BOUNDARY RIGIDITY WITH PARTIAL DATA

PLAMEN STEFANOV, GUNTHER UHLMANN, AND ANDRAS VASY

Abstract. We study the boundary rigidity problem with partial data consisting of determining locally the Riemannian metric of a Riemannian manifold with boundary from the distance function measured at pairs of points near a fixed point on the boundary. We show that one can recover uniquely a conformal factor near a strictly convex point where we have the information. In particular, this implies that we can determine locally the isotropic sound speed of a medium by measuring the travel times of waves joining points close to a convex point on the boundary.

The local results leads to a global lens rigidity uniqueness result assuming that the manifold is foliated by strictly convex hypersurfaces.

1. Introduction and main results

Travel time tomography deals with the problem of determining the sound speed or index of refraction of a medium by measuring the travel times of waves going through the medium. This type of inverse problem, also called the inverse kinematic problem, arose in geophysics in an attempt to determine the substructure of the Earth by measuring the travel times of seismic waves at the surface. We consider an anisotropic index of refraction, that is the sound speed depends on the direction. The Earth is generally isotropic. More recently it has been realized, by measuring these travel times, that the inner core of the Earth exhibits anisotropic behavior with the fast direction parallel to the Earth's spin axis, see [4]. In the human body, muscle tissue is anisotropic. As a model of anisotropy, we consider a Riemannian metric $g = (g_{ij})$. The problem is to determine the metric from the lengths of geodesics joining points on the boundary.

This leads to the general question of whether given a compact Riemannian manifold with boundary (M, g) one can determine the Riemannian metric in the interior knowing the boundary distance function joining points on the boundary $d_g(x, y)$, with $x, y \in \partial M$. This is known as the *boundary rigidity problem*. Of course, isometries preserve distance, so that the boundary rigidity problem is whether two metrics that have the same boundary distance function are the same up to isometry. Examples can be given of manifolds that are not boundary rigid. Such examples show that the boundary rigidity problem should be considered under some restrictions on the geometry of the manifold. The most usual of such restrictions is simplicity of the metric. A Riemannian manifold (M, g) (or the metric g) is called *simple* if the boundary ∂M is strictly convex and any two points $x, y \in M$ are joined by a unique minimizing geodesic. Michel conjectured [19] that every simple compact Riemannian manifold with boundary is boundary rigid.

Simple surfaces with boundary are boundary rigid [27]. In higher dimensions, simple Riemannian manifolds with boundary are boundary rigid under some a-priori constant curvature on the manifold or special symmetries [1], [11]. Several local results near the Euclidean metric are known [36], [17],

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[2]. The most general result in this direction is the generic local (with respect to the metric) one proven in [38]. Surveys of some of the results can be found in [14], [39], [7].

In this paper, we consider the boundary rigidity problem in the class of metrics conformal to a given one and with partial data, that is, we know the boundary distance function for points on the boundary near a given point. Partial data problems arise naturally in applications since in many cases one doesn't have access to the whole boundary. We prove the first result on the determination of the conformal factor locally near the boundary from partial data without assuming analyticity. We develop a novel method to attack partial data non-linear problems that will likely have other applications.

We now describe the known results with full data on the boundary. Let us fix the metric g_0 and let c be a positive smooth function on the compact manifold with boundary M. The problem is whether we can determine c from $d_{c^{-2}g_0}(x,y)$, $x,y\in\partial M$. Notice that in this case the problem is not invariant under changes of variables that are the identity at the boundary so that we expect to be able to recover c under appropriate a-priori conditions. This was proven by Mukhometov in two dimensions [21], and in [22] in higher dimensions for the case of simple metrics. Of particular importance in applications is the case of an isotropic sound speed that is when we are in a bounded domain of Euclidean space and g_0 is the Euclidean metric. This is the isotropic case. This problem was considered by Herglotz [12] and Wieckert and Zoeppritz [46] for the case of a spherical symmetric sound speed. They found a formula to recover the sound speed from the boundary distance function assuming $\frac{d}{dr}(\frac{r}{c(r)})>0$. Notice that this condition is equivalent in the radial case to non-trapping and is more general than simplicity.

From now on we will call d the function $d_{c^{-2}g_0}$.

The partial data problem, that we will also call the *local boundary rigidity problem*[1], in this case is whether knowledge of the distance function on part of the boundary determines the sound speed c locally. Given another smooth $\tilde c$, here and below we define $\tilde L$, $\tilde\ell$ and $\tilde d$ in the same way but related to $\tilde c$. We prove the following uniqueness result:

Theorem 1.1. *Let $n=\dim M\ge 3$, let $c>0$, $\tilde c>0$ be smooth and let ∂M be strictly convex with respect to both $g=c^{-2}g_0$ and $\tilde g=\tilde c^{-2}g_0$ near a fixed $p\in\partial M$. Let $d(p_1,p_2)=\tilde d(p_1,p_2)$ for p_1, p_2 on ∂M near p. Then $c=\tilde c$ in M near p.*

As mentioned earlier, this is the only known result for the boundary rigidity problem with partial data except in the case that the metrics are assumed to be real-analytic [17]. The latter follows from determination of the jet of the metric at a convex point from the distance function known near p.

The boundary rigidity problem is closely connected to the lens rigidity one. To define the latter, we first introduce the manifolds $\partial_\pm SM$, defined as the sets of all vectors (x,v) with $x\in\partial M$, v unit in the metric g, and pointing outside/inside M. We define the lens relation

$$L:\partial_-SM\longrightarrow\partial_+SM$$

in the following way: for each $(x,v)\in\partial_-SM$, $L(x,v)=(y,w)$, where y is the exit point, and w the exit direction, if exist, of the maximal unit speed geodesic $\gamma_{x,v}$ in the metric g, issued from (x,v). Let

$$\ell:\partial_-SM\longrightarrow\mathbb{R}\cup\infty$$

be its length, possibly infinite. If $\ell<\infty$, we call M non-trapping.

[1]It is local in the sense that $d(x,y)$ is known locally and depends on c locally; the term *local* has been used before to indicate that the metric is a priori close to a fixed one.

The *lens rigidity* problem is whether the lens relation L (and possibly, ℓ) determine g (and the topology of M) up to an isometry as above. The lens rigidity problem with partial data for a sound speed is whether we can determine the speed near some p from L known near $S_p\partial M$. For general metrics, we want to recover isometric copies of the metrics locally, as above.

We assume that ∂M is strictly convex at $p \in \partial M$ w.r.t. g. Then the boundary rigidity and the lens rigidity problems with partial data are equivalent: knowing d near (p,p) is equivalent to knowing L in some neighborhood of $S_p\partial M$. The size of that neighborhood however depends on a priori bounds of the derivatives of the metrics with which we work. This equivalence was first noted by Michel [19], since the tangential gradients of $d(x,y)$ on $\partial M \times \partial M$ give us the tangential projections of $-v$ and w, see also [35, sec. 2]. Note that local knowledge of ℓ is not needed for either problems, and in fact, ℓ can be recovered locally from either d or ℓ.

Vargo [45] proved that real-analytic manifolds satisfying an additional mild condition are lens rigid. Croke has shown that if a manifold is lens rigid, a finite quotient of it is also lens rigid [7]. He has also shown that the torus is lens rigid [3]. G. Uhlmann and P. Stefanov have shown lens rigidity locally near a generic class of non-simple manifolds [41]. The only result we know for the lens rigidity problem with partial data is for real-analytic metric satisfying a mild condition [41]. While in [41], the lens relation is assumed to be known on a subset only, the geodesics issued from that subset cover the whole manifold. In contrast, in this paper, we have localized information.

We state below an immediate corollary of our main result for this problem. For the partial data problem instead of assuming $d = \tilde d$ locally, we can assume that $L = \tilde L$ in a neighborhood of $S_p\partial M$. To reduce this problem to Theorem 1.1 directly, we need to assume first that $c = \tilde c$ on ∂M near p to make the definition of $\partial_\pm SM$ independent of the choice of the speed but in fact, one can redefine the lens relation in a way to remove that assumption, see [41].

Corollary 1.1. *Let M, c, $\tilde c$ be as in Theorem 1.1 with $c = \tilde c$ on ∂M near p. Let $L = \tilde L$ near $S_p\partial M$. Then $c = \tilde c$ in M near p.*

Remark 1.1. The theorem or its corollary does not preclude the existence of an infinite set of speeds c_j all having the same boundary distance function (or lens data) in $U \times U$, where $U \subset \partial M$ is some fixed small set but not coinciding in any fixed neighborhood of p. One such case is when the maximal neighborhood of U, which can be covered with strictly convex surfaces, which continuously deform U, shrinks when $j \to \infty$. Then the theorem does not imply existence of a fixed neighborhood of p, where all speeds are equal. If one assumes that a priori, the sound speeds have uniformly bounded derivatives of some finite order near p, this situation does not arise.

The linearization of the boundary rigidity and lens rigidity problem is the tensor tomography problem, i.e., recovery of a tensor field up to "potential fields" from integrals along geodesics joining points on the boundary. It has been extensively studied in the literature for both simple and non-simple manifolds [10, 24, 25, 26, 20, 28, 33, 30, 37, 40, 43]. See the book [31] and [26] for a recent survey. The *local* tensor tomography problem has been considered in [15] for functions and real-analytic metrics and in [16] for tensors of order two and real-analytic metrics. Those results can also be thought of as support theorems of Helgason type. The only known results for the local problem for smooth metrics and integrals of functions is [44].

Now we use a layer stripping type argument to obtain a global result which is different from Mukhometov's for simple manifolds.

Definition 1.1. *Let (M,g) be a compact Riemannian manifold with boundary. We say that M satisfies the foliation condition by strictly convex hypersurfaces if M is equipped with a smooth function $\rho : M \to [0,\infty)$ which level sets $\Sigma_t = \rho^{-1}(t)$, $t < T$ with some $T > 0$ are strictly convex* *viewed from $\rho^{-1}((0,t))$ for g, $d\rho$ is non-zero on these level sets, and $\Sigma_0 = \partial M$ and $M \setminus \cup_{t\in[0,T)}\Sigma_t$ has empty interior.*

The statement of the global result on lens rigidity is as follows:

Theorem 1.2. *Let $n = \dim M \geq 3$, let $c > 0$, $\tilde c > 0$ be smooth and equal on ∂M, let ∂M be strictly convex with respect to both $g = c^{-2}g_0$ and $\tilde g = \tilde c^{-2}g_0$. Assume that M can be foliated by strictly convex hypersurfaces for g. Then if $L = \tilde L$ on $\partial_- SM$, we have $c = \tilde c$ in M.*

A more general foliation condition is formulated in [42]. In particular, Σ_0 does not need to be ∂M and one can have several such foliations with the property that the closure of their union is M.

