

# INVERSE PROBLEMS IN SPACETIME I: INVERSE PROBLEMS FOR EINSTEIN EQUATIONS

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**Abstract:** *We consider inverse problems for the coupled Einstein equations and the matter field equations on a 4-dimensional globally hyperbolic Lorentzian manifold  $(M, g)$ . We give a positive answer to the question: Do the active measurements, done in a neighborhood  $U \subset M$  of a freely falling observed  $\mu = \mu([s_-, s_+])$ , determine the conformal structure of the spacetime in the minimal causal diamond-type set  $V_g = J_g^+(\mu(s_-)) \cap J_g^-(\mu(s_+)) \subset M$  containing  $\mu$ ?*

*More precisely, we consider the Einstein equations coupled with the scalar field equations and study the system  $\text{Ein}(g) = T$ ,  $T = T(g, \phi) + \mathcal{F}_1$ , and  $\square_g \phi - \mathcal{V}'(\phi) = \mathcal{F}_2$ , where the sources  $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$  correspond to perturbations of the physical fields which we control. The sources  $\mathcal{F}$  need to be such that the fields  $(g, \phi, \mathcal{F})$  are solutions of this system and satisfy the conservation law  $\nabla_j T^{jk} = 0$ . Let  $(\hat{g}, \hat{\phi})$  be the background fields corresponding to the vanishing source  $\mathcal{F}$ . We prove that the observation of the solutions  $(g, \phi)$  in the set  $U$  corresponding to sufficiently small sources  $\mathcal{F}$  supported in  $U$  determine  $V_{\hat{g}}$  as a differentiable manifold and the conformal structure of the metric  $\hat{g}$  in the domain  $V_{\hat{g}}$ . The methods developed here have potential to be applied to a large class of inverse problems for non-linear hyperbolic equations encountered e.g. in various practical imaging problems.*

**Keywords:** Inverse problems, active measurements, Lorentzian manifolds, non-linear hyperbolic equations, Einstein equations, scalar fields.

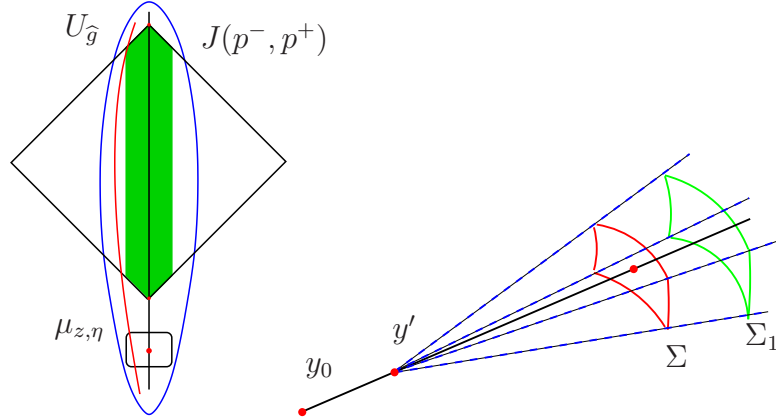
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## 1. INTRODUCTION AND MAIN RESULTS

We consider inverse problems for the non-linear Einstein equations coupled with matter field equations. In this paper, we consider for the matter fields the simplest possible model, the scalar field equations and study the perturbations of a globally hyperbolic Lorentzian manifold  $(M, \hat{g})$  of dimension  $(1 + 3)$ , where the metric signature of  $\hat{g}$  is  $(-, +, +, +)$ .

Roughly speaking, we study the following problem: Can an observer in a space-time determine the structure of the surrounding space-time by doing measurements near its world line? More precisely, when  $U_{\hat{g}}$  is a neighborhood of a time-like geodesic  $\hat{\mu}$ , we assume that we can control sources supported in an open neighborhood  $W_{\hat{g}} \subset U_{\hat{g}}$  of  $\hat{\mu}$  and measure the physical fields in the set  $U_{\hat{g}}$ . We ask, can the properties of the metric (the metric itself or its conformal class) be determined in a suitable larger set  $J(p^-, p^+)$ ,  $p^\pm = \hat{\mu}(s_\pm)$  that is not contained in the set  $U_{\hat{g}}$ , see Fig. 1(Left). This paper considers inverse problems for active measurements and the corresponding problem for passive measurements is studied in the second part of this paper, [57].



**FIGURE 1. Left:** This is a schematic figure in  $\mathbb{R}^{1+1}$ . The black vertical line is the freely falling observer  $\hat{\mu}([-1, 1])$ . The rounded black square is  $\pi(\mathcal{U}_{z_0, \eta_0})$  that is is a neighborhood of  $z_0$ , and the red curve passing through  $z \in \pi(\mathcal{U}_{z_0, \eta_0})$  is the time-like geodesic  $\mu_{z, \eta}([-1, 1])$ . The boundary of the domain  $U_{\hat{g}}$  where we observe waves is shown on blue. The green area is the set  $W_{\hat{g}} \subset U_{\hat{g}}$  where sources are supported, and the black “diamond” is the set  $J(p^-, p^+) = J_{\hat{g}}^+(p^-) \cap J_{\hat{g}}^-(p^+)$ .

**Right:** This is a schematic figure in the space  $\mathbb{R}^3$ . It describes the location of a distorted plane wave (or a piece of a spherical wave)  $\dot{u}$  at different time moments. This wave propagates near the geodesic  $\gamma_{x_0, \zeta_0}((0, \infty)) \subset \mathbb{R}^{1+3}$ ,  $x_0 = (y_0, t_0)$  and is singular on a subset of a

light cone emanated from  $x' = (y', t')$ . The piece of the distorted plane wave is sent from the surface  $\Sigma \subset \mathbb{R}^3$ , it starts to propagate, and at a later time its singular support is the surface  $\Sigma_1$ .

1.0.1. *Notations.* Let  $(M, g)$  be a  $C^\infty$ -smooth (1+3)-dimensional time-orientable Lorentzian manifold. For  $x, y \in M$  we say that  $x$  is in the chronological past of  $y$  and denote  $x \ll y$  if  $x \neq y$  and there is a time-like path from  $x$  to  $y$ . If  $x \neq y$  and there is a causal path from  $x$  to  $y$ , we say that  $x$  is in the causal past of  $y$  and denote  $x < y$ . If  $x < y$  or  $x = y$  we denote  $x \leq y$ . The chronological future  $I^+(p)$  of  $p \in M$  consist of all points  $x \in M$  such that  $p \ll x$ , and the causal future  $J^+(p)$  of  $p$  consist of all points  $x \in M$  such that  $p \leq x$ . One defines similarly the chronological past  $I^-(p)$  of  $p$  and the causal past  $J^-(p)$  of  $p$ . For a set  $A$  we denote  $J^\pm(A) = \cup_{p \in A} J^\pm(p)$ . We also denote  $J(p, q) := J^+(p) \cap J^-(q)$  and  $I(p, q) := I^+(p) \cap I^-(q)$ . If we need to emphasize the metric  $g$  which is used to define the causality, we denote  $J^\pm(p)$  by  $J_g^\pm(p)$  etc.

Let  $\gamma_{x,\xi}(t) = \gamma_{x,\xi}^g(t) = \exp_x(t\xi)$  denote a geodesics in  $(M, g)$ . The projection from the tangent bundle  $TM$  to the base point of a vector is denoted by  $\pi : TM \rightarrow M$ . Let  $L_x M$  denote the light-like directions of  $T_x M$ , and  $L_x^+ M$  and  $L_x^- M$  denote the future and past pointing light-like vectors, respectively. We also denote  $\mathcal{L}_g^+(x) = \exp_x(L_x^+ M) \cup \{x\}$  the union of the image of the future light-cone in the exponential map of  $(M, g)$  and the point  $x$ .

By [9], an open time-orientable Lorentzian manifold  $(M, g)$  is globally hyperbolic if and only if there are no closed causal paths in  $M$  and for all  $q^-, q^+ \in M$  such that  $q^- < q^+$  the set  $J(q^-, q^+) \subset M$  is compact. We assume throughout the paper that  $(M, g)$  is globally hyperbolic.

When  $g$  is a Lorentzian metric, having eigenvalues  $\lambda_j(x)$  and eigenvectors  $v_j(x)$  in some local coordinates, we will use also the corresponding Riemannian metric, denoted by  $g^+$  which has the eigenvalues  $|\lambda_j(x)|$  and the eigenvectors  $v_j(x)$  in the same local coordinates. Let  $B_{g^+}(x, r) = \{y \in M; d_{g^+}(x, y) < r\}$ .

1.0.2. *Perturbations of a global hyperbolic metric.* Let  $(M, \hat{g})$  be a  $C^\infty$ -smooth globally hyperbolic Lorentzian manifold. We will call  $\hat{g}$  the background metric on  $M$  and consider its small perturbations. A Lorentzian metric  $g_1$  dominates the metric  $g_2$ , if all vectors  $\xi$  that are light-like or time-like with respect to the metric  $g_2$  are time-like with respect to the metric  $g_1$ , and in this case we denote  $g_2 < g_1$ . As  $(M, \hat{g})$  is globally hyperbolic, it follows from [36] that there is a Lorentzian metric  $\tilde{g}$  such that  $(M, \tilde{g})$  is globally hyperbolic and  $\hat{g} < \tilde{g}$ . One can assume that the metric  $\tilde{g}$  is smooth. We use the positive definite Riemannian metric  $\hat{g}^+$  to define norms in the spaces  $C_b^k(M)$  of functions with bounded  $k$  derivatives and the Sobolev spaces  $H^s(M)$ .

By [10], the globally hyperbolic manifold  $(M, \tilde{g})$  has an isometry  $\Phi$  to the smooth product manifold  $(\mathbb{R} \times N, \tilde{h})$ , where  $N$  is a 3-dimensional manifold and the metric  $\tilde{h}$  can be written as  $\tilde{h} = -\beta(t, y)dt^2 + \kappa(t, y)$  where  $\beta : \mathbb{R} \times N \rightarrow (0, \infty)$  is a smooth function and  $\kappa(t, \cdot)$  is a Riemannian metric on  $N$  depending smoothly on  $t \in \mathbb{R}$ , and the submanifolds  $\{t'\} \times N$  are  $C^\infty$ -smooth Cauchy surfaces for all  $t' \in \mathbb{R}$ . We define the smooth time function  $\mathbf{t} : M \rightarrow \mathbb{R}$  by setting  $\mathbf{t}(x) = t$  if  $\Phi(x) \in \{t\} \times N$ . Let us next identify these isometric manifolds, that is, we denote  $M = \mathbb{R} \times N$ .

For  $t \in \mathbb{R}$ , let  $M(t) = (-\infty, t) \times N$  and, for a fixed  $t_0 > 0$  and  $t_1 > t_0$ , let  $M_j = M(t_j)$ ,  $j = 0, 1$ . Let  $r_0 > 0$  be sufficiently small and  $\mathcal{V}(r_0)$  be the set of metrics  $g$  on  $M_1 = (-\infty, t_1) \times N$ , which  $C_b^8(M_1)$ -distance to  $\hat{g}$  is less than  $r_0$  and coincide with  $\hat{g}$  in  $M(0) = (-\infty, 0) \times N$ .

1.0.3. *Observation domain  $U$ .* For  $g \in \mathcal{V}(r_0)$ , let  $\mu_g : [-1, 1] \rightarrow M_1$  be a freely falling observer, that is, a time-like geodesic on  $(M, g)$ . Let  $-1 < s_{-3} < s_{-2} < s_{-1} < s_{+1} < s_{+2} < s_{+3} < 1$  be such that  $p^- = \mu_g(s_{-1}) \in \{0\} \times N$  and let  $p^+ = \mu_g(s_{+1})$ . Below, we denote  $s_\pm = s_{\pm 1}$  and  $\hat{\mu} = \mu_{\hat{g}}$ .

When  $z_0 = \hat{\mu}(s_{-2}) \in M(0)$  and  $\eta_0 = \partial_s \hat{\mu}(s_{-2})$ , we denote by  $\mathcal{U}_{z_0, \eta_0}(h)$  the open  $h$ -neighborhood of  $(z_0, \eta_0)$  in the Sasaki metric of  $(TM, \hat{g}^+)$ . We use below a small parameter  $\hat{h} > 0$ . For  $(z, \eta) \in \mathcal{U}_{z_0, \eta_0}(2\hat{h})$  we define on  $(M, g)$  a freely falling observer  $\mu_{g, z, \eta} : [-1, 1] \rightarrow M$ , such that  $\mu_{g, z, \eta}(s_{-2}) = z$ , and  $\partial_s \mu_{g, z, \eta}(s_{-2}) = \eta$ . We assume that  $\hat{h}$  is so small that  $\pi(\mathcal{U}_{z_0, \eta_0}(2\hat{h})) \subset M(0)$  and for all  $g \in \mathcal{V}(r_0)$  and  $(z, \eta) \in \mathcal{U}_{z_0, \eta_0}(2\hat{h})$  the geodesic  $\mu_{g, z, \eta}([-1, 1]) \subset M$  is well defined and time-like and satisfies

$$(1) \quad \mu_{g, z, \eta}(s_{-j-1}) \in I_g^-(\mu_{g, z_0, \eta_0}(s_{-j})), \quad \mu_{g, z, \eta}(s_{j+1}) \in I_g^+(\mu_{g, z_0, \eta_0}(s_j)),$$

for  $j = 1, 2$ . We denote, see Fig. 1(Left),  $\mathcal{U}_{z_0, \eta_0} = \mathcal{U}_{z_0, \eta_0}(\hat{h})$  and

$$(2) \quad U_g = \bigcup_{(z, \eta) \in \mathcal{U}_{z_0, \eta_0}} \mu_{g, z, \eta}([-1, 1]), \quad \hat{U} = U_{\hat{g}}.$$

## 1.1. Formulation of the inverse problem.

1.1.1. *Inverse problems for non-linear wave equations.* The solution of inverse problems is often done by constructing the coefficients of the equations using invariant methods, e.g. using travel time coordinates. Thus in the topical studies of the subject, inverse problems are formulated invariantly, that is, on manifolds, see e.g. [2, 7, 29, 31, 39, 40, 62]. Many physical models lead to non-linear differential equations. In small perturbations, these equations can be approximated by linear equations, and most of the previous results on hyperbolic inverse problems in the multi-dimensional case concern linear models. Moreover, the existing uniqueness results are limited to the time-independent or real-analytic coefficients [2, 6, 7, 31, 54] as these results are based on

Tataru's unique continuation principle [83, 84]. Such unique continuation results have been shown to fail for general metric tensors which are not analytic in the time variable [1]. Even some linear inverse problems are not uniquely solvable. In fact, the counterexamples for these problems have been used in the so-called transformation optics. This has led to models for fixed frequency invisibility cloaks, see e.g. [42] and references therein. These applications give one more motivation to study inverse problems.

Earlier studies on inverse problems for non-linear equations have concerned parabolic equations [50], elliptic equations [51, 52, 81, 82], and 1-dimensional hyperbolic equations [73]. The present paper differs from the earlier studies in that in our approach we do not consider the non-linearity as a perturbation, which effect is small with special solutions, but as a tool that helps us to solve the inverse problem. Indeed, the non-linearity makes it possible to solve a non-linear inverse problem which linearized version is not yet solved. This is the key novel feature of the paper.

1.1.2. *Einstein equations.* Below, we use the Einstein summation convention. The roman indexes  $i, j, k$  etc. run usually over indexes of spacetime variables as the greek letters are reserved to other indexes in sums. The Einstein tensor of a Lorentzian metric  $g = g_{jk}(x)$  is

$$\text{Ein}_{jk}(g) = \text{Ric}_{jk}(g) - \frac{1}{2}(g^{pq} \text{Ric}_{pq}(g))g_{jk}.$$

Here,  $\text{Ric}_{pq}(g)$  is the Ricci curvature of the metric  $g$ . We define the divergence of a 2-covariant tensor  $T_{jk}$  to be  $(\text{div}_g T)_k = \nabla_n(g^{nj}T_{jk})$ .

Let us consider the Einstein equations in the presence of matter,

$$(3) \quad \text{Ein}_{jk}(g) = T_{jk},$$

$$(4) \quad \text{div}_g T = 0,$$

for a Lorentzian metric  $g$  and a stress-energy tensor  $T$  related to the distribution of mass and energy. We recall that by Bianchi's identity  $\text{div}_g(\text{Ein}(g)) = 0$  and thus the equation (4), called the conservation law for the stress-energy tensor, follows automatically from (3).

1.1.3. *Reduced Einstein tensor.* Let  $m \geq 5$ ,  $t_1 > t_0 > 0$  and  $g' \in \mathcal{V}(r_0)$  be a  $C^m$ -smooth metric that satisfy the Einstein equations  $\text{Ein}(g') = T'$  on  $M(t_1)$ . When  $r_0$  above is small enough, there is a diffeomorphism  $f : M(t_1) \rightarrow f(M(t_1)) \subset M$  that is a  $(g', \widehat{g})$ -wave map  $f : (M(t_1), g') \rightarrow (M, \widehat{g})$  and satisfies  $M(t_0) \subset f(M(t_1))$ . Here,  $f : (M(t_1), g') \rightarrow (M, \widehat{g})$  is a wave map, see [14, Sec. VI.7.2 and App. III, Thm. 4.2], if

$$(5) \quad \square_{g', \widehat{g}} f = 0 \quad \text{in } M(t_1),$$

$$(6) \quad f = Id, \quad \text{in } (-\infty, 0) \times N,$$

where

$$\square_{g', \hat{g}} f = (g')^{jk} \left( \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^k} f^A - \Gamma_{jk}^n(x) \frac{\partial}{\partial x^n} f^A + \hat{\Gamma}_{BC}^A(f(x)) \frac{\partial}{\partial x^j} f^B \frac{\partial}{\partial x^k} f^C \right),$$

and  $\hat{\Gamma}_{BC}^A$  denotes the Christoffel symbols of metric  $\hat{g}$  and  $\Gamma_{kl}^j$  are the Christoffel symbols of metric  $g'$ , see [14, formula (VI.7.32)]. The wave map has the property that  $g = f_* g'$  satisfies  $\text{Ein}(g) = \text{Ein}_{\hat{g}}(g)$ , where  $\text{Ein}_{\hat{g}}(g)$  is the  $\hat{g}$ -reduced Einstein tensor, see also formula (23) below,

$$(\text{Ein}_{\hat{g}} g)_{pq} = -\frac{1}{2} g^{jk} \hat{\nabla}_j \hat{\nabla}_k g_{pq} + \frac{1}{4} (g^{nm} g^{jk} \hat{\nabla}_j \hat{\nabla}_k g_{nm}) g_{pq} + P_{pq}(g, \hat{\nabla} g),$$

where  $\hat{\nabla}_j$  is the covariant differentiation with respect to the metric  $\hat{g}$  and  $P_{pq}$  is a polynomial function of  $g_{nm}$ ,  $g^{nm}$ , and  $\hat{\nabla}_j g_{nm}$  with coefficients depending on the metric  $\hat{g}_{nm}$  and its derivatives. Considering the wave map  $f$  as a transformation of coordinates, we see that  $g = f_* g'$  and  $T = f_* T'$  satisfy the  $\hat{g}$ -reduced Einstein equations

$$(7) \quad \text{Ein}_{\hat{g}}(g) = T \quad \text{on } M(t_0).$$

In the literature, the above is often stated by saying that the reduced Einstein equations (7) is the Einstein equations written with the wave-gauge corresponding to the metric  $\hat{g}$ . The equation (7) is a quasi-linear hyperbolic system of equations for  $g_{jk}$ . We emphasize that a solution of the reduced Einstein equations can be a solution of the original Einstein equations only if the stress energy tensor satisfies the conservation law  $\nabla_j^g T^{jk} = 0$ . It is usual also to assume that the energy density is non-negative. For instance, the weak energy condition requires that  $T_{jk} X^j X^k \geq 0$  for all time-like vectors  $X$ . Next, we couple the Einstein equations with matter fields and formulate the direct problem for the  $\hat{g}$ -reduced Einstein equations.

