, s/\varepsilon), \varepsilon \in (0,1)$ are uniformly bounded in $L^2(W)$, the terms in (99), involving η_2, tend to zero. Second, as $v_\varepsilon(s)$ converge in strong topology of $L^2(W)$ to w, the first integral in (100) tends to $\int_W Q^*(s)w(s)\overline{\phi(s)}\,ds$. Third, the inner product of $\widehat{f}(s)$ and $\varepsilon\phi_1$ in the last integral in (100) goes to zero. Fourth, the functions $\psi_\varepsilon(s) = \sigma^*(s, s/\varepsilon)\eta_1(s, s/\varepsilon)$ two-scale converge to the function $\psi(s,t) = \sigma^*(s,t)\eta_1(s,t)$. Since $\psi(x,y) \in L^2(\Omega; C^0_\#(T))$, we have by [1, Lem. 1.3] that

$$(100) \qquad \lim_{\varepsilon\to 0}\|\psi_\varepsilon\|_{L^2(W)} = \|\psi\|_{L^2(W\times T)}.$$

Furthermore, as $\nabla v_{\varepsilon_j}(s)$ two-scale converges to $\nabla_s w(s) + \nabla_t w_1(s,t)$, it follows from (100) and [1, Thm. 1.8], that

$$\int_W \sigma^*(s, \frac{s}{\varepsilon_j})[(\nabla_s + ib^*(s))v_{\varepsilon_j}(s)] \cdot \overline{\eta_1(s, \frac{s}{\varepsilon_j})}\,ds \to$$
$$\int_W \int_T \sigma^*(s,t)[\nabla_s w(s) + \nabla_t w_1(s,t)] \cdot \overline{\eta_1(s,t)}\,dsdt,$$

as $j \to \infty$. Summarizing, we see that

$$(101) \qquad \int_W \int_T \sigma^*(s,t)[\nabla_s w(s) + \nabla_t w_1(s,t) + ib^*(s)w(s)]$$
$$\cdot \overline{[\nabla_s\phi(s) + ib^*(s)\phi(x) + (\nabla_t\phi_1)(s,t)]}\,dsdt +$$
$$+ \int_W \int_T Q^*(s)w(s)\overline{\phi(s)}\,dsdt = \int_W \widehat{f}(s)\overline{\phi(s)}\,ds.$$

Taking $\phi(s) = 0$ in (101) and varying $\phi_1(s,t)$ over all test functions, integration by parts with respect to t shows that

$$(102) \quad \nabla_t \cdot \sigma^*(s,t)[\nabla_s w(s) + \nabla_t w_1(s,t) + ib^*(s)w(s)] = 0 \text{ a.e. in } W \times T.$$

Since $\nabla_s w(s) + ib^*(s)w(s)$ depends only on s and thus can be considered as a constant vector in the t-variable, we see that

$$(103) \quad w_1(s,t) = \sum_{j=1}^{3}\left(\frac{\partial w}{\partial s^j}(s) + ib_j^*(s)w(s)\right)W^j(s,t) \quad \text{a.e. in } W \times T,$$

where W^j are solutions of the cell equations (83).

On the other hand, taking $\phi_1(s,t) = 0$ in (101) and varying $\phi(s)$ over all test functions, we see, using integration by parts with respect to s, that

$$-(\nabla_s + ib^*(s)) \cdot \left(\int_T \sigma^*(s,t)[\nabla_s w(s) + \nabla_t w_1(s,t) + ib^*(s)w(s)]dt \right) +$$

(104) $+Q^*(s)w(s) = \widehat{f}(s)$ a.e. in W.

Comparing the above integral with formula (86) for σ_{hom}, we see that (103) and (104) imply that

$$-(\nabla_s + ib^*(s)) \cdot \sigma_{hom}^*(s)(\nabla_s + ib^*(s))w(s) + Q^*(s)w(s) = \widehat{f}(s) \quad \text{in } W,$$

or equivalently, in the Cartesian coordinates

(105) $$-(\nabla + ib) \cdot \sigma_{hom}(\nabla + ib)w + Qw = \widetilde{f} \quad \text{in } \Omega'.$$

Since $\Omega \setminus \{O\}$ can be covered with coordinate neighborhoods Ω' used above, the equation (105) is valid in the domain $\Omega \setminus \{O\}$. Moreover, since all the mass densities σ_{hom} and σ_ε are the same near the origin we see that all functions $v_{\varepsilon_j}(x)$ satisfy the equation (105) near the origin. As $v_{\varepsilon_j}(x)$ converge weakly in $H_0^1(\Omega)$ to $w(x)$, the equation (105) is valid in Ω. This means that $w(x)$, as well as $v(x)$, is the solution of the equation (93). By the Lax-Milgram Theorem, equation (93) has a unique solution, and thus $w = v$.

Summarizing, we have now shown that an arbitrary sequence of the original family $v_\varepsilon(x)$ has a subsequence that weakly converges, in $H_0^1(\Omega)$, to the solution $v(x)$ of equations (93). This shows that the whole family $v_\varepsilon(x)$ of the solutions of the equations (92) converge weakly to $v(x)$ in $H_0^1(\Omega)$. $\square$

3.2. **Approximation of the mass densities σ_R.** Next we apply the above results to approximate the mass densities σ_R for a fixed $R > 1$. In the forthcoming analysis, we will choose the function h to be

(106) $$h(s,r') = \frac{a(s)}{1 + b(s)p(r')},$$

where $p(r')$ is a fixed positive, smooth 1-periodic function of r' and $a(s), b(s)$ are positive. For any $x \in \Omega$ and a given function $p(r')$, the pair $(\omega_1(x), \omega_2(x))$ in (87), (88) depends only on the values of functions $(a(s(x)), b(s(x)))$. By choosing $(a(s), b(s))$ appropriately, it is possible to achieve any value

$$(\omega_1(x), \omega_2(x)) \in \mathbb{R}_+^2, \quad \text{with } \omega_1(x) \leq \omega_2(x).$$

We can choose $a(x) = a_R(x)$ and $b(x) = b_R(x)$ so that $\sigma_{hom}(x)$ corresponds to the approximate cloaking mass density, that is,

(107) $\sigma_{hom}(x) = \sigma_R(x), \quad x \in \Omega.$

In sequel, we will denote by $\sigma_R(x, r') = h_R(x, r')I$ the isotropic mass densities for which the homogenized mass densities σ_{hom} satisfy (107). Observe that, for $R > 1$, we can choose $h_R(s, r')$ so that

(108) $0 < c_1(R) \leq h_R(s, r') \leq c_2, \quad c_1(R) \to 0$ as $R \searrow 1.$

Let $\lambda < 0$ and $\mathcal{B}_{R,m,\varepsilon}$ be the operators

$$\mathcal{B}_{R,m,\varepsilon}u = -\nabla^b \cdot \sigma_{R,\varepsilon}\nabla^b u + g_m^{1/2}qu,$$
$$\mathcal{D}(\mathcal{B}_{R,m,\varepsilon}) = \{u \in H_0^1(\Omega) : -\nabla^b \cdot \sigma_{R,\varepsilon}\nabla^b u \in L^2(\Omega)\}.$$

Lemma 3.2. *For any $\lambda < 0$ and $f \in L^2(\Omega)$,*

$$\lim_{\varepsilon \to 0}(\mathcal{B}_{R,m,\varepsilon} - \lambda g_m^{1/2})^{-1}f = (\mathcal{B}_{R,m} - \lambda g_m^{1/2})^{-1}f,$$

where $\mathcal{B}_{R,m}$ is as in (72), and the limit holds both in the sense of the strong topology in $L^2(\Omega)$ and weak topology in $H^1(\Omega)$.

Proof. This follows immediately from Prop. 3.1 by taking $Q = (q - \lambda)g_m^{1/2}$. □

As in the previous section, one can analyze the convergence of the resolvents in more detail. To this end, introduce operators $\mathcal{A}_{R,m,\varepsilon} = g_m^{-1/2}\mathcal{B}_{R,m,\varepsilon}$ that is,

(109) $\mathcal{A}_{R,m,e} = -g_m^{-1/2}\nabla^b \cdot \sigma_{R,\varepsilon}\nabla^b u + qu,$

$$\mathcal{D}(\mathcal{A}_{R,m,\varepsilon}) = \mathcal{D}(\mathcal{B}_{R,m,\varepsilon}) = \{u \in H_0^1(\Omega) : -\nabla^b \cdot \sigma_{R,\varepsilon}\nabla^b u \in L^2(\Omega)\}.$$

Clearly, the operators $\mathcal{A}_{R,m,e}$ are self-adjoint in $L^2(\Omega, g_m^{1/2}dx)$ and

(110) $(\mathcal{A}_{R,m,\varepsilon} - \lambda)^{-1}f = (\mathcal{B}_{R,m,\varepsilon} - \lambda g_m^{1/2})^{-1}(g_m^{1/2}f)$

for $\lambda \notin \text{spec}(\mathcal{A}_{R,m,\varepsilon})$ and $f \in L^2(\Omega)$, cf. (72), (73).

Lemma 3.3. *Let $R > 1$, $m \in \mathbb{Z}_+$ and $K \subset \mathbb{C}$ be compact and such that $dist(K, spec(\mathcal{A}_{R,m})) \geq \delta_0 > 0$. Then*

(i) There is an $\varepsilon_0 = \varepsilon_0(K, R, m)$ such that such that for $0 < \varepsilon < \varepsilon_0$

$$\|(\mathcal{A}_{R,m,\varepsilon} - \lambda)^{-1}\|_{L^2(\Omega, g_m^{1/2}dx) \to L^2(\Omega, g_m^{1/2}dx)} \leq \frac{2}{\delta_0}, \quad \lambda \in K,$$

(ii) For any $f \in L^2(\Omega)$,

(111) $\lim_{\varepsilon \to 0}(\mathcal{A}_{R,m,\varepsilon} - \lambda)^{-1}f = (\mathcal{A}_{R,m} - \lambda)^{-1}f$

in the strong topology of $L^2(\Omega)$ and weak topology of $H_0^1(\Omega)$, with convergence being uniform for $\lambda \in K$.

Proof. (i) Since the $\mathcal{A}_{R,m,\varepsilon}$ are self–adjoint operators on $L^2(\Omega, g_m^{1/2}dx)$, it is enough to show that there is $\varepsilon_0 > 0$ such that the operators $\mathcal{A}_{R,m,\varepsilon} - \mu$ are invertible for $|\lambda - \mu| \leq \delta_0/2$ if $\varepsilon < \varepsilon_0$.