Spherically symmetric $c(x)$ under the condition considered by Herglotz and Wieckert and Zoeppritz satisfy the foliation condition of the theorem. Other examples of non-simple metrics that satisfy the condition are the tubular neighborhood of a closed geodesic in negative curvature. These have trapped geodesics. Also the rotationally symmetric spaces on the ball with convex spheres can be far from simple. It follows from the result of [29], that manifolds with no focal points satisfy the foliation condition. It would be interesting to know whether this is also the case for simple manifolds. As it was mentioned earlier manifolds satisfying the foliation condition are not necessarily simple.

The linearization of the non-linear problem with partial data considered in Theorem 1.1 was considered in [44], where uniqueness and stability were shown. This corresponds to integrating functions along geodesics joining points in a neighborhood of p. The method of proof of Theorem relies on using an identity proven in [36] to reduce the problem to a "pseudo-linear" one: to show uniqueness when one integrates the function $f =$ and its derivatives on the geodesics for the metric g joining points near p, with weight depending non-linearly on both g and $\tilde g$. Notice that this is not a proof by linearization, and unlike the problem with full data, an attempt to do such a proof is connected with essential difficulties. The proof of uniqueness for this linear transform follows the method of [44] introducing an artificial boundary and using Melrose' scattering calculus. In section 2, we do the reduction to a "pseudo-linear problem", and in section 3, we show uniqueness for the "pseudo-linear" problem. In section 4, we finish the proofs of the main theorems.

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2. Reducing the non-linear problem to a pseudo-linear one

We recall the known fact [17] that one can determine the jet of c at any boundary point p at which ∂M is convex (not necessarily strictly) from the distance function d known near (p,p). For a more general result not requiring convexity, see [41]. Since the result in [17] is formulated for general metrics, and the reconstruction of the jet is in boundary normal coordinates, we repeat the proof in this (simpler) situation of recovery a conformal factor. As in [17], we say that ∂M is convex near $p \in \partial M$, if for any two distinct points $p_1, p_2 \in \partial M$, close enough to p, there exists a geodesic $\gamma : [0,1] \to M$ joining them such that its length is $d(p_1,p_2)$ and all the interior of γ belongs to the interior of M. Of course, strict convexity (positive second fundamental form at p) implies convexity.

Lemma 2.1. *Let c and $\tilde c$ be smooth and let ∂M be convex at p with respect to g and $\tilde g$. Let $d = \tilde d$ near (p,p). Then $\partial^\alpha c = \partial^\alpha \tilde c$ on ∂M near p for any multiindex α.*

Proof. Let $V \subset \partial M$ be a neighborhood of p on ∂M such that for any $p_1, p_2 \in V$, we have the property guaranteeing convexity at p. Let x^n be a boundary normal coordinate related to g; i.e.,

$x^n(q) = \mathrm{dist}(q, \partial M)$, and $x^n \geq 0$ in M. We can complete x^n to a local coordinate system (x', x^n), where x' parameterizes ∂M near p.

It is enough to prove

$$\partial_{x^n}^k c = \partial_{x^n}^k \tilde{c} \quad \text{in } V,\ k = 0, 1, \dots. \tag{1}$$

For $k = 0$, this follows easily by taking the limit in $d(p, q) = \tilde{d}(p, q)$, $\partial M \ni q \to p$; and this can be done for any $p \in V$. Let l be the first value of k for which (1) fails. Without loss of generality, we may assume that it fails at p, and $\partial_{x^n}^l (c - \tilde{c}) > 0$ at p. Then $\partial_{x^n}^k (c - \tilde{c}) = 0$ in V, $k = 0, \dots, l - 1$. Consider the Taylor expansion of $c - \tilde{c}$ w.r.t. x^n with x' close enough to $x'(p)$. We get $c - \tilde{c} > 0$ in some neighborhood of p in M minus the boundary.

Now, let $\gamma(p, q)$ be a minimizing geodesic in the metric g connecting p and q when $q \in \partial M$ as well, close enough to p, see also [17]. Let $If(p, q)$ be the geodesic ray transform of the tensor field f defined as an integral of $f_{ij}\dot{\gamma}^i\dot{\gamma}^j$ along $\gamma(p, q)$. All geodesics here are parameterized by a parameter in $[0, 1]$. Then $I(g - \tilde{g}) > 0$ by what we proved above. On the other hand,

$$0 < I(g - \tilde{g}) = d^2(p, q) - I\tilde{g} \leq d^2(p, q) - \tilde{I}\tilde{g} = d^2(p, q) - \tilde{d}^2(p, q) = 0$$

because $\tilde{\gamma}(p, q)$ minimizes integrals of g along curves connecting p and q. This a contradiction. $\square$

The starting point is an identity in [36]. We will repeat the proof.

Let V, $\tilde{V}$ be two vector fields on a manifold M (which will be replaced later with S^*M). Denote by $X(s, X^{(0)})$ the solution of $\dot{X} = V(X)$, $X(0) = X^{(0)}$, and we use the same notation for $\tilde{V}$ with the corresponding solution are denoted by $\tilde{X}$. Then we have the following simple statement.

Lemma 2.2. *For any $t > 0$ and any initial condition $X^{(0)}$, if $\tilde{X}\big(\cdot, X^{(0)}\big)$ and $X\big(\cdot, X^{(0)}\big)$ exist on the interval $[0, t]$, then*

$$\tilde{X}\Big(t, X^{(0)}\Big) - X\Big(t, X^{(0)}\Big) = \int_0^t \frac{\partial \tilde{X}}{\partial X^{(0)}}\Big(t - s, X(s, X^{(0)})\Big)\Big(V - \tilde{V}\Big)\Big(X(s, X^{(0)})\Big)\, ds. \tag{2}$$

Proof. Set

$$F(s) = \tilde{X}\Big(t - s, X(s, X^{(0)})\Big).$$

Then

$$F'(s) = -\tilde{V}\Big(\tilde{X}(t - s, X(s, X^{(0)}))\Big) + \frac{\partial \tilde{X}}{\partial X^{(0)}}\Big(t - s, X(s, X^{(0)})\Big)\, V\Big(X(s, X^{(0)})\Big).$$

The proof of the lemma would be complete by the fundamental theorem of calculus

$$F(t) - F(0) = \int_0^t F'(s)\, ds$$

if we show the following

$$\tilde{V}\Big(\tilde{X}(t - s, X(s, X^{(0)}))\Big) = \frac{\partial \tilde{X}}{\partial X^{(0)}}\Big(t - s, X(s, X^{(0)})\Big)\, \tilde{V}\Big(X(s, X^{(0)})\Big). \tag{3}$$

Indeed, (3) follows from

$$0 = \frac{d}{d\tau}\bigg|_{\tau=0} X(T - \tau, X(\tau, Z)) = -V(X(T, Z)) + \frac{\partial X}{\partial X^{(0)}}(T, Z)V(Z), \quad \forall T,$$

after setting $T = t - s$, $Z = X(s, X^{(0)})$. $\square$

Let c, $\tilde c$ be two speeds. Then the corresponding metrics are $g = c^{-2}dx^2$, and $\tilde g = \tilde c^{-2}dx^2$. The corresponding Hamiltonians and Hamiltonian vector fields are

$$H = \frac{1}{2}c^2 g_0^{ij}\xi_i\xi_j, \qquad V = \left(c^2 g_0^{-1}\xi, -\frac{1}{2}\partial_x c^2|\xi|^2_{g_0}\right),$$

and the same ones related to $\tilde c$. We used the notation $|\xi|^2_{g_0} := g_0^{ij}\xi_i\xi_j$.

We change the notation at this point. We denote points in the phase space T^*M, in a fixed coordinate system, by $z = (x, \xi)$. We denote the bicharacteristic with initial point z by $Z(t, z) = (X(t, z), \Xi(t, z))$.

Then we get the identity already used in [36]

$$\tilde Z(t, z) - Z(t, z) = \int_0^t \frac{\partial \tilde Z}{\partial z}(t-s, Z(s, z))\big(V - \tilde V\big)(Z(s, z))\, ds. \tag{4}$$

We can naturally think of the lens relation L and the travel time ℓ as functions on the cotangent bundle instead of the tangent one. Then we get the following.

Proposition 2.1. *Assume*

$$L(x_0, \xi^0) = \tilde L(x_0, \xi^0), \quad \ell(x_0, \xi^0) = \tilde\ell(x_0, \xi^0) \tag{5}$$

for some $z_0 = (x_0, \xi^0) \in \partial_- S^*M$. *Then*

$$\int_0^{\ell(z_0)} \frac{\partial \tilde Z}{\partial z}(\ell(z_0) - s, Z(s, z_0))\big(V - \tilde V\big)(Z(s, z_0))\, ds = 0.$$

2.1. **Linearization near** $c = 1$ **and** g **Euclidean.** As a simple exercise, let $c = 1$, $g_{ij} = \delta_{ij}$ and linearize for $\tilde c$ near 1 first under the assumption that $\tilde c = 1$ outside Ω. Then

$$Z(s, z) = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} z, \quad \frac{\partial Z(s, z)}{\partial z} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, \tag{6}$$

and we get the following formal linearization of (4)

$$\int \left(f\xi - \frac{1}{2}(t-s)(\partial_x f),\, -\frac{1}{2}\partial_x f\right)(x + s\xi, \xi)\, \mathrm{d}s = 0, \tag{7}$$

where

$$f := c^2 - \tilde c^2. \tag{8}$$

We integrate over the whole line $s \in \mathbb{R}$ because the integrand vanishes outside the interval $[0, \ell(x, \xi)]$. The last n components of (7) imply

$$\int \partial_x f(x + s\xi)\, \mathrm{d}s = 0. \tag{9}$$

Now, assume that this holds for all (x, ξ). Then $\partial_x f = 0$, and since $f = 0$ on ∂M, we get $f = 0$.