1.1.4. *The initial value problem with sources.* We consider metric and physical fields on a Lorentzian manifold  $(M, g)$ . This is an informal discussion. We aim to study an inverse problem with active measurements. As measurements cannot be implemented in Vacuum (as the Einstein equations is uniquely solvable with fixed initial data), we have to add matter fields in the model. We consider the coupled system of the Einstein equations and the equations for  $L$  scalar fields  $\phi = (\phi_\ell)_{\ell=1}^L$  with some sources  $\mathcal{F}^1$  and  $\mathcal{F}^2$ .

Let  $\hat{g}$  and  $\hat{\phi} = (\hat{\phi}_\ell)_{\ell=1}^L$  be  $C^\infty$ -background fields on  $M$ . Consider

$$(8) \quad \text{Ein}_{\hat{g}}(g) = T, \quad T_{jk} = \mathbf{T}_{jk}(g, \phi) + \mathcal{F}_{jk}^1, \quad \text{in } M_0 = (-\infty, t_0) \times N,$$

$$\mathbf{T}_{jk}(g, \phi) = \left( \sum_{\ell=1}^L (\partial_j \phi_\ell \partial_k \phi_\ell - \frac{1}{2} g_{jk} g^{pq} \partial_p \phi_\ell \partial_q \phi_\ell) \right) - V(\phi) g_{jk},$$

$$\square_g \phi_\ell - V'_\ell(\phi) = \mathcal{F}_\ell^2, \quad \ell = 1, 2, 3, \dots, L,$$

$$g = \hat{g} \text{ and } \phi_\ell = \hat{\phi}_\ell \text{ in } M_0 \setminus J_g^+(p^-),$$

where  $\mathcal{F}^1$  and  $\mathcal{F}^2$  are supported in  $U_g^+ \cap J_g^+(p^-)$ ,  $V \in C^\infty(\mathbb{R}^L)$ , and  $V'_\ell(s) = \frac{\partial}{\partial s_\ell} V(s)$ ,  $s = (s_1, s_2, \dots, s_L)$ . A typical model is  $V(s) = \sum_{\ell=1}^L \frac{1}{2} m^2 s_\ell^2$ . Above,  $\square_g \phi = |\det(g)|^{-\frac{1}{2}} \partial_p (|\det(g)|^{\frac{1}{2}} g^{pq} \partial_q \phi)$ . We assume that the background fields  $\widehat{g}$  and  $\widehat{\phi}$  satisfy the equations (8) with  $\mathcal{F}^1 = 0$  and  $\mathcal{F}^2 = 0$ . Note that above  $J_g^+(p^-) \cap M_0 \subset J_{\widehat{g}}^+(p^-)$  when  $g \in \mathcal{V}(r_0)$  and  $r_0$  is small enough.

To obtain a physically meaningful model, we need to assume that the physical conservation law in relativity,

$$(9) \quad \nabla_p (g^{pk} T_{kj}) = 0, \quad \text{for } j = 1, 2, 3, 4, \text{ where } T_{kj} = \mathbf{T}_{kj}(g, \phi) + \mathcal{F}_{kj}^1$$

is satisfied. Here  $\nabla = \nabla^g$  is the connection corresponding to  $g$ . As will be noted in Subsection 3.1.1, the reduced Einstein tensor  $\text{Ein}_{\widehat{g}}(g)$  is equal to the Einstein tensor  $\text{Ein}(g)$  when  $(g, \phi)$  satisfies the system (8) and the conservation law (9). We mainly need local existence results<sup>1</sup> for the system (8). The global existence problem for the related systems has recently attracted much interest in the mathematical community and many important results been obtained, see e.g. [21, 25, 59, 61, 63, 64].

We encounter above the difficulty that the source  $\mathcal{F} = (\mathcal{F}^1, \mathcal{F}^2)$  in (8) has to satisfy the condition (9) that depends on the solution  $g$  of (8). This makes the formulation of active measurements in relativity difficult. In Appendix A we consider a model where the source term  $\mathcal{F}^1$  corresponds to e.g. fluid fields consisting of particles whose 4-velocity vectors are controlled and  $\mathcal{F}^2$  contains a term corresponding to a secondary source function that adapts the changes of  $g, \phi$ , and  $\mathcal{F}^1$  so that the physical conservation law (9) is satisfied. This model is considered in detail in [58]. However, in this paper we replace the adaptive source functions by a general assumption of microlocal linearization stability that does not fix the physical model for the source fields  $\mathcal{F}$ .

1.1.5. *Definition of measurements.* For  $r > 0$  let, see Fig. 1(Left),

$$(10) \quad W_g(r) = \bigcup_{s_- < s < s_+ - r} I_g(\mu_g(s), \mu_g(s+r)),$$

and let  $r_1 > 0$  be so small that  $W_g(2r_1) \subset U_g$  for all  $g \in \mathcal{V}(r_0)$ . We denote  $W_g = W_g(r_1)$ . We use Fermi-type coordinates: Let  $Z_j(s)$ ,  $j = 1, 2, 3, 4$  be a parallel frame of linearly independent time-like vectors at  $\mu_g(s)$  such that  $Z_1(s) = \dot{\mu}_g(s)$ . Let  $\Phi_g : (t_j)_{j=1}^4 \mapsto \exp_{\mu_g(t_1)}(\sum_{j=2}^4 t_j Z_j(t_1))$ . We assume that  $r_1 > 0$  is so small that  $\Psi_g = \Phi_g^{-1}$  defines coordinates in  $W_g(2r_1)$ . We define the norm-like functions

$$\mathcal{N}_{\widehat{g}}^{(k)}(g) = \|(\Psi_g)_* g - (\Psi_{\widehat{g}})_* \widehat{g}\|_{C_b^k(\overline{\Psi_{\widehat{g}}(W_{\widehat{g}})})}, \quad \mathcal{N}_g^{(k)}(\mathcal{F}) = \|(\Psi_g)_* \mathcal{F}\|_{C_b^k(\overline{\Psi_g(W_g)})},$$

<sup>1</sup>In this paper we do not use optimal smoothness for the solutions in classical  $C^k$  spaces or Sobolev space  $W^{k,p}$  but just suitable smoothness for which the non-linear wave equations can be easily analyzed using  $L^2$ -based Sobolev spaces.

where  $k \in \mathbb{N}$ , that measures the  $C^k$  distance of  $g$  from  $\widehat{g}$  and  $\mathcal{F}$  from zero in the Fermi-type coordinates. As we have assumed that the background metric  $\widehat{g}$  and the field  $\widehat{\phi}$  are  $C^\infty$ -smooth, we can consider as smooth sources as we wish. Thus we use below smoothness assumptions on the sources that are far from the optimal ones.

Let us define the source-observation 4-tuples corresponding to measurements in  $U_g$  with sources  $\mathcal{F}$  supported in  $W_g$ . Let  $\varepsilon > 0$  and  $k_0 \geq 8$  and define

$$(11) \quad \mathcal{D}(\widehat{g}, \widehat{\phi}, \varepsilon) = \{[(U_g, g|_{U_g}, \phi|_{U_g}, \mathcal{F}|_{U_g})] \ ; \ (g, \phi, \mathcal{F}) \text{ are } C^{k_0+3}\text{-smooth} \\ \text{solutions of (8) and (9) with } \mathcal{F} \in C_0^{k_0+3}(W_g; \mathcal{B}^L),$$

$$J_g^+(\text{supp}(\mathcal{F})) \cap J_g^-(\text{supp}(\mathcal{F})) \subset W_g, \mathcal{N}_g^{(k_0)}(\mathcal{F}) < \varepsilon, \mathcal{N}_g^{(k_0)}(g) < \varepsilon\}.$$

Above, the sources  $\mathcal{F}$  above are considered as sections of the bundle  $\mathcal{B}^L$ , where  $\mathcal{B}^L$  is a vector bundle on  $M$  that is the product bundle of the bundle of symmetric  $(0, 2)$ -tensors and the trivial vector bundle with the fiber  $\mathbb{R}^L$ . Also,  $[(U_g, g, \phi, \mathcal{F})]$  denotes the equivalence class of all Lorentzian manifolds  $(U', g')$  and functions  $\phi' = (\phi'_\ell)_{\ell=1}^L$  and the tensors  $\mathcal{F}'$  defined on a  $C^\infty$ -smooth manifold  $U'$ , such that there is  $C^\infty$ -smooth diffeomorphism  $\Psi : U' \rightarrow U_g$  satisfying  $\Psi_*g' = g$ ,  $\Psi_*\phi'_\ell = \phi_\ell$ , and  $\Psi_*\mathcal{F}' = \mathcal{F}$ .

In many inverse problems one considers a Dirichlet-to-Neumann map or, equivalently to that, the Cauchy data set that is the graph of the Dirichlet-to-Neumann map. Similarly, the source-observation 4-tuples  $\mathcal{D}(\widehat{g}, \widehat{\phi}, \varepsilon)$  could be considered as graph of a “source-to-field” map but due to the conservation law the source-to-field map could be defined only on a subset of sources supported in  $U_g$ . To avoid the difficulties related to the fact that we do not have a good characterization for this subset, nor do we know the wave map coordinates in  $U_g$ , we do not define a “source-to-field” map but use the data set  $\mathcal{D}(\widehat{g}, \widehat{\phi}, \varepsilon)$ .

Note that  $[(U_{\widehat{g}}, \widehat{g}, \widehat{\phi}, 0)]$  is the only element in  $\mathcal{D}(\widehat{g}, \widehat{\phi}, \varepsilon)$  for which the  $\mathcal{F}$ -component is zero. Thus the collection  $\mathcal{D}(\widehat{g}, \widehat{\phi}, \varepsilon)$  determines the isometry type of  $(U_{\widehat{g}}, \widehat{g})$ .

1.1.6. *Linearized equations.* We need also to consider the linearized version of the equations (8) that have the form (in local coordinates)

$$(12) \quad \square_{\widehat{g}} \dot{g}_{jk} + A_{jk}(\dot{g}, \dot{\phi}, \partial \dot{g}, \partial \dot{\phi}) = f^1, \quad \text{in } M_0, \\ \square_{\widehat{g}} \dot{\phi}_\ell + B_\ell(\dot{g}, \dot{\phi}, \partial \dot{g}, \partial \dot{\phi}) = f^2, \quad \ell = 1, 2, 3, \dots, L,$$

where  $A_{jk}$  and  $B_\ell$  are first order linear differential operators which coefficients depend on  $\widehat{g}$  and  $\widehat{\phi}$ . When  $g_\varepsilon$  and  $\phi_\varepsilon$  are solutions of (8) with source  $\mathcal{F}_\varepsilon$  depending smoothly on  $\varepsilon \in \mathbb{R}$  such that  $(g_\varepsilon, \phi_\varepsilon, \mathcal{F}_\varepsilon)|_{\varepsilon=0} = (\widehat{g}, \widehat{\phi}, 0)$ , then  $(\dot{g}, \dot{\phi}, f) = (\partial_\varepsilon g_\varepsilon, \partial_\varepsilon \phi_\varepsilon, \partial_\varepsilon \mathcal{F}_\varepsilon)|_{\varepsilon=0}$  solve (12).

Let us consider the concept of the *linearization stability (LS)* for the source problems, cf. [11, 15, 35, 37], and references therein: Let  $s_0 > 4$



and consider a  $C^{s_0+4}$ -smooth source  $f = (f^1, f^2)$  that is supported in  $W_{\widehat{g}}$  and satisfies the linearized conservation law

$$(13) \quad \frac{1}{2} \widehat{g}^{pk} \widehat{\nabla}_p f_{kj}^1 + \sum_{\ell=1}^L f_\ell^2 \partial_j \widehat{\phi}_\ell = 0, \quad j = 1, 2, 3, 4.$$

Let  $(\dot{g}, \dot{\phi})$  be the solution of the linearized Einstein equations (12) with source  $f$ . We say that  $f$  has the LS-property in  $C^{s_0}(M_0)$  if there are  $\varepsilon_0 > 0$  and a family  $\mathcal{F}_\varepsilon = (\mathcal{F}_\varepsilon^1, \mathcal{F}_\varepsilon^2)$  of sources, supported in  $W_{g_\varepsilon}$  for all  $\varepsilon \in [0, \varepsilon_0)$ , and functions  $(g_\varepsilon, \phi_\varepsilon)$  that all depend smoothly on  $\varepsilon \in [0, \varepsilon_0)$  in  $C^{s_0}(M_0)$  such that

$$(14) \quad (g_\varepsilon, \phi_\varepsilon) \text{ satisfies the equations (8) and the conservation law (9),}$$

$$(g_\varepsilon, \phi_\varepsilon, \mathcal{F}_\varepsilon)|_{\varepsilon=0} = (\widehat{g}, \widehat{\phi}, 0), \text{ and } (\dot{g}, \dot{\phi}, f) = (\partial_\varepsilon g_\varepsilon, \partial_\varepsilon \phi_\varepsilon, \partial_\varepsilon \mathcal{F}_\varepsilon)|_{\varepsilon=0}.$$

In this case, we say that  $f = (f^1, f^2)$  has the LS-property with the family  $\mathcal{F}_\varepsilon$ ,  $\varepsilon \in [0, \varepsilon_0)$ .

Note that above (13) is obtained by linearization of the conservation law (9).

The above linearization stability, concerning the local existence of the solutions in  $M_0 = (-\infty, t_0) \times N$ , is valid under quite general conditions, see [12]. Below we will require that for some sources  $f$ , supported in a neighborhood  $V$  of a point  $y$ , we can find functions  $\mathcal{F}_\varepsilon$  that are also supported in the set  $V$ . The conditions when this happen are considered below and in Appendix A and [58].

Next, we consider sources that are conormal distributions. When  $Y \subset U_{\widehat{g}}$  is a 2-dimensional space-like submanifold, consider local coordinates defined in  $V \subset M_0$  such that  $Y \cap V \subset \{x \in \mathbb{R}^4; x^j b_j^1 = 0, x^j b_j^2 = 0\}$ , where  $b_j^1, b_j^2 \in \mathbb{R}$ . Next we slightly abuse the notation by identifying  $x \in V$  with its coordinates  $X(x) \in \mathbb{R}^4$ . We denote  $f \in \mathcal{I}^n(Y)$ ,  $n \in \mathbb{R}$ , if in the above local coordinates,  $f$  can be written as

$$(15) \quad f(x^1, x^2, x^3, x^4) = \operatorname{Re} \int_{\mathbb{R}^2} e^{i(\theta_1 b_m^1 + \theta_2 b_m^2)x^m} \sigma_f(x, \theta_1, \theta_2) d\theta_1 d\theta_2,$$

where  $\sigma_f(x, \theta) \in S_{0,1}^n(V; \mathbb{R}^2)$ ,  $\theta = (\theta_1, \theta_2)$  is a classical symbol. A function  $c(x, \theta)$  that is  $n$ -positive homogeneous in  $\theta$ , i.e.,  $c(x, s\theta) = s^n c(x, \theta)$  for  $s > 0$ , is the principal symbol of  $f$  if there is  $\phi \in C_0^\infty(\mathbb{R}^2)$  being 1 near zero such that  $\sigma_f(x, \theta) - (1 - \phi(\theta))c(x, \theta) \in S_{0,1}^{n-1}(V; \mathbb{R}^2)$ . When  $\eta = \theta_1 b_1 + \theta_2 b_2 \in N_x^* Y$ , we say that  $\widetilde{c}(x, \eta) = c(x, \theta)$  is the value of the principal symbol of  $f$  at  $(x, \eta) \in N^* Y$ .

We need a condition that we call *microlocal linearization stability*:

**Assumption  $\mu$ -LS (Microlocal linearization stability):** Let  $n_0 \in \mathbb{Z}_-$ ,  $Y \subset U_{\widehat{g}}$  be a 2-dimensional space-like submanifold,  $V \subset U_{\widehat{g}}$  an open local coordinate neighborhood of  $y \in Y$  with the coordinates  $X : V \rightarrow \mathbb{R}^4$ ,  $X^j(x) = x^j$  such that  $X(Y \cap V) \subset \{x \in \mathbb{R}^4; x^j b_j^1 = 0, x^j b_j^2 = 0\}$ .

Let, in addition,  $(y, \eta) \in N^*Y$  be a light-like covector,  $\mathcal{W} \subset N^*Y$  be a conic neighborhood of  $(y, \eta)$ ,  $(c_{jk})_{j,k=1}^4$  be a symmetric  $(0, 2)$  tensor at  $y$  that satisfies

$$(16) \quad \widehat{g}^{jk}(y)\eta_l c_{kj} = 0, \quad \text{for all } j = 1, 2, 3, 4,$$

and  $(d_\ell)_{\ell=1}^L \in \mathbb{R}^L$ . Then, for any  $n \in \mathbb{Z}_-$ ,  $n \leq n_0$  there are  $f_{jk}^1 \in \mathcal{I}^n(Y)$ ,  $(j, k) \in \{1, 2, 3, 4\}^2$ , and  $f_\ell^2 \in \mathcal{I}^n(Y)$ ,  $\ell = 1, 2, \dots, L$ , supported in  $V$  with symbols that are in  $S^{-\infty}$  outside the neighborhood  $\mathcal{W}$  of  $(y, \eta)$ . The principal symbols of  $f_{jk}^1(x)$  and  $f_\ell^2(x)$  in the  $X$ -coordinates, denoted by  $\widetilde{f}_{jk}^1$  and  $\widetilde{f}_\ell^2$ , respectively, are at  $(y, \eta)$  equal to  $\widetilde{f}_{jk}^1(y, \eta) = c_{jk}$  and  $\widetilde{f}_\ell^2(y, \eta) = d_\ell$ . Moreover, the source  $f = (f^1, f^2)$  satisfies the linearized conservation law (13) and  $f$  has the LS property (14) in  $C^{s_1}(M_0)$ ,  $s_1 \geq 13$ , with a family  $\mathcal{F}_\varepsilon$ ,  $\varepsilon \in [0, \varepsilon_0)$  such that  $\mathcal{F}_\varepsilon$  are supported in  $V$ .