Assume that there are $\varepsilon_n \searrow 0$, $\mu_n \to \mu$, $|\lambda - \mu| \leq \delta_0/2$, and u_n, with $\|u_n\|_{L^2(\Omega, g_m^{1/2}dx)} = 1$, such that

$$\mathcal{A}_{R,m,\varepsilon_n} u_n - \mu_n u_n = 0.$$

Rewrite this as

$$(112) \quad -\nabla \cdot \sigma_{R,\varepsilon_n} \nabla u_n = f_n := -g_m^{1/2} q u_n + \mu_n g_m^{1/2} u_n, \quad u_n|_{\partial\Omega} = 0.$$

Taking into account the uniform boundedness of the right hand side of (112) in $L^2(\Omega)$, it follows from (92), (95), with $Q = 0$, that $\|u_n\|_{H_0^1(\Omega)} \leq C$, for some $C > 0$ independent ε_n.

Thus, up to a subsequence, there exists $u \in H_0^1(\Omega)$ such that $u_n \to u$ as $n \to \infty$ in the weak topology of $H^1(\Omega)$ and strong topology of $L^2(\Omega)$. This shows, in particular, that $\|u\|_{L^2(\Omega, g_m^{1/2}dx)} = 1$.

Let us show that u is an eigenfunction of $\mathcal{A}_{R,m}$, see (73), corresponding to the eigenvalue μ; as $|\lambda - \mu| \leq \delta_0/2$ with $\lambda \in K$ and $\mathrm{dist}(K, \mathrm{spec}(\mathcal{A}_{R,m})) > \delta > 0$, this would yield a contradiction. We compare u_n with v_n, the solution to

$$(113) \quad -\nabla^b \cdot \sigma_{R,\varepsilon_n} \nabla^b v_n = \widetilde{f}_n := -g_m^{1/2} q u + \mu g_m^{1/2} u, \quad v_n|_{\partial\Omega} = 0.$$

Letting $\mu_n \to \mu$, (113) and the weak convergence $u_n \to u$ in $H^1(\Omega)$ imply that

$$\lim_{n \to \infty} \|f_n - \widetilde{f}_n\|_{L^2(\Omega)} = 0.$$

Appealing again to (108), with $R > 1$ fixed, we see that $u_n - v_n \to 0$ in the strong topology of $H_0^1(\Omega)$.

Using Proposition 3.1 with $Q = 0$, we see that $v_n \rightharpoonup v$ in $H^1(\Omega)$, where

$$-\nabla^b \cdot \sigma_R \nabla^b v = -g_m^{1/2} q u + \mu g_m^{1/2} u, \quad v|_{\partial\Omega} = 0.$$

Summarizing, we have that $u = v$, showing that u is an eigenfunction of $\mathcal{A}_R$ corresponding to the eigenvalue λ. This proves claim (i).

(ii) Compare the solutions of

$$-\nabla^b \cdot \sigma_{R,\varepsilon} \nabla^b u_{m,\varepsilon} = f_{m,\varepsilon} := f - g_m^{1/2} q u_{m,\varepsilon} + \lambda g_m^{1/2} u_{m,\varepsilon}, \quad u_{m,\varepsilon}|_{\partial\Omega} = 0.$$

with those of

$$-\nabla^b \cdot \sigma_R \nabla^b u = f_m := f - g_m^{1/2} qu + \lambda g_m^{1/2} u, \quad u|_{\partial\Omega} = 0.$$

It follows from claim (i) together with (95), that the $u_{m,\varepsilon}$ are uniformly bounded in $H_0^1(\Omega)$. Choosing a weakly converging subsequence, we show, by similar reasoning to the above, that the limit is u_m. This proves

$$\lim_{\varepsilon\to 0}(\mathcal{B}_{R,m,\varepsilon} - \lambda g_m^{1/2})^{-1} f = (\mathcal{B}_{R,m} - \lambda g_m^{1/2})^{-1} f$$

This, (110), and the boundedness of the multiplication operator $f \mapsto g_m^{1/2} f$ in $L^2(\Omega)$ yields (111).   □

For $\mu \notin \mathrm{spec}\,(\mathcal{A}_{R,m,\varepsilon})$ we denote by $N_{R,m,\varepsilon}(\mu)$ the subspace of $L^2(\Omega)$ spanned by the eigenfunctions of $\mathcal{A}_{R,m,\varepsilon}$ corresponding to the eigenvalues in $(-\infty, \mu)$, and by $P_{R,m,\varepsilon}^\mu$ the orthoprojectors in $L^2(\Omega, g_m^{1/2} dx)$ onto $N_{R,m,\varepsilon}(\mu)$.

Lemma 3.4. *Let $R > 1$, $m \in \mathbb{Z}_+$, and $\mu \notin \mathrm{spec}\,(\mathcal{A}_{R,m})$. Then*

(i) There is an $\varepsilon_{R,m} > 0$ such that $\mu \notin \mathrm{spec}\,(\mathcal{A}_{R,m,\varepsilon})$ for $0 < \varepsilon < \varepsilon_{R,m}$.

(ii) For $f \in L^2(\Omega)$ we have

$$(114) \qquad\qquad \lim_{\varepsilon\to 0} P_{R,m,\varepsilon}^\mu f = P_{R,m}^\mu f,$$

in the strong topology of $L^2(\Omega)$ and weak topology of $H_0^1(\Omega)$. Moreover,

$$(115) \qquad\qquad \lim_{\varepsilon\to 0} \dim(N_{R,m,\varepsilon}(\mu)) = \dim(N_{R,m}(\mu)).$$

Remark. We note that the strong convergence of the resolvents does *not* generally imply the stability of the resolvent set, *cf.* [33, Sect. VIII.1.2].

Proof. (i) Take $K = \{\mu\}$ and $\delta_0 = \mathrm{dist}(K, \mathrm{spec}(\mathcal{A}_{R,m}))$. Lemma 3.3 (i) implies that, for sufficiently small ε, $\mathrm{dist}(K, \mathrm{spec}(\mathcal{A}_{R,m,\varepsilon})) > \delta_0/2$.

(ii) Relation (114) follows immediately from (111) and the Riesz formula for the projectors, *cf.* (62). For the proof of (115), let us consider $\lambda_0 \in \mathrm{spec}\,(\mathcal{A}_{R,m})$ and $a, b \in \mathbb{R}$, $a < \lambda_0 < b$ such that $[a,b] \cap \mathrm{spec}\,(\mathcal{A}_{R,m}) = \{\lambda_0\}$. Then by (i) there are $\theta(\varepsilon)$, $\theta(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that

$$\mathrm{spec}\,(\mathcal{A}_{R,m,\varepsilon}) \cap [a + \theta(\varepsilon), \lambda_0 - \theta(\varepsilon)) = \emptyset,$$
$$\mathrm{spec}\,(\mathcal{A}_{R,m,\varepsilon}) \cap (\lambda_0 + \theta(\varepsilon), b - \theta(\varepsilon)] = \emptyset.$$

Let $P^\varepsilon = P_{R,m,\varepsilon}^{b-\theta(\varepsilon)} - P_{R,m,\varepsilon}^{a+\theta(\varepsilon)}$ and $P^0 = P_{R,m}^b - P_{R,m}^a$. Then, by (114), $\lim_{\varepsilon \to 0} P^\varepsilon f = P^0 f$ in the strong topology of $L^2(\Omega)$ and weak topology of $H_0^1(\Omega)$.

To establish (115), it is sufficient to show that

$$(116) \qquad \lim_{\varepsilon \to 0} \dim(\mathrm{Ran}\,(P^\varepsilon)) = \dim(\mathrm{Ran}\,(P^0)).$$

Indeed, if we show (116) for each eigenvalue of $\mathcal{A}_{R,m}$ smaller than μ, equality (115) follows.

To show (116), we note that the ranges of the orthoprojectors P^ε, P^0 are also subspaces of $H_0^1(\Omega) \subset L^2(\Omega)$. We show first that

$$(117) \qquad \liminf_{\varepsilon \to 0} \dim(\mathrm{Ran}\,(P^\varepsilon)) \geq \dim(\mathrm{Ran}\,(P^0)).$$

On the contrary, if this does not hold, there is a sequence $\varepsilon(n) \to 0$ such that

$$(118) \qquad \dim(\mathrm{Ran}\,(P^{\varepsilon(n)})) < \dim(\mathrm{Ran}\,(P^0)) = k_0.$$

Denote by η_k, $k = 1, \ldots, k_0$, an $L^2(\Omega)$–orthonormal basis in $\mathrm{Ran}\,(P^0)$. (Note that as $\mathrm{Ran}\,(P^0) \subset H_0^1(\Omega)$ is finite dimensional all norms are equivalent). Introduce $\eta_{n,k} = P^\varepsilon \eta_k \in \mathrm{Ran}\,(P^\varepsilon)$, $k = 1, \ldots, k_0$. Consider the Gram-Schmidt matrices, $G^n = [G_{kl}^n]_{k,l=1}^{k_0}$,

$$G_{kl}^n = \int_\Omega \eta_{n,k}(x) \overline{\eta}_{n,l}(x)\, dx.$$

By (114) and orthonormality of η_k, the matrix G^n is invertible for sufficiently large n. Thus, $\eta_{n,k}$, $k = 1, \ldots, k_0$ are linearly independent. This implies that $\dim(\mathrm{Ran}\,(P^{\varepsilon(n)})) \geq k_0$, contradicting (118). This proves (117).

Assume next that

$$(119) \qquad \limsup_{\varepsilon \to 0} \dim(\mathrm{Ran}\,(P^\varepsilon)) > \dim(\mathrm{Ran}\,(P^0)).$$

Then there is a sequence $\varepsilon(n)$, such that

$$\dim(\mathrm{Ran}\,(P^{\varepsilon(n)})) > \dim((\mathrm{Ran}\,(P^0)).$$

Thus, there are u_n such that

$$(120) \qquad u_n \in \mathrm{Ran}\,(P^{\varepsilon(n)}), \quad \|u_n\|_{L^2(\Omega, g_m^{1/2} dx)} = 1,$$

$$(121) \qquad (u_n, v)_{L^2(\Omega, g_m^{1/2} dx)} = 0, \text{ for all } v \in \mathrm{Ran}\,(P^0).$$

Then

$$u_n = \sum_{k=1}^{k_n} u_{n,k}\psi_{n,k}, \quad \sum_{k=1}^{k_n} |u_{n,k}|^2 = 1, \quad k_n = \dim(\mathrm{Ran}\,(P^{\varepsilon(n)})),$$

where $\psi_{n,k}$ are $L^2(\Omega, g_m^{1/2}dx)$–orthonormal eigenfunctions of $\mathcal{A}_{R,m,\varepsilon(n)}$,

$$\mathcal{A}_{R,m,\,\varepsilon(n)}\psi_{n,k} = \lambda_{n,k}\psi_{n,k}, \quad |\lambda_{n,k} - \lambda_0| \le \theta(\varepsilon(n))$$