2.2. **The general case.** We take the second n-dimensional component on (4) and use the fact that $c^2|\xi|^2_g = 1$ on the bicharacteristics related to c. We assume that both geodesics extend to $t \in [0, \ell(x, \xi)]$. We want to emphasize that the bicharacteristics on the energy level $H = 1/2$, related to c, do not necessarily stay on the same energy level for the Hamiltonian $\tilde H$. We get

$$\begin{aligned} &\int \frac{\partial \tilde\Xi}{\partial x}(\ell(z) - s, Z(s, z))(f g_0^{-1}\xi)(Z(s, z))\, \mathrm{d}s \\ &- \frac{1}{2}\int \frac{\partial \tilde\Xi}{\partial \xi}(\ell(z) - s, Z(s, z))(\partial_x(f g_0^{-1})\xi\cdot\xi)(Z(s, z))\, \mathrm{d}s = 0 \end{aligned} \tag{10}$$

for any $z \in \partial_- SM$ for which (5) holds. As before, we integrate over $s \in \mathbb{R}$ because the support of the integrand vanishes for $s \notin [0, \ell(x,\xi)]$ (for that, we extend the bicharacteristics formally outside so that they do not come back). Write

$$\partial_x (f g_0^{-1}) = g_0^{-1} \partial_x f + (\partial_x g_0^{-1}) f$$

to get

$$\begin{aligned} &\int \frac{\partial \tilde{\Xi}}{\partial x}(\ell(z) - s, Z(s,z))(f g_0^{-1}\xi)(Z(s,z))\,\mathrm{d}s \\ &- \frac{1}{2}\int \frac{\partial \tilde{\Xi}}{\partial \xi}(\ell(z) - s, Z(s,z))\left(\left(g_0^{-1}\partial_x f + f(\partial_x g_0^{-1}\right)\xi \cdot \xi\right)(Z(s,z))\,\mathrm{d}s = 0. \end{aligned} \tag{11}$$

One of terms on the r.h.s. above involves $g_0^{-1}\xi \cdot \xi$ which equals c^{-2} on the bicharacteristics of H on the level 1/2.

Introduce the exit times $\tau(x,\xi)$ defined as the minimal (and the only) $t \geq 0$ so that $X(t,x,\xi) \in \partial M$. They are well defined near $S_p \partial M$, if ∂M is strictly convex at p. We need to write $\frac{\partial \tilde{Z}}{\partial z}(\ell(z) - s, Z(s,z))$ as a function of $(x,\xi) = Z(s,z)$. We have

$$\frac{\partial \tilde{Z}}{\partial z}(\ell(z) - s, Z(s,z)) = \frac{\partial \tilde{Z}}{\partial z}(\tau(x,\xi),(x,\xi)).$$

Then we get, with f as in (8),

$$J_i f(\gamma) := \int \Big(A_i^j(X(t),\Xi(t))(\partial_{x^j} f)(X(t)) + B_i(X(t),\Xi(t)) f(X(t))\Big)\,\mathrm{d}t = 0 \tag{12}$$

for any bicharacteristic $X((t),\Xi(t))$ (related to the speed c) in our set, where

$$\begin{aligned} A_i^j(x,\xi) &= -\frac{1}{2}\frac{\partial \tilde{\Xi}_i}{\partial \xi_j}(\tau(x,\xi),(x,\xi))c^{-2}(x), \\ B_i(x,\xi) &= \frac{\partial \tilde{\Xi}_i}{\partial x^j}(\tau(x,\xi),(x,\xi)) g_0^{ik}(x)\xi_k - \frac{1}{2}\frac{\partial \tilde{\Xi}_i}{\partial \xi_j}(\tau(x,\xi),(x,\xi))(\partial_{x^j} g_0^{-1}(x))\xi \cdot \xi. \end{aligned} \tag{13}$$

A major inconvenience with this representation is that the exit time function $\tau(x,\xi)$ (recall that we assume strong convexity) becomes singular at $(x,\xi) \in T^*\partial M$. More precisely, the normal derivative w.r.t. x when ξ is tangent to ∂M has a square root type of singularity. On the other hand, we have some freedom to extend the flow outside M since we know that the jets of c and $\tilde{c}$ at ∂M are the same near p: therefore, any smooth local extension of c is also a smooth extension of $\tilde{c}$. Then for any $(x,\xi) \in \partial_- S^*M$ close enough to $S^*_{x_0}M$, the bicharacteristics originating from it will be identical once they exist T^*M but are still close enough to it. Similarly, instead of starting from $T^*\partial M$, we could start at points and codirections close to it, but away from $\bar{M}$.

With this in mind, we push the boundary away a bit. Let x_0 represent the point p near which we work, in a fixed coordinate system. Let $S(x_0,r)$ be the sphere in the metric $c^{-2}\mathrm{d}x^2$ centered at x_0 with radius $0 < r \ll 1$. For (x,ξ) with x in the geodesic ball $B(x_0,r)$, redefine $\tau(x,\xi)$ to be the travel time from (x,ξ) to $S(x_0,r)$. Let $U_- \subset \partial_- SB(x_0,r)$ be the set of all points on $S(x_0,r)$ and incoming unit directions so that the corresponding geodesic in the metric g is close enough to one tangent to ∂M at x_0. Similarly, let U_+ be the set of such pairs with outgoing directions. Redefine the scattering relation L locally to act from U_- to U_+, and redefine ℓ similarly, see Figure 1. Then $L = \tilde{L}$ and $\ell = \tilde{\ell}$ on U_-. We can apply the construction above by replacing $\partial_\pm SM$ locally by $U_\pm$. The advantage we have now that on U_-, the travel time τ is non-singular. Equalities (12), (13) are preserved then.

$=$

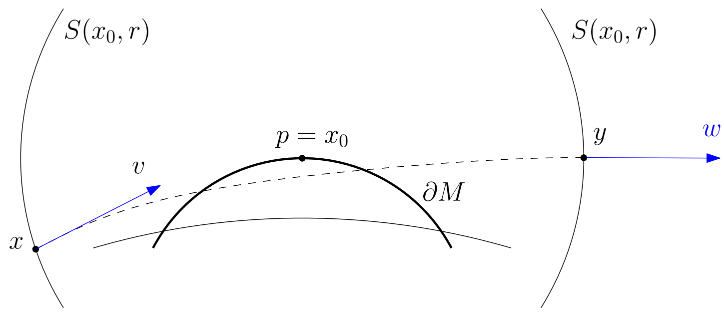

FIGURE 1. The redefined lens relation $(x, v) \mapsto (y, w)$

We now have

$$
\begin{aligned}
A_i^j(x_0,\xi) &= -\frac{1}{2}\frac{\partial \tilde{\Xi}_i}{\partial \xi_j}(r,(x_0,\xi))c^{-2}(x_0),\\
B_i(x_0,\xi) &= \frac{\partial \tilde{\Xi}_i}{\partial x^j}(r,(x_0,\xi))g_0^{ik}(x_0)\xi_k - \frac{1}{2}\frac{\partial \tilde{\Xi}_i}{\partial \xi_j}(r,(x,\xi))(\partial_{x^j} g_0^{-1}(x_0))\xi\cdot\xi.
\end{aligned}
\tag{14}
$$

Since $c = \tilde{c}$ outside M, and by the strict convexity, near $S_p^*\partial M$,

$$
A_i^j(x,\xi) = -\frac{1}{2}c^{-2}(x)\delta_i^j, \quad \forall (x,\xi) \in S^*\partial M. \tag{15}
$$

2.3. **A new linear problem.** The arguments above lead to the following linear problem:

Problem. Assume (12) holds with some f supported in M, for all geodesics close to the ones originating from $S^*_{x_0}\partial M$ (i.e. initial point x_0 and all unit initial co-directions tangent to ∂M). Assume that ∂M is strictly convex at x_0 w.r.t. the speed c. Assume (15). Is it true that $f = 0$ near x_0?

We show below in Proposition 3.3, that the answer is affirmative. Note that this reduces the original non-linear problem to a linear one but this is not a linearization.

3. LINEAR ANALYSIS

We first recall the setting introduced in [44] in our current notation. There the scalar X-ray transform along geodesics was considered, namely for $\beta \in SM$,

$$
(If)(\beta) = \int_{\mathbb{R}} f(\gamma_\beta(t))\,dt,
$$

where γ_β is the geodesic with lift to SM having starting point $\beta \in SM$. Here M is assumed to have a strictly convex boundary, which can be phrased as the statement that if ρ is a defining function for ∂M, then $-\frac{d^2}{dt^2}(\rho\circ\gamma_\beta)|_{t=0} \geq C > 0$ whenever $\beta \in T\partial M$. One then considers a point $p \in \partial M$, and another function $\tilde{\mathsf{x}}$, denoted in [44] by $\tilde{x}$, such that $\tilde{\mathsf{x}}(p) = 0$, $d\tilde{\mathsf{x}}(p) = -d\rho(p)$, and the level sets of $\tilde{\mathsf{x}}$ near the 0 are strictly concave when viewed from the superlevel sets (which are on the side of p when talking about the c-level set with $\mathsf{c} < 0$), i.e. $\frac{d^2}{dt^2}(\tilde{\mathsf{x}}\circ\gamma_\beta)|_{t=0} \geq C > 0$ if $\beta \in T\{\tilde{\mathsf{x}} = \mathsf{c}\}$, i.e. if $\frac{d}{dt}(\tilde{\mathsf{x}}\circ\gamma_\beta)|_{t=0} = 0$. For $\mathsf{c} > 0$, we denote $M \cap \{\mathsf{x} > -\mathsf{c}\}$ by Ω_{c}; we assume that $\mathsf{c}_0 > 0$ is such that $\overline{\Omega_{\mathsf{c}_0}}$ is compact on M, and the concavity assumption holds on Ω_{c_0}. Then it was shown that the X-ray transform I restricted to $\beta \in S\overline{\Omega_{\mathsf{c}}}$ such that γ_β leaves Ω_{c} with both endpoints on ∂M

(i.e. at $\rho = 0$) is injective if $\mathsf{c} > 0$ is sufficiently small, and indeed one has a stability estimate for f in terms of If on exponentially weighted spaces.