**1.2. Main results.** Our main result is the following uniqueness theorem for the inverse problem for the Einstein-scalar field equations:

**Theorem 1.1.** *Let  $(M^{(1)}, \widehat{g}^{(1)})$  and  $(M^{(2)}, \widehat{g}^{(2)})$  be globally hyperbolic manifolds and  $(\widehat{g}^{(j)}, \widehat{\phi}^{(j)})$ ,  $j = 1, 2$  satisfy Einstein-scalar field equations (8) with vanishing sources  $\mathcal{F}^1 = 0$  and  $\mathcal{F}^2 = 0$ . Also, assume that there are neighborhoods  $U_{\widehat{g}^{(j)}}$ ,  $j = 1, 2$  of the time-like geodesics  $\widehat{\mu}_j \subset M^{(j)}$  where the Assumption  $\mu$ -LS is valid. Moreover, assume that, for some  $\varepsilon > 0$ , the sets  $\mathcal{D}(\widehat{g}^{(1)}, \widehat{\phi}^{(1)}, \varepsilon)$  and  $\mathcal{D}(\widehat{g}^{(2)}, \widehat{\phi}^{(2)}, \varepsilon)$  are the same, that is, the measurements done in  $U_{\widehat{g}^{(1)}}$  and  $U_{\widehat{g}^{(2)}}$  coincide. Let  $p_j^- = \widehat{\mu}_j(s_-)$  and  $p_j^+ = \widehat{\mu}_j(s_+)$ . Then there is a diffeomorphism  $\Psi : I_{\widehat{g}^{(1)}}(p_1^-, p_1^+) \rightarrow I_{\widehat{g}^{(2)}}(p_2^-, p_2^+)$  such that the metric  $\Psi^*\widehat{g}^{(2)}$  is conformal to  $\widehat{g}^{(1)}$ .*

The theorem above says that the data  $\mathcal{D}(\widehat{g}, \widehat{\phi}, \varepsilon)$  determine uniquely the manifold  $I_{\widehat{g}}(p^-, p^+) \subset M$  and the conformal type of  $\widehat{g}$  in  $I_{\widehat{g}}(p^-, p^+)$ . Reconstruction of the conformal structure of the manifold provides naturally less information than finding the whole metric structure, but the conformal structure is crucial for many questions of analysis and physics, see e.g. [16, 38]. Roughly speaking, the above result means that if the manifold  $(M_0, \widehat{g})$  is unknown, then the source-to-observation pairs corresponding to freely falling sources which are near a freely falling observer  $\mu_{\widehat{g}}$  and the measurements of the metric tensor and the scalar fields in a neighborhood  $U_{\widehat{g}}$  of  $\mu_{\widehat{g}}$ , determine the metric tensor up to conformal transformation in the set  $I_{\widehat{g}}(p^-, p^+)$ . In this paper we present the proof of this result, but mention for the convenience of the reader that extended versions of some technical computations discussed briefly can be found in the preprint [56].

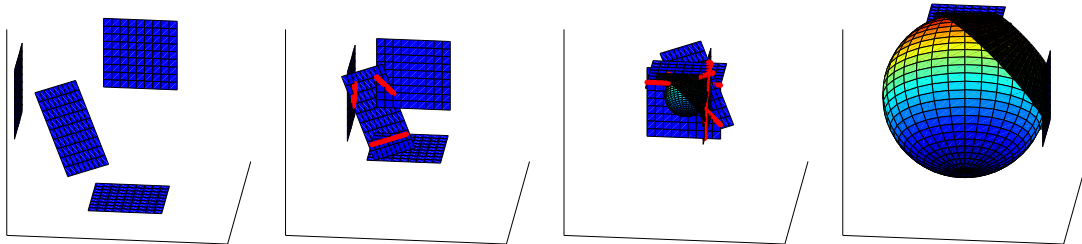
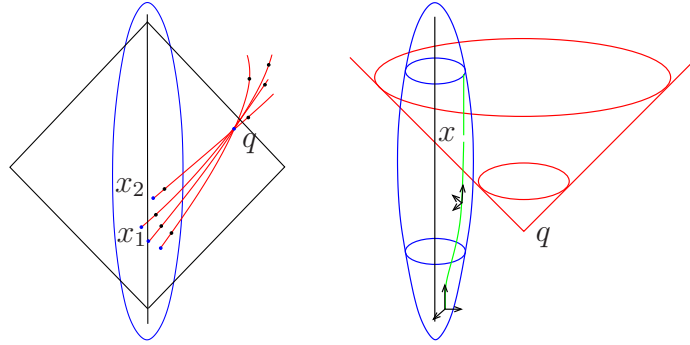


FIGURE 2. Four plane waves propagate in space. When the planes intersect, the non-linearity of the hyperbolic system produces new waves. The four figures show the waves before the interaction of the waves start, when 2-wave interactions have started, when all 4 waves have just interacted, and later after the interaction. **Left:** Plane waves before interacting. **Middle left:** The 2-wave interactions (red line segments) appear but do not cause new propagating singularities. **Middle right and Right:** All plane waves have intersected and new waves have appeared. The 3-wave interactions cause new conic waves (black surface). Only one such wave is shown in the figure. The 4-wave interaction causes a point source in spacetime that sends a spherical wave in all future light-like directions. This spherical wave is essential in our considerations. For an animation on these interactions, see the supplementary video [88].

**The outline of the proof of Theorem 1.1:** We consider 4 sources that send distorted plane waves from  $W_{\hat{g}}$  that propagate near geodesics  $\gamma_{x_j, \xi_j}$ , see Fig. 1(Right). Due to the non-linearity of the Einstein equations, these waves interact and may produce a point source in the space-time, see Fig. 2 on the interaction of waves. All four waves interact if the geodesics  $\gamma_{x_j, \xi_j}$  intersect at a single point  $q$  of the space-time, see Fig. 3(Left). We show in Sec. 3, that if the intersection of these geodesics happens before the conjugate points of the geodesics (i.e. the caustics of the waves), then there are some sources satisfying (13) such that the produced spherical plane wave has a non-vanishing singularity on the future light cone emanating from  $q$ , see Fig. 3(Right). Thus we can observe in  $U_{\hat{g}}$  the set of the earliest light observations of the point  $q$ ,  $\mathcal{E}_{U_{\hat{g}}}(q)$ . By varying the starting directions  $(x_j, \xi_j)$  of the geodesics we can observe the collection of sets of the earliest light observations for all points  $q \in I(p^-, p^+)$ . This data determine uniquely the topological, differentiable and the conformal structures on  $I(p^-, p^+)$ , as is shown in the second part of this paper, [57]. In the proof we have to deal with several technical difficulties: First, the wave produced by the 4th order interaction consists of many terms which could cancel each other. In Sec. 3, we show that the principal symbol of this wave, considered in the wave map coordinates, does not vanish in a generic situation. Second, we do not know the wave map coordinates in  $U_{\hat{g}}$  and thus we

have to consider observations in normal coordinates. This change of coordinates is a gauge transform where some of the produced singularities may vanish. However, in Sec. 4 we show that some singularities persists and can be observed. Third, caustics may produce singularities whose interactions are difficult to analyze. We avoid this by using global Lorentzian geometry (in Sec. 2) and show that no caustics affect in the earliest observations from the interaction of four colliding, distorted plane waves when these waves are produced by appropriate sources and the collision of the waves is observed before a certain time. We use this in Sec. 5 to give a step-by-step construction of the diamond set  $J^+(p^-) \cap J^-(p^+)$ .



**FIGURE 3. Left:** The four light-like geodesics  $\gamma_{x_j, \xi_j}([0, \infty))$ ,  $j = 1, 2, 3, 4$  starting at the blue points  $x_j$  intersect at  $q$  before the first cut points of  $\gamma_{x_j, \xi_j}([t_0, \infty))$ , denoted by black points. The points  $\gamma_{x_j, \xi_j}(t_0)$  are also shown as black points. **Right:** The future light cone  $\mathcal{L}_g^+(q)$  emanating from the point  $q$  is shown as a red cone. The set of the light observation points  $\mathcal{P}_{U_{\hat{g}}}(q)$ , see Definition 2.4, is the intersection of the set  $\mathcal{L}_g^+(q)$  and the set  $U_{\hat{g}}$ . The green curve is the geodesic  $\mu = \mu_{\hat{g}, z, \eta}$ . This geodesic intersects the future light cone  $\mathcal{L}_g^+(q)$  at the point  $x$ . The black vectors are the frame  $(Z_j)$  that is obtained using parallel translation along the geodesic  $\mu$ . Near the intersection point  $x$  we detect singularities in normal coordinates centered at  $x$  and associated to the frame  $(Z_j)$ .

We want to point out that by the main theorem, if we have two non-conformal spacetimes, a generic measurement gives different results on these manifolds. In particular, this implies that perfect space-time cloaking, in sense of light rays, see [33, 66], with a smooth metric in a globally hyperbolic universe is not possible.

The assumptions of Theorem 1.1 are valid in many cases. For instance, consider the a case when the background fields vary sufficiently:

**Condition A:** Assume that at any  $x \in \overline{U_{\hat{g}}}$  there is a permutation  $\sigma : \{1, 2, \dots, L\} \rightarrow \{1, 2, \dots, L\}$ , denoted  $\sigma_x$ , such that the  $5 \times 5$

matrix  $[B_{jk}^\sigma(\widehat{\phi}(x), \nabla\widehat{\phi}(x))]_{j,k \leq 5}$  is invertible, where

$$\begin{aligned} B_{j\ell}^\sigma(\widehat{\phi}(x), \nabla\widehat{\phi}(x)) &= \frac{\partial}{\partial x^j} \widehat{\phi}_{\sigma(\ell)}(x), \quad \text{for } j \leq 4, \ell = 1, 2, 3, 4, 5, \\ B_{jk}^\sigma(\widehat{\phi}(x), \nabla\widehat{\phi}(x)) &= \widehat{\phi}_{\sigma(\ell)}(x), \quad \text{for } j = 5, \ell = 1, 2, 3, 4, 5. \end{aligned}$$

When the Condition A is valid, also the condition  $\mu$ -LS is valid, see Appendix A and [58]. Very roughly speaking, Condition A means that the background fields vary so much that one could implement a measurement with some suitable sources  $\mathcal{F}$  by taking the needed “energy” from the varying  $\phi_\ell$  fields.

Theorem 1.1 can in some cases be improved so that also the conformal factor of the metric tensor can be reconstructed. Indeed, Theorem 1.1 and Corollary 1.3 of [57] imply that if  $V \subset I_{\widehat{g}}(p^-, p^+)$  is Vacuum, i.e., Ricci-flat, and all points  $x \in V$  can be connected by a curve  $\alpha \subset V^{int}$  to points of  $U_{\widehat{g}}$ . Then under the assumptions of Theorem 1.1, the whole metric tensor  $g$  in  $V$  can be reconstructed.

Theorem 1.1 deals with an inverse problem for “near field” measurements. We remark that for inverse problems for linear equations the measurements of the Dirichlet-to-Neumann map or the “near field” measurements are equivalent to scattering or “far field” information [8]. Analogous considerations for non-linear equations have not yet been done but are plausible. On related inverse scattering problems, see [38, 69].

Also, one can ask if one can make an approximate image of the space-time doing only one measurement. In general, in many inverse problems several measurements can be packed together to one measurement. For instance, for the wave equation with a time-independent simple metric this is done in [45]. Similarly, Theorem 1.1 and its proof make it possible to do approximate reconstructions in a suitable class of manifolds with only one measurement, see Remark 5.1.

The techniques considered in this paper can be used also to study inverse problems for non-linear hyperbolic systems encountered in applications. For instance, in medical imaging, in the the recently developed Ultrasound Elastography imaging technique the elastic material parameters are reconstructed by sending (s-polarized) elastic waves that are imaged using (p-polarized) elastic waves, see e.g. [46, 67]. This imaging method uses interaction of waves and is based on the non-linearity of the system.

## 2. GEOMETRY OF THE OBSERVATION TIMES AND THE CUT POINTS

Let us consider points  $x, y \in M$ . If  $x < y$ , we define the time separation function  $\tau(x, y) \in [0, \infty)$  to be the supremum of the lengths  $L(\alpha) = \int_0^1 \sqrt{-g(\dot{\alpha}(s), \dot{\alpha}(s))} ds$  of the piecewise smooth causal paths

$\alpha : [0, 1] \rightarrow M$  from  $x$  to  $y$ . If the condition  $x < y$  does not hold, we define  $\tau(x, y) = 0$ .

Since  $M$  is globally hyperbolic, the time separation function  $(x, y) \mapsto \tau(x, y)$  is continuous in  $M \times M$  by [74, Lem. 14.21] and the sets  $J^\pm(q)$  are closed by [74, Lem. 14.22]. Also, any points  $x, y \in M$ ,  $x < y$  can be connected by a causal geodesic whose length is  $\tau(x, y)$  by [74, Prop. 14.19].

When  $(x, \xi)$  is a light-like vector, we define  $\mathcal{T}(x, \xi)$  to be the length of the maximal interval on which  $\gamma_{x, \xi} : [0, \mathcal{T}(x, \xi)) \rightarrow M$  is defined. Below, to simplify notations, we sometimes use the notation  $\gamma_{x, \xi}([0, \infty))$  for the geodesic  $\gamma_{x, \xi}([0, \mathcal{T}(x, \xi))$ .

When  $(x, \xi_+)$  is a future pointing light-like vector, and  $(x, \xi_-)$  is a past pointing light-like vector, we define the modified cut locus functions, c.f. [5, Def. 9.32],

$$(17) \quad \begin{aligned} \rho_g(x, \xi_+) &= \sup\{s \in [0, \mathcal{T}(x, \xi_+)); \tau(x, \gamma_{x, \xi_+}(s)) = 0\}, \\ \rho_g(x, \xi_-) &= \sup\{s \in [0, \mathcal{T}(x, \xi_-)); \tau(\gamma_{x, \xi_-}(s), x) = 0\}. \end{aligned}$$

The point  $\gamma_{x, \xi}(s)|_{s=\rho(x, \xi)}$  is called the cut point on the geodesic  $\gamma_{x, \xi}$ .

Using [5, Thm. 9.33], we see that the function  $\rho_g(x, \xi)$  is lower semi-continuous on a globally hyperbolic Lorentzian manifold  $(M, g)$ .

Below, in this section, we consider the manifold  $(M, \widehat{g})$  and denote by  $\gamma_{x, \xi}$  the geodesics of  $(M, \widehat{g})$  and  $\rho(x, \xi) = \rho_{\widehat{g}}(x, \xi)$ . Also, we denote  $\mu_{\widehat{g}, z_0, \eta_0} = \widehat{\mu}$ ,  $p_{\pm} = \mu_{\widehat{g}}(s_{\pm})$ ,  $p_{+2} = \widehat{\mu}(s_{+2})$ , and  $p_{-2} = \widehat{\mu}(s_{-2})$ . Recall that by (1),  $\mu_{\widehat{g}, z_0, \eta_0}(s_{\pm j}) \in I^\mp(\mu_{\widehat{g}, z, \eta}(s_{\pm(j+1)}))$ ,  $j = 1, 2$ , for all  $(z, \eta) \in \mathcal{U}_{z_0, \eta_0}$ .

**Definition 2.1.** Let  $\mu = \mu_{\widehat{g}, z, \eta}$ ,  $(z, \eta) \in \mathcal{U}_{z_0, \eta_0}$ . For  $x \in J^+(\mu(-1)) \cap J^-(\mu(+1))$  we define  $f_\mu^\pm(x) \in [-1, 1]$  by setting

$$\begin{aligned} f_\mu^+(x) &= \inf(\{s \in (-1, 1); \tau(x, \mu(s)) > 0\} \cup \{1\}), \\ f_\mu^-(x) &= \sup(\{s \in (-1, 1); \tau(\mu(s), x) > 0\} \cup \{-1\}). \end{aligned}$$

We need the following simple properties of these functions.

**Lemma 2.2.** Let  $\mu = \mu_{\widehat{g}, z, \eta}$ ,  $(z, \eta) \in \mathcal{U}_{z_0, \eta_0}$ , and  $x \in J^-(p_{+2}) \cap J^+(p_{-2})$ .

(i) The function  $s \mapsto \tau(x, \mu(s))$  is non-decreasing on the interval  $s \in [-1, 1]$  and strictly increasing on  $s \in [f_\mu^+(x), 1]$ .

(ii) We have that  $s_{-3} < f_\mu^+(x) < s_{+3}$ .

(iii) Let  $y = \mu(f_\mu^+(x))$ . Then  $\tau(x, y) = 0$ . Also, if  $x \notin \mu$ , there is a light-like geodesic  $\gamma([0, s])$  in  $M$  from  $x$  to  $y$  with no conjugate points on  $\gamma([0, s])$ .

(iv) The maps  $f_\mu^+ : J^-(p_{+2}) \cap J^+(p_{-2}) \rightarrow [-1, 1]$  is continuous.

The analogous results hold for  $f_\mu^- : J^-(p_{+2}) \cap J^+(p_{-2}) \rightarrow [-1, 1]$ .

**Proof.** (i) and (ii) follows from the definition of  $f_\mu^+$  and the fact that  $p_{\pm 2} \in I^\mp(\mu(s_{\pm 3}))$  by (1). Claim (iii) follows from [74, Lem. 10.51].

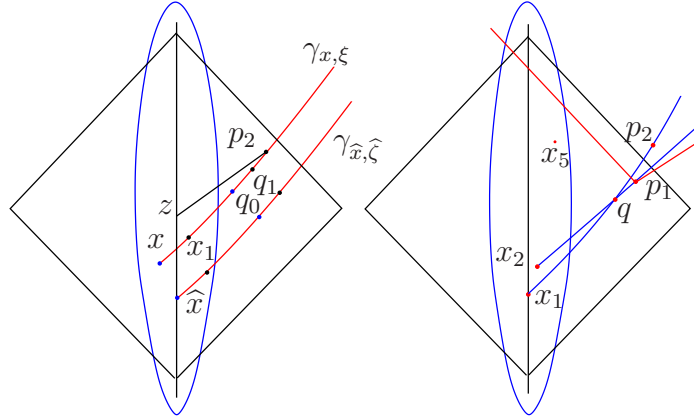
(iv) Assume that  $x_j \rightarrow x$  in  $J^-(p_{+2}) \cap J^+(p_{-2})$  as  $j \rightarrow \infty$ . Let  $s_j = f_\mu^+(x_j)$  and  $s = f_\mu^+(x)$ . As  $\tau$  is continuous, for any  $\varepsilon > 0$  we have  $\lim_{j \rightarrow \infty} \tau(x_j, \mu(s + \varepsilon)) = \tau(x, \mu(s + \varepsilon)) > 0$  and thus for  $j$  large enough  $s_j \leq s + \varepsilon$ . Thus  $\limsup_{j \rightarrow \infty} s_j \leq s$ . Assume next that  $\liminf_{j \rightarrow \infty} s_j = \tilde{s} < s$  and denote  $\varepsilon = \tau(\mu(\tilde{s}), \mu(s)) > 0$ . Then by the reverse triangle inequality, [74, Lem. 14.16],  $\liminf_{j \rightarrow \infty} \tau(x_j, \mu(s)) \geq \liminf_{j \rightarrow \infty} \tau(\mu(s_j), \mu(s)) \geq \varepsilon$ , and as  $\tau$  is continuous in  $M \times M$ , we obtain  $\tau(x, \mu(s)) \geq \varepsilon$ , which is not possible since  $s = f_\mu^+(x)$ . Hence  $s_j \rightarrow s$  as  $j \rightarrow \infty$ , proving (iv). The analogous results for  $f_\mu^-$  follow similarly.  $\square$

Let  $W \subset M$ . We define the earliest points of set  $W$  on the curve  $\mu_{z,\eta} = \mu_{z,\eta}([-1, 1])$ , and in the set  $U$ , respectively, to be

$$(18) \quad \begin{aligned} \mathbf{e}_{z,\eta}(W) &= \{\mu_{z,\eta}(\inf\{s \in [-1, 1]; \mu_{z,\eta}(s) \in W\})\}, \text{ if } \mu_{z,\eta} \cap W \neq \emptyset, \\ \mathbf{e}_{z,\eta}(W) &= \emptyset, \text{ if } \mu_{z,\eta} \cap W = \emptyset. \end{aligned}$$

Below, we use for a pair  $(x, \xi) \in L^+M$  the notation

$$(19) \quad (x(h), \xi(h)) = (\gamma_{x,\xi}(h), \dot{\gamma}_{x,\xi}(h)).$$



**FIGURE 4. Left.** Figure shows the situation in Lemma 2.3. The point  $\hat{x} = \hat{\mu}(r_1)$  is on the time-like geodesic  $\hat{\mu}$  shown as a black line. The black diamond is the set  $J_{\hat{g}}(p^-, p^+)$ ,  $(x, \xi)$  is a light-like direction close to  $(\hat{x}, \hat{\xi})$ , and  $x_1 = \gamma_{x,\xi}(t_0) = x(t_0)$ . The points  $q_0 = \gamma_{x,\xi}(\rho(x, \xi))$  and  $q_1 = \gamma_{x(t_0),\xi(t_0)}(\rho(x(t_0), \xi(t_0)))$  are the first cut point on  $\gamma_{x,\xi}$  corresponding to the points  $x$  and  $x_1$ , respectively. The blue and black points on  $\gamma_{\hat{x},\hat{\xi}}$  are the corresponding cut points on  $\gamma_{\hat{x},\hat{\xi}}$ . Also,  $z = \hat{\mu}(r_2)$ ,  $r_2 = f_\mu^-(p_2)$ . **Right:** The figure shows the configuration in formulas (68) and (69). We send light-like geodesics  $\gamma_{x_j,\xi_j}([t_0, \infty))$  from  $x_j$ ,  $j = 1, 2, 3, 4$ . The boundary  $\partial\mathcal{V}((\vec{x}, \vec{\xi}), t_0)$  is denoted by red line segments. We assume the these geodesics intersect at the point  $q$  before their first cut points  $p_j$ .