Therefore,

$$(122)\, \mathcal{A}_{R,m,\varepsilon(n)}u_n - \lambda_0 u_n = f_n := \sum_{k=1}^{k_n} u_{n,k}(\lambda_{n,k} - \lambda_0)\psi_{n,k} \to 0 \text{ as } n \to \infty$$

in $L^2(\Omega, g_m^{1/2}dx)$, so that

$$\int_\Omega \left(\sigma_{R,\varepsilon(n)}\nabla^b u_n \cdot \overline{\nabla^b u_n} + g_m^{1/2}(q - \lambda_0)|u_n|^2 \right)\, dx$$
$$= (f_n, u_n)_{L^2(\Omega, g_m^{1/2}dx)} \to 0 \text{ as } n \to \infty.$$

Together with (120) and the fact that $q \in L^\infty(\Omega)$, this implies that the forms $\int_\Omega \sigma_{R,\varepsilon(n)}\nabla^b u_n \cdot \overline{\nabla^b u_n}\, dx$ are uniformly bounded in n. Recalling (95), we see that with some $c > 0$,

$$\|u_n\|_{H_0^1(\Omega)} < c, \quad \text{for all } n \in \mathbb{Z}_+.$$

Restricting to a subsequence of the $\varepsilon(n)$, assume there exists $u_0 \in H_0^1(\Omega)$ such that

$$(123)\ \lim_{n\to\infty} u_n = u_0 \quad \text{weakly in } H_0^1(\Omega) \text{ and strongly in } L^2(\Omega),$$

so that also $\|u_0\|_{L^2(\Omega, g_m^{1/2}dx)} = 1$. Let us show that $\mathcal{A}_{R,m}u_0 = \lambda_0 u_0$,, contradicting the fact that $u_0 \perp \mathrm{Ran}\,(P^0)$ in $L^2(\Omega, g_m^{1/2}dx)$, which follows from (120). Let v_n be the solutions to

$$-\nabla^b \cdot \sigma_{R,\varepsilon(n)}\nabla^b v_n = g_m^{1/2}(\lambda_0 - q)u_0, \quad v_n|_{\partial\Omega} = 0.$$

From (95), together with (122), (123), we see that $u_n - v_n \to 0$ as $n \to \infty$ in $L^2(\Omega, g_m^{1/2}dx)$. On the other hand, by Prop. 3.1 with $Q = 0$, $v_n \to v_0$ in $L^2(\Omega)$ as $n \to \infty$, where v_0 is the solution to

$$-\nabla^b \cdot \sigma_R \nabla^b v_0 = g_m^{1/2}(\lambda_0 - q)u_0, \quad v_n|_{\partial\Omega} = 0.$$

Hence, $v_0 = u_0 \ne 0$ is an eigenfunction of $\mathcal{A}_{R,m}$ and belongs in $\mathrm{Ran}\,(P^0)$. This proves the claim. □

4. Approximate acoustic cloaking

In this section, we show that, for $\lambda \notin \operatorname{spec}(\mathcal{A}_1)$, it is possible to approximate the solutions to the singular, anisotropic acoustic equation

$$\mathcal{A}_1 u - \lambda u = f, \quad f \in L^2(\Omega),$$

by solutions to certain non-singular, isotropic acoustic equations. Namely,

Theorem 4.1. *Let $K \subset \mathbb{C}$ be a compact set such that $K \cap \operatorname{spec}(\mathcal{A}_1) = \emptyset$, and $\lambda \in K$. Then, for $f \in L^2(\Omega)$,*

$$\lim_{R \to 1} \lim_{m \to \infty} \lim_{\varepsilon \to 0} (\mathcal{A}_{R,m,\varepsilon} - \lambda)^{-1} f = (\mathcal{A}_1 - \lambda)^{-1} f,$$

in the strong topology of $L_g^2(\Omega)$ and weak topology of $H_g^1(\Omega)$, uniformly with respect to $\lambda \in K$.

Proof. By Lemma 3.3, (111), it follows that

(124) $$\lim_{\varepsilon \to 0} (\mathcal{A}_{R,m,\varepsilon} - \lambda)^{-1} f = (\mathcal{A}_{R,m} - \lambda)^{-1} f$$

in the strong topology of $L^2(\Omega)$ and weak topology of $H^1(\Omega)$, uniformly in $\lambda \in K$. Using Lemma 2.11 we obtain

(125) $$\lim_{m \to \infty} (\mathcal{A}_{R,m} - \lambda)^{-1} f = (\mathcal{A}_R - \lambda)^{-1} f$$

in the strong topology of $H^1(\Omega)$, uniformly in $\lambda \in K$. Using (8) and (9), the convergences in (124) and (125) are valid also in $L_g^2(\Omega)$ and $H_g^1(\Omega)$.

As $f \in L^2(\Omega) \subset L_g^2(\Omega)$ we have by Lemma 2.8

$$\lim_{R \searrow 1} (\mathcal{A}_R - \lambda)^{-1} f = (\mathcal{A}_1 - \lambda)^{-1} f,$$

in the strong topology of $L_g^2(\Omega)$ and weak topology of $H_g^1(\Omega)$, uniformly with respect to $\lambda \in K$. This equation, together with (124), (125), proves the claim. $\square$

Let us turn our attention to the convergence of the DN maps. For $R > 1$ and $\varepsilon > 0$, the DN map for the acoustic equation,

(126) $$-\nabla^b \cdot \sigma_{R,\varepsilon} \nabla^b u + g^{1/2} q u - \lambda g_m^{1/2} u = 0, \quad u|_{\partial\Omega} = h,$$

is the operator

(127) $$\Lambda_{R,\varepsilon,m}^\lambda : h \mapsto (\partial_\nu + i\nu \cdot b) u_{R,\varepsilon,m}^h |_{\partial\Omega}.$$

Here $u_{R,\varepsilon,m}^\lambda$ is the unique solution to (126) assuming $\lambda \notin \operatorname{spec}(\mathcal{A}_{R,\varepsilon,m})$ and ∂_ν is the normal (radial) derivative at $\partial\Omega$.

Recall that by Lemma 2.3 the DN map Λ_{out}^{λ} defined in (42) corresponds to the boundary measurements for equation (10). In the following we assume, for technical simplicity, that $b \in C^1(\Omega, \mathbb{R}^3)$.

Corollary 4.2. *Assume that the magnetic field b is C^1-smooth on $\overline{\Omega}$ and $h \in H^{3/2}(\partial\Omega)$, $[a,b] \cap spec(\mathcal{A}_1) = \emptyset$. Then, uniformly with respect to $\lambda \in [a,b]$,*

$$(128) \qquad \lim_{R \to 1} \lim_{m \to \infty} \lim_{\varepsilon \to 0} \left(\Lambda_{R,m,\varepsilon}^{\lambda} h - \Lambda_1^{\lambda} h\right) = 0,$$

where the convergence is strong in the $H^{1/2}(\partial\Omega)$ topology.

Proof. Let $h^e \in H^2(\Omega)$, $\mathrm{supp}(h^e) \subset \{x : 7/3 \leq |x| \leq 3\}$, be an extension of h, that is,

$$(129) \qquad h^e|_{\partial\Omega} = h, \quad \|h^e\|_{H^2(\Omega)} \leq C, \quad \text{if } \|h^e\|_{H^{3/2}(\partial\Omega)} \leq 1.$$

Then the solution $u_{R,m,\varepsilon}^{\lambda}(h)$ to (127) may be represented as

$$u_{R,m,\varepsilon}^{\lambda}(h) = h^e + v_{R,m,\varepsilon}^{\lambda}(h), \quad u_1^{\lambda}(h) = h^e + v_1^{\lambda}(h);$$

where

$$(130) \quad v_1^{\lambda} = -(\mathcal{A}_1 - \lambda)^{-1} F^{\lambda}(h),$$
$$v_{R,m,\varepsilon}^{\lambda}(h) = -(\mathcal{B}_{R,m,\varepsilon} - \lambda g_m^{1/2})^{-1} F^{\lambda}(h) = -(\mathcal{A}_{R,m,\varepsilon} - \lambda)^{-1} F^{\lambda}(h),$$
$$F^{\lambda}(h) = -\nabla^b \cdot \nabla^b h^e - \lambda h^e \in L^2(\Omega), \quad \mathrm{supp}(F^{\lambda}(h)) \subset \overline{\Omega} \setminus B(\tfrac{7}{3}).$$

Here we use the fact that, in the layer $\overline{\Omega} \setminus B(\tfrac{7}{3})$, $\sigma_{R,\varepsilon}^{ij} = \sigma_1^{ij} = \delta^{ij}$, $\quad g_m = g = 1$. By Theorem 4.1,

$$\lim_{R \to 1} \lim_{m \to \infty} \lim_{\varepsilon \to 0} \|v_{R,m,\varepsilon}^{\lambda}(h) - v_1^{\lambda}(h)\|_{L^2(\{7/3 \leq |x| \leq 3\})} = 0.$$

It then follows from [19, Thm. 9.13] that

$$\lim_{R \to 1} \lim_{m \to \infty} \lim_{\varepsilon \to 0} \|v_{R,m,\varepsilon}^{\lambda}(h) - v_1^{\lambda}(h)\|_{H^2(\{8/3 \leq |x| \leq 3\})} = 0,$$

so that, by trace theorem for Sobolev spaces, we have

$$(131) \quad \lim_{R \to 1} \lim_{m \to \infty} \lim_{\varepsilon \to 0} (\partial_\nu + i\nu \cdot b)\left(u_{R,m,\varepsilon}^{\lambda}(h) - u_1^{\lambda}(h)\right)|_{\partial\Omega} = 0,$$

strongly in $H^{1/2}(\partial\Omega)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Our next goal is to show the existence of a sequence of non-singular isotropic mass densities and non-singular bulk moduli, uniformly bounded (in n) from above but not from below, with the corresponding acoustic operators approximating the singular acoustic operator $\mathcal{A}_1 - \lambda$.

Theorem 4.3. *There exist sequences* $R(n) \to 1$, $m(n) \to \infty$, *and* $\varepsilon(n) \to 0$ *such that, for any* $f \in L^2(\Omega)$, $\lambda \in \mathbb{C}$, $\mu \in \mathbb{R}$, $\lambda, \mu \notin spec(\mathcal{A}_1)$,

(132) $\displaystyle\lim_{n\to\infty} (\mathcal{A}_{R(n),m(n)\varepsilon(n)} - \lambda)^{-1} f = (\mathcal{A}_1 - \lambda)^{-1} f \quad in \ \ L_g^2(\Omega),$

(133) $\displaystyle\lim_{n\to\infty} P_{R(n),m(n),\varepsilon(n)}^\mu f = P_1^\mu f \quad in \ \ L_g^2(\Omega),$

(134) $\displaystyle\lim_{n\to\infty} dim(N_{R(n),m(n),\varepsilon(n)}(\mu)) = dim(N_1(\mu)).$

For compact $K \subset \mathbb{C}$ *with* $K \cap spec(\mathcal{A}) = \emptyset$, *for* n *sufficiently large one has* $K \cap spec(\mathcal{A}_{R(n),m(n)\varepsilon(n)}) = \emptyset$ *and the limits are uniform in* $\lambda \in K$ *and* $\mu \in K \cap \mathbb{R}$.