To explain this in detail, let $\mathsf{x} = \mathsf{x}_\mathsf{c} = \tilde{\mathsf{x}} + \mathsf{c}$ be the boundary defining function of the artificial boundary, $\tilde{\mathsf{x}} = -\mathsf{c}$, that we introduced; indeed it is convenient to work in $\tilde{M}$, a $\mathcal{C}^\infty$ manifold extending M across the boundary, and defining $\hat{\Omega} = \hat{\Omega}_\mathsf{c} = \{\mathsf{x}_\mathsf{c} > 0\}$ as the extension of Ω, so $\overline{\hat{\Omega}}$ is a smooth manifold with boundary, with only the artificial boundary being a boundary face. Then one writes $\beta = (\lambda, \omega) = \lambda\partial_\mathsf{x} + \omega\partial_\mathsf{y} \in S_{\mathsf{x},\mathsf{y}}\tilde{M}$ relative to a product decomposition $(-\mathsf{c}_0, \mathsf{c}_0)_\mathsf{x} \times U_\mathsf{y}$ of $\tilde{M}$ near p. The concavity condition becomes that for β whose λ-component vanishes,

$$2\alpha(\mathsf{x}, \mathsf{y}, 0, \omega) = \frac{d^2}{dt^2}(\mathsf{x} \circ \gamma_\beta)|_{t=0} \geq 2C > 0,$$

with a new $C > 0$, see the discussion preceding Equation (3.1) in [44]. For $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$, $\chi \geq 0$, $\chi(0) > 0$, one considers the map

$$L_0 v(\mathsf{x}, \mathsf{y}) = \int_{\mathbb{R}} \int_{\mathbb{S}^{n-2}} \mathsf{x}^{-2} \chi(\lambda/\mathsf{x}) v(\mathsf{x}, \mathsf{y}, \lambda, \omega)\, d\lambda\, d\omega$$

defined for v a function on $S_{\overline{\hat{\Omega}}}\tilde{M}$. This differs from [44] in that the weight x^{-2} differs by 1 from the weight x^{-1} used in [44]; this simply has the effect of removing an x^{-1} in [44, Proposition 3.3], as compared to the proposition stated below. If c is sufficiently small, or instead χ has sufficiently small support, for $(\mathsf{x}, \mathsf{y}) \in \overline{\Omega}$, I only integrates over points in $\beta \in S\overline{\hat{\Omega}}$ such that γ_β leaves Ω_c with both endpoints on ∂M, i.e. over β corresponding to Ω_c-*local geodesics* – the set of the latter is denoted by $\mathcal{M}_\mathsf{c}$. We refer to the discussion around [44, Equation (3.1)] for more detail, but roughly speaking the concavity of the level sets of x means that the geodesics that are close to being tangent to the foliation, with 'close' measured by the distance from the artificial boundary, $\mathsf{x} = 0$, then they cannot reach $\mathsf{x} = 0$ (or reach again, in case they start there) without reaching $\mathsf{x} = \mathsf{c}'$ for some fixed $\mathsf{c}' > 0$; notice that the geodesics involved in the integration through a point on the level set $\mathsf{x} = \tilde{\mathsf{c}}$ make an angle $\lesssim \tilde{\mathsf{c}}$ with the tangent space of the level set due to the compact support of χ. Then we consider the map $P = L_0 \circ I$. The main technical result of [44], whose notation involving the so-called scattering Sobolev spaces $H_{\mathrm{sc}}^{s,r}(\overline{\hat{\Omega}})$ and scattering pseudodifferential operators $\Psi_{\mathrm{sc}}^{s,r}(\overline{\hat{\Omega}})$ is explained below, was:

Proposition 3.1. *(See* [44, Proposition 3.3 and Lemma 3.6]*.) For $F > 0$ let*

$$P_F = e^{-F/x} P e^{F/x} : \mathcal{C}_c^\infty(\overline{\hat{\Omega}}) \to \mathcal{C}^\infty(\overline{\hat{\Omega}}).$$

Then $P_F \in \Psi_{\mathrm{sc}}^{-1,0}(\overline{\hat{\Omega}})$.

Further, if $\mathsf{c} > 0$ is sufficiently small, then for suitable choice of $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$ with $\chi(0) = 1$, $\chi \geq 0$, P is elliptic in $\Psi_{\mathrm{sc}}^{-1,0}(\overline{\hat{\Omega}})$ on a neighborhood of $\overline{\Omega}$.

Shrinking c further if needed, P_F satisfies the estimate

$$\|v\|_{H_{\mathrm{sc}}^{s,r}(\overline{\hat{\Omega}})} \leq C \|P_F v\|_{H_{\mathrm{sc}}^{s+1,r}(\overline{\hat{\Omega}})} \tag{16}$$

for v supported in $M \cap \overline{\hat{\Omega}}$.

We now briefly explain the role of the so-called scattering pseudodifferential operators and the corresponding Sobolev spaces (which are typically used to study phenomena 'at infinity') in our problem (where there is no obvious 'infinity'); we refer to [44, Section 2] for a more thorough exposition. These concepts were introduced by Melrose, [18], in a general geometric setting, but on $\mathbb{R}^n$ these operators actually correspond to a special case of Hörmander's Weyl calculus [13], also studied earlier by Shubin [34] and Parenti [23]. So consider the reciprocal spherical coordinate map, $(0,\epsilon)_{\mathsf{x}} \times \mathbb{S}^{n-1}_\theta \to \mathbb{R}^n$, with $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ (unit sphere), and map $(\mathsf{x},\theta) \mapsto \mathsf{x}^{-1}\theta \in \mathbb{R}^n$. This map is a diffeomorphism onto its range, and it provides a compactification of $\mathbb{R}^n$ (the so called radial or geodesic compactification) by adding $\{0\}_{\mathsf{x}} \times \mathbb{S}^{n-1}_\theta$ as infinity to $\mathbb{R}^n$, to obtain $\overline{\mathbb{R}^n}$, which is now diffeomorphic to a ball. Now for general U_{y} above, we may regard, at least locally[2] U_{y} also as a coordinate chart in $\mathbb{S}^{n-1}$, and thus obtain an identification of $\overline{\Omega}$ with a region intersection $\overline{\mathbb{R}^n}$, thus our artificial boundary $\mathsf{x} = 0$ corresponds to infinity at $\mathbb{R}^n$. In particular, notions from $\mathbb{R}^n$ can now be transferred to a neighborhood of our artificial boundary. Since the relevant vector fields on $\mathbb{R}^n$ are generated by translation invariant vector fields, which are complete under the exponential map, the transferred analysis replaces the incomplete geometry of standard vector fields on $\overline{\hat{\Omega}}$ by a complete one. Concretely, these vector fields, when transferred, become linear combinations of $\mathsf{x}^2\partial_{\mathsf{x}}$ and $\mathsf{x}\partial_{\mathsf{y}_j}$, with smooth coefficients. In particular, these are the vector fields with respect to which Sobolev regularity is measured. Thus, $H^{s,r}_{\mathrm{sc}}(\overline{\hat{\Omega}})$ is the so-called *scattering Sobolev space*, which is locally, under the above identification, just the standard weighted Sobolev space $H^{s,r}(\mathbb{R}^n)$, see [44, Section 2], while $\Psi^{s,r}_{\mathrm{sc}}(\overline{\hat{\Omega}})$ is Melrose's *scattering pseudodifferential algebra*, which locally, again under this identification, simply corresponds to quantizing symbols a with $|D^\alpha_z D^\beta_\zeta a| \leq C\langle z\rangle^{r-|\alpha|}\langle\zeta\rangle^{s-|\beta|}$ on $T^*\mathbb{R}^n = \mathbb{R}^n_z \times \mathbb{R}^n_\zeta$, see again [44, Section 2] for more detail. Note that ellipticity in this algebra, called full ellipticity, is *both* in the sense as $|z| \to \infty$ and $|\zeta| \to \infty$, i.e. modulo symbols one order lower *and* with an extra order of decay as $|z| \to \infty$.

Notice that (16) implies the estimate

$$\|f\|_{e^{F/\mathsf{x}}H^{s,r}_{\mathrm{sc}}(\overline{\hat{\Omega}})} \leq C\|Pf\|_{e^{F/\mathsf{x}}H^{s+1,r}_{\mathrm{sc}}(\overline{\hat{\Omega}})}$$

for the unconjugated operator, valid when f is supported in $M \cap \overline{\hat{\Omega}}$. Rewriting P as $L_0 \circ I$, this gives that for $\delta > 0$, $s \geq -1$,

$$\|f\|_{e^{(F+\delta)/\mathsf{x}}H^{s}(\overline{\hat{\Omega}})} \leq C\|If\|_{e^{F/\mathsf{x}}H^{s+1}(\mathcal{M}_{\mathsf{c}})};$$

see the discussion in [44] after Lemma 3.6.

After this recollection, we continue by generalizing (12) to regard the functions $\partial_{x_j} f$ and f entering into it as independent unknowns, while restricting the transform to the region of interest $\Omega = \Omega_{\mathsf{c}}$. So let $\tilde{J}_i$ be defined by

$$\tilde{J}_i(u_0, u_1, \ldots, u_n)(\beta) := \int_{\gamma_\beta} \Big(A^j_i(X(t), \Xi(t)) u_j(X(t)) + B_i(X(t), \Xi(t)) u_0(X(t)) \Big)\, \mathrm{d}t,$$

where γ_β is the geodesic with lift to $S\Omega$ having starting point $\beta \in S\Omega$. Let $\tilde{J} = (\tilde{J}_1, \ldots, \tilde{J}_n)$. This is a vector valued version of the geodesic X-ray transform considered in [44], and described above, sending functions on Ω with values in $\mathbb{C}^{n+1}$ to functions with values in $\mathbb{C}^n$. We then define L as a map from $\mathbb{C}^n$-valued functions on $S\Omega$ to $\mathbb{C}^n$ valued functions on Ω by

$$Lv(\mathsf{x}, \mathsf{y}) = \int_{\mathbb{R}} \int_{\mathbb{S}^{n-2}} \mathsf{x}^{-2} \chi(\lambda/\mathsf{x}) v(\mathsf{x}, \mathsf{y}, \lambda, \omega)\, d\lambda\, d\omega$$

as in [44]; this is a diagonal operator: $L = L_0 \otimes \mathrm{Id}$. Then we consider the map $P = L \circ \tilde{J}$. The main technical result here is:

[2] That is, possibly at the cost of shrinking it; in fact all concepts below are essentially local within $\mathbb{S}^{n-1}$, thus even in full generality one can reduce scattering objects to (conic regions near infinity in) $\mathbb{R}^n$ this way, much as standard Sobolev spaces and pseudodifferential operators are so reduceable to subsets of $\mathbb{R}^n$ with compact closure

Proposition 3.2. *For $\digamma > 0$, let*

$$P_\digamma = e^{-\digamma/\mathsf{x}} P e^{\digamma/\mathsf{x}} : \mathcal{C}_c^\infty(\overline{\hat{\Omega}}; \mathbb{C}^{n+1}) \to \mathcal{C}^\infty(\overline{\hat{\Omega}}; \mathbb{C}^n).$$

Then $P_\digamma \in \Psi_{\mathrm{sc}}^{-1,0}(\overline{\hat{\Omega}}; \mathbb{C}^{n+1}, \mathbb{C}^n)$.