Later, we will consider wave packets sent from a point  $x$  that propagate near a geodesic  $\gamma_{x,\xi}([0, \infty))$ . These waves may have singularities near the conjugate points of the geodesic and due to this we analyze

next how the conjugate points move along a geodesic when the initial point of the geodesic is moved from  $x$  to  $\gamma_{x,\xi}(t_0)$ .

**Lemma 2.3.** *There are  $\vartheta_1, \kappa_1, \kappa_2 > 0$  such that for all  $\hat{x} = \hat{\mu}(r_1)$  with  $r_1 \in [s_-, s_+]$ ,  $\hat{\zeta} \in L^+_{\hat{x}}M$ ,  $\|\hat{\zeta}\|_{\hat{g}^+} = 1$ ,  $t_0 \in [\kappa_1, 4\kappa_1]$ , and  $(x, \xi) \in L^+M$  satisfying  $d_{\hat{g}^+}((\hat{x}, \hat{\zeta}), (x, \xi)) \leq \vartheta_1$  the following holds:*

- (i)  $0 < t \leq 5\kappa_1$ , then  $\gamma_{x,\xi}(t) \in U_{\hat{g}}$  and  $f_{\hat{\mu}}^-(\gamma_{\hat{x},\hat{\zeta}}(t)) = r_1$ .
- (ii) Assume that  $t_2 \in [t_0 + \rho(\gamma_{x,\xi}(t_0), \dot{\gamma}_{x,\xi}(t_0)), \mathcal{T}(x, \xi))$  and  $p_2 = \gamma_{x,\xi}(t_2) \in J^-(\hat{\mu}(s_{+2}))$ . Then  $r_2 = f_{\hat{\mu}}^-(p_2)$  satisfies  $r_2 - r_1 > 2\kappa_2$ .

Note that above in (ii) we can choose  $t_2 = t_0 + \rho(\gamma_{x,\xi}(t_0), \dot{\gamma}_{x,\xi}(t_0))$  in which case  $p_2$  is the first cut point  $q_1$  of  $\gamma_{x,\xi}([t_0, \infty))$ , see Fig. 4(Left).

**Proof.** Let  $B = \{(\hat{x}, \hat{\zeta}) \in L^+M; \hat{x} \in \hat{\mu}([s_-, s_+]), \|\hat{\zeta}\|_{\hat{g}^+} = 1\}$ . Since  $B$  is compact, the positive and lower semi-continuous function  $\rho(x, \xi)$  obtains its minimum on  $B$ . Hence we see that (i) holds when  $\kappa_1 \in (0, \frac{1}{5} \inf\{\rho(\hat{x}, \hat{\zeta}); (\hat{x}, \hat{\zeta}) \in B\})$  is small enough.

(ii) Let  $K$  be the compact set  $K = \{(x, \xi) \in L^+M; d_{\hat{g}^+}((x, \xi), B) \leq \vartheta_1\}$ . Also, let  $T_+(x, \xi) = \sup\{t \geq 0; \gamma_{x,\xi}(t) \in J^-(p_{+2})\}$  and

$$K_0 = \{(x, \xi) \in K; \rho(x(\kappa_1), \xi(\kappa_1)) + \kappa_1 \leq T_+(x, \xi)\}, \quad K_1 = K \setminus K_0.$$

Using [74, Lem. 14.13], we see that  $T_+(x, \xi)$  is bounded in  $K$ . Note that for  $t_0 \geq \kappa_1$  and  $a > t_0$  the geodesic  $\gamma_{x,\xi}([t_0, a])$  can have a cut point only if  $\gamma_{x,\xi}([\kappa_1, a])$  has a cut point and thus  $t_0 + \rho(x(t_0), \xi(t_0)) \geq \kappa_1 + \rho(x(\kappa_1), \xi(\kappa_1))$ . If  $K_0 = \emptyset$ , the claim is valid as the condition  $p_2 \in J^-(p_{+2})$  does not hold for any  $(x, \xi) \in K_1$ . Thus it is enough consider the case when  $K_0 \neq \emptyset$ .

We can also assume that  $\vartheta_1 > 0$  is so small that for all  $(x, \xi) \in K$  we have  $f_{\hat{\mu}}^-(x) > s_{-2}$ . Then, by Lemma 2.2, the map  $L : G_0 = \{(x, \xi, t) \in K \times \mathbb{R}_+; \rho(x(\kappa_1), \xi(\kappa_1)) + \kappa_1 \leq t \leq T_+(x, \xi)\} \rightarrow \mathbb{R}$ , defined by  $L(x, \xi, t) = f_{\hat{\mu}}^-(\gamma_{x,\xi}(t)) - f_{\hat{\mu}}^-(x)$ , is continuous. Since  $\rho(x, \xi)$  is lower semi-continuous and  $T_+(x, \xi)$  is upper semi-continuous and bounded we have that the sets  $K_0$  and  $G_0$  are compact.

For  $(x, \xi, t) \in G_0$ , the geodesic  $\gamma_{x,\xi}([\kappa_1, t])$  has a cut point in which case we see, for  $y = \gamma_{x,\xi}(t)$ , that  $\tau(x, y) > 0$ . Thus, for  $z_1 = \hat{\mu}(f_{\hat{\mu}}^-(x))$ , we have  $\tau(z_1, y) \geq \tau(z_1, x) + \tau(x, y) \geq \tau(x, y) > 0$ . This shows that  $L(x, \xi, t) > 0$ . Since  $G_0$  is compact and  $L$  is continuous and strictly positive,  $\varepsilon_1 := \inf\{L(x, \xi, t); (x, \xi, t) \in G_0\} > 0$ .

As  $f_{\hat{\mu}}^-$  is continuous and  $\hat{\mu}([-1, 1])$  is compact, we have that, by making  $\vartheta_1$  smaller if necessary, we can assume that if  $\hat{x} \in \hat{\mu}$  and  $d_{\hat{g}^+}(x, \hat{x}) \leq \vartheta_1$  then  $|f_{\hat{\mu}}^-(x) - f_{\hat{\mu}}^-(\hat{x})| < \varepsilon_1/2$ . Let  $\kappa_2 = \varepsilon_1/4$ . Then,  $\rho(x(\kappa_1), \xi(\kappa_1)) + \kappa_1 < t_2 < \mathcal{T}_+(x, \xi)$  so that  $r_2 = f_{\hat{\mu}}^-(p_2)$  and  $r_3 = f_{\hat{\mu}}^-(x)$  satisfies  $r_2 - r_3 \geq \varepsilon_1$  and  $r_2 - r_1 > \varepsilon_1/2$ . This proves the claim.  $\square$

Note that for proving the unique solvability of the inverse problem we need to consider two manifolds,  $(M^{(1)}, \hat{g}^{(1)})$  and  $(M^{(2)}, \hat{g}^{(2)})$  with



the same data. For these manifolds, we can choose  $\vartheta_1, \kappa_j$  so that they are the same for both manifolds.

2.0.1. *Geometric results on the light observation sets.* Let us first consider  $M$  with a fixed metric  $\widehat{g}$ . Denote below in this subsection  $U = U_{\widehat{g}}$ . See (18) for the notations we use.

**Definition 2.4.** *We define the light-observation set of the point  $q \in M$  to be  $\mathcal{P}_U(q) = (\mathcal{L}_{\widehat{g}}^+(q) \cup \{q\}) \cap U = \{\gamma_{q,\eta}(r) \in M; r \geq 0, \eta \in L_q^+M, \gamma_{q,\eta}(r) \in U\}$ , see Fig. 3(Right). The set of the earliest light observations of  $q$  is  $\mathcal{E}_U(q) = \bigcup_{(z,\eta) \in \mathcal{U}_{z_0,\eta_0}} \mathbf{e}_{z,\eta}(\mathcal{P}_U(q))$ , that is,*

$$\mathcal{E}_U(q) = \{\gamma_{q,\eta}(r) \in M; r \in [0, \rho(q, \eta)], \eta \in L_q^+M, \gamma_{q,\eta}(r) \in U\} \subset \mathcal{P}_U(q).$$

Below, when  $X$  is a set, let  $P(X) = 2^X = \{Z; Z \subset X\}$  denote the power set of  $X$ . When  $\Phi : U_1 \rightarrow U_2$  is a map, we say that the power set extension of  $\Phi$  is the map  $\widetilde{\Phi} : 2^{U_1} \rightarrow 2^{U_2}$  given by  $\widetilde{\Phi}(U') = \{\Phi(z); z \in U'\}$  for  $U' \subset U$ . We need the following theorem proven in [57] with  $V = I_{\widehat{g}}^+(p^-) \cap I_{\widehat{g}}^-(p^+) \subset I_{\widehat{g}}^-(\mu_{\widehat{g}}(s_{+2})) \setminus I_{\widehat{g}}^-(\mu_{\widehat{g}}(s_{-2}))$ .

**Theorem 2.5.** *Let  $(M_j, \widehat{g}_j)$ ,  $j = 1, 2$  be two open,  $C^\infty$ -smooth, globally hyperbolic Lorentzian manifolds of dimension  $(1 + 3)$  and let  $p_j^+, p_j^- \in M_j$  be the points of a time-like geodesic  $\mu_{\widehat{g}_j}([-1, 1]) \subset M_j$ ,  $p_j^\pm = \mu_{\widehat{g}_j}(s_\pm)$ . Let  $U_j \subset M_j$  be a neighborhood of  $\mu_{\widehat{g}_j}([s_-, s_+])$  and  $V_j = I_{\widehat{g}_j}^-(p_j^+) \cap I_{\widehat{g}_j}^+(p_j^-) \subset M_j$ . Then*

(i) *The map  $\mathcal{E}_{U_1} : q \mapsto \mathcal{E}_{U_1}(q)$  is injective in  $V_1$ .*

(ii) *Let us denote by  $\mathcal{E}_{U_j}(V_j) = \{\mathcal{E}_{U_j}(q); q \in V_j\} \subset 2^{U_j}$  the collections of the sets of the earliest light observations on the manifold  $(M_j, \widehat{g}_j)$  of the points in the set  $V_j$ . Assume that there is a conformal diffeomorphism  $\Phi : U_1 \rightarrow U_2$  such that  $\Phi(\mu_1(s)) = \mu_2(s)$ ,  $s \in [s_-, s_+]$  and the power set extension  $\widetilde{\Phi}$  of  $\Phi$  satisfies*

$$\widetilde{\Phi}(\mathcal{E}_{U_1}(V_1)) = \mathcal{E}_{U_2}(V_2).$$

*Then there is a diffeomorphism  $\Psi : V_1 \rightarrow V_2$  such the metric  $\Psi^*\widehat{g}_2$  is conformal to  $\widehat{g}_1$  and  $\Psi|_{V_1 \cap U_1} = \Phi$ .*

### 3. ANALYSIS OF THE EINSTEIN EQUATIONS IN WAVE COORDINATES

#### 3.1. Asymptotic analysis of the reduced Einstein equations.

3.1.1. *The reduced Einstein tensor.* For the Einstein equations, we will consider a smooth background metric  $\widehat{g}$  on  $M$  and the smooth metric  $\widetilde{g}$  for which  $\widehat{g} < \widetilde{g}$  and  $(M, \widetilde{g})$  is globally hyperbolic. We also use the notations defined in Section 1.0.1. In particular, we identify  $M = \mathbb{R} \times N$  and consider the metric tensor  $g$  on  $M_0 = (-\infty, t_0) \times N$ ,  $t_0 > 0$  that coincide with  $\widehat{g}$  in  $(-\infty, 0) \times N$ . Recall also that we consider a freely falling observer  $\widehat{\mu} = \mu_{\widehat{g}} : [-1, 1] \rightarrow M_0$  for which  $\widehat{\mu}(s_-) = p^- \in [0, t_0] \times$

$N$ . We denote  $L_x^+ M_0 = L_x^+(M_0, \widehat{g})$  and  $L^+ M_0 = L^+(M_0, \widehat{g})$ , and the cut locus function on  $(M_0, \widehat{g})$  by  $\rho(x, \xi) = \rho_{\widehat{g}}(x, \xi)$ . Also, recall that  $\widehat{U} = U_{\widehat{g}}$  the neighborhood of the geodesic  $\widehat{\mu} = \mu_{\widehat{g}}$ . We denote by  $\gamma_{x, \xi}(t)$  the geodesics of  $(M_0, \widehat{g})$ .

Following [34] we recall that

$$(20) \quad \text{Ric}_{jk}(g) = \text{Ric}_{jk}^{(h)}(g) + \frac{1}{2}(g_{jq} \frac{\partial \Gamma^q}{\partial x^k} + g_{kq} \frac{\partial \Gamma^q}{\partial x^j})$$

where  $\Gamma^q = g^{mn} \Gamma_{mn}^q$ ,

$$(21) \quad \text{Ric}_{jk}^{(h)}(g) = -\frac{1}{2} g^{pq} \frac{\partial^2 g_{jk}}{\partial x^p \partial x^q} + P_{jk},$$

$$P_{jk} = g^{ab} g_{pq} \Gamma_{jb}^p \Gamma_{ka}^q + \frac{1}{2} \left( \frac{\partial g_{jk}}{\partial x^a} \Gamma^a + g_{kl} \Gamma_{ab}^l g^{aq} g^{bd} \frac{\partial g_{qd}}{\partial x^j} + g_{jl} \Gamma_{ab}^l g^{aq} g^{bd} \frac{\partial g_{qd}}{\partial x^k} \right).$$

Note that  $P_{jk}$  is a polynomial of  $g_{pq}$  and  $g^{pq}$  and first derivatives of  $g_{pq}$ .

The  $\widehat{g}$ -reduced Einstein tensor  $\text{Ein}_{\widehat{g}}(g)$  and Ricci tensor  $\text{Ric}_{\widehat{g}}(g)$  are

$$(22) \quad (\text{Ric}_{\widehat{g}}(g))_{jk} = \text{Ric}_{jk} g - \frac{1}{2} (g_{jn} \widehat{\nabla}_k F^n + g_{kn} \widehat{\nabla}_j F^n)$$

$$= \text{Ric}_{jk}^{(h)}(g) + \frac{1}{2} (g_{jq} \frac{\partial}{\partial x^k} (g^{ab} \widehat{\Gamma}_{ab}^q) + g_{kq} \frac{\partial}{\partial x^j} (g^{ab} \widehat{\Gamma}_{ab}^q)),$$

$$(23) \quad (\text{Ein}_{\widehat{g}}(g))_{jk} = (\text{Ric}_{\widehat{g}}(g))_{jk} - \frac{1}{2} (g^{ab} (\text{Ric}_{\widehat{g}} g)_{ab}) g_{jk},$$

where  $F^n$  are the harmonicity functions given by

$$(24) \quad F^n = \Gamma^n - \widehat{\Gamma}^n, \quad \text{where } \Gamma^n = g^{jk} \Gamma_{jk}^n, \quad \widehat{\Gamma}^n = g^{jk} \widehat{\Gamma}_{jk}^n,$$

where  $\Gamma_{jk}^n$  and  $\widehat{\Gamma}_{jk}^n$  are the Christoffel symbols for  $g$  and  $\widehat{g}$ , respectively. The harmonicity functions  $F^n$  of the solution  $(g, \phi)$  of the equations (8) vanish when the conservation law (9) is valid, see [78, eq. (14.8)]. Thus by (22), the conservation law implies that the solutions of the reduced Einstein equations (7) satisfy of the Einstein equations (3).

**3.1.2. Local existence of solutions.** Let us consider the solutions  $(g, \phi)$  of the equations (8) with source  $\mathcal{F}$ . To consider their local existence, let us denote  $u := (g, \phi) - (\widehat{g}, \widehat{\phi})$ .

It follows from by [4, Cor. A.5.4] that  $\mathcal{K}_j = J_{\widehat{g}}^+(p^-) \cap \overline{M}_j$  is compact. Since  $\widehat{g} < \widetilde{g}$ , we see that if  $r_0$  above is small enough, for all  $g \in \mathcal{V}(r_0)$ , see subsection 1.0.2, we have  $g|_{\mathcal{K}_1} < \widetilde{g}|_{\mathcal{K}_1}$ . In particular, we have  $J_{\widehat{g}}^+(p^-) \cap M_1 \subset J_{\widetilde{g}}^+(p^-)$ .

Let us assume that  $\mathcal{F}$  is small enough in the norm  $C_b^4(M_0)$  and that it is supported in a compact set  $\mathcal{K} = J_{\widetilde{g}}^+(p^-) \cap [0, t_0] \times N \subset \overline{M}_0$ . Then we can write the equations (8) for  $u$  in the form

$$(25) \quad P_{g(u)}(u) = \mathcal{F}, \quad x \in M_0,$$

$$u = 0 \text{ in } (-\infty, 0) \times N, \text{ where}$$

$$P_{g(u)}(u) := g^{jk}(x; u) \partial_j \partial_k u(x) + H(x, u(x), \partial u(x)).$$

Here, the notation  $g^{jk}(x; u)$  is used to indicate that the metric depends on the solution  $u$ . More precisely, as the metric and the scalar field are  $(g, \phi) = u + (\widehat{g}, \widehat{\phi})$ , we have  $(g^{jk}(x; u))_{j,k=1}^4 = (g_{jk}(x))^{-1}$ . Moreover, above  $(x, v, w) \mapsto H(x, v, w)$  is a smooth function which is a second order polynomial in  $w$  with coefficients being smooth functions of  $v, \widehat{g}$ , and the derivatives of  $\widehat{g}$ , [85]. Note that when the norm of  $\mathcal{F}$  in  $C_b^4(M_0)$  is small enough, we have  $\text{supp}(u) \cap M_0 \subset \mathcal{K}$ . We note that one could also consider non-compactly supported sources or initial data, see [22]. Also, the scalar field-Einstein system can be considered with much less regularity that is done below, see [17, 18].