Proof. Let $\{f_p : p = 1, 2, \dots\}$ be a dense set in $L^2(\Omega)$. Let $K_n \subset \mathbb{C}$, $n = 1, 2, \dots$, be a sequence of compact sets so that

(135) $K_n \cap \mathrm{spec}(\mathcal{A}_1) = \emptyset, \quad K_n \subset K_{n+1}^{int},$

$$\bigcup_{n=1}^\infty K_n = \mathbb{C} \setminus \mathrm{spec}(\mathcal{A}_1), \quad \mathrm{dist}(\mathrm{spec}(\mathcal{A}_1), K_n) \geq n^{-1/2}.$$

As $L^2(\Omega) \subset L_g^2(\Omega)$, it follows from Lemmas 2.8 and 2.9, that for any $n \in \mathbb{Z}_+$, there exists a $R(n) > 1$ such that

(136) $\|(\mathcal{A}_{R(n)} - \lambda)^{-1} f_p - (\mathcal{A}_1 - \lambda)^{-1} f_p\|_g < \dfrac{1}{n},$

$\dim(N_{R(n)}(\mu)) = \dim(N_1(\mu)), \quad \mathrm{dist}(\mathrm{spec}\,(\mathcal{A}_{R(n)}), K_n) \geq n^{-1/2}/2,$

$\|P_{R(n)}^\mu - P_1^\mu\|_{L_g^2 \to L_g^2} \leq \dfrac{1}{n}, \quad \|P_{R(n)}^\mu f_p - P_1^\mu f_p\|_{L_g^2} \leq \dfrac{1}{n},$

for all $p = 1, \dots, p_n$, $\lambda \in K_n$, $\mu \in K_n \cap \mathbb{R}$. Here p_n is defined so that

$$\{f \in L^2(\Omega) : \|f\|_{L^2(\Omega)} \leq n\} \subset N_{1/n}(\{f_p\}_{p=1}^{p_n}),$$

where $N_\varepsilon(S)$ denotes the $\varepsilon-$neighborhoods of S in $L^2(\Omega)$.

Having chosen $R(n)$, using (8) and Lemma 2.11 and Corollary 2.12, we choose $m(n)$ such that

(137) $\|(\mathcal{A}_{R(n),m(n)} - \lambda)^{-1} f_p - (\mathcal{A}_{R(n)} - \lambda)^{-1} f_p\|_g < \dfrac{1}{n},$

$\dim(N_{R(n),m(n)}(\mu)) = \dim(N_1(\mu)),$

$\mathrm{dist}(\mathrm{spec}\,(\mathcal{A}_{R(n),m(n)}), K_n) \geq n^{-1/2}/3,$

$\|P_{R(n),m(n)}^\mu - P_{R(m)}^\mu\|_{L^2 \to L_g^2} \leq \dfrac{1}{n}, \quad \|P_{R(n),m(n)}^\mu f_p - P_{R(n)}^\mu f_p\|_{L_g^2} \leq \dfrac{1}{n},$

for all $p = 1, \dots, p_n$, $\lambda \in K_n$, $\mu \in K_n \cap \mathbb{R}$. Having chosen $R(n)$ and $m(n)$, one can now use (8) and Lemmas 3.3 and 3.4 to choose $\varepsilon(n) > 0$

so that,

$$(138) \qquad \|(\mathcal{A}_{R(n),m(n),\varepsilon(n)}-\lambda)^{-1}f_p-(\mathcal{A}_{R(n),m(n)}-\lambda)^{-1}f_p\|_g<\frac{1}{n},$$

$$\dim(N_{R(n),m(n),\varepsilon(n)}(\mu)) = \dim(N_{R(n),m(n)}(\mu)),$$

$$\mathrm{dist}(\mathrm{spec}\,(\mathcal{A}_{R(n),m(n),\varepsilon(n)}),K_n) \geq n^{-1/2}/4,$$

$$\|P^\mu_{R(n),m(n),\varepsilon(n)}f_p - P^\mu_{R(n),m(n)}f_p\|_{L^2_g} \leq \frac{1}{n},$$

for all $p = 1,\dots,p_n$, $\lambda \in K_n$ and $\mu \in K_n \cap \mathbb{R}$; and,

$$(139) \qquad \|(\mathcal{A}_{R,m,\varepsilon} - \lambda)^{-1}\|_{L^2(\Omega,g_m^{1/2}dx)\to L^2(\Omega,g_m^{1/2}dx)} \leq 6n^{1/2}, \quad \lambda \in K_n,$$

cf. (137) and Lemma 3.3 (i).

Clearly, equations (136), (137) and (138) imply that

$$(140) \qquad \|(\mathcal{A}_{R(n),m(n),\varepsilon(n)}-\lambda)^{-1}f_p-(\mathcal{A}_1-\lambda)^{-1}f_p\|_g<\frac{3}{n},$$

$$\dim(N_{R(n),m(n),\varepsilon(n)}(\mu)) = \dim(N_1(\mu)),$$

$$\|P^\mu_{R(n),m(n),\varepsilon(n)}f_p - P^\mu_1 f_p\|_{L^2_g} \leq \frac{3}{n},$$

for all $p = 1,\dots,p_n$, $\lambda \in K_n$, $\mu \in K_n \cap \mathbb{R}$.

In particular, (140) implies (134) due to $\cup_n K_n = \mathbb{C} \setminus \mathrm{spec}(\mathcal{A}_1)$. It remains to show that, for $f \in L^2(\Omega)$ and a compact set K such that $K \cap \mathrm{spec}(\mathcal{A}_1) = \emptyset$, one has that $(\mathcal{A}_{R(n),m(n),\varepsilon(n)} - \lambda)^{-1}f$ converge to $(\mathcal{A}_1 - \lambda)^{-1}f$ in $L^2_g(\Omega)$ and that convergence is uniform for all $\lambda \in K$. Indeed, this will provide equation (132). Using the Riesz formula for $P^\mu_{R(n),m(n),\varepsilon(n)}$, P^μ_1, this will also prove equation (133).

Let $n_0 \in \mathbb{Z}_+$ be such that $K \subset K_n$ for all $n \geq n_0$. Furthermore, for $n \geq n_0$, there exists $f_{p(n)}$ with $p(n) \leq p_n$ such that

$$(141) \qquad \|f - f_{p(n)}\|_{L^2(\Omega)} < \frac{1}{n}.$$

Clearly, for $\lambda \in K$,

$$(142) \quad \|(\mathcal{A}_{R(n),m(n),\varepsilon(n)} - \lambda)^{-1}f - (\mathcal{A}_1 - \lambda)^{-1}f\|_g$$

$$\leq \|(\mathcal{A}_{R(n),m(n),\varepsilon(n)} - \lambda)^{-1}f_{p(n)} - (\mathcal{A}_1 - \lambda)^{-1}f_{p(n)}\|_g$$

$$+\|(\mathcal{A}_{R(n),m(n),\varepsilon(n)} - \lambda)^{-1}(f_{p(n)} - f)\|_g + \|(\mathcal{A}_1 - \lambda)^{-1}(f_{p(n)} - f)\|_g.$$

By (140), the first term in the right-hand side of (142) is bounded by $3/n$. To estimate the second term, we use the estimates (66), (139), and (141), which imply that this term s bounded by $12(2/n)^{1/2}$ for

$n \geq n_0$. Finally, by (8) and (135),

$$\|(\mathcal{A}_1 - \lambda)^{-1}\|_{L^2 \to L_g^2} \leq \sqrt{8}\, \|(\mathcal{A}_1 - \lambda)^{-1}\|_{L_g^2 \to L_g^2} \leq \frac{\sqrt{8}}{\mathrm{dist}\,(\lambda, \mathrm{spec}\,(\mathcal{A}_1))} \leq \sqrt{8n},$$

for $\lambda \in K$. Thus by (141), the third term on the right-hand side of (142) is bounded by $(8/n)^{1/2}$ for $n \geq n_0$.

Summarizing, we see that the left-hand side of (142) tends to 0, as $n \to \infty$, uniformly for $\lambda \in K$, thus proving (132). $\square$

In the sequel, we will use abbreviated notations

$$(143) \qquad \sigma_{(n)} := \sigma_{R(n),\varepsilon(n)}, \quad g_{(n)} := g_{m(n)}, \quad \mathcal{A}_{(n)} := \mathcal{A}_{R(n),m(n),\varepsilon(n)}$$

for the sequences $R(n), m(n), \varepsilon(n)$ obtained in Theorem 4.3. For simplicity, denote by $\Lambda_{(n)}^\lambda$, rather than $\Lambda_{R(n),m(n),\varepsilon(n)}^\lambda$, the corresponding DN map.

Regarding the convergence of the DN-maps, by the same arguments as in proof of Corollary 4.2, we obtain

Corollary 4.4. *Assume that the magnetic field b is C^1-smooth in $\overline{\Omega}$. Let $K \subset \mathbb{C}$ be compact and such that $K \cap spec(\mathcal{A}_1) = \emptyset$. Then, for any $h \in H^{3/2}(\partial\Omega)$,*

$$(144) \qquad\qquad \lim_{n \to \infty} \Lambda_{(n)}^\lambda h = \Lambda_{out}^\lambda h,$$

where the limit is in $H^{1/2}(\partial\Omega)$, uniformly with respect to $\lambda \in K$.

In particular, when b and $q|_{\Omega \setminus B(1)}$ vanish, Λ_{out}^λ corresponds to the measurements on the boundary of a homogeneous ball. Thus (144) means that the isotropic material parameters $\sigma_{(n)}$ and $g_{(n)}$ approximate an acoustic invisibility cloak as $n \to \infty$.

5. Approximate quantum cloaking

The results of the previous sections can now be used to obtain approximate quantum cloaking at a fixed energy, for any potential $Q \in L^\infty(\Omega)$ supported inside the cloaked region $B(1)$.

In sequel, we assume that the conductivities σ_R satisfy (23) and that also $\sigma_R \in C^2(\overline{\Omega}; \mathbb{R}^2)$. This yields that also $\sigma_{R,\varepsilon}, \sigma_{(n)} \in C^2(\overline{\Omega}; \mathbb{R}^2)$.