Further, if c is sufficiently small and (15) holds, then for suitable choice of $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$ with $\chi(0) = 1$, $\chi \geq 0$, if we write $P_\digamma = (P_0, \tilde{P})$, with P_0 corresponding to the first component, $\tilde{P}$ the last n components, in the domain space, then $\tilde{P}$ is elliptic in $\Psi_{\mathrm{sc}}^{-1,0}(\overline{\hat{\Omega}}; \mathbb{C}^n, \mathbb{C}^n)$ in a neighborhood of $\overline{\Omega}$.

Proof. This is simply a vector valued version of [44, Proposition 3.3] and [44, Lemma 3.6], recalled above in Proposition 3.1. In particular, to show $P \in \Psi_{\mathrm{sc}}^{-1,0}(\overline{\hat{\Omega}}; \mathbb{C}^{n+1}, \mathbb{C}^n)$, it suffices to show that P is a matrix of pseudodifferential operators $P_{ij} \in \Psi_{\mathrm{sc}}^{-1,0}(\overline{\hat{\Omega}})$, $i = 1, 2, \ldots, n$, $j = 0, 1, 2, \ldots, n$. But for $j > 0$, with $j = 0$ being completely analogous, $P_{ij} w$ has the form

$$\int_{\mathbb{R}} \int_{\mathbb{S}^{n-2}} \mathsf{x}^{-2} \chi(\lambda/\mathsf{x}) \int A_i^j(X_{\mathsf{x},\mathsf{y},\lambda,\omega}(t), \Xi_{\mathsf{x},\mathsf{y},\lambda,\omega}(t)) w(X_{\mathsf{x},\mathsf{y},\lambda,\omega}(t))\, dt\, d\lambda\, d\omega$$

The only difference from [44, Proposition 3.3] then is the presence of the weight factor

$$A_i^j(X_{\mathsf{x},\mathsf{y},\lambda,\omega}(t), \Xi_{\mathsf{x},\mathsf{y},\lambda,\omega}(t)).$$

It is convenient to rewrite this via the metric identification, say by g_0, in terms of tangent vectors. Changing the notation for the flow, in our coordinates $(\mathsf{x}, \mathsf{y}, \lambda, \omega)$, writing now

$$(\gamma_{\mathsf{x},\mathsf{y},\lambda,\omega}(t), \gamma'_{\mathsf{x},\mathsf{y},\lambda,\omega}(t)) = (\mathsf{X}_{\mathsf{x},\mathsf{y},\lambda,\omega}(t), \mathsf{Y}_{\mathsf{x},\mathsf{y},\lambda,\omega}(t), \Lambda_{\mathsf{x},\mathsf{y},\lambda,\omega}(t), \Omega_{\mathsf{x},\mathsf{y},\lambda,\omega}(t))$$

for the lifted geodesic $\gamma_{\mathsf{x},\mathsf{y},\lambda,\omega}(t)$,

$$\tilde{A}_i^j(\mathsf{X}_{\mathsf{x},\mathsf{y},\lambda,\omega}(t), \mathsf{Y}_{\mathsf{x},\mathsf{y},\lambda,\omega}(t), \Lambda_{\mathsf{x},\mathsf{y},\lambda,\omega}(t), \Omega_{\mathsf{x},\mathsf{y},\lambda,\omega}(t))$$

replaces $A_i^j(X_{\mathsf{x},\mathsf{y},\lambda,\omega}(t), \Xi_{\mathsf{x},\mathsf{y},\lambda,\omega}(t))$. As in [44, Proposition 3.3] one rewrites the integral in terms of coordinates $(\mathsf{x}, \mathsf{y}, \mathsf{x}', \mathsf{y}')$ on the left and right factors of $\overline{\hat{\Omega}}$ (i.e. one explicitly expresses the Schwartz kernel), using that the map

$$\Gamma_+ : S\tilde{M} \times [0, \infty) \to [\tilde{M} \times \tilde{M}; \mathrm{diag}],\ \Gamma_+(\mathsf{x}, \mathsf{y}, \lambda, \omega, t) = ((\mathsf{x}, \mathsf{y}), \gamma_{\mathsf{x},\mathsf{y},\lambda,\omega}(t))$$

is a local diffeomorphism, and similarly for Γ_- in which $(-\infty, 0]$ takes the place of $[0, \infty)$; see the discussion around [44, Equation (3.2)-(3.3)]. Here $[\tilde{M} \times \tilde{M}; \mathrm{diag}]$ is the *blow-up* of $\tilde{M}$ at the diagonal $z = z'$, which essentially means the introduction of spherical/polar coordinates, or often more conveniently projective coordinates, about it. Concretely, writing the (local) coordinates from the two factors of $\tilde{M}$ as (z, z'),

$$z, |z - z'|, \frac{z - z'}{|z - z'|} \tag{17}$$

give (local) coordinates on this space. Further,

$$(\mathsf{x}, \mathsf{y}, \lambda, \omega, t) \mapsto \gamma'_{\mathsf{x},\mathsf{y},\lambda,\omega}(t)$$

is a smooth map $SM \times \mathbb{R} \to \mathbb{R}^n$ so composing it with $\Gamma_\pm^{-1}$ from the right, one can re-express the integral giving $P_{ij} w$ away from the boundary as

$$\int w(z') |z - z'|^{-n+1} b\Big(z, \frac{z - z'}{|z - z'|}, |z - z'|\Big)\, dz'$$

as in [44, Equation (3.7)], with

$$b\Big(z,\frac{z-z'}{|z-z'|},0\Big)=\tilde\chi\Big(z,\frac{z'-z}{|z'-z|}\Big)\tilde A_i^j\Big(z,\frac{z'-z}{|z'-z|}\Big)\sigma\Big(z,\frac{z'-z}{|z'-z|}\Big)$$

with $\sigma>0$, bounded below by a positive constant, a weight factor, and where $\chi(\lambda/\mathsf{x})$ is written as $\tilde\chi(\mathsf{x},\mathsf{y},\lambda,\omega)$. Recall from [44, Section 2] that coordinates on Melrose's scattering double space, on which the Schwartz kernels of elements of $\Psi_{\mathrm{sc}}^{s,r}(\overline{\hat\Omega})$ are conormal to the diagonal, near the lifted scattering diagonal, are (with $\mathsf{x}\geq 0$)

$$\mathsf{x},\ \mathsf{y},\ X=\frac{\mathsf{x}-\mathsf{x}'}{\mathsf{x}^2},\ Y=\frac{\mathsf{y}-\mathsf{y}'}{\mathsf{x}}.$$

Further, it is convenient to write coordinates on $[\tilde M\times\tilde M;\mathrm{diag}]$ in the region of interest (see the beginning of the paragraph of Equation (3.10) in [44]), namely (the lift of) $|\mathsf{x}-\mathsf{x}'|<C|\mathsf{y}-\mathsf{y}'|$, as

$$\mathsf{x},\mathsf{y},|\mathsf{y}-\mathsf{y}'|,\frac{\mathsf{x}-\mathsf{x}'}{|\mathsf{y}-\mathsf{y}'|},\frac{\mathsf{y}-\mathsf{y}'}{|\mathsf{y}-\mathsf{y}'|},$$

with the norms being Euclidean norms,[3] instead of (17); we write $\Gamma_\pm$ in terms of these. Note that these are $\mathsf{x},\mathsf{y},\mathsf{x}|Y|,\frac{\mathsf{x}|X|}{|Y|},\hat Y$. Then, similarly, near the boundary as in [44, Equation (3.13)], one obtains the Schwartz kernel

$$\begin{aligned}K^\flat(\mathsf{x},\mathsf{y},X,Y)=\sum_\pm e^{-\digamma X/(1+\mathsf{x}X)}\chi\Big(\frac{X}{|Y|}+|Y|\tilde\Lambda_\pm\Big(\mathsf{x},\mathsf{y},\mathsf{x}|Y|,\frac{\mathsf{x}|X|}{|Y|},\hat Y\Big)\Big)\\ \tilde A_i^j\Big(\Gamma_\pm^{-1}\Big(\mathsf{x},\mathsf{y},\mathsf{x}|Y|,\frac{\mathsf{x}X}{|Y|},\hat Y\Big)\Big)|Y|^{-n+1}J_\pm\Big(\mathsf{x},\mathsf{y},\frac{X}{|Y|},|Y|,\hat Y\Big),\end{aligned}\tag{18}$$

with the density factor J smooth, positive, $=1$ at $\mathsf{x}=0$. Here

$$\mathsf{x},\mathsf{y},|Y|,\frac{X}{|Y|},\hat Y$$

are valid coordinates on the blow-up of the scattering diagonal in[4] $|Y|>\epsilon|X|$, $\epsilon>0$, which is the case automatically on the support of the kernel due to the argument of χ, cf. the discussion after [44, Equation(3.12)], so the argument of $\tilde A_i^j\circ\Gamma_\pm^{-1}$ is smooth *on this blown up space.* We can evaluate this argument: for instance, by [44, Equation(3.10)],

$$(\Lambda\circ\Gamma_\pm^{-1})\Big(\mathsf{x},\mathsf{y},\mathsf{x}|Y|,\frac{\mathsf{x}X}{|Y|},\hat Y\Big)=\mathsf{x}\frac{X}{|Y|}+\mathsf{x}|Y|\tilde\Lambda_\pm\Big(\mathsf{x},\mathsf{y},\mathsf{x}|Y|,\frac{\mathsf{x}X}{|Y|},\hat Y\Big)$$

with $\tilde\Lambda$ smooth, while the subsequent equation in the same location gives

$$(\Omega\circ\Gamma_\pm^{-1})\Big(\mathsf{x},\mathsf{y},\mathsf{x}|Y|,\frac{\mathsf{x}X}{|Y|},\hat Y\Big)=\hat Y+\mathsf{x}|Y|\tilde\Omega_\pm\Big(\mathsf{x},\mathsf{y},\mathsf{x}|Y|,\frac{\mathsf{x}X}{|Y|},\hat Y\Big)$$

with $\tilde\Omega$ smooth. This proves the first part of the proposition as in [44, Proposition 3.3].