Let  $s_0 \geq 4$  be an even integer. Below we will consider the solutions  $u = (g - \widehat{g}, \phi - \widehat{\phi})$  and the sources  $\mathcal{F}$  as sections of the bundle  $\mathcal{B}^L$  on  $M_0$ . We will consider these functions as elements of the section-valued Sobolev spaces  $H^s(M_0; \mathcal{B}^L)$  etc. Below, we omit the bundle  $\mathcal{B}^L$  in these notations and denote  $H^s(M_0; \mathcal{B}^L) = H^s(M_0)$ . We use the same convention for the spaces

$$E^s = \bigcap_{j=0}^s C^j([0, t_0]; H^{s-j}(N)), \quad s \in \mathbb{N}.$$

Note that  $E^s \subset C^p([0, t_0] \times N)$  when  $0 \leq p < s - 2$ . Local existence results for (25) follow from the standard techniques for quasi-linear equations developed e.g. in [47] or [55], or [78, Section 9]. These yield that when  $\mathcal{F}$  is supported in the compact set  $\mathcal{K}$  and  $\|\mathcal{F}\|_{E^{s_0}} < c_0$ , where  $c_0 > 0$  is small enough, there exists a unique function  $u$  satisfying equation (25) on  $M_0$  with the source  $\mathcal{F}$ . Moreover,  $\|u\|_{E^{s_0}} \leq C_1 \|\mathcal{F}\|_{E^{s_0}}$ . For a detailed analysis, see Appendix B in [56].

**3.1.3. Asymptotic expansion for the non-linear wave equation.** Let us consider a small parameter  $\varepsilon > 0$  and the sources  $\mathcal{F} = \mathcal{F}_\varepsilon$ , depending smoothly on  $\varepsilon \in [0, \varepsilon_0)$ , with  $\mathcal{F}_\varepsilon|_{\varepsilon=0} = 0$ ,  $\partial_\varepsilon \mathcal{F}_\varepsilon|_{\varepsilon=0} = \mathbf{f}$  and  $\mathbf{f} = (\mathbf{f}^1, \mathbf{f}^2)$ , for which the equations (25) have a solution  $u_\varepsilon$ . Denote  $\vec{h} = (h_{(j)})_{j=1}^4$ , where  $h_{(j)} = \partial_\varepsilon^j \mathcal{F}_\varepsilon|_{\varepsilon=0}$  so that  $\mathbf{f} = h_{(1)}$ . Below, we always assume that  $\mathcal{F}_\varepsilon$  is supported in  $\mathcal{K}$  and  $\mathcal{F}_\varepsilon \in E^s$ , where  $s \geq s_0 + 10$  is an odd integer. We consider the solution  $u = u_\varepsilon$  of (25) with  $\mathcal{F} = \mathcal{F}_\varepsilon$  and write it in the form

$$(26) \quad u_\varepsilon(x) = \sum_{j=1}^4 \varepsilon^j w^j(x) + w^{res}(x, \varepsilon).$$

To obtain the equations for  $w^j$ , we use the representation (23) for the  $\widehat{g}$ -reduced Einstein tensor. Below, we use the notation  $w^j = ((w^j)_{pq})_{p,q=1}^4, ((w^j)_\ell)_{\ell=1}^L$  where  $((w^j)_{pq})_{p,q=1}^4$  is the  $g$ -component of  $w^j$  and  $((w^j)_\ell)_{\ell=1}^L$  is the  $\phi$ -component of  $w^j$ . Below, we use also the notation where the components of  $w = ((g_{pq})_{p,q=1}^4, (\phi_\ell)_{\ell=1}^L)$  are re-enumerated so that  $w$  is represented as a  $(10 + L)$ -dimensional vector,

i.e., we write  $w = (w_m)_{m=1}^{10+L}$  (cf. Voigt notation). We have that  $w^j$ ,  $j = 1, 2, 3, 4$  are given by

$$\begin{aligned}
(27) \quad w^j &= (g^j, \phi^j) = \mathbf{Q}_{\widehat{g}} \mathcal{H}^j, \quad j = 1, 2, 3, 4, \text{ where} \\
\mathcal{H}^1 &= h_{(1)}, \\
\mathcal{H}^2 &= (2\widehat{g}^{jp} w_{pq}^1 \widehat{g}^{qk} \partial_j \partial_k w^1, 0) + \mathcal{A}^{(2)}(w^1, \partial w^1) + h_{(2)}, \\
\mathcal{H}^3 &= (\mathcal{G}_3, 0) + \mathcal{A}^{(3)}(w^1, \partial w^1, w^2, \partial w^2) + h_{(3)}, \\
\mathcal{G}_3 &= -6\widehat{g}^{jl} w_{li}^1 \widehat{g}^{ip} w_{pq}^1 \widehat{g}^{qk} \partial_j \partial_k w^1 + \\
&\quad + 3\widehat{g}^{jp} w_{pq}^2 \widehat{g}^{qk} \partial_j \partial_k w^1 + 3\widehat{g}^{jp} w_{pq}^1 \widehat{g}^{qk} \partial_k \partial_j w^2,
\end{aligned}$$

and

$$\begin{aligned}
(28) \quad \mathcal{H}^4 &= (\mathcal{G}_4, 0) + \mathcal{A}^{(4)}(w^1, \partial w^1, w^2, \partial w^2, w^3, \partial w^3) + h_{(4)}, \\
\mathcal{G}_4 &= 24\widehat{g}^{js} w_{sr}^1 \widehat{g}^{rl} w_{li}^1 \widehat{g}^{ip} w_{pq}^1 \widehat{g}^{qk} \partial_k \partial_j w^1 + 6\widehat{g}^{jp} w_{pq}^2 \widehat{g}^{qk} \partial_k \partial_j w^2 + \\
&\quad - 18\widehat{g}^{jl} w_{li}^1 \widehat{g}^{ip} w_{pq}^2 \widehat{g}^{qk} \partial_k \partial_j w^1 - 12\widehat{g}^{jl} w_{li}^1 \widehat{g}^{ip} w_{pq}^1 \widehat{g}^{qk} \partial_k \partial_j w^2 + \\
&\quad + 3\widehat{g}^{jp} w_{pq}^3 \widehat{g}^{qk} \partial_k \partial_j w^1 + 3\widehat{g}^{jp} w_{pq}^1 \widehat{g}^{qk} \partial_k \partial_j w^3.
\end{aligned}$$

Moreover,  $\mathbf{Q}_{\widehat{g}} = (\square_{\widehat{g}} + V(x, D))^{-1}$  is the causal inverse of the operator  $\square_{\widehat{g}} + V(x, D)$  where  $V(x, D)$  is a first order differential operator with coefficients depending on  $\widehat{g}$  and its derivatives and  $\mathcal{A}^{(\alpha)}$ ,  $\alpha = 2, 3, 4$  denotes a sum of a multilinear operators of orders  $m$ ,  $2 \leq m \leq \alpha$  having at a point  $x$  the representation

$$\begin{aligned}
(29) \quad &(\mathcal{A}^{(\alpha)}(v^1, \partial v^1, v^2, \partial v^2, v^3, \partial v^3))(x) \\
&= \sum \left( a_{abcijkP_1P_2P_3pq}^{(\alpha)}(x) (v_a^1(x))^i (v_b^2(x))^j (v_c^3(x))^k \right. \\
&\quad \left. \cdot P_1(\partial v^1(x)) P_2(\partial v^2(x)) P_3(\partial v^3(x)) \right)
\end{aligned}$$

where  $(v_a^1(x))^i$  denotes the  $i$ -th power of  $a$ -th component of  $v^1(x)$  and the sum is taken over the indexes  $a, b, c, p, q, n$ , and integers  $i, j, k$ . The homogeneous monomials  $P_d(y) = y^{\beta_d}$ ,  $\beta_d = (b_1, b_2, \dots, b_{4(10+L)}) \in \mathbb{N}^{4(10+L)}$ ,  $d = 1, 2, 3$  having orders  $|\beta_d|$ , respectively, where  $P_d(y) = 1$  for  $d > \alpha$ , and

$$(30) \quad i + 2j + 3k + |\beta_1| + 2|\beta_2| + 3|\beta_3| = \alpha,$$

$$(31) \quad |\beta_1| + |\beta_2| + |\beta_3| \leq 2.$$

Here, by (30), the term  $\mathcal{A}^{(\alpha)}$  produces a term of order  $O(\varepsilon^\alpha)$  when  $v^j = w^j$  and condition (31) means that  $\mathcal{A}^{(\alpha)}(v^1, \partial v^1, v^2, \partial v^2, v^3, \partial v^3)$  contain only terms where the sum of the powers of derivatives of  $v^1, v^2$ , and  $v^3$  is at most two.

By [14, App. III, Thm. 3.7], or alternatively, the proof of [47, Lemma 2.6] adapted for manifolds, we see that the estimate  $\|\mathbf{Q}_{\widehat{g}} H\|_{E^{s_1+1}} \leq C_{s_1} \|H\|_{E^{s_1}}$  holds for all  $H \in E^{s_1}$ ,  $s_1 \in \mathbb{Z}_+$  that are supported in  $\mathcal{K}_0 =$

$J_g^+(p^-) \cap \overline{M}_0$ . Note that we are interested only on the local solvability of the Einstein equations.

Consider next an even integer  $s \geq s_0 + 4$  and  $\mathcal{F}_\varepsilon \in E^s$  that depends smoothly on  $\varepsilon$ . Then, by defining  $w^j$  via the equations (27)-(28) with  $h_{(j)} \in E^s$ ,  $j \leq 4$  and using results of [47] we obtain that in (26) we have  $w^j = \partial_\varepsilon^j u_\varepsilon|_{\varepsilon=0} \in E^{s+2-j}$ ,  $j = 1, 2, 3, 4$  and  $\|w^{res}(\cdot, \varepsilon)\|_{E^{s-4}} \leq C\varepsilon^5$ .

### 3.2. Distorted plane wave solutions for the linearized equations.

3.2.1. *Lagrangian distributions.* Let us recall the definition of the conormal and Lagrangian distributions that we will use below. Let  $X$  be a manifold of dimension  $n$  and  $\Lambda \subset T^*X \setminus \{0\}$  be a Lagrangian submanifold. Let  $\phi(x, \theta)$ ,  $\theta \in \mathbb{R}^N$  be a non-degenerate phase function that locally parametrizes  $\Lambda$ . We say that a distribution  $u \in \mathcal{D}'(X)$  is a Lagrangian distribution associated with  $\Lambda$  and denote  $u \in \mathcal{I}^m(X; \Lambda)$ , if in local coordinates  $u$  can be represented as an oscillatory integral,

$$(32) \quad u(x) = \int_{\mathbb{R}^N} e^{i\phi(x, \theta)} a(x, \theta) d\theta,$$

where  $a(x, \theta) \in S^{m+n/4-N/2}(X; \mathbb{R}^N)$ , see [41, 49, 70].

In particular, when  $S \subset X$  is a submanifold, its conormal bundle  $N^*S = \{(x, \xi) \in T^*X \setminus \{0\}; x \in S, \xi \perp T_x S\}$  is a Lagrangian submanifold. If  $u$  is a Lagrangian distribution associated with  $\Lambda_1$  where  $\Lambda_1 = N^*S$ , we say that  $u$  is a conormal distribution.

Let us next consider the case when  $X = \mathbb{R}^n$  and let  $(x^1, x^2, \dots, x^n) = (x', x'', x''')$  be the Euclidean coordinates with  $x' = (x_1, \dots, x_{d_1})$ ,  $x'' = (x_{d_1+1}, \dots, x_{d_1+d_2})$ ,  $x''' = (x_{d_1+d_2+1}, \dots, x_n)$ . If  $S_1 = \{x' = 0\} \subset \mathbb{R}^n$ ,  $\Lambda_1 = N^*S_1$  then  $u \in \mathcal{I}^m(X; \Lambda_1)$  can be represented by (32) with  $N = d_1$  and  $\phi(x, \theta) = x' \cdot \theta$ .

Next we recall the definition of  $\mathcal{I}^{p,l}(X; \Lambda_1, \Lambda_2)$ , the space of the distributions  $u$  in  $\mathcal{D}'(X)$  associated to two cleanly intersecting Lagrangian manifolds  $\Lambda_1, \Lambda_2 \subset T^*X \setminus \{0\}$ , see [24, 41, 70]. These classes have been widely used in the study of inverse problems, see [19, 32]. Let us start with the case when  $X = \mathbb{R}^n$ .

Let  $S_1, S_2 \subset \mathbb{R}^n$  be the linear subspaces of codimensions  $d_1$  and  $d_1 + d_2$ , respectively,  $S_2 \subset S_1$ , given by  $S_1 = \{x' = 0\}$ ,  $S_2 = \{x' = x'' = 0\}$ . Let us denote  $\Lambda_1 = N^*S_1$ ,  $\Lambda_2 = N^*S_2$ . Then  $u \in \mathcal{I}^{p,l}(\mathbb{R}^n; N^*S_1, N^*S_2)$  if and only if

$$u(x) = \int_{\mathbb{R}^{d_1+d_2}} e^{i(x' \cdot \theta' + x'' \cdot \theta'')} a(x, \theta', \theta'') d\theta' d\theta'',$$

where the symbol  $a(x, \theta', \theta'')$  belongs in the product type symbol class  $S^{\mu_1, \mu_2}(\mathbb{R}^n; (\mathbb{R}^{d_1} \setminus 0) \times \mathbb{R}^{d_2})$  that is the space of function  $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$  that satisfy

$$(33) \quad |\partial_x^\gamma \partial_{\theta'}^\alpha \partial_{\theta''}^\beta a(x, \theta', \theta'')| \leq C_{\alpha\beta\gamma K} (1 + |\theta'| + |\theta''|)^{\mu_1 - |\alpha|} (1 + |\theta''|)^{\mu_2 - |\beta|}$$

for all  $x \in K$ , multi-indexes  $\alpha, \beta, \gamma$ , and compact sets  $K \subset \mathbb{R}^n$ . Above,  $\mu_1 = p + l - d_1/2 + n/4$  and  $\mu_2 = -l - d_2/2$ .

When  $X$  is a manifold of dimension  $n$  and  $\Lambda_1, \Lambda_2 \subset T^*X \setminus \{0\}$  are two cleanly intersecting Lagrangian manifolds, we define the class  $\mathcal{I}^{p,l}(X; \Lambda_1, \Lambda_2) \subset \mathcal{D}'(X)$  to consist of locally finite sums of distributions of the form  $u = Au_0$ , where  $u_0 \in \mathcal{I}^{p,l}(\mathbb{R}^n; N^*S_1, N^*S_2)$  and  $S_1, S_2 \subset \mathbb{R}^n$  are the linear subspace of codimensions  $d_1$  and  $d_1 + d_2$ , respectively, such that  $S_2 \subset S_1$ , and  $A$  is a Fourier integral operator of order zero with a canonical relation  $\Sigma$  for which  $\Sigma \circ (N^*S_1)' \subset \Lambda_1'$  and  $\Sigma \circ (N^*S_2)' \subset \Lambda_2'$ . Here, for  $\Lambda \subset T^*X$  we denote  $\Lambda' = \{(x, -\xi) \in T^*X; (x, \xi) \in \Lambda\}$ , and for  $\Sigma \subset T^*X \times T^*X$  we denote  $\Sigma' = \{(x, \xi, y, -\eta); (x, \xi, y, \eta) \in \Sigma\}$ .

In most cases, below  $X = M$ . We denote then  $\mathcal{I}^p(M; \Lambda_1) = \mathcal{I}^p(\Lambda_1)$  and  $\mathcal{I}^{p,l}(M; \Lambda_1, \Lambda_2) = \mathcal{I}^{p,l}(\Lambda_1, \Lambda_2)$ . Also,  $\mathcal{I}(\Lambda_1) = \cup_{p \in \mathbb{R}} \mathcal{I}^p(\Lambda_1)$ .

By [41, 70], microlocally away from  $\Lambda_1$  and  $\Lambda_0$ ,

$$(34) \quad \mathcal{I}^{p,l}(\Lambda_0, \Lambda_1) \subset \mathcal{I}^{p+l}(\Lambda_0 \setminus \Lambda_1) \quad \text{and} \quad \mathcal{I}^{p,l}(\Lambda_0, \Lambda_1) \subset \mathcal{I}^p(\Lambda_1 \setminus \Lambda_0),$$

respectively. Thus the principal symbol of  $u \in \mathcal{I}^{p,l}(\Lambda_0, \Lambda_1)$  is well defined on  $\Lambda_0 \setminus \Lambda_1$  and  $\Lambda_1 \setminus \Lambda_0$ . We denote  $\mathcal{I}(\Lambda_0, \Lambda_1) = \cup_{p,q \in \mathbb{R}} \mathcal{I}^{p,q}(\Lambda_0, \Lambda_1)$ .

Below, when  $\Lambda_j = N^*S_j$ ,  $j = 1, 2$  are conormal bundles of smooth cleanly intersecting submanifolds  $S_j \subset M$  of codimension  $m_j$ , where  $\dim(M) = n$ , we use the traditional notations,

$$(35) \quad \mathcal{I}^\mu(S_1) = \mathcal{I}^{\mu+m_1/2-n/4}(N^*S_1), \quad \mathcal{I}^{\mu_1, \mu_2}(S_1, S_2) = \mathcal{I}^{p,l}(N^*S_1, N^*S_2),$$

where  $p = \mu_1 + \mu_2 + m_1/2 - n/4$  and  $l = -\mu_2 - m_2/2$ , and call such distributions the conormal distributions associated to  $S_1$  or product type conormal distributions associated to  $S_1$  and  $S_2$ , respectively. By [41],  $\mathcal{I}^\mu(X; S_1) \subset L_{loc}^p(X)$  for  $\mu < -m_1(p-1)/p$ ,  $1 \leq p < \infty$ .

For the wave operator  $\square_g$  on the globally hyperbolic manifold  $(M, g)$ ,  $\text{Char}(\square_g)$  is the set of light-like co-vectors with respect to  $g$ . For  $(x, \xi) \in \text{Char}(\square_g)$ ,  $\Theta_{x,\xi}$  denotes the bicharacteristic of  $\square_g$ . Then,  $(y, \eta) \in \Theta_{x,\xi}$  if and only if there is  $t \in \mathbb{R}$  such that for  $a = \eta^\sharp$  and  $b = \xi^\sharp$  we have  $(y, a) = (\gamma_{x,b}^g(t), \dot{\gamma}_{x,b}^g(t))$  where  $\gamma_{x,b}^g$  is a light-like geodesic with respect to the metric  $g$  with the initial data  $(x, b) \in LM$ . Here, we use notations  $(\xi^\sharp)^j = g^{jk}\xi_k$  and  $(b^\sharp)_j = g_{jk}b^k$ .