Let $E \in \mathbb{R}$ be a given energy level and

$$(145) \qquad\qquad q^E(x) := \frac{1}{4}Q(x) + \frac{3}{4}E\chi_1(x),$$

where χ_1 is the indicator function of the ball $B(1)$. In this section we assume that E and Q are such that E is not in the spectrum of the operator $\mathcal{A}_1 = \mathcal{A}_1^E$ defined in (19) using the potential q^E. Observe that $\mathrm{spec}(\mathcal{A}_1^E) = \mathrm{spec}(A_{out}) \cup \mathrm{spec}(A_{in}^E)$, where A_{out} and A_{in}^E are defined by (40) in (41) using potential q^E. Note that A_{out} is independent of E and $A_{in}^E = \frac{1}{4}S_{in} + \frac{3}{4}E$, where S_{in} is the Schrödinger operator

$$S_{in} = -\Delta + Q, \quad \mathcal{D}(S_{in}) = \{v_2 \in H^2(B(1)) : \ \partial_\nu v|_{\partial B(1)} = 0\}.$$

Observe that

(146) $\quad E \notin \mathrm{spec}(\mathcal{A}_1^E) \quad \Longleftrightarrow \quad E \notin \mathrm{spec}(S_{in})$ and $E \notin \mathrm{spec}(A_{out})$.

Then for n large enough, $E \notin \mathrm{spec}(\mathcal{A}_{(n)})$. Next, consider the solutions u_n of

(147) $\quad \left(-g_{(n)}^{-1/2}\nabla^b \cdot \sigma_{(n)}\nabla^b + q^E - E \right) u_n = 0 \quad \text{in} \ \ \Omega, \quad u_n|_{\partial\Omega} = h.$

This equation can be converted to a Schrödinger equation with magnetic potential using the gauge transformation,

(148) $$\psi_n(x) = \sigma_{(n)}^{1/2}(x)u_n(x).$$

Then

(149) $\quad -\sigma_{(n)}^{-1/2}\nabla^b \cdot \sigma_{(n)}\nabla^b(\sigma_{(n)}^{-1/2}\psi_n) = -\nabla^b \cdot \nabla^b \psi_n + W_{(n)}\psi_n,$

where

(150) $\quad W_{(n)} = \sigma_{(n)}^{-1/2}\Delta \left(\sigma_{(n)}^{1/2}\right), \quad \mathrm{supp}(W_{(n)}) \subset \overline{B(2)} \setminus B(1).$

Thus, using the transformation (148) we see that the acoustic equation (147) for u_n is equivalent to the Schrödinger equation for ψ_n,

(151) $\quad \left(-\nabla^b \cdot \nabla^b + W_{(n)} + \frac{g_{(n)}^{1/2}}{\sigma_{(n)}} \left(q^E - E \right) \right) \psi_n = 0, \quad \psi_n|_{\partial\Omega} = h,$

where for the boundary condition we use $\psi_n|_{\partial\Omega} = u_n|_{\partial\Omega} = h$, since $\sigma_{(n)} = 1$ near $\partial\Omega$.

Next, define the *cloaking potential*

(152) $\quad V_n^E(x) \ := \ W_{(n)}(x) + \dfrac{g_{(n)}^{1/2}(x)}{\sigma_{(n)}(x)} \left(\dfrac{3E}{4}\chi_1(x) - E \right) + E$

$\qquad\qquad = \ W_{(n)}(x) + E(1 - \chi_1(x))\left(1 - \dfrac{g_{(n)}^{1/2}(x)}{\sigma_{(n)}(x)} \right),$

where we have used the fact that $g_{(n)}^{1/2}(x) = 8$, $\sigma_{(n)}(x) = 2$ for $|x| < 1$. Thus $V_n^E = 0$ in $B(1)$. Clearly, V_n^E vanishes also near $\partial\Omega$. Then (151) can be written as

$$(153) \quad \left(-\nabla^b \cdot \nabla^b + V_n^E(x) + Q - E\right)\psi_n = 0, \quad \psi_n|_{\partial\Omega} = h.$$

Now we are ready to prove our main result concerning approximate cloaking in quantum mechanics.

Theorem 5.1. *Assume that $Q \in L^\infty(\Omega)$ is a function supported in $B(1)$, $b \in C^1(\Omega, \mathbb{R}^3)$, and $E \in \mathbb{R}$ are such that $E \notin spec(A_{out}) \cup spec(S_{in})$. Then for any $h \in H^{3/2}(\partial\Omega)$,*

$$(154) \qquad \lim_{n\to\infty} \Lambda_{V_n^E+Q}^E h = \Lambda_{out}^E h \quad in \ H^{1/2}(\partial\Omega).$$

Here $\Lambda_{V_n^E+Q}^E$ are the DN maps,

$$\Lambda_{V_n^E+Q}^E : h \mapsto \partial_\nu \psi_n|_{\partial\Omega}; \ where \, (-\nabla^b \cdot \nabla^b + V_n^E + Q - E)\psi_n = 0, \ \psi_n|_{\partial\Omega} = h,$$

while the DN-map Λ_{out}^E corresponds to the operator A_{out} with $\kappa_1 = 0$, see (40), (42).

Proof.   By the hypotheses of the Theorem, it follows that $E \notin$ spec$((\mathcal{A}_1^E)$, where $\mathcal{A}_1$ defined by (19) with magnetic potential b and the potential q^E given by (145). Thus, the Dirichlet problem (147) is uniquely solvable for large n.   As the gauge transformation (148) is the identity map near $\partial\Omega$, we see that $\partial_\nu \psi_n|_{\partial\Omega} = \partial_\nu u_n|_{\partial\Omega}$ and $\psi_n|_{\partial\Omega} = u_n|_{\partial\Omega}$. The DN maps for the Schrödinger equation (153) and for equation (147) thus coincide, and the assertion follows from Corollary 4.4.   □

Note that Theorem 5.1 is of a very different nature than the well-known results from the classical theory of spectral convergence, since the cloaking potentials V_n^E do not tend to 0 as $n \to \infty$. On the contrary as seen from the construction of $\sigma_{R,\varepsilon}$ in Sec. 3.2 and definition (150), (152), $\sup_x |V_n^E(x)| \to \infty$ as $n \to \infty$. Moreover, V_n^E is of a highly oscillatory nature in $B(2) \setminus B(1)$ with quasiperiod tending to 0 as $n \to \infty$.

Theorem 5.1 has two important physical consequences; see [25, 26] for further discussion and applications. Consider separately two cases:

- Suppose that $b = 0$. Since Q is supported in $\overline{B}(1)$, the operator $A_{out} = -\Delta$ is the free Schrödinger operator. Then, from a physical point of view, the potentials $V_n^E + Q$ can be considered as *almost transparent potentials* at energy E. Also, the

V_n^E, which depend on E but are independent of Q, serve as *approximate invisibility cloaks* for two-body scattering in quantum mechanics. As all measurement devices have limited precision, we can interpret this as saying that, given a specific device, one can design, for a given energy level E, a potential to cloak an object *i.e.*, an arbitrary potential, from any single-particle measurements made at this energy E.

• Now suppose that $Q = 0$, while $b \in C^1(\Omega, \mathbb{R}^3)$, $b \neq 0$. We now have $A_{out} = -\nabla^{\beta_1} \cdot \nabla^{\beta_1}$, and, due to the transformation rules for the magnetic potentials, β_1 is in general no longer bounded near O. Thus, the potentials V_n^E act as devices which give an external observer the illusion that, as $n \to \infty$, the magnetic field is unbounded near O. In particular, for $b(x) = B_0 \times x$, $B_0 \in \mathbb{R}^3$, corresponding to a homogeneous magnetic field, the illusion $\beta_1(x)$ has a singularity of the order $|x|^{-1}$ at O; see [25] for details.

6. DN map near exceptional values of E

Theorem 5.1 shows that the behavior of $\Lambda_{V_n^E+Q}^E$, when E is far from $\mathrm{spec}(\mathcal{A}_{in}^E)$ and n is large, well approximates the behavior of Λ_{out}^E. This situation changes dramatically when E is close to an eigenvalue of the cloaked region.

For simplicity, let us consider the case of an $E^0 \in \mathbb{R}$ which, while being in the resolvent set of A_{out}, is also a simple eigenvalue of $A_{in}^{E^0}$, that is, $E^0 \in \mathrm{spec}(S_{in})$, see (146). This implies that E^0 is a simple eigenvalue of $\mathcal{A}_1^{E^0}$. The corresponding eigenfunction u^0 then satisfies $\mathrm{supp}\,(u^0) \subset \overline{B}(1)$, *i.e.*, u^0 is a trapped state supported in the cloaked region, *cf.* Lemma 2.3 and (43).

In the following, let $d \in (0,1)$ be such that $\mathrm{dist}(E^0, spec(\mathcal{A}_1^{E^0}) \backslash E^0) > d$.

Theorem 6.1. *Let $Q \in L^\infty(\Omega)$ be a function supported in $\overline{B}(1)$, $b \in C^1(\Omega, \mathbb{R}^3)$, and $E^0 \in \mathbb{R}$ be an eigenvalue of $\mathcal{A}_1^{E^0}$ corresponding to potentials b and q^{E_0} defined in (145). Assume that E^0 has multiplicity one, and let u^0 be the corresponding eigenfunction supported in $\overline{B}(1)$. Then*

(i) There is a sequence $E_{(n)}$, $E_{(n)} \to E^0$ as $n \to \infty$, such that $E_{(n)}$ are simple Dirichlet eigenvalues of the Schrödinger operators $-\nabla^b \cdot \nabla^b + V_{(n)}^{E_{(n)}} + Q$. Moreover, the $L^2(\Omega)$-normalized eigenfunctions $\phi_{(n)}$ of these

Schrödinger operators for the eigenvalues $E_{(n)}$ satisfy, for any $\rho > 2$,

$$(155) \qquad \lim_{n \to \infty} \phi_{(n)}|_{\overline{\Omega} \setminus B(\rho)} = 0, \quad in \ C^1(\overline{\Omega} \setminus B(\rho)).$$

(ii) Let $h \in H^{3/2}(\partial\Omega)$ and $\psi_{(n)}(h)$ be the solution to (153) for some E with $0 < |E - E_{(n)}| < d/2$. Then

$$(156) \quad (\partial_\nu + i\nu \cdot b) \, \psi_{(n)}(h)|_{\partial\Omega} \ = \ \frac{\alpha^0_{(n)}(h)}{E - E_{(n)}} \, (\partial_\nu + i\nu \cdot b) \, \psi_{(n)}|_{\partial\Omega} + p^E_{(n)}(h),$$

$$(157) \qquad\qquad \alpha^0_{(n)}(h) \ = \ \int_{\partial\Omega} h \frac{\overline{\partial\psi_{(n)}}}{\partial\nu} \, dS,$$

and functions $p^E_{(n)}(h)$ are uniformly bounded, as $n \to \infty$, in $H^{1/2}(\partial\Omega)$) for $\|h\|_{H^{3/2}} \leq 1$ and $E \in (E^0 - d/4, E^0 + d/4)$.