To prove the second part, note that in view of (18) (which just needs to be evaluated at $\mathsf{x}=0$), [44, Lemma 3.5] is replaced by the statement that the boundary principal symbol of P_{ij} in $\Psi_{\mathrm{sc}}^{-1,0}(\overline{\hat\Omega})$ is twice the (X,Y)-Fourier transform of

$$e^{-\digamma X}\chi\Big(\frac{X-\alpha(0,\mathsf{y},0,\hat Y)|Y|^2}{|Y|}\Big)\tilde A_i^j\Big(0,\mathsf{y},0,0,\hat Y\Big)|Y|^{-n+1},\tag{19}$$

[3] This is an example of partial projective coordinates for a blow-up.

[4] This is another example of partial projective coordinates for a blow-up.

while for P_{i0} it is twice the (X,Y)-Fourier transform of

$$e^{-\digamma X}\chi\Big(\frac{X-\alpha(0,\mathsf{y},0,\hat{Y})|Y|^2}{|Y|}\Big)\tilde{B}_i\Big(0,\mathsf{y},0,0,\hat{Y}\Big)|Y|^{-n+1},$$

with $\tilde{B}_i$ defined analogously to $\tilde{A}_i^j$. (Recall that $2\alpha(\mathsf{x},\mathsf{y},\lambda,\omega)$ is the x component of $\gamma''_{\mathsf{x},\mathsf{y},\lambda,\omega}(0)$, and the convexity assumption on x is that α is positive; see [44] above Equation (3.1).) For $A_i^j=-\frac{1}{2}c^{-2}(x_0)\delta_i^j$, see (15), the invertibility of the principal symbol, with values in $n\times n$ matrices, of the principal symbol of $\tilde{P}$ follows when χ is chosen as in [44, Lemma 3.6], for it is $-\frac{1}{2}c^{-2}(x_0)$ times the boundary symbol in [44, Lemma 3.6] times the $n\times n$ identity matrix. In general, due to the perturbation stability of the property of invertibility, the same follows for c sufficiently small. $\square$

Corollary 3.1. *With the notation of Proposition 3.2, there is $\tilde{\mathsf{c}}>0$ such that if $0<\mathsf{c}<\tilde{\mathsf{c}}$, then $P_\digamma$ satisfies the estimate*

$$\|u\|_{H^{s,r}_{\mathrm{sc}}(\overline{\hat{\Omega}})}\leq C\|P_\digamma u\|_{H^{s+1,r}_{\mathrm{sc}}(\overline{\hat{\Omega}})}+C\|u_0\|_{H^{s,r}_{\mathrm{sc}}(\overline{\hat{\Omega}})} \tag{20}$$

for u supported in $M\cap\overline{\hat{\Omega}}$, with the constant C uniform in c.

Proof. By the density of elements of $\dot{\mathcal{C}}^\infty(M\cap\overline{\hat{\Omega}})$ in $H^{s,r}_{\mathrm{sc}}(\overline{\hat{\Omega}})$ supported in $M\cap\overline{\hat{\Omega}}$, it suffices to consider $u\in\dot{\mathcal{C}}^\infty(M\cap\overline{\hat{\Omega}})$ to prove (20).

Consider $s=0$, $r=0$. Let $\Lambda\in\Psi^{1,0}_{\mathrm{sc}}(\Omega)$ be elliptic and invertible (one can e.g. locally identify Ω with $\mathbb{R}^n$; then on the Fourier transform side multiplication by $\langle\xi\rangle$ works). Thus, $\Lambda P_\digamma$, with Λ acting diagonally on $\dot{\mathcal{C}}^\infty(\Omega;\mathbb{C}^n)$, is in $\Psi^{0,0}_{\mathrm{sc}}(\overline{\hat{\Omega}};\mathbb{C}^{n+1},\mathbb{C}^n)$ and $\Lambda\tilde{P}\in\Psi^{0,0}_{\mathrm{sc}}(\overline{\hat{\Omega}};\mathbb{C}^n,\mathbb{C}^n)$ is elliptic in $\Psi^{0,0}_{\mathrm{sc}}(\overline{\hat{\Omega}};\mathbb{C}^n,\mathbb{C}^n)$ *locally in a neighborhood of* Ω. This implies, as presented in [44] after Lemma 3.6, relying on the arguments at the end of Section 2 there, that if $\mathsf{c}>0$ is sufficiently small then $\Lambda\tilde{P}$ satisfies

$$\|w\|_{L^2_{\mathrm{sc}}(\overline{\hat{\Omega}})}\leq C_0\|\Lambda\tilde{P}w\|_{L^2_{\mathrm{sc}}(\overline{\hat{\Omega}})} \tag{21}$$

for w supported in $\overline{\Omega}$. Here $L^2_{\mathrm{sc}}(\overline{\hat{\Omega}})=H^{0,0}_{\mathrm{sc}}(\overline{\hat{\Omega}})$ is the L^2 space relative to a non-degenerate *scattering density* — the latter are equivalent to the lifted Lebesgue measure from $\mathbb{R}^n$, thus are bounded multiples of $\frac{d\mathsf{x}\,d\mathsf{y}}{\mathsf{x}^{n+1}}$.

We recall the essential part of this argument briefly. One considers the whole family of domains $\overline{\hat{\Omega}}_{\mathsf{c}}$, which can be identified with each other locally in the region of interest by the maps $\Phi_{\mathsf{c}}(\tilde{\mathsf{x}},\mathsf{y})=(\tilde{\mathsf{x}}+\mathsf{c},\mathsf{y})$, i.e. simply translation in the $\tilde{\mathsf{x}}$-coordinate, so instead of considering a family of spaces with an operator on each of them, one can consider a fixed space, denote this by $\overline{\hat{\Omega}}_0$, with a continuous family of operators, T_{c}, namely $T_{\mathsf{c}}=(\Phi_{\mathsf{c}}^{-1})^*\Lambda\tilde{P}\Phi_{\mathsf{c}}^*$. Notice that we are interested in the region Ω_{c}, and that there is a continuous function f on $\mathbb{R}$ with $f(0)=0$ and $\mathsf{x}_{\mathsf{c}}\leq f(\mathsf{c})$ on Ω_{c}. Correspondingly, in the translated space, $\tilde{\mathsf{x}}\leq f(\mathsf{c})$ on the image of Ω_{c}; notice that this region shrinks as $\mathsf{c}>0$ goes to 0. On the other hand, there is a fixed open set $O\subset\overline{\hat{\Omega}}$, a neighborhood of x_0, on which the operators T_{c} are elliptic in $\Psi^{0,0}_{\mathrm{sc}}(\overline{\hat{\Omega}})$ for $0\leq|\mathsf{c}|<\mathsf{c}_0$. Let K_0 be a compact subset of O, still including a neighborhood of x_0, $\phi\in\mathcal{C}^\infty_c(O)$ be identically 1 on a neighborhood of K. Then the elliptic parametrix construction (which is local, and uniform in c by the continuity) produces a parametrix family $G_{\mathsf{c}}\in\Psi^{0,0}_{\mathrm{sc}}(\overline{\hat{\Omega}};\mathbb{C}^n,\mathbb{C}^n)$, $G_{\mathsf{c}}T_{\mathsf{c}}=\mathrm{Id}+E_{\mathsf{c}}$, where $E_{\mathsf{c}}\in\Psi^{0,0}_{\mathrm{sc}}(\overline{\hat{\Omega}};\mathbb{C}^n,\mathbb{C}^n)$ only, but $\phi E_{\mathsf{c}}\phi\Psi^{-\infty,-\infty}_{\mathrm{sc}}(\overline{\hat{\Omega}};\mathbb{C}^n,\mathbb{C}^n)$, uniformly in c. Then (multiplying the parametrix identity by ϕ from left and right and applying to v) for v supported in K, $(\mathrm{Id}+\phi E_{\mathsf{c}}\phi)v=\phi G_{\mathsf{c}}T_{\mathsf{c}}v$. Now, the Schwartz kernel of $\phi E_{\mathsf{c}}\phi$ is Schwartz, i.e. is bounded by $C(\mathsf{x}\mathsf{x}')^N$ for any N, uniformly in c. (Here we write, say, the Schwartz kernel relative to scattering densities, but as N is arbitrary, this makes little difference.) For $\mathsf{c}>0$, let ϕ_{c} be supported, say, in $\tilde{\mathsf{x}}\leq 2f(\mathsf{c})$, identically 1 near the region $\tilde{\mathsf{x}}\leq f(\mathsf{c})$; one may assume that $\phi\equiv 1$ on $\operatorname{supp}\phi_{\mathsf{c}}$ by making $\mathsf{c}>0$ small. Then, by Schur's lemma, $\phi_{\mathsf{c}}\phi E_{\mathsf{c}}\phi\phi_{\mathsf{c}}$, acting say on $L^2_{\mathrm{sc}}(\overline{\hat{\Omega}})$ (i.e. the L^2-space relative to scattering densities) is bounded by $f(\mathsf{c})^N$ for any N. Thus, there is $\mathsf{c}_1>0$ such that $\mathrm{Id}+\phi_{\mathsf{c}}\phi E_{\mathsf{c}}\phi\phi_{\mathsf{c}}=\mathrm{Id}+\phi_{\mathsf{c}}E_{\mathsf{c}}\phi_{\mathsf{c}}$ is invertible for $0<\mathsf{c}<\mathsf{c}_1$ on $L^2_{\mathrm{sc}}(\overline{\hat{\Omega}})$. In particular, for v supported in $\tilde{\mathsf{x}}\leq f(\mathsf{c})$, so $\phi_{\mathsf{c}}v=v$, $(\mathrm{Id}+\phi_{\mathsf{c}}E_{\mathsf{c}}\phi_{\mathsf{c}})v=\phi_{\mathsf{c}}G_{\mathsf{c}}T_{\mathsf{c}}v$, so inverting the factor on the left and then undoing the transformation Φ_{c} gives the desired conclusion (21).