Let  $P = \square_g + B^0 + B^j \partial_j$ , where  $B^0$  is a scalar function and  $B_j$  is a vector field. Then  $P$  is a classical pseudodifferential operator of real principal type and order  $m = 2$  on  $M$ , and [70], see also [60],  $P$  has a parametrix  $Q \in \mathcal{I}^{p,l}(\Delta'_{T^*M}, \Lambda_P)$ ,  $p = \frac{1}{2} - m$ ,  $l = -\frac{1}{2}$ , where  $\Delta_{T^*M} = N^*(\{(x, x); x \in M\})$  and  $\Lambda_g \subset T^*M \times T^*M$  is the Lagrangian manifold associated to the canonical relation of the operator  $P$ , that is,

$$(36) \quad \Lambda_g = \{(x, \xi, y, -\eta); (x, \xi) \in \text{Char}(P), (y, \eta) \in \Theta_{x,\xi}\},$$

where  $\Theta_{x,\xi} \subset T^*M$  is the bicharacteristic of  $P$  containing  $(x, \xi)$ . When  $(M, g)$  is a globally hyperbolic manifold, the operator  $P$  has a causal inverse operator, see e.g. [4, Thm. 3.2.11]. We denote it by  $P^{-1}$  and by

[70], we have  $P^{-1} \in \mathcal{I}^{-3/2, -1/2}(\Delta'_{T^*M}, \Lambda_g)$ . We will repeatedly use the fact (see [41, Prop. 2.1]) that if  $F \in \mathcal{I}^p(\Lambda_0)$  and  $\Lambda_0$  intersects  $\text{Char}(P)$  transversally so that all bicharacteristics of  $P$  intersect  $\Lambda_0$  only finitely many times, then  $(\square_g + B^0 + B^j \partial_j)^{-1} F \in \mathcal{I}^{p-3/2, -1/2}(\Lambda_0, \Lambda_1)$  where  $\Lambda'_1 = \Lambda_g \circ \Lambda'_0$  is called the flowout from  $\Lambda_0$  on  $\text{Char}(P)$ , that is,

$$\Lambda_1 = \{(x, -\xi); (x, \xi, y, -\eta) \in \Lambda_g, (y, \eta) \in \Lambda_0\}.$$

3.2.2. *The linearized Einstein equations and the linearized conservation law.* We will below consider sources  $\mathcal{F} = \varepsilon \mathbf{f}(x)$  and solution  $u_\varepsilon$  satisfying (25), where  $\mathbf{f} = (\mathbf{f}^{(1)}, \mathbf{f}^{(2)})$ .

We consider the linearized Einstein equations and the linearized wave  $w^1 = \partial_\varepsilon u_\varepsilon|_{\varepsilon=0}$  in (26) that we denote by  $u^{(1)} = w^1$ . It satisfies the linearized Einstein equations (12) that we write as

$$(37) \quad \square_{\widehat{g}} u^{(1)} + V(x, \partial_x) u^{(1)} = \mathbf{f},$$

where  $v \mapsto V(x, \partial_x)v$  is a linear first order partial differential operator with coefficients depending on  $\widehat{g}$  and its derivatives.

Assume that  $Y \subset M_0$  is a 2-dimensional space-like submanifold and consider local coordinates defined in  $V \subset M_0$ . Moreover, assume that in these local coordinates  $Y \cap V \subset \{x \in \mathbb{R}^4; x^j b_j = 0, x^j b'_j = 0\}$ , where  $b'_j \in \mathbb{R}$  and let  $\mathbf{f} = (\mathbf{f}^{(1)}, \mathbf{f}^{(2)}) \in \mathcal{I}^{n+1}(Y)$ ,  $n \leq n_0 = -17$ , be defined by

$$(38) \quad \mathbf{f}(x^1, x^2, x^3, x^4) = \text{Re} \int_{\mathbb{R}^2} e^{i(\theta_1 b_m + \theta_2 b'_m)x^m} \sigma_{\mathbf{f}}(x, \theta_1, \theta_2) d\theta_1 d\theta_2.$$

Here, we assume that  $\sigma_{\mathbf{f}}(x, \theta)$ ,  $\theta = (\theta_1, \theta_2)$  is a  $\mathcal{B}^L$ -valued classical symbol and we denote the principal symbol of  $\mathbf{f}$  by  $c(x, \theta)$ , or component-wise,  $((c_{jk}^{(1)}(x, \theta))_{j,k=1}^4, (c_\ell^{(2)}(x, \theta))_{\ell=1}^L)$ . When  $x \in Y$  and  $\xi = (\theta_1 b_m + \theta_2 b'_m) dx^m$  so that  $(x, \xi) \in N^*Y$ , we denote the value of the principal symbol of  $\mathbf{f}$  at  $(x, \xi)$  by  $\widetilde{c}(x, \xi) = c(x, \theta)$ , that is component-wise,  $\widetilde{c}_{jk}^{(1)}(x, \xi) = c_{jk}^{(1)}(x, \theta)$  and  $\widetilde{c}_\ell^{(2)}(x, \xi) = c_\ell^{(2)}(x, \theta)$ . We say that this is the principal symbol of  $\mathbf{f}$  at  $(x, \xi)$ , associated to the phase function  $\phi(x, \theta_1, \theta_2) = (\theta_1 b_m + \theta_2 b'_m)x^m$ . The above defined principal symbols can be defined invariantly, see [44].

We will below consider what happens when  $\mathbf{f} = (\mathbf{f}^{(1)}, \mathbf{f}^{(2)}) \in \mathcal{I}^{n+1}(Y)$  satisfies the *linearized conservation law* (13). Roughly speaking, these four linear conditions imply that the principal symbol of the source  $\mathbf{f}$  satisfies four linear conditions. Furthermore, the linearized conservation law implies that also the linearized wave  $u^{(1)}$  produced by  $\mathbf{f}$  satisfies four linear conditions that we call the linearized harmonicity conditions, and finally, the principal symbol of the wave  $u^{(1)}$  has to satisfy four linear conditions. Next we explain these conditions in detail.

When (13) is valid, we have

$$(39) \quad \widehat{g}^{lk} \xi_l \widetilde{c}_{kj}^{(1)}(x, \xi) = 0, \quad \text{for } j \leq 4 \text{ and } \xi \in N_x^*Y.$$

We say that this is the *linearized conservation law for the principal symbols*.

3.2.3. *The harmonicity condition for the linearized solutions.* Assume that  $(g, \phi)$  satisfy equations (8) and the conservation law (9) is valid. The conservation law (9) and the  $\widehat{g}$ -reduced Einstein equations (8) imply, see e.g. [14, 78], that the harmonicity functions  $\Gamma^j = g^{nm}\Gamma_{nm}^j$  satisfy

$$(40) \quad g^{nm}\Gamma_{nm}^j = g^{nm}\widehat{\Gamma}_{nm}^j.$$

Next we denote  $u^{(1)} = (g^1, \phi^1) = (\dot{g}, \dot{\phi})$ , see (27), and discuss the implications of (40) for the metric component  $\dot{g}$  of the solution of the linearized Einstein equations.

Let us next do calculations in local coordinates of  $M_0$  and denote  $\partial_k = \frac{\partial}{\partial x^k}$ . Direct calculations show that  $h^{jk} = g^{jk}\sqrt{-\det(g)}$  satisfies  $\partial_k h^{kq} = -\Gamma_{kn}^q h^{nk}$ . Then (40) is equivalent to

$$(41) \quad \partial_k h^{kq} = -\widehat{\Gamma}_{kn}^q h^{nk}.$$

We call (41) the *harmonicity condition*, cf. [48].

Taking the derivative of (41) with respect to  $\varepsilon$ , we see that satisfies

$$(42) \quad \widehat{\nabla}_a(\dot{g}^{ab} - \frac{1}{2}\widehat{g}^{ab}\widehat{g}_{qp}\dot{g}^{pq}) = 0, \quad b = 1, 2, 3, 4.$$

We call (42) the *linearized harmonicity condition* for  $\dot{g}$ .

3.2.4. *Properties of the principal symbols of the waves.* Let  $K \subset M_0$  be a light-like submanifold of dimension 3 that in local coordinates  $X : V \rightarrow \mathbb{R}^4$ ,  $x^k = X^k(y)$  is given by  $K \cap V \subset \{x \in \mathbb{R}^4; b_k x^k = 0\}$ , where  $b_k \in \mathbb{R}$  are constants. Assume that the solution  $u^{(1)} = (\dot{g}, \dot{\phi})$  of the linear wave equation (37) with the right hand side vanishing in  $V$  is such that  $u^{(1)} \in \mathcal{I}^\mu(K)$  with  $\mu \in \mathbb{R}$ . Below we use  $\mu = n - \frac{1}{2}$  where  $n \in \mathbb{Z}_-$ ,  $n \leq n_0 = -17$ . Let us write  $\dot{g}_{jk}$  as an oscillatory integral using a phase function  $\varphi(x, \theta) = b_k x^k \theta$ , and a classical symbol  $\sigma_{\dot{g}_{jk}}(x, \theta) \in S_{cl}^n(\mathbb{R}^4, \mathbb{R})$ ,

$$(43) \quad \dot{g}_{jk}(x^1, x^2, x^3, x^4) = \operatorname{Re} \int_{\mathbb{R}} e^{i(\theta b_m x^m)} \sigma_{\dot{g}_{jk}}(x, \theta) d\theta,$$

where  $n = \mu + \frac{1}{2}$ . We denote the (positively homogeneous) principal symbol of  $\dot{g}_{jk}$  by  $a_{jk}(x, \theta)$ . When  $x \in K$  and  $\xi = \theta b_k dx^k$  so that  $(x, \xi) \in N^*K$ , we denote the value of  $a_{jk}$  at  $(x, \theta)$  by  $\widetilde{a}_{jk}(x, \xi)$ , that is,  $\widetilde{a}_{jk}(x, \xi) = a_{jk}(x, \theta)$ .

Then, if  $\dot{g}_{jk}$  satisfies the linearized harmonicity condition (40), its principal symbol  $\widetilde{a}_{jk}(x, \xi)$  satisfies

$$(44) \quad -\widehat{g}^{mn}(x)\xi_m v_{nj} + \frac{1}{2}\xi_j(\widehat{g}^{pq}(x)v_{pq}) = 0, \quad v_{pq} = \widetilde{a}_{pq}(x, \xi),$$



where  $j = 1, 2, 3, 4$  and  $\xi = \theta b_k dx^k \in N_x^*K$ . If (44) holds, we say that the *harmonicity condition for the symbol* is satisfied for  $\tilde{a}(x, \xi)$  at  $(x, \xi) \in N^*K$ .

**3.2.5. Distorted plane waves satisfying a linear wave equation.** Next we consider a distorted plane wave whose singular support is concentrated near a geodesic. These waves, sketched in Fig. 1(Right), propagate near the geodesic  $\gamma_{x_0, \zeta_0}([t_0, \infty))$  and are singular on a surface  $K(x_0, \zeta_0; t_0, s_0)$ , defined below in (45), that is a subset of the light cone  $\mathcal{L}_{\hat{g}}^+(x')$ ,  $x' = \gamma_{x_0, \zeta_0}(t_0)$ . The parameter  $s_0$  gives a “width” of the wave packet and when  $s_0 \rightarrow 0$ , its singular support tends to the set  $\gamma_{x_0, \zeta_0}([2t_0, \infty))$ . Next we will define these wave packets.

We define the 3-submanifold  $K(x_0, \zeta_0; t_0, s_0) \subset M_0$  associated to  $(x_0, \zeta_0) \in L^+(M_0, \hat{g})$ ,  $x_0 \in U_{\hat{g}}$  and parameters  $t_0, s_0 \in \mathbb{R}_+$  as

$$(45) \quad K(x_0, \zeta_0; t_0, s_0) = \{\gamma_{x', \eta}(t) \in M_0; \eta \in \mathcal{W}, t \in (0, \infty)\},$$

where  $(x', \zeta') = (\gamma_{x_0, \zeta_0}(t_0), \dot{\gamma}_{x_0, \zeta_0}(t_0))$  and  $\mathcal{W} \subset L_{x'}^+(M_0, \hat{g})$  is a neighborhood of  $\zeta'$  consisting of vectors  $\eta \in L_{x'}^+(M_0)$  satisfying  $\|\eta - \zeta'\|_{\hat{g}^+} < s_0$ . Note that  $K(x_0, \zeta_0; t_0, s_0) \subset \mathcal{L}_{\hat{g}}^+(x')$  is a subset of the light cone starting at  $x' = \gamma_{x_0, \zeta_0}(t_0)$  and that it is singular at the point  $x'$ . Let  $S = \{x \in M_0; \mathbf{t}(x) = \mathbf{t}(\gamma_{x_0, \zeta_0}(2t_0))\}$  be a Cauchy surface which intersects  $\gamma_{x_0, \zeta_0}(\mathbb{R})$  transversally at the point  $\gamma_{x_0, \zeta_0}(2t_0)$ . When  $t_0 > 0$  is small enough,  $Y(x_0, \zeta_0; t_0, s_0) = S \cap K(x_0, \zeta_0; t_0, s_0)$  is a smooth 2-dimensional space-like surface that is a subset of  $U_{\hat{g}}$ .

Let  $\Lambda(x_0, \zeta_0; t_0, s_0)$  be the Lagrangian manifold that is the flowout from  $N^*Y(x_0, \zeta_0; t_0, s_0) \cap N^*K(x_0, \zeta_0; t_0, s_0)$  on  $\text{Char}(\square_{\hat{g}})$  in the future direction. When  $K^{reg} \subset K = K(x_0, \zeta_0; t_0, s_0)$  is the set of points  $x$  that have a neighborhood  $W$  such that  $K \cap W$  is a smooth 3-dimensional submanifold, we have  $N^*K^{reg} \subset \Lambda(x_0, \zeta_0; t_0, s_0)$ . Below, we represent locally the elements  $w \in \mathcal{B}_x$  in the fiber of the bundle  $\mathcal{B}$  as a  $(10 + L)$ -dimensional vector,  $w = (w_m)_{m=1}^{10+L}$ .

**Lemma 3.1.** *Let  $n \leq n_0 = -17$  be an integer,  $t_0, s_0 > 0$ ,  $Y = Y(x_0, \zeta_0; t_0, s_0)$ ,  $K = K(x_0, \zeta_0; t_0, s_0)$ ,  $\Lambda_1 = \Lambda(x_0, \zeta_0; t_0, s_0)$ , and  $(y, \xi) \in N^*Y \cap \Lambda_1$ . Assume that  $\mathbf{f} = (\mathbf{f}_1, \mathbf{f}_2) \in \mathcal{I}^{n+1}(Y)$ , is a  $\mathcal{B}^L$ -valued conormal distribution that is supported in a neighborhood  $V \subset M_0$  of  $\gamma_{x_0, \zeta_0} \cap Y = \{\gamma_{x_0, \zeta_0}(2t_0)\}$  and has a  $\mathbb{R}^{10+L}$ -valued classical symbol. Denote the principal symbol of  $\mathbf{f}$  by  $\tilde{f}(y, \xi) = (\tilde{f}_k(y, \xi))_{k=1}^{10+L}$ , and assume that the symbol of  $\mathbf{f}$  vanishes near the light-like directions in  $N^*Y \setminus N^*K$ .*

Let  $u^{(1)} = (\dot{g}, \dot{\phi})$  be a solution of the linear wave equation (37) with the source  $\mathbf{f}$ . Then  $u^{(1)}$ , considered as a vector valued Lagrangian distribution on the set  $M_0 \setminus Y$ , satisfies  $u^{(1)} \in \mathcal{I}^{n-1/2}(M_0 \setminus Y; \Lambda_1)$ , and its principal symbol  $\tilde{a}(x, \eta) = (\tilde{a}_j(x, \eta))_{j=1}^{10+L}$  at  $(x, \eta) \in \Lambda_1$ , is given by

$$(46) \quad \tilde{a}_j(x, \eta) = \sum_{k=1}^{10+L} R_j^k(x, \eta, y, \xi) \tilde{f}_k(y, \xi),$$

where the pairs  $(y, \xi)$  and  $(x, \eta)$  are on the same bicharacteristics of  $\square_{\widehat{g}}$ , and  $y \ll x$ . Observe that  $((y, \xi), (x, \eta)) \in \Lambda'_{\widehat{g}}$ , and in addition,  $(y, \xi) \in N^*Y \cap N^*K$ . Moreover, the matrix  $(R_j^k(x, \eta, y, \xi))_{j,k=1}^{10+L}$  is invertible.

We call the solution  $u^{(1)}$  a distorted plane wave that is associated to the submanifold  $K(x_0, \zeta_0; t_0, s_0)$ .

**Proof.** It follows from [70] that the causal inverse of the scalar wave operator  $\square_{\widehat{g}} + V(x, D)$ , where  $V(x, D)$  is a 1st order differential operator, satisfies  $(\square_{\widehat{g}} + V(x, D))^{-1} \in \mathcal{I}^{-3/2, -1/2}(\Delta'_{T^*M_0}, \Lambda_{\widehat{g}})$ . Here,  $\Delta'_{T^*M_0}$  is the conormal bundle of the diagonal of  $M_0 \times M_0$  and  $\Lambda_{\widehat{g}}$  is the flow-out of the canonical relation of  $\square_{\widehat{g}}$ . A geometric representation for its kernel is given in [60]. An analogous result holds for the matrix valued wave operator,  $\square_{\widehat{g}}I + V(x, D)$ , when  $V(x, D)$  is a 1st order differential operator, that is,  $(\square_{\widehat{g}}I + V(x, D))^{-1} \in \mathcal{I}^{-3/2, -1/2}(\Delta'_{T^*M_0}, \Lambda_{\widehat{g}})$ , see [70] and [28]. By [41, Prop. 2.1], this yields  $u^{(1)} \in \mathcal{I}^{n-1/2}(\Lambda_1)$  and the formula (46) where  $R = (R_j^k(x, \eta, y, \xi))_{j,k=1}^{10+L}$  is obtained by solving a system of ordinary differential equation along a bicharacteristic curve. Making similar considerations for the adjoint of the  $(\square_{\widehat{g}}I + V(x, D))^{-1}$ , i.e., considering the propagation of singularities using reversed causality, we see that the matrix  $R$  is invertible.  $\square$

Below, let  $(y, \xi) \in N^*Y \cap \Lambda_1$  and  $(x, \eta) \in T^*M_0$  be a light-like covector such that  $(x, \eta) \in \Theta_{y, \xi}$ ,  $x \notin Y$  and,  $y \ll x$ .

Let  $\mathcal{B}_x^L$  be the fiber of the bundle  $\mathcal{B}^L$  at  $x$  and  $\mathfrak{S}_{x, \eta}$  be the space of the elements in  $\mathcal{B}_x^L$  satisfying the harmonicity condition for the symbols (44) at  $(x, \eta)$ . Let  $(y, \xi) \in N^*Y$  and  $\mathfrak{C}_{y, \xi}$  be the set of elements  $b$  in  $\mathcal{B}_y^L$  that satisfy the linearized conservation law for symbols, i.e., (39).