Proof. For a given potential Q, the potential q^E defined in (145) depends on E. Thus we start by analyzing how the eigenvalues and eigenfunctions of $\mathcal{A}^E_1$ and $\mathcal{A}^E_{(n)}$ change relative to the variation of E. Denote by $\lambda(k, E)$, $k = 1, 2, \ldots$, the eigenvalues of $\mathcal{A}^E_1$, numbered in increasing order and taking multiplicity into account. The dependence of these eigenvalues on E then follows from (146). Similarly, let $\lambda_{(n)}(k, E)$ be the eigenvalues of $\mathcal{A}^E_{(n)}$, where $\mathcal{A}^E_{(n)}$ is the operator of form (143) with $q = q^E$. Observe that

$$(158) \quad \mathcal{A}^E_1 = \mathcal{A}^{E^0}_1 + (q^E - q^{E^0}), \quad \mathcal{A}^E_{(n)} = \mathcal{A}^{E^0}_{(n)} + (q^E - q^{E^0}).$$

Using Kato-Rellich formula [33, Thm. VII.3.6] and the fact that $\|q^{\widetilde{E}} - q^E\|_{L^\infty} = \frac{3}{4}|\widetilde{E} - E|$, see (145), we obtain that

$$(159) \qquad\qquad |\lambda_{(n)}(k, \widetilde{E}) - \lambda_{(n)}(k, E)| \leq \frac{3}{4}|\widetilde{E} - E|.$$

In the future we will consider only the value $k = k^0$ such that $\lambda(k^0, E^0) = E^0$, writing, *e.g.*, $\lambda_{(n)}(E)$ for $\lambda_{(n)}(k^0, E)$. Next, consider the spectral projectors for $\mu = E^0 - d/2$ or $\mu = E^0 + d/2$. Let $P^\mu_1(E)$ be the Riesz projectors for the operators $\mathcal{A}^E_1$ and $P^\mu_{(n)}(E)$ be the projectors for $\mathcal{A}^E_{(n)}$. They are defined analogously to (62) using a contour $\Gamma \subset \mathbb{C}$ that surrounds all of the eigenvalues smaller than μ. We can assume that Γ is such a contour that, for n large enough, the distance from Γ to the eigenvalues of the operators $\mathcal{A}^{E^0}_1$ and $\mathcal{A}^{E^0}_{(n)}$ is more than $d/4$. Then the norm of $(\mathcal{A}^{E^0}_{(n)} - z)^{-1}$ in $L^2(\Omega, g^{1/2}_{(n)}dx)$ is bounded by $4/d$. Thus,

assuming that $|E - E^0| < d/8$, we obtain, using the formula

$$(\mathcal{A}_{(n)}^E - z)^{-1} - (\mathcal{A}_{(n)}^{E^0} - z)^{-1}$$
$$= (\mathcal{A}_{(n)}^{E^0} - z)^{-1} \left(\left(I - (q^E - q^{E^0})(\mathcal{A}_{(n)}^{E^0} - z)^{-1} \right)^{-1} - I \right)$$

in (62), the estimate

$$(160) \qquad \|P_{(n)}^\mu(E) - P_{(n)}^\mu(E^0)\|_{L^2(g_{(n)}^{1/2}dx) \to L^2(g_{(n)}^{1/2}dx)} \leq C_\mu |E - E^0|,$$

where C_μ depends only on the choice of Γ.

As E^0 is the only eigenvalue of $\mathcal{A}_1^{E^0}$ in the interval $(E^0 - d, E^0 + d)$ and has the multiplicity one, it follows from Theorem 4.3 that when n is large enough, then $\mathcal{A}_{(n)}^{E^0}$ has only one eigenvalue E_n^0 in the interval $(E^0 - 3d/4,\ E^0 + 3d/4)$ and $|E_n^0 - E^0| < d/4$. Moreover, the eigenvalue E_n^0 is simple.

Let us show that there are $E_{(n)}$, $E_{(n)} \to E^0$, such that

$$(161) \qquad \lambda_{(n)}(E_{(n)}) = E_{(n)}.$$

Observe that, by Theorem 4.3, $\lambda_{(n)}(E^0) \to E^0$ as $n \to \infty$. Together with (159) this implies that for any $\varepsilon > 0$ there is $n(\varepsilon)$ such that for $n > n(\varepsilon)$,

$$\lambda_{(n)}(\cdot) : [E^0 - \varepsilon,\ E^0 + \varepsilon] \to [E^0 - \varepsilon,\ E^0 + \varepsilon].$$

As $\lambda_{(n)}(\cdot)$ is a contraction, see (159), we conclude by the Banach fixed point theorem that there is a unique $E_{(n)}$ satisfying (161).

Returning to equation (160) we see that

$$(162) \qquad \lim_{n \to \infty} \|P_{(n)}^\mu(E_{(n)}) - P_{(n)}^\mu(E^0)\|_{L^2(g_{(n)}^{1/2}dx) \to (g_{(n)}^{1/2}dx)} = 0,$$

Combining this with (133) (for $q = q^{E^0}$) and embedding (66), we see

$$(163) \qquad \lim_{n \to \infty} P_{(n)}^\mu(E_{(n)})f = P_1^\mu(E^0)f \quad \text{in} \quad L_g^2(\Omega)$$

Let $\mu = E + d/2$ and $\nu = E - d/2$. For large n, the operators $P_{(n)}^\mu(E_{(n)}) - P_{(n)}^\nu(E_{(n)})$ are the orthoprojectors, in $L^2(\Omega, g_{(n)}^{1/2}dx)$ onto the eigenspace of $\mathcal{A}_{(n)}^{E_{(n)}}$ corresponding to the eigenvalue $E_{(n)}$. Using (162) we see that for n large enough, the eigenvalue $E_{(n)}$ has multiplicity one. Denote by $\widetilde{u}_{(n)}$ the eigenfunction corresponding to $E_{(n)}$, normalized in $L_g^2(\Omega)$. Now u^0 is supported in $B(1)$ and thus

$$\|u^0\|_g = \|u^0\|_{L^2(\Omega, g_{(n)}^{1/2}dx)} = 1,\ n > 0, \quad \|u^0\|_{L^2(\Omega)} = \frac{1}{\sqrt{8}}.$$

Using (163) we see that

$$\left(P_{(n)}^\mu(E_{(n)}) - P_{(n)}^\nu(E_{(n)})\right) u^0 = a_n \widetilde{u}_{(n)} \xrightarrow[n\to\infty]{} \left(P_1^\mu(E^0) - P_1^\nu(E^0)\right) u^0 = u^0,$$

in $L_g^2(\Omega)$, where $a_n = |a_n| e^{i\alpha_n}$ and $|a_n| \to 1$ as $n \to \infty$.

Denoting $u_{(n)} = e^{i\alpha_n} \widetilde{u}_{(n)}$, we see that

(164) $$\lim_{n\to\infty} u_{(n)} = u^0 \quad \text{in } L_g^2(\Omega).$$

Since $u^0(x) = 0$, $|x| > 1$, this implies

(165) $$\lim_{n\to\infty} \|u_{(n)}|_{\Omega\setminus B(2)}\|_{L^2(\Omega\setminus B(2))} = 0.$$

Observe that $\sigma_{(n)} = \gamma_0$, $g_{(n)} = 1$ and $V_{(n)}^E = 0$ in $\Omega \setminus B(2)$. Thus it follows from (165) that the functions $u_{(n)}$ satisfy

$$-\nabla^b \cdot \nabla^b u_{(n)} = E_{(n)} u_{(n)} \quad \text{in } \Omega \setminus B(2), \quad u_{(n)}|_{\partial\Omega} = 0,$$

where the right side $E_{(n)} u_{(n)} \to 0$ in $L^2(\Omega \setminus B(2))$ as $n \to \infty$. Since $b \in C^1(\Omega \setminus B(2))$, standard elliptic regularity results [19] imply that

(166) $$\lim_{n\to\infty} \|u_{(n)}\|_{C^1(\Omega\setminus B(\rho))} = 0,$$

for any $2 < \rho < 3$. Using the transformation (148) to define $\phi_n(x) = \sigma_{(n)}^{1/2}(x) u_n(x)$, we see that

$$(-\nabla^b \cdot \nabla^b + V_n^{E_{(n)}} + Q)\phi_n = E_{(n)}\phi_n, \quad \phi_n|_{\partial\Omega} = 0.$$

This proves that ϕ_n is an eigenfunction of the Schrödinger operator $-\nabla^b \cdot \nabla^b + V_n^{E_{(n)}} + Q$ for the eigenvalue $E_{(n)}$. Moreover, since $\sigma_{(n)} = 2$ in $B(1)$, it follows from (8) and (164) that

$$\liminf_{n\to\infty} \|\phi_n\|_{L^2(\Omega)} \geq \sqrt{2} \liminf_{n\to\infty} \|u_n\|_{L^2(B(1))} \geq \sqrt{2}\|u^0\|_{L^2(B(1))} = \frac{1}{2}.$$

This inequality and equation (166) together imply (155). Thus we have proven (i).

Next, consider (ii). We start with the boundary-value problem for the acoustic equation

$$\left(-g_{(n)}^{-1/2}\nabla^b \cdot \sigma_{(n)}\nabla^b + V_n^E + q^E + E\right) v_n^E(h) = 0, \quad v_n^E(h)|_{\partial\Omega} = h.$$

Clearly

(167) $$v_n^E(h) = a_n^E(h) u_{(n)} + w_n^E(h), \quad \text{with } \left(w_n^E(h), u_{(n)}\right)_{L^2(\Omega, g_{(n)}^{1/2}dx)} = 0,$$

where, using the notations introduced in (129), (130), we have

$$(168) \quad a_n^E(h) = \left(h^e,\, u_{(n)}\right)_{L^2(g_{(n)}^{1/2}dx)} + \frac{1}{E - E_{(n)}} \left(F^E(h),\, u_{(n)}\right)_{L^2(g_{(n)}^{1/2}dx)},$$

$$w_n^E(h) = h^e - (h^e,\, u_{(n)})_{L^2(g_{(n)}^{1/2}dx)}\, u_{(n)} - \widetilde{w}_n^E(h),$$

$$\widetilde{w}_n^E(h) = (\mathcal{A}_{(n)} - E)^{-1} \left(F^E(h) - (F^E(h),\, u_{(n)})_{L^2(g_{(n)}^{1/2}dx)}\, u_{(n)} \right).$$