Thus, with $u=(u_0,\tilde{u})$, we have, with all norms being $L^2_{\mathrm{sc}}(\overline{\hat{\Omega}})$-norms,

$$\|\Lambda P_\digamma u\|^2=\|\Lambda P_0u_0+\Lambda\tilde{P}\tilde{u}\|^2=\|\Lambda\tilde{P}\tilde{u}\|^2+\langle\Lambda\tilde{P}\tilde{u},\Lambda P_0u_0\rangle+\langle\Lambda P_0u_0,\Lambda\tilde{P}\tilde{u}\rangle+\langle\Lambda P_0u_0,\Lambda P_0u_0\rangle.$$

By (21), $\|\Lambda\tilde{P}\tilde{u}\|\geq C_0\|\tilde{u}\|$, $C_0>0$. On the other hand, $\|\Lambda P_0u_0\|\leq C_1\|u_0\|$, $\|\Lambda\tilde{P}\tilde{u}\|\leq C_1\|\tilde{u}\|$ as elements of $\Psi^{0,0}_{\mathrm{sc}}(\Omega)$ are L^2-bounded. Using the Cauchy-Schwartz inequality, for $\delta>0$,

$$|\langle\Lambda P_0u_0,\Lambda\tilde{P}\tilde{u}\rangle|\leq\frac{\delta}{2}\|\tilde{u}\|^2+\frac{C_1^2}{2\delta}\|u_0\|^2.$$

Thus, the last three terms are bounded by $\delta\|\tilde{u}\|^2+(1+\delta^{-1})C_1^2\|u_0\|^2$ in absolute value, so we conclude that, with $\delta=C_0^2/2$,

$$\frac{C_0^2}{2}\|\tilde{u}\|^2\leq\|\Lambda P_\digamma u\|^2+C_1^2(1+2C_0^{-2})\|u_0\|^2, \tag{22}$$

completing the proof of the corollary if $s=r=0$.

The general case follows via conjugating $P_\digamma$ by an elliptic, invertible, element of $\Psi^{s,r}_{\mathrm{sc}}(\overline{\hat{\Omega}})$, which is thus an isomorphism from $H^{s,r}_{\mathrm{sc}}(\overline{\hat{\Omega}})$ to $L^2=H^{0,0}_{\mathrm{sc}}(\overline{\hat{\Omega}})$. Note that such a conjugation does not change the principal symbol, thus the ellipticity. □

We now remark that the even simpler setting of the *scalar transform with a positive weight A on S^*M*, I_A, which was not considered in [44], and which can be considered a special case of $\tilde{J}$ with $(u_0,u_1,\ldots,u_n)$ simply replaced by u. Thus, for consistency with the above notation, let $\tilde{A}(\mathsf{x},\mathsf{y},\lambda,\omega)$ be the weight on SM induced by the metric identification, and let

$$(I_Af)(\beta)=\int_{\mathbb{R}}\tilde{A}(\gamma_\beta(t),\gamma'_\beta(t))f(\gamma_\beta(t))\,dt,\ \beta\in SM.$$

Then the above argument gives that $P_\digamma\in\Psi^{-1,0}_{\mathrm{sc}}(\overline{\hat{\Omega}})$. If at $S_{\partial M}M$, the weight A is independent of the momentum variable ξ, it further gives that $P_\digamma$ is elliptic in a neighborhood of $\overline{\Omega}$. More generally, a modification of the argument of [44] due to H. Zhou [47] allows one to show that the principal symbol is fully elliptic in $\overline{\Omega}$ in the scattering sense merely assuming that A is positive (but *not* the independence condition just mentioned). To see this, one has to Fourier transform (19) in (X,Y) with $\tilde{A}_i^j$ replaced by $\tilde{A}$. The X-Fourier transform is unaffected by the presence of $\tilde{A}$, and gives, as in [44, Equation (3.16)],

$$|Y|^{2-n}e^{-\alpha(\digamma+i\xi)|Y|^2}\hat{\chi}(\xi-i\digamma)\tilde{A}(0,\mathsf{y},0,\hat{Y}).$$

Replacing χ with a Gaussian, $\chi(s)=e^{-s^2/(2\nu)}$, $\nu=\digamma^{-1}\alpha$, which does not have compact support, but an approximation argument (in symbols of order -1) will give this desired property, one can compute that this is, up to a constant factor,

$$\digamma^{-1/2}\alpha^{1/2}|Y|^{2-n}e^{-\digamma^{-1}(\xi^2+\digamma^2)\alpha|Y|^2/2}\tilde{A}(0,\mathsf{y},0,\hat{Y}).$$

We need to compute the Fourier transform in Y. Following [47], one expresses this in polar coordinates in Y:

$$\begin{aligned}\digamma^{-1/2}&\int_{\mathbb{S}^{n-2}}\int_0^\infty e^{-i|Y|\hat{Y}\cdot\eta}\alpha^{1/2}e^{-\digamma^{-1}(\xi^2+\digamma^2)\alpha|Y|^2/2}\tilde{A}(0,\mathsf{y},0,\hat{Y})\,d|Y|\,d\hat{Y}\\&=\frac{1}{2}\digamma^{-1/2}\int_{\mathbb{S}^{n-2}}\int_{\mathbb{R}} e^{-it\hat{Y}\cdot\eta}\alpha^{1/2}e^{-\digamma^{-1}(\xi^2+\digamma^2)\alpha t^2/2}\tilde{A}(0,\mathsf{y},0,\hat{Y})\,dt\,d\hat{Y},\end{aligned}$$

which in turn becomes, up to a constant factor,

$$(\xi^2+\digamma^2)^{-1/2}\int_{\mathbb{S}^{n-2}} e^{-\digamma|\hat{Y}\cdot\eta|^2/(2(\xi^2+\digamma^2)\alpha)}\tilde{A}(0,\mathsf{y},0,\hat{Y})\,d\hat{Y}.$$

The integrand is now positive, which gives the desired ellipticity at $\mathsf{x}=0$. (One also needs to check the ellipticity as $(\xi,\eta)\to\infty$; this is standard, see [44, 47].) One proceeds with an approximation argument as in [44, 47] to complete the proof of the ellipticity. Thus, the above argument gives the estimate

$$\|f\|_{H^{s,r}_{\mathrm{sc}}(\bar{\tilde{\Omega}})}\leq C\|P_\digamma f\|_{H^{s+1,r}_{\mathrm{sc}}(\bar{\tilde{\Omega}})}.$$

As discussed in [44] after Lemma 3.6, this yields the following corollary:

Corollary 3.2. *The weighted scalar transform with a positive weight A on S^*M, with $\tilde{A}$ the associated weight on SM,*

$$(I_Af)(\beta)=\int_{\mathbb{R}}\tilde{A}(\gamma_\beta(t),\gamma'_\beta(t))f(\gamma_\beta(t))\,dt,$$

satisfies that for $\digamma>0$ there is $\mathsf{c}_0>0$ such that for $\delta>0$, $0<\mathsf{c}<\mathsf{c}_0$, $s\geq-1$, we have

$$\|f\|_{e^{(\digamma+\delta)/\mathsf{x}}H^s(\Omega_{\mathsf{c}})}\leq C\|I_Af\|_{e^{\digamma/\mathsf{x}}H^{s+1}(\mathcal{M}_{\mathsf{c}})}.$$

We now return to the actual case of interest and apply Corollary 3.1 with $u_0=e^{-\digamma/\mathsf{x}}f$, $u_j=e^{-\digamma/\mathsf{x}}\partial_jf$. If we show that given $\tilde{\delta}>0$ there is $\mathsf{c}_0>0$ such that $\|u_0\|_{L^2}\leq\tilde{\delta}\|\tilde{u}\|_{L^2}$, i.e. $e^{-\digamma/\mathsf{x}}f$ is bounded by a small multiple of a derivative of f times $e^{-\digamma/\mathsf{x}}$ in L^2, when f is supported in $\overline{\Omega_{\mathsf{c}}}$, then for $\tilde{\delta}>0$ sufficiently small (20) proves that if $P_\digamma u$ vanishes, then so does u, i.e. in this case so does f, for the u_0 term can then be absorbed into the left hand side of (20):

$$(1-\tilde{\delta}C)\|u\|_{H^{0,0}_{\mathrm{sc}}(\bar{\tilde{\Omega}})}\leq C\|P_\digamma u\|_{H^{1,0}_{\mathrm{sc}}(\bar{\tilde{\Omega}})}. \tag{23}$$

Further, rewriting this by removing the weights $e^{-\digamma/\mathsf{x}}$, and estimating the norms in terms if the standard L^2-based space, cf. the discussion after [44, Lemma 3.6] already referenced above,[5] gives, for $\delta>0$,

$$\|f\|_{e^{(\digamma+\delta)/\mathsf{x}}H^1(\Omega_{\mathsf{c}})}\leq C\|\tilde{J}(f,\partial_1f,\ldots,\partial_nf)\|_{e^{\digamma/\mathsf{x}}H^1(\mathcal{M}_{\mathsf{c}})}.$$

But this can now be easily done: let V be a smooth vector field with $V\mathsf{x}=0$, so V is tangent to the boundary of Ω_{c} for every c, and make the non-degeneracy assumption that, for some $\mathsf{c}_1>0$, there is a continuous $T:[0,\mathsf{c}_1)\to\mathbb{R}$ such that $T(0)=0$ and the V-flow takes every point in $\overline{\Omega_{\mathsf{c}}}$ to ∂M in time $\leq T(\mathsf{c})$ (i.e. outside the original manifold). Then the Poincaré inequality for V gives

$$\|v\|_{L^2_{\mathrm{sc}}}\leq C_2T(\mathsf{c})\|Vv\|_{L^2_{\mathrm{sc}}}, \tag{24}$$

for v vanishing outside M, hence the constant is small if $T(\mathsf{c})>0$ is small. (Here the L^2 space we need is the scattering L^2-space, L^2_{sc}, which is $\mathsf{x}^{(n+1)/2}$ times the standard L^2-space, but the extra weight does not affect the argument, since V commutes with multiplication by powers of x.)