Let  $n \leq n_0$  and  $t_0, s_0 > 0$ ,  $Y = Y(x_0, \zeta_0; t_0, s_0)$ ,  $K = K(x_0, \zeta_0; t_0, s_0)$ ,  $\Lambda_1 = \Lambda(x_0, \zeta_0; t_0, s_0)$ , and  $b_0 \in \mathfrak{C}_{x, \xi}$ . By Condition  $\mu$ -SL, there is a conormal distribution  $\mathbf{f} \in \mathcal{I}^{n+1}(Y) = \mathcal{I}^{n+1}(N^*Y)$  such that  $\mathbf{f}$  satisfies the linearized conservation law (13) and the principal symbol  $\widetilde{f}$  of  $\mathbf{f}$ , defined on  $N^*Y$ , satisfies  $\widetilde{f}(y, \xi) = b_0$ . Moreover, by Condition  $\mu$ -SL there is a family of sources  $\mathcal{F}_\varepsilon$ ,  $\varepsilon \in [0, \varepsilon_0)$  such that  $\partial_\varepsilon \mathcal{F}_\varepsilon|_{\varepsilon=0} = \mathbf{f}$  and a solution  $u_\varepsilon + (\widehat{g}, \widehat{\phi})$  of the Einstein equations with the source  $\mathcal{F}_\varepsilon$  that depend smoothly on  $\varepsilon$  and  $u_\varepsilon|_{\varepsilon=0} = 0$ . Then  $\dot{u} = \partial_\varepsilon u_\varepsilon|_{\varepsilon=0} \in \mathcal{I}^{n-1/2}(M_0 \setminus Y; \Lambda_1)$ .

As  $\dot{u} = (\dot{g}, \dot{\phi})$  satisfies the linearized harmonicity condition (40), the principal symbol  $\widetilde{a}(x, \eta) = (\widetilde{a}_1(x, \eta), \widetilde{a}_2(x, \eta))$  of  $\dot{u}$  satisfies  $\widetilde{a}(x, \eta) \in \mathfrak{S}_{x, \eta}$ . This shows that the map  $R = R(x, \eta, y, \xi)$ , given by  $R : \widetilde{f}(y, \xi) \mapsto \widetilde{a}(x, \eta)$  that is defined in Lemma 3.1, satisfies  $R : \mathfrak{C}_{y, \xi} \rightarrow \mathfrak{S}_{x, \eta}$ . Since  $R$  is one-to-one and the linear spaces  $\mathfrak{C}_{y, \xi}$  and  $\mathfrak{S}_{x, \eta}$  have the same dimension, we see that

$$(47) \quad R : \mathfrak{C}_{y, \xi} \rightarrow \mathfrak{S}_{x, \eta}$$

is a bijection. Hence, when  $\mathbf{f} \in \mathcal{I}^{n+1}(Y)$  varies so that the linearized conservation law (39) for the principal symbols is satisfied, the principal symbol  $\tilde{a}(x, \eta)$  at  $(x, \eta)$  of the solution  $\dot{u}$  of the linearized Einstein equation achieves all values in the  $(L + 6)$  dimensional space  $\mathfrak{S}_{x, \eta}$ .

Below, we denote  $\mathbf{f} \in \mathcal{I}_C^{n+1}(Y(x_0, \zeta_0; t_0, s_0))$  when the principal symbols of  $\mathbf{f}$  satisfies the linearized conservation law for principal symbols, that is, equation (39).

### 3.3. Microlocal analysis of the non-linear interaction of waves.

Next we consider the interaction of four  $C^k$ -smooth waves having conormal singularities, where  $k \in \mathbb{Z}_+$  is sufficiently large. Interaction of such waves produces a ‘‘corner point’’ in the spacetime. On related microlocal tools to consider scattering by corners, see [86, 87]. Earlier considerations of interaction of three waves has been done by Melrose and Ritter [71, 72] and Rauch and Reed, [76] for non-linear hyperbolic equations in  $\mathbb{R}^{1+2}$  where the non-linearity appears in the lower order terms. Recently, the interaction of two strongly singular waves has been studied by Luc and Rodnianski [65].

*3.3.1. Interaction of non-linear waves on a general manifold.* Next, we introduce a vector of four  $\varepsilon$  variables denoted by  $\vec{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \in \mathbb{R}^4$ . Let  $s_0, t_0 > 0$  and consider  $u_{\vec{\varepsilon}} = (g_{\vec{\varepsilon}} - \widehat{g}, \phi_{\vec{\varepsilon}} - \widehat{\phi})$  where  $v_{\vec{\varepsilon}} = (g_{\vec{\varepsilon}}, \phi_{\vec{\varepsilon}})$  solve the equations (8) with  $\mathcal{F} = \mathbf{f}_{\vec{\varepsilon}}$  where

$$(48) \quad \mathbf{f}_{\vec{\varepsilon}} := \sum_{j=1}^4 \varepsilon_j \mathbf{f}_j, \quad \mathbf{f}_j \in \mathcal{I}_C^{n+1}(Y(x_j, \zeta_j; t_0, s_0)),$$

and  $(x_j, \zeta_j)$  are light-like vectors with  $x_j \in U_{\widehat{g}}$ . Moreover, we assume that for some  $0 < r_2 < r_1$  and  $s_- + r_2 < s' < s_+$  the sources satisfy

$$(49) \quad \begin{aligned} \text{supp}(\mathbf{f}_j) \cap J_{\widehat{g}}^+(\text{supp}(\mathbf{f}_k)) &= \emptyset, \quad \text{for all } j \neq k, \\ \text{supp}(\mathbf{f}_j) &\subset I_{\widehat{g}}(\mu_{\widehat{g}}(s' - r_2), \mu_{\widehat{g}}(s')), \quad \text{for all } j = 1, 2, 3, 4, \end{aligned}$$

where  $r_1$  is the parameter used to define  $W_{\widehat{g}} = W_{\widehat{g}}(r_1)$ , see (10). The first condition implies that the supports of the sources are causally independent.

The sources  $\mathbf{f}_j$  give raise to  $\mathcal{B}^L$ -section valued solutions of the linearized wave equations, which we denote by

$$u_j := u_j^{(1)} = \partial_{\varepsilon_j} u_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0} = \mathbf{Q} \mathbf{f}_j \in \mathcal{I}(\Lambda(x_j, \zeta_j; t_0, s_0)),$$

where  $\mathbf{Q} = \mathbf{Q}_{\widehat{g}}$ . In the following we use the notations  $\partial_{\vec{\varepsilon}}^1 u_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0} := \partial_{\varepsilon_1} u_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0}$ ,  $\partial_{\vec{\varepsilon}}^2 u_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0} := \partial_{\varepsilon_1} \partial_{\varepsilon_2} u_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0}$ ,  $\partial_{\vec{\varepsilon}}^3 u_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0} := \partial_{\varepsilon_1} \partial_{\varepsilon_2} \partial_{\varepsilon_3} u_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0}$ , and

$$\partial_{\vec{\varepsilon}}^4 u_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0} := \partial_{\varepsilon_1} \partial_{\varepsilon_2} \partial_{\varepsilon_3} \partial_{\varepsilon_4} u_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0}.$$

Next we denote the waves produced by the  $\ell$ -th order interaction by

$$(50) \quad \mathcal{M}^{(\ell)} := \partial_{\vec{\varepsilon}}^{\ell} u_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0}, \quad \ell \in \{1, 2, 3, 4\}.$$

Below, we use the notations  $\mathcal{B}_j^\beta$ ,  $j = 1, 2, 3, 4$  and  $S_n^\beta$ ,  $n = 1, 2$  to denote operators of the form

$$(51) \quad \mathcal{B}_j^\beta : (v_p)_{p=1}^{10+L} \mapsto (b_p^{r,(j,\beta)}(x) \partial_x^{k(\beta,j)} v_r(x))_{p=1}^{10+L}, \text{ and} \\ S_n^\beta = \mathbf{Q} \text{ or } S_n^\beta = I, \text{ or } S_n^\beta = [\mathbf{Q}, a_n^\beta(x) D^\alpha], \quad \alpha = \alpha(\beta, n), \quad |\alpha| \leq 4$$

and the coefficients  $b_p^{r,(j,\beta)}(x)$  and  $a_j^\beta(x)$  depend on the derivatives of  $\widehat{g}$ . Here,  $k(\beta, j) = k_j^\beta = \text{ord}(\mathcal{B}_j^\beta)$  are the orders of  $\mathcal{B}_j^\beta$  and  $\beta$  is just an index running over a finite set  $J_4 \subset \mathbb{Z}_+$ .

Computing the  $\varepsilon_j$  derivatives of the equations (27), (28), and (29) with the sources  $\mathbf{f}_\varepsilon$ , and taking into account the condition (49), we obtain:

$$(52) \quad \mathcal{M}^{(4)} = \mathbf{Q}\mathcal{F}^{(4)}, \quad \mathcal{F}^{(4)} = \sum_{\sigma \in \Sigma(4)} \sum_{\beta \in J_4} (\mathcal{G}_\sigma^{(4),\beta} + \widetilde{\mathcal{G}}_\sigma^{(4),\beta}),$$

where  $\Sigma(4)$  is the set of permutations, that is, the set of the bijections  $\sigma : \{1, 2, 3, 4\} \rightarrow \{1, 2, 3, 4\}$ . Then, we have

$$(53) \quad \mathcal{G}_\sigma^{(4),\beta} = \mathcal{B}_4^\beta u_{\sigma(4)} \cdot S_2^\beta (\mathcal{B}_3^\beta u_{\sigma(3)} \cdot S_1^\beta (\mathcal{B}_2^\beta u_{\sigma(2)} \cdot \mathcal{B}_1^\beta u_{\sigma(1)}))$$

where the orders of the differential operators satisfy  $k_4^\beta + k_3^\beta + k_2^\beta + k_1^\beta + |\alpha(\beta, 1)| + |\alpha(\beta, 2)| \leq 6$  and  $k_4^\beta + k_3^\beta + |\alpha(\beta, 2)| \leq 2$  and

$$(54) \quad \widetilde{\mathcal{G}}_\sigma^{(4),\beta} = \mathcal{S}_2^\beta (\mathcal{B}_4^\beta u_{\sigma(4)} \cdot \mathcal{B}_3^\beta u_{\sigma(3)}) \cdot \mathcal{S}_1^\beta (\mathcal{B}_2^\beta u_{\sigma(2)} \cdot \mathcal{B}_1^\beta u_{\sigma(1)}),$$

where  $k_4^\beta + k_3^\beta + k_2^\beta + k_1^\beta + |\alpha(\beta, 1)| + |\alpha(\beta, 2)| \leq 6$ ,  $k_4^\beta + k_3^\beta + |\alpha(\beta, 2)| \leq 4$ , and  $k_1^\beta + k_2^\beta + |\alpha(\beta, 1)| \leq 4$ .

We denote below  $\vec{S}_\beta = (S_1^\beta, S_2^\beta)$  and  $\mathcal{M}_\sigma^{(4),\beta} = \mathbf{Q}_g \mathcal{G}_\sigma^{(4),\beta}$  and  $\widetilde{\mathcal{M}}_\sigma^{(4),\beta} = \mathbf{Q}_{\widehat{g}} \widetilde{\mathcal{G}}_\sigma^{(4),\beta}$ . Let us explain how the terms above appear in the Einstein equations: By taking  $\partial_\varepsilon$  derivatives of the wave  $u_\varepsilon$  we obtain terms similar to (27) and (28). In particular, we have that  $\mathcal{G}_\sigma^{(4),\beta}$  and  $\widetilde{\mathcal{G}}_\sigma^{(4),\beta}$  can be written in the form

$$(55) \quad \mathcal{G}_\sigma^{(4),\beta} = A_3^\beta [u_{\sigma(4)}, S_2^\beta (A_2^\beta [u_{\sigma(3)}, S_1^\beta (A_1^\beta [u_{\sigma(2)}, u_{\sigma(1)}])]),$$

$$(56) \quad \widetilde{\mathcal{G}}_\sigma^{(4),\beta} = A_3^\beta [S_2^\beta (A_2^\beta [u_{\sigma(4)}, u_{\sigma(3)}]), S_1^\beta (A_1^\beta [u_{\sigma(2)}, u_{\sigma(1)}])],$$

where  $A_j^\beta[V, W]$  are 2nd order multilinear operators of the form

$$(57) \quad A[V, W] = \sum_{|\alpha|+|\gamma| \leq 2} a_{\alpha\gamma}(x) (\partial_x^\alpha V(x)) \cdot (\partial_x^\gamma W(x)).$$

By commuting derivatives and the operator  $\mathbf{Q}_{\widehat{g}}$  we obtain (53) and (54). Below, the terms of particular importance are the bilinear form that is given for  $V = (v_{jk}, \phi)$  and  $W = (w^{jk}, \phi')$  by

$$(58) \quad A_1[V, W] = -\widehat{g}^{jb} w_{ab} \widehat{g}^{ak} \partial_j \partial_k v_{pq}.$$

3.3.2. *On the singular support of the non-linear interaction of three waves.* Let us next consider four light-like future pointing directions  $(x_j, \xi_j)$ ,  $j = 1, 2, 3, 4$ , and use below the notations, see (19),

$$(\vec{x}, \vec{\xi}) = ((x_j, \xi_j))_{j=1}^4, \quad (\vec{x}(h), \vec{\xi}(h)) = ((x_j(h), \xi_j(h)))_{j=1}^4.$$

We will consider the case when we send distorted plane waves propagating on surfaces  $K_j = K(x_j, \xi_j; t_0, s_0)$ ,  $t_0, s_0 > 0$ , cf. (45), and these waves interact.

Next we consider the 3-interactions of the waves. Let  $\mathcal{X}((\vec{x}, \vec{\xi}); t_0, s_0)$  be set of all light-like vectors  $(x, \xi) \in L^+M_0$  that are in the normal bundles  $N^*(K_{j_1} \cap K_{j_2} \cap K_{j_3})$  with some  $1 \leq j_1 < j_2 < j_3 \leq 4$ .

Moreover, we define  $\mathcal{Y}((\vec{x}, \vec{\xi}); t_0, s_0)$  to be the set of all  $y \in M_0$  such that there are  $(z, \zeta) \in \mathcal{X}((\vec{x}, \vec{\xi}); t_0, s_0)$ , and  $t \geq 0$  such that  $\gamma_{z, \zeta}(t) = y$ . Finally, let

$$(59) \quad \mathcal{Y}((\vec{x}, \vec{\xi}); t_0) = \bigcap_{s_0 > 0} \mathcal{Y}((\vec{x}, \vec{\xi}); t_0, s_0).$$

The three wave interaction happens then on  $\pi(\mathcal{X}((\vec{x}, \vec{\xi}); t_0, s_0))$  and, roughly speaking, this interaction sends singularities to  $\mathcal{Y}((\vec{x}, \vec{\xi}); t_0, s_0)$ .

For instance in Minkowski space, when three plane waves (whose singular supports are hyperplanes) collide, the intersections of the hyperplanes is a 1-dimensional space-like line  $K_{123} = K_1 \cap K_2 \cap K_3$  in the 4-dimensional space-time. This corresponds to a point moving continuously in time. Roughly speaking, the point seems to move at a higher speed than light (i.e. it appears like a tachyonic, moving point source) and produces a (conic) shock wave type of singularity (see Fig. 2 where the interaction time is only finite). In this paper we do not analyze carefully the singularities produced by the three wave interaction near  $\mathcal{Y}((\vec{x}, \vec{\xi}); t_0, s_0)$ . Our goal is to consider the singularities produced by the four wave interaction in the domain  $M_0 \setminus \mathcal{Y}((\vec{x}, \vec{\xi}); t_0, s_0)$ .

3.3.3. *Gaussian beams.* Our aim is to consider interactions of 4 waves that produce a new source, and to this end we use test sources that produce gaussian beams.

Let  $y \in U_{\hat{g}}$  and  $\eta \in T_y M$  be a future pointing light-like vector. We choose a complex function  $p \in C^\infty(M_0)$  such that  $\text{Im } p(x) \geq 0$  and  $\text{Im } p(x)$  vanishes only at  $y$ ,  $p(y) = 0$ ,  $d(\text{Re } p)|_y = \eta^\sharp$ ,  $d(\text{Im } p)|_y = 0$  and the Hessian of  $\text{Im } p$  at  $y$  is positive definite. To simplify notations, we use below complex sources and waves. The physical linearized waves can be obtained by taking the real part of these complex waves. We use a large parameter  $\tau \in \mathbb{R}_+$  and define a test source

$$(60) \quad F_\tau(x) = F_\tau(x; y, \eta), \quad \text{where} \quad F_\tau(x; y, \eta) = \tau^{-1} \exp(i\tau p(x))h(x)$$

where  $h$  is section on  $\mathcal{B}^L$  supported in a small neighborhood  $W$  of  $y$ . The construction of  $p(x)$  and  $F_\tau$  is discussed in detail in [56].

We consider both the usual causal solutions and the solutions of the adjoint wave equation for which time is reversed, that is, we use the anti-causal parametrix  $\mathbf{Q}^* = \mathbf{Q}_{\hat{g}}^*$  instead of the usual causal parametrix  $\mathbf{Q} = (\square_{\hat{g}} + V(x, D))^{-1}$ .

For a function  $v : M_0 \rightarrow \mathbb{R}$  the large  $\tau$  asymptotics of the  $L^2$ -inner products  $\langle F_\tau, v \rangle_{L^2(M_0)}$  can be used to verify if the point  $(y, \eta^\sharp) \in T^*M_0$  belongs in the wave front set  $\text{WF}(v)$  of  $v$ , see e.g. [75]. Below, we will use the fact that when  $v$  is of the form  $v = \mathbf{Q}_{\hat{g}} b$ , we have  $\langle F_\tau, v \rangle_{L^2(M_0)} = \langle \mathbf{Q}_{\hat{g}}^* F_\tau, b \rangle_{L^2(M_0)}$  where  $\mathbf{Q}_{\hat{g}}^* F_\tau$  is a Gaussian beam solution to the adjoint wave equation. We will use this to analyze the singularities of  $\mathcal{M}^{(4)}$  at  $y \in U_{\hat{g}}$ .

The wave  $u_\tau = \mathbf{Q}^* F_\tau$  satisfies by [75], see also [54],

$$(61) \quad \|u_\tau - u_\tau^N\|_{C^k(J(p^-, p^+))} \leq C_N \tau^{-n_{N,k}}$$

where  $n_{N,k} \rightarrow \infty$  as  $N \rightarrow \infty$  and  $u_\tau^N$  is a formal Gaussian beam of order  $N$  having the form

$$(62) \quad u_\tau^N(x) = e^{i\tau\varphi(x)} a(x, \tau), \quad a(x, \tau) = \sum_{n=0}^N U_n(x) \tau^{-n}.$$

Here  $\varphi(x) = A(x) + iB(x)$  and  $A(x)$  and  $B(x)$  are real-valued functions,  $B(x) \geq 0$ , and  $B(x)$  vanishes only on  $\gamma_{y,\eta}(\mathbb{R})$ , and for  $z = \gamma_{y,\eta}(t)$ , and  $\zeta = \dot{\gamma}_{y,\eta}^b(t)$ ,  $t < 0$  we have  $dA|_z = \zeta$ ,  $dB|_z = 0$ . Moreover, the Hessian of  $B$  at  $z$  restricted to the orthocomplement of  $\zeta$  is positive definite. The functions  $h$  and  $U_n$  defined above can be chosen to be smooth so that  $h$  is supported in any neighborhood  $V$  of  $y$  and  $U_N$  are supported in any neighborhoods of  $\gamma_{y,\eta}((-\infty, 0])$ . Below,  $a(x, \tau)$  is the symbol and  $U_0(y)$  is the principal symbol of the gaussian beam  $u_\tau(x)$ .