Since $F^E(h) - (F^E(h),\, u_{(n)})\, u_{(n)}$ and $u_{(n)}$ are orthogonal in $L^2(\Omega, g_{(n)}^{1/2}dx)$, and $\mathrm{dist}(E, \mathrm{spec}(\mathcal{A}_{(n)}) \setminus \{E_{(n)}\}) > d/2$, it follows from (129) that, for n large enough,

$$\|\widetilde{w}_n^E(h)\|_{L^2(\Omega, g_{(n)}^{1/2}dx)} \leq C,$$

where C is independent of $E \in (E^0 - d/4, E^0 + d/4)$ and h satisfying $\|h\|_{H^{3/2}(\partial\Omega)} \leq 1$. Note that in $\Omega \setminus B(2)$ the function $\widetilde{w}_n^E(h)$ satisfies the equation

$$-\nabla^b \cdot \nabla^b \widetilde{w}_n^E(h) = E\widetilde{w}_n^E(h) + F^E(h) - (F^E(h),\, u_{(n)})_{L^2(g_m^{1/2}dx)}\, u_{(n)},$$

$$\widetilde{w}_n^E(h)|_{\partial\Omega} = 0.$$

Thus, by boundary elliptic regularity, [19, Thm. 9.13],

$$\|\widetilde{w}_n^E(h)\|_{H^2(\Omega \setminus B(\rho))} \leq C_\rho, \quad 2 < \rho < 3.$$

This inequality, (129) and (130) together imply that $p_{(n)}^E(h) = (\partial_\nu + i\nu \cdot b)w_n^E(h)$ satisfies

$$(169) \qquad \|p_{(n)}^E(h)\|_{H^{1/2}(\partial\Omega)} \leq C_0, \quad \text{for } \|h\|_{H^{3/2}(\partial\Omega)} \leq 1,$$

if n is large and $|E - E_{(n)}| \leq d/2$. Finally, integration by parts shows that

$$a_n^E(h) = \frac{1}{E - E_{(n)}} \int_{\partial\Omega} h \frac{\partial \overline{u_{(n)}}}{\partial \nu}\, dS = \frac{1}{E - E_{(n)}} \int_{\partial\Omega} h \frac{\partial \overline{\phi_{(n)}}}{\partial \nu}\, dS.$$

The desired equation (156) follows from the above equation together with (167), (169) and (157), if we take into the account the relation (148) between $\psi_{(n)}^E(h)$ and $v_n^E(h)$. $\qquad \square$

Remark. Theorem 6.1 means that, away from the cloaking structure, the eigenfunctions $\phi_{(n)}$ converge to zero as $n \to \infty$, *i.e.*, the $\phi_{(n)}$ represent *almost trapped modes*, effectively vanishing near $\partial\Omega$. Physically speaking, we can say that if E^0 is an eigenvalue of the Schrödinger operator with Neumann boundary condition in the cloaked region $B(1)$, and we connect the interior and the exterior via the cloaking potential V_n^E in the layer $B(R_{(n)}) \setminus B(1)$, with $R_{(n)} \to 1$ as $n \to \infty$, a particle

under the influence of the combined potential on $B(3)$ is still largely confined (modulo standard tunneling) to the interior region, with a slight shift of the energy of the eigenmode from E^0 to $E_{(n)}$.

Moreover, for energies E close to the values $E_{(n)}$ the presence of the cloaked region is very clearly seen in the boundary measurements of the DN map, so that the invisibility effect is compromised. On the other hand, at energies which are away from the $E_{(n)}$, the DN map for the potential $V_{(n)}^E + Q$ well approximates Λ_{out}^E, and thus the potential Q is approximately cloaked.
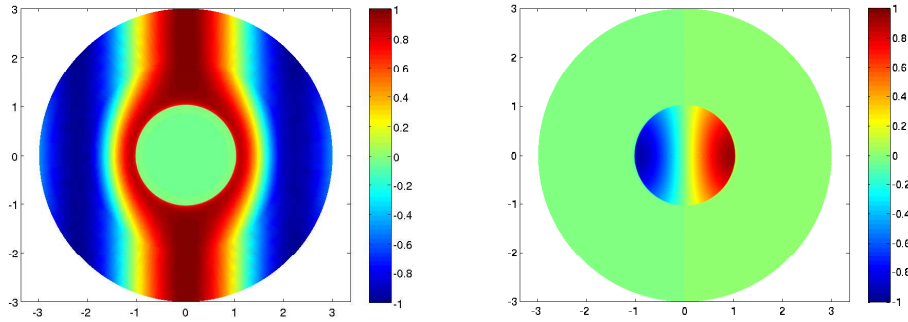
7. Numerical results

Figure 1. **Solutions of acoustic equations.** Left: Solution u^{tot} of the scattering problem (170) restricted to ball $B(3)$, when a plane wave scatters from an approximate cloak in the case (175), *i.e.*, when k^2 is away from the exceptional values $E_{(n)}$. Right: Almost trapped eigenfunction u of the acoustic operator (172), with Dirichlet boundary condition, $h = 0$, in the case (176), *i.e.*, when k^2 is equal to the exceptional value $E_{(n)}$.

Next we consider scattering problems for the Helmholtz and Schrödinger equations in the case when the magnetic potential vanishes, *i.e.*, $b = 0$.

The scattering problem for Helmholtz equation is

$$(170) \qquad (\nabla \cdot \sigma_{(n)} \nabla + k^2(1 + \alpha(x))g_{(n)}^{1/2})u^{tot} = 0 \quad \text{in} \ \ \mathbb{R}^3,$$
$$u^{tot}(x,k) = u^{in}(x,k) + u^{sc}(x,k)$$

and for the Schrödinger equation

$$(171) \qquad (-\nabla \cdot \nabla + V_n^E + Q - E)\psi^{tot} = 0 \quad \text{in} \ \ \mathbb{R}^3,$$
$$\psi^{tot}(x,E) = \psi^{in}(x,E) + \psi^{sc}(x,E),$$

where $E > 0$, $k = E^{1/2}$, the incident fields are $u^{in}(x, k) = \psi^{in}(x, E) = e^{ik\omega \cdot x}$, $|\omega| = 1$ and the scattered fields satisfy the radiation condition

$$\lim_{r \to \infty} r(\frac{\partial}{\partial r} - ik)u^{sc}(x, k) = \lim_{r \to \infty} r(\frac{\partial}{\partial r} - iE^{1/2})\psi^{sc}(x, E) = 0, \quad r = |x|.$$

In following we consider $\alpha(x)$ that corresponds to a real bounded potential Q supported in $B(1)$, that is, $\alpha(x) = -(E^{-1}Q(x) + 3)/4$. We assume that $Q(x)$ is such that $1 + \alpha(x) \geq c_0 > 0$.

We consider also the solutions of the boundary value problems in Ω (note that we denote these solutions by u and ψ, without using superscripts),

(172) $\quad \left(\nabla \cdot \sigma_{(n)}\nabla + k^2(1 + \alpha(x))g_{(n)}^{1/2}\right)u = 0 \quad \text{in } \Omega, \quad u|_{\partial\Omega} = h$

and for the Schrödinger equation

(173) $\quad\quad (-\nabla \cdot \nabla + V_n^E + Q - E)\psi = 0 \quad \text{in } \Omega, \quad u|_{\partial\Omega} = h.$

The solutions of these scattering and boundary value problems are related through a gauge transformation,

(174) $\quad\quad \psi^{tot}(x) = \sigma_{(n)}^{1/2}(x)u^{tot}(x), \quad \psi(x) = \sigma_{(n)}^{1/2}(x)u(x).$

The computations are made without reference to physical units; for simplicity, we use $E = 2$. The cloak corresponds to the parameter $R = 1.005$ and inside the cloak we have located a spherically symmetric potential;

$$Q(x) = Q_{in}\chi_{[0,R]}(|x|), \quad \text{that is,} \quad \alpha(x) = -Q_{in}(4E)^{-1}\chi_{[0,R]}(|x|) - 3/4.$$

To illustrate the approximate cloaking, we used

(175) $\quad\quad\quad\quad\quad\quad\quad\quad Q_{in} = 1,$

and to obtain an almost trapped state,

(176) $\quad\quad\quad\quad\quad\quad\quad\quad Q_{in} = -2.576.$

In our numerical solution we have approximated $\sigma_{(n)}$ by a piecewise constant function consisting of 30 layers in the region $R < r < 2$. The values of the conductivity in these layers are chosen as in the above sections. This corresponds to the case when the cloaking potential V_n^E is a weighted sum of delta functions, and their derivatives, on spheres.

In the numerical solution of the problem, we represent the solution u^{tot} and u in terms of spherical harmonics Y_m^n and Bessel functions up to order $N = 7$ in each layer where the cloaking conductivity is constant. The transmission condition on the boundaries of these layers are
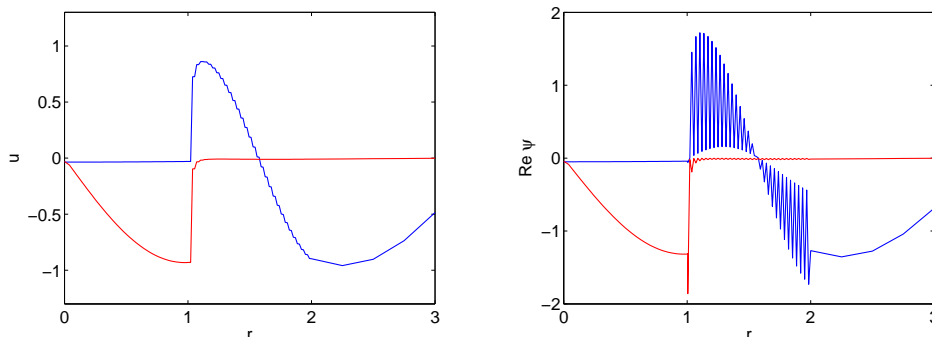
FIGURE 2. **Radial profiles of solutions of acoustic and Schrödinger equations.** Left: The solution u^{tot} of the scattering problem (170) on the line segment $L = \{(x, 0, 0) : \ x \in [0, 3]\}$ in the case of Fig. 1(left), where k^2 is far from the exceptional values $E_{(n)}$, is shown with the blue curve. Also, the eigenfunction u of acoustic operator (173) on the line segment L in the case of Fig. 1(right), where k^2 is the exceptional value $E_{(n)}$, is shown with the red curve. Right: The solutions ψ^{tot} and ψ of the Schrödinger equations (171) and (173) on the line segment L, obtained from the solutions on the left via gauge transformations (174).

solved numerically by solving linear equations. After this we compute the solution ψ^{tot} and ψ of the Schrödinger equation using the gauge transformation (174).

REFERENCES

[1] G. Allaire, Homogenization and two-scale convergence, *SIAM J. Math. Anal.* **23** (1992), no. 6, 1482–1518.

[2] G. Allaire, A. Damlamian and U. Hornung, Two-scale convergence on periodic surfaces and applications, In A. Bourgeat, C. Carasso, S. Luckhaus and A. Mikelic (eds.), *Mathematical Modelling of Flow through Porous Media*, 15-25, Singapore, World Scientific, 1995.

[3] H. Attouch, *Variational convergence for functions and operators*, Appl. Math. Series, Pitman, Boston, MA, 1984.