[5]The δ loss is actually just the loss of a power of x, due to change of the measure.

To see (24), we recall a standard proof of the local Poincaré inequality: in order to reduce confusion with the notation, let $(z_1, ..., z_n) = (z', z_n)$ be the coordinates, $z_1 = 0$ being the boundary of Ω (so x would be z_1), and assume that the flow of ∂_{z_n} flows from every point in Ω to outside the region in 'time' $\leq \delta$. To normalize the argument, assume that $z_n \geq 0$ in Ω, and we want to estimate v in $z_n \leq \delta$. We assume that the L^2 space is given by a density $F(z')\,|dz|$. Then, for $v \in \mathcal{C}^\infty(\mathbb{R}^n_+)$, $\mathbb{R}^n_+ = [0,\infty)_{z_1} \times \mathbb{R}^{n-1}$, with support in $z_n \geq 0$, by the fundamental theorem of calculus and the Cauchy-Schwartz inequality,

$$|v(z', z_n)| = \Big| \int_0^{z_n} \partial_n v(z', t)\, dt \Big| \leq \Big(\int_0^{z_n} 1\, dt \Big)^{1/2} \Big(\int_0^{z_n} |\partial_n v(z', t)|^2\, dt \Big)^{1/2}$$
$$\leq \delta^{1/2} \Big(\int_0^{\delta} |\partial_n v(z', t)|^2\, dt \Big)^{1/2}.$$

Squaring both sides, multiplying by $F(z')$, and integrating in z', z_n (to $z_n = \delta$) gives

$$\int_{z_n \leq \delta} |v(z', z_n)|^2\, F(z')\, dz \leq \delta^2 \int_{t \leq \delta} |\partial_n v(z', t)|^2\, dt\, F(z')\, dz'.$$

This says that actually

$$\|v\|_{L^2(\mathbb{R}^n_+; F(z')\,|dz|)} \leq \delta \|\partial_n v\|_{L^2(\mathbb{R}^n_+; F(z')\,|dz|)},$$

proving the claim (using $F(z') = z_1^{-n-1}$) in view of the quasi-isometry invariance (which gives a constant factor) of the bound (24). Even if there is more complicated topology, so there are no global coordinates and vector fields as stated, dividing up the problem into local pieces and adding them together gives the desired result: taking steps of size δ, one needs T/δ steps to cover the set, using cutoff functions to localize is easily seen to give a bound proportional to T.

On the other hand, in view of the strict convexity of the boundary, one can construct such a V and T. With $v = e^{-\digamma/\mathsf{x}} f$, this is exactly the desired conclusion since $V(e^{-\digamma/\mathsf{x}} f) = e^{-\digamma/\mathsf{x}} V f$.

In summary we have proved that with $L^2(\Omega) = L^2(\Omega_{\mathsf{c}})$ the standard L^2-space now (as the exponential weight $e^{-\digamma/\mathsf{x}}$ maps such f to $L^2_{\mathrm{sc}}(\hat{\overline{\Omega}})$, see also the discussion after Lemma 3.6 in [44]):

Proposition 3.3. *There is $\mathsf{c}_0 > 0$ such that for $0 < \mathsf{c} < \mathsf{c}_0$, if $f \in L^2(\Omega_{\mathsf{c}})$ and $\tilde{J}(f, \partial_1 f, \dots, \partial_n f) = 0$, then $f = 0$. In fact, for $\digamma > 0$ there is $\mathsf{c}_0 > 0$ such that for $\delta > 0$, $0 < \mathsf{c} < \mathsf{c}_0$, we have*

$$\|f\|_{e^{(\digamma+\delta)/\mathsf{x}} H^1(\Omega_{\mathsf{c}})} \leq C \|\tilde{J}(f, \partial_1 f, \dots, \partial_n f)\|_{e^{\digamma/\mathsf{x}} H^1(\mathcal{M}_{\mathsf{c}})}.$$

4. Proof of Theorem 1.1 and Theorem 1.2

Proof of Theorem 1.1. Let c and $\tilde{c}$ be as in the theorem. Redefine the scattering relation $\mathcal{L}$ as in Figure 1. By Proposition 2.1, we get $J_j f(\gamma) = 0$, see (12) for all geodesics close enough to the ones tangent to ∂M at p. The weights are given by (14), in the new parameterization, with the ellipticity condition (15) satisfied. Then Proposition 3.3 implies $f = 0$ in a neighborhood of p, where $f = c^2 - \tilde{c}^2$ as in (8). $\square$

Proof of Theorem 1.2. Theorem 1.2 is now an easy consequence of Theorem 1.1 using a layer stripping argument. Let $f = c^2 - \tilde{c}^2$. Assume $f \neq 0$, then $\operatorname{supp} f$ has non-empty interior. On the other hand, let $\tau = \inf_{\operatorname{supp} f} \rho$; if $\tau = T$ we are done, for then $\operatorname{supp} f \subset M \setminus \cup_{t \in [0,T)} \Sigma_t$. Thus, suppose $\tau < T$, so $f \equiv 0$ on Σ_t for $t < \tau$, but there exists $x \in \Sigma_\tau \cap \operatorname{supp} f$ (since $\operatorname{supp} f$ is closed). We will show below how to use Corollary 1.1 on $M_\tau := \rho^{-1}(\tau, \infty)$ to conclude that a neighborhood of x is disjoint from $\operatorname{supp} f$ to obtain a contradiction.

All we need to show is that the lens relations L_τ and $\tilde{L}_\tau$ on Σ_τ coincide. Note that $\Sigma_\tau = \partial M_\tau$ is strictly convex for $\tilde{g}$ as well because the second fundamental form for $\tilde{g}$ can be computed by taking derivatives from the exterior $\rho < \tau$, where $g = \tilde{g}$. Fix $(x_\tau, v_\tau) \in \partial_- SM_\tau$, see Figure 2. The geodesic $\gamma_{x_\tau, v_\tau}(s)$ cannot hit Σ_τ again for negative "times" s because otherwise, we would get a contradiction with the strict convexity at Σ_t, where t corresponds to the smallest value of ρ on that geodesic between two contacts with Σ_τ. Since $c = \tilde{c}$ outside M_τ, $\gamma_{x_\tau, v_\tau}(s)$ and $\tilde{\gamma}_{x_\tau, v_\tau}(s)$ coincide outside M_τ for $s < 0$. We prove below the following claim: this negative geodesic ray must be non-trapping, i.e., γ_{x_τ, v_τ} would hit ∂M for a finite negative time s at some point and direction $(x, v) \in \partial_- SM$. In the same way, we show that the same hold for the positive part, $s > 0$, of a geodesic issued from $L_\tau(x_\tau, y_\tau) =: (y_\tau, w_\tau) \in \partial_+ SM_\tau$; and the corresponding point on $\partial_+ SM$ will be denoted by (y, w). Then, since $L(x, v) = (y, w)$, we would also get $L_\tau(x_\tau, v_\tau) = (y_\tau, w_\tau) = \tilde{L}_\tau(x_\tau, v_\tau)$.

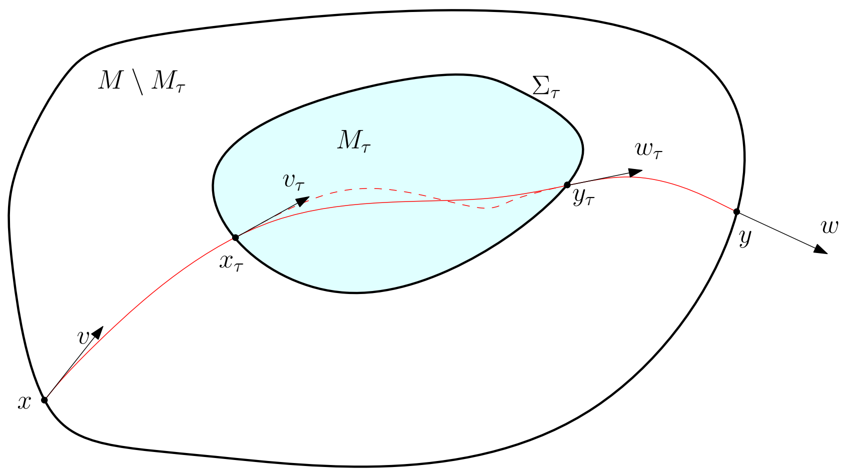

FIGURE 2. One can recover the lens relation on Σ_τ knowing that on ∂M.

To prove the claim, assume that $s \mapsto \gamma_{x_\tau, v_\tau}(s)$ extends to the whole $(-\infty, 0)$ in $M \setminus M_\tau$, and in particular, it never reaches ∂M for $s < 0$. By the arguments above (no critical points), $\rho \circ \gamma_{x_\tau, v_\tau}$ is a strictly increasing function for $s < 0$. On the other hand, it is bounded by below, so it has a limit $\hat{\tau} \geq 0$, as $s \to \infty$, which is also its infimum. By compactness, there exists a sequence $s_j \to -\infty$ (we can start with $s_j = -j$ and take a subsequence) so that $(x_j, v_j) := (\gamma_{x_\tau, v_\tau}(s), \dot{\gamma}_{x_\tau, v_\tau}(s))$ converges to some $(\hat{x}, \hat{v}) \in S(M \setminus M_\tau)$. Next, $\rho(x_j) \searrow \hat{\tau}$. The limit $\hat{v}$ must be tangent to $\Sigma_{\hat{\tau}}$ at $\hat{x}$, because we can easily obtain a contradiction with the strict convexity if it is not. For the next step however, we only need to know that $(\hat{x}, \hat{v}) \in \overline{\partial_- SM_{\hat{\tau}}}$. Now, by the strict convexity of $\Sigma_{\hat{\tau}}$ again, there exists $\delta > 0$ so that $\gamma_{\hat{x}, \hat{v}}(s)$ would hit $\Sigma_{\hat{\tau} - \delta}$ for some negative time. This property is preserved under a small perturbation of $(\hat{x}, \hat{v})$; and therefore applies to (x_j, v_j) for $j \gg 1$. This contradicts the choice of $\hat{\tau}$ however. Note that to cover the possibility that $\hat{\tau} = 0$, we need to extend first g and ρ in a small neighborhood of M first.

$\square$

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