[4] N. Aronszajn, A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order. *J. Math. Pures Appl.* **36** (1957), 235–249

[5] A. Bensoussan, J.L. Lions and G. Papanicolaou, *Asymptotic analysis for periodic structures*, North-Holland, Amsterdam, 1978.

[6] Y. Berezanskii, The uniqueness theorem in the inverse problem of spectral analysis for the Schrödinger equation (Russian), *Trudy Moskov. Mat. Obsch.*, **7** (1958), 1–62.

[7] A. Bukhgeim, Recovering a potential from Cauchy data, *J. Inverse Ill-Posed Prob.* **16** (2008), 19–33.

[8] S. Chanillo, A problem in electrical prospection and an n-dimensional Borg-Levinson theorem, *Proc. Amer. Math. Soc.* **108** (1990), 761–767.

[9] H.-Y. Chen and C.T. Chan, Transformation media that rotate electromagnetic fields, *Appl. Phys. Lett.* **90** (2007), 241105.

[10] H.-Y. Chen and C.T. Chan, Acoustic cloaking in three dimensions using acoustic metamaterials, *Appl. Phys. Lett.* **91** (2007), 183518.

[11] H.-Y. Chen and C.T. Chan, Electromagnetic wave manipulation using layered systems, *Phys. Rev. B* **78** (2008), 054204.

[12] H.-S. Chen, B.-I. Wu, B. Zhang and J.A. Kong, Electromagnetic wave interactions with a metamaterial cloak, *Phys. Rev. Lett.*, **99** (2007), 063903.

[13] A. Cherkaev, *Variational methods for structural optimization*, Appl. Math. Sci., **140**, Springer-Verlag, 2000.

[14] S. Cummer and D. Schurig, One path to acoustic cloaking, *New J. Phys.* **9** (2007), 45.

[15] S. Cummer, *et al.*, Scattering theory derivation of a 3D acoustic cloaking shell, *Phys. Rev. Lett.*, **100** (2008), 024301.

[16] G. Dal Maso, *An Introduction to Γ-convergence*, Prog. in Nonlinear Diff. Eq. and their Appl., **8**. Birkhauser Boston, Inc., Boston, 1993.

[17] L. Dolin, To the possibility of comparison of three-dimensional electromagnetic systems with nonuniform anisotropic filling (in Russian), *Izv. VUZov. Radiofizika*, **4** (1961), 964–967.

[18] M. Farhat, S. Guenneau, A.B. Movchan and S. Enoch, Achieving invisibility over a finite range of frequencies, *Optics Express*, **16** (2008), 5656.

[19] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd ed., Springer-Verlag, Berlin, 1983.

[20] A. Greenleaf, Y. Kurylev, M. Lassas and G. Uhlmann, Full-wave invisibility of active devices at all frequencies, *Comm. Math. Phys.*, **279** (2007), 749–789.

[21] A. Greenleaf, Y. Kurylev, M. Lassas and G. Uhlmann, Electromagnetic wormholes and virtual magnetic monopoles from metamaterials, *Phys. Rev. Lett.*, **99** (2007), 183901.

[22] A. Greenleaf, Y. Kurylev, M. Lassas and G. Uhlmann: Improvement of cylindrical cloaking with the SHS lining. *Optics Express* **15** (2007), 12717-12734.

[23] A. Greenleaf, Y. Kurylev, M. Lassas and G. Uhlmann, Electromagnetic wormholes via handlebody constructions, *Comm. Math. Phys.*, **281** (2008), 369-385.

[24] A. Greenleaf, Y. Kurylev, M. Lassas and G. Uhlmann, Comment on "Scattering theory derivation of a 3D acoustic cloaking shell", arXiv:0801.3279 (2008).

[25] A. Greenleaf, Y. Kurylev, M. Lassas and G. Uhlmann, Isotropic transformation optics: approximate acoustic and quantum cloaking, arXiv:0806.0085 (2008); *New J. Phys.* **10** (2008), 115024.

[26] A. Greenleaf, Y. Kurylev, M. Lassas and G. Uhlmann, Approximate quantum cloaking and almost trapped states, arXiv:0806.0368 (2008); *Phys. Rev. Lett.* **101** (2008), 220404.

[27] A. Greenleaf, M. Lassas, and G. Uhlmann, The Calderón problem for conormal potentials, I: Global uniqueness and reconstruction, *Comm. Pure Appl. Math.* **56** (2003), no. 3, 328–352.

[28] A. Greenleaf, M. Lassas, and G. Uhlmann, Anisotropic conductivities that cannot detected by EIT, *Physiolog. Meas.* (special issue on Impedance Tomography), **24** (2003), 413-420.

[29] A. Greenleaf, M. Lassas, and G. Uhlmann, On nonuniqueness for Calderón's inverse problem, *Math. Res. Lett.* **10** (2003), 685-693.

[30] P. Grinevich and R. Novikov, Transparent potentials at fixed energy in dimension two. Fixed-energy dispersion relations for the fast decaying potentials, *Comm. Math. Phys.* **174** (1995), 409–446.

[31] A. Hendi, J. Henn and U. Leonhardt, Ambiguities in the scattering tomography for central potentials, *Phys. Rev. Lett.* **97** (2006), 073902.

[32] Y. Huang, Y. Feng and T. Jiang, Electromagnetic cloaking by layered structure of homogeneous isotropic materials, *Optics Expresss*, **15** (2007), 11133-11141.

[33] T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, 1980.

[34] H. Koch and D. Tataru, Carleman estimates and unique continuation for second-order elliptic equations with nonsmooth coefficients. *Comm. Pure Appl. Math.* **54** (2001), 339–360.

[35] R. Kohn, D. Onofrei, M. Vogelius and M. Weinstein, in preparation.

[36] R. Kohn, H. Shen, M. Vogelius, and M. Weinstein, Cloaking via change of variables in electrical impedance tomography, *Inver. Prob.*, **24** (2008), 015016.

[37] A. Kufner and B. Opic, *Hardy-type inequalities*. Pitman Research Notes in Mathematics Series **219**, Longman, 1990.

[38] Y. Kurylev, M. Lassas and E. Somersalo, Maxwell's equations with a polarization independent wave velocity: direct and inverse problems, *Jour. Math. Pures Appl.* **86** (2006), 237–270.

[39] R. Lavine and A. Nachman, The Faddeev-Lippmann-Schwinger equation in multidimensional quantum inverse scattering, in *Inverse problems: an interdisciplinary study (Montpellier, 1986)*, 169–174, Adv. Electron. Electron Phys., Suppl. 19, Academic Press, London, 1987.

[40] H. Leinfelder and C. G. Simader, Schrödinger operators with singular magnetic vector potentials, *Math Zeit.* **176** (1981), 1-19.

[41] U. Leonhardt, Optical conformal mapping, *Science* **312** (2006), 1777-1780.

[42] U. Leonhardt and T. Philbin, General relativity in electrical engineering, *New J. Phys.* **8** (2006), 247.

[43] R. Lipton, Homogenization and field concentrations in heterogeneous media, *SIAM J. Math. Anal.*, **38** (2006), 1048–1059.

[44] Y. Luo, *et al.*, Design and analytically full-wave validation of the invisibility cloaks, concentrators, and field rotators created with a general class of transformations, *Phys. Rev. B* **77** (2008), 125127.

[45] G. Milton, *The Theory of Composites*, Cambridge Univ. Press, 2001.

[46] G. Milton, M. Briane, and J. Willis, On cloaking for elasticity and physical equations with a transformation invariant form, *New J. Phys.*, **8** (2006), 248.

[47] A. Nachman, Reconstructions from boundary measurements, *Ann. Math.* **128** (1988), 531–576.

[48] A. Nachman, Global uniqueness for a two-dimensional inverse boundary value problem, *Ann. Math.* **143** (1996), 71–96.

[49] A. Nachman, J. Sylvester and G. Uhlmann, An n-dimensional Borg-Levinson theorem, *Comm. Math. Phys.* **115** (1988), 595–605.

[50] R. Newton, Construction of potentials from the phase shifts at fixed energy, *J. Math. Phys.* **3** (1962), 75–82.

[51] G. Nguetseng, A general convergence result for a functional related to the theory of homogenization, *SIAM J. Math. Anal.*, **20** (1989), 608.

[52] A. Norris, Acoustic cloaking theory, *Proc. Royal Soc. A*, **464** (2008), 2411–2434.

[53] P. Ola, L. Päivärinta and E. Somersalo, An inverse boundary value problem in electrodynamics, *Duke Math. Jour.* **70** (1993), 617-653.

[54] J.B. Pendry, D. Schurig, and D.R. Smith, Controlling Electromagnetic Fields, *Science* **312** (2006), 1780 - 1782.

[55] M. Rahm, *et al.*, Optical Design of Reflectionless Complex Media by Finite Embedded Coordinate Transformations, *Phys. Rev. Lett.* **100** (2008), 063903.

[56] M. Reed and B. Simon, *Methods of modern mathematical physics. I.* Academic Press, New York, 1980. xv+400 pp.

[57] T. Regge, Introduction to complex orbital method, *Nuovo Cimento* **14** (1959), 951–976.

[58] Z. Ruan, M. Yan, C. Neff and M. Qiu, Ideal cylindrical cloak: perfect but sensitive to tiny perturbations, *Phys. Rev. Lett.* **99** (2007), 113903.

[59] P. Sabatier, Asymptotic properties of the potentials in the inverse-scattering problem at fixed energy, *J. Math. Phys.* **7** (1966), 1515–1531.

[60] B. Simon, Maximal and minimal Schrödinger forms, *J. Operator Theory* **1** (1979), 37-47.

[61] Z. Sun and G. Uhlmann, Generic uniqueness for an inverse boundary value problem, *Duke Math. Journal* **62** (1991), 131–155.

[62] J. Sylvester and G. Uhlmann, A global uniqueness theorem for an inverse boundary value problem, Ann. Math. **125** (1987), 153–169.

[63] A. Ward and J. Pendry, Refraction and geometry in Maxwell's equations, *Jour. Modern Opt.*, **43** (1996), 773–793.

[64] R. Weder, A rigorous analysis of high-order electromagnetic invisibility cloaks, *Jour. Phys. A: Math. and Theor.*, **41** (2008), 065207.

[65] S. Zhang, D. Genov, C. Sun and X. Zhang, Cloaking of matter waves, *Phys. Rev. Lett.* **100** (2008), 123002.

Allan Greenleaf, Department of Mathematics, University of Rochester, Rochester, NY 14627,USA, `allan@math.rochester.edu`

Yaroslav Kurylev, Department of Mathematical Sciences, University College, London, WC1E 6BT, UK, `y.kurylev@ucl.ac.uk`

Matti Lassas, Institute of Mathematics, Helsinki University of Technology, FIN-02015, Finland, `mjlassas@math.tkk.fi`

Gunther Uhlmann, Department of Mathematics, University of Washington, Seattle, WA 98195, `gunther@math.washington.edu`