*3.3.4. Indicator function for singularities produced by interaction of four waves.* Let  $y \in U_{\hat{g}}$  and  $\eta \in L_{x_0}^+(M_0, \hat{g})$  be a future pointing light-like vector. We will next check whether  $(y, \eta^b) \in \text{WF}(\mathcal{M}^{(4)})$ .

Using the function  $\mathcal{M}^{(4)}$  defined in (52) with four sources  $\mathbf{f}_j \in \mathcal{I}_C^{n+1}(Y(x_j, \zeta_j; t_0, s_0))$ ,  $j \leq 4$ , that produce the pieces of plane waves  $u_j \in \mathcal{I}^n(\Lambda(x_j, \xi_j; t_0, s_0))$  in  $M_0 \setminus Y(x_j, \zeta_j; t_0, s_0)$ , and the source  $F_\tau$  in (60), we define the indicator functions

$$(63) \quad \Theta_\tau^{(4)} = \langle F_\tau, \mathcal{M}^{(4)} \rangle_{L^2(U)} = \sum_{\beta \in J_4} \sum_{\sigma \in \Sigma(4)} (T_{\tau,\sigma}^{(4),\beta} + \tilde{T}_{\tau,\sigma}^{(4),\beta}).$$

Here, the terms  $T_{\tau,\sigma}^{(\ell),\beta}$  and  $\tilde{T}_{\tau,\sigma}^{(\ell),\beta}$  correspond, for  $\sigma$  and  $\beta$ , to different types of interactions the four waves  $u_{(j)}$  can have. To define the terms  $T_{\tau,\sigma}^{(\ell),\beta}$  and  $\tilde{T}_{\tau,\sigma}^{(\ell),\beta}$  appearing above, we use generic notations where we drop the index  $\beta$ , that is, we denote  $\mathcal{B}_j = \mathcal{B}_j^\beta$  and  $S_j = S_j^\beta$ . With these notations, using the decompositions (52), (53), and (54) and the fact

that  $u_\tau = \mathbf{Q}^* F_\tau$ , we define

$$\begin{aligned} T_{\tau,\sigma}^{(4),\beta} &= \langle F_\tau, \mathbf{Q}(\mathcal{B}_4 u_{\sigma(4)} \cdot S_2(\mathcal{B}_3 u_{\sigma(3)} \cdot S_1(\mathcal{B}_2 u_{\sigma(2)} \cdot \mathcal{B}_1 u_{\sigma(1)})) \rangle_{L^2(M_0)} \\ (64) \quad &= \langle (\mathcal{B}_3 u_{\sigma(3)}) \cdot S_2^*((\mathcal{B}_4 u_{\sigma(4)}) \cdot u_\tau), S_1(\mathcal{B}_2 u_{\sigma(2)} \cdot \mathcal{B}_1 u_{\sigma(1)}) \rangle_{L^2(M_0)}, \end{aligned}$$

$$\begin{aligned} \widetilde{T}_{\tau,\sigma}^{(4),\beta} &= \langle F_\tau, \mathbf{Q}(S_2(\mathcal{B}_4 u_{\sigma(4)} \cdot \mathcal{B}_3 u_{\sigma(3)}) \cdot S_1(\mathcal{B}_2 u_{\sigma(2)} \cdot \mathcal{B}_1 u_{\sigma(1)})) \rangle_{L^2(M_0)} \\ (65) \quad &= \langle S_2(\mathcal{B}_4 u_{\sigma(4)} \cdot \mathcal{B}_3 u_{\sigma(3)}) \cdot u_\tau, S_1(\mathcal{B}_2 u_{\sigma(2)} \cdot \mathcal{B}_1 u_{\sigma(1)}) \rangle_{L^2(M_0)}. \end{aligned}$$

When  $\sigma$  is the identity, we will omit it in our notations and denote  $T_\tau^{(4),\beta} = T_{\tau,id}^{(4),\beta}$ , etc. It turns out later, in Prop. 3.5, that the term  $\langle F_\tau, \mathbf{Q}(\mathcal{B}_4 u_4 \mathbf{Q}(\mathcal{B}_3 u_3, \mathbf{Q}(\mathcal{B}_2 u_2 \mathcal{B}_1 u_1))) \rangle$ , where the orders  $k_j$  of  $\mathcal{B}_j$  are  $k_1 = 6$ ,  $k_2 = k_3 = k_4 = 0$  is the term that is crucial for our considerations. We enumerate this term with  $\beta = \beta_1 := 1$ , i.e.,

$$(66) \quad \vec{S}_{\beta_1} = (\mathbf{Q}, \mathbf{Q}) \text{ and } k_1^{\beta_1} = 6, k_2^{\beta_1} = k_3^{\beta_1} = k_4^{\beta_1} = 0.$$

This term arises from the term  $\langle F_\tau, \mathcal{G}_\sigma^{(4),\beta} \rangle$ , such that  $\mathcal{G}_\sigma^{(4),\beta}$  is of the form (55), where  $\sigma = id$  and all quadratic forms  $A_3^\beta$ ,  $A_2^\beta$ , and  $A_1^\beta$  are of the form (58). More precisely,  $\langle F_\tau, \mathcal{G}_\sigma^{(4),\beta} \rangle$  is the sum of  $T_{\tau,id}^{(4),\beta_1}$  and terms analogous to that (including terms with commutators of  $\mathbf{Q}$  and differential operators), where

$$(67) \quad T_{\tau,id}^{(4),\beta_1} = -\langle \mathbf{Q}^*((F_\tau)_{nm}), u_4^{rs} \cdot \mathbf{Q}(u_3^{ab} \cdot \mathbf{Q}(u_2^{ik} \cdot \partial_r \partial_s \partial_a \partial_b \partial_i \partial_k u_1^{nm})) \rangle.$$

Here,  $u_j^{ik} = \widehat{g}^{in}(x) \widehat{g}^{km}(x) (u_j(x))_{nm}$ ,  $j = 1, 2, 3, 4$  and  $(u_j(x))_{nm}$  is the metric part of the wave  $u_j(x) = (((u_j(x))_{nm})_{n,m=1}^4, ((u_j(x))_\ell)_{\ell=1}^L)$ .

**3.3.5. Properties of the indicator functions on a general manifold.** Next we analyze the indicator function for sources  $\mathbf{f}_j \in \mathcal{I}_C^{n+1}(Y(x_j, \zeta_j; t_0, s_0))$ ,  $j \leq 4$ , and the source  $F_\tau$  related to  $(y, \eta) \in L^+ M_0$ ,  $y \in U_{\widehat{g}}$ . We denote in the following  $(x_5, \xi_5) = (y, \eta)$ .

**Definition 3.2.** *Let  $t_0 > 0$ . We say that the geodesics corresponding to vectors  $(\vec{x}, \vec{\xi}) = ((x_j, \xi_j))_{j=1}^4$  intersect and the intersection takes place at the point  $q \in M_0$  if there are  $t_j \in (0, \mathbf{t}_j)$ ,  $\mathbf{t}_j = \rho(x_j(t_0), \xi_j(t_0))$  such that  $q = \gamma_{x_j, \xi_j}(t_j)$  for all  $j = 1, 2, 3, 4$ .*

Let  $\Lambda_q^+$  be the Lagrangian manifold

$$\Lambda_q^+ = \{(x, \xi) \in T^* M_0; x = \gamma_{q,\zeta}(t), \xi^\sharp = \dot{\gamma}_{q,\zeta}(t), \zeta \in L_q^+ M_0, t > 0\}$$

Note that the projection of  $\Lambda_q^+$  on  $M_0$  is the light cone  $\mathcal{L}_{\widehat{g}}^+(q)$ .

Next we consider  $x_j \in U_{\widehat{g}}$  and  $\xi_j \in L_{x_j}^+ M_0$ , and  $\vartheta_1, t_0 > 0$  such that  $(\vec{x}, \vec{\xi}) = ((x_j, \xi_j))_{j=1}^4$  satisfy, see Fig. 4(Right),

- $$(68) \quad \begin{aligned} (i) \quad &\gamma_{x_j, \xi_j}([0, t_0]) \subset W_{\widehat{g}}, x_j(t_0) \notin J_{\widehat{g}}^+(x_k(t_0)), \text{ for } j, k \leq 4, j \neq k, \\ (ii) \quad &\text{For all } j, k \leq 4, d_{\widehat{g}^+}((x_j, \xi_j), (x_k, \xi_k)) < \vartheta_1, \\ (iii) \quad &\text{There is } \widehat{x} \in \widehat{\mu} \text{ such that for all } j \leq 4, d_{\widehat{g}^+}(\widehat{x}, x_j) < \vartheta_1, \end{aligned}$$

Above,  $(x_j(h), \xi_j(h))$  are defined in (19). We denote

$$(69) \quad \mathcal{V}((\vec{x}, \vec{\xi}), t_0) = M_0 \setminus \bigcup_{j=1}^4 J_{\hat{g}}^+(\gamma_{x_j(t_0), \xi_j(t_0)}(\mathbf{t}_j)),$$

where  $\mathbf{t}_j := \rho(x_j(t_0), \xi_j(t_0))$ . Note that two geodesics  $\gamma_{x_j(t_0), \xi_j(t_0)}([0, \infty))$  can intersect only once in  $\mathcal{V}((\vec{x}, \vec{\xi}), t_0)$ . We will analyze the 4-wave interaction of waves in the set  $\mathcal{V}((\vec{x}, \vec{\xi}), t_0)$  where all observed singularities are produced before the geodesics  $\gamma_{x_j(t_0), \xi_j(t_0)}([0, \infty))$  have conjugate points, that is, before the waves  $u_{(j)}$  have caustics. Below we use  $\sim$  for the terms that have the same asymptotics up to an error  $O(\tau^{-N})$  for all  $N > 0$ , as  $\tau \rightarrow \infty$ .

**Proposition 3.3.** *Let  $(\vec{x}, \vec{\xi}) = ((x_j, \xi_j))_{j=1}^4$  be future pointing light-like vectors satisfying (68) and  $t_0 > 0$ . Let  $x_5 \in \mathcal{V}((\vec{x}, \vec{\xi}), t_0) \cap U_{\hat{g}}$  and  $(x_5, \xi_5)$  be a future pointing light-like vector such that  $x_5 \notin \mathcal{V}((\vec{x}, \vec{\xi}), t_0) \cup \bigcup_{j=1}^4 \gamma_{x_j, \xi_j}(\mathbb{R})$ , see (59) and (69). When  $n \in \mathbb{Z}_+$  is large enough and  $s_0 > 0$  is small enough, the function  $\Theta_\tau^{(4)}$ , see (63), corresponding to  $\mathbf{f}_j \in \mathcal{I}_C^{n+1}(Y(x_j, \zeta_j; t_0, s_0))$ ,  $j \leq 4$ , and the source  $F_\tau$ , see (60), corresponding to  $(x_5, \xi_5)$  satisfy*

- (i) *If the geodesics corresponding to  $(\vec{x}, \vec{\xi})$  either do not intersect or intersect at  $q$  and  $(x_5, \xi_5) \notin \Lambda_q^+$ , then  $|\Theta_\tau^{(4)}| \leq C_N \tau^{-N}$  for all  $N > 0$ .*
- (ii) *If the geodesics corresponding to  $(\vec{x}, \vec{\xi})$  intersect at  $q$ ,  $q = \gamma_{x_j, \xi_j}(t_j)$  and the vectors  $\dot{\gamma}_{x_j, \xi_j}(t_j)$ ,  $j = 1, 2, 3, 4$  are linearly independent, and there are  $t_5 < 0$  and  $\xi_5 \in L_{x_5}^+ M_0$  such that  $q = \gamma_{x_5, \xi_5}(t_5)$ , then, with  $m = -4n + 2$ ,*

$$(70) \quad \Theta_\tau^{(4)} \sim \sum_{k=m}^{\infty} s_k \tau^{-k}, \quad \text{as } \tau \rightarrow \infty.$$

Moreover, let  $b_j = (\dot{\gamma}_{x_j, \xi_j}(t_j))^b$ ,  $j = 1, 2, 3, 4, 5$ , and  $\mathbf{b} = (b_j)_{j=1}^5 \in (T_q^* M_0)^5$ . Let  $w_j$  be the principal symbols of the waves  $u_j = \mathbf{Q}f_j$  at  $(q, b_j)$  for  $j \leq 4$ . Also, let  $w_5$  be the principal symbol of  $u_\tau = \mathbf{Q}F_\tau$  at  $(q, b_5)$ , and  $\mathbf{w} = (w_j)_{j=1}^5$ . Then there is a real-analytic function  $\mathcal{G}(\mathbf{b}, \mathbf{w})$  such that the leading order term in (70) satisfies

$$(71) \quad s_m = \mathcal{G}(\mathbf{b}, \mathbf{w}).$$

- (iii) *Under the assumptions in (ii), the point  $x_5$  has a neighborhood  $V$  such that  $\mathcal{M}^4$  in  $V$  satisfies  $\mathcal{M}^4|_V \in \mathcal{I}(\Lambda_q^+)$ .*

**Proof.** Below, to simplify notations, we denote  $K_j = K(x_j, \xi_j; t_0, s_0)$  and  $K_{12} = K_1 \cap K_2$  and  $K_{124} = K_1 \cap K_2 \cap K_4$ , etc, and  $\mathcal{Y} = \mathcal{Y}((\vec{x}, \vec{\xi}), t_0)$ . We will denote  $\Lambda_j = \Lambda(x_j, \xi_j; t_0, s_0)$ .

Let us assume that  $s_0$  is so small that either the intersection  $\gamma_{x_5, \xi_5}(\mathbb{R}_-) \cap (\bigcap_{j=1}^4 K_j)$  is empty or it consists of a point  $q$  and if the intersection



happens, then  $K_j$  intersect transversally at  $q$ , as  $b_j$  are linearly independent.

We consider local coordinates  $Z : W_0 \rightarrow \mathbb{R}^4$  such that  $W_0 \subset \mathcal{V}((\vec{x}, \vec{\xi}), t_0)$  and if  $K_j$  intersects  $W_0$ , we assume that  $K_j \cap W_0 = \{x \in W_0; Z^j(x) = 0\}$ . Thus,  $Z(q) = (0, 0, 0, 0)$  if  $q \in W_0$ . Also, let  $Y : W_1 \rightarrow \mathbb{R}^4$  be local coordinates that have the same properties as  $Z$ . We will denote  $z^j = Z^j(x)$  and  $y^j = Y^j(x)$ . Let  $\Phi_0 \in C_0^\infty(W_0)$  and  $\Phi_1 \in C_0^\infty(W_1)$ .

Let us next consider the map  $\mathbf{Q}^* : C_0^\infty(W_1) \rightarrow C^\infty(W_0)$ . By [70],  $\mathbf{Q}^* \in I(W_1 \times W_0; \Delta'_{T^*M_0}, \Lambda_{\widehat{g}})$  is an operator with a classical symbol and its canonical relation  $\Lambda'_{\mathbf{Q}^*}$  is the union of two cleanly intersecting Lagrangian manifolds,  $\Lambda'_{\mathbf{Q}^*} = \Lambda'_{\widehat{g}} \cup \Delta_{T^*M_0}$ , see [70]. Let  $\varepsilon_2 > \varepsilon_1 > 0$  and  $B_{\varepsilon_1, \varepsilon_2}$  be a pseudodifferential operator on  $M_0$  which is microlocally smoothing operator i.e., the full symbol vanishes in local coordinates, outside in the conic  $\varepsilon_2$ -neighborhood  $\mathcal{V}_2 \subset T^*M_0$  of the set of the light-like covectors  $L^*M_0$ , and for which  $(I - B_{\varepsilon_1, \varepsilon_2})$  is microlocally smoothing operator in the  $\varepsilon_1$ -neighborhood  $\mathcal{V}_1$  of  $L^*M_0$ . Here, the conic neighborhoods are defined using the Hausdorff distance of the intersection of the sets with the  $\widehat{g}^+$ -unit sphere bundle on which we use the Sasaki metric determined by  $\widehat{g}^+$ . The parameters  $\varepsilon_1$  and  $\varepsilon_2$  are chosen later in the proof. Let us decompose the operator  $\mathbf{Q}^* = \mathbf{Q}_1^* + \mathbf{Q}_2^*$  where  $\mathbf{Q}_1^* = \mathbf{Q}^*(I - B_{\varepsilon_1, \varepsilon_2})$  and  $\mathbf{Q}_2^* = \mathbf{Q}^*B_{\varepsilon_1, \varepsilon_2}$ . Then  $\mathbf{Q}_2^* \in I(W_1 \times W_0; \Delta'_{T^*M_0}, \Lambda_{\widehat{g}})$  is an operator whose Schwartz kernel is a paired Lagrangian distribution, similarly to  $\mathbf{Q}^*$ , see Sec. 3.2.1. There is a neighborhood  $\mathcal{W}_2(\varepsilon_2)$  of  $L^*M_0 \times L^*M_0 \subset (T^*M_0)^2$  such that the Schwartz kernel of the operator  $\mathbf{Q}_2^*$  satisfies

$$(72) \quad \text{WF}(\mathbf{Q}_2^*) \subset \mathcal{W}_2 = \mathcal{W}_2(\varepsilon_2)$$

and  $\mathcal{W}_2(\varepsilon_2)$  tends to the set  $L^*M_0 \times L^*M_0$  as  $\varepsilon_2 \rightarrow 0$ . Moreover,  $\mathbf{Q}_1^* \in I(W_1 \times W_0; \Delta'_{T^*M_0})$  is a pseudodifferential operator with a classical symbol.

In the case when  $p = 1$  we can write  $\mathbf{Q}_p^*$  as

$$(73) \quad (\mathbf{Q}_1^*v)(z) = \int_{\mathbb{R}^{4+4}} e^{i(y-z) \cdot \xi} \sigma_{\mathbf{Q}_1^*}(z, y, \xi) v(y) dy d\xi.$$

Here,  $\sigma_{\mathbf{Q}_1^*}(z, y, \xi) \in S_{cl}^{-2}(W_1 \times W_0; \mathbb{R}^4)$  is a classical symbol with the principal symbol

$$(74) \quad q_1(z, y, \xi) = \chi(z, \xi) (\widehat{g}^{jk}(z) \xi_j \xi_k)^{-1}$$

where  $\chi(z, \xi) \in C^\infty$  vanishes in a neighborhood of light-like co-vectors.

Next we start to consider the terms of the type  $T_\tau^{(4), \beta}$  and  $\widetilde{T}_\tau^{(4), \beta}$  defined in (64) and (65). In these terms, we can represent the gaussian beam  $u_\tau(y)$  in  $W_1$  in the form

$$(75) \quad u_\tau(y) = e^{i\tau\varphi(y)} a_5(y, \tau)$$

where the function  $\varphi : W_1 \rightarrow \mathbb{C}$  is a complex phase function having a non-negative imaginary part such that  $\text{Im} \varphi$  vanishes exactly on the

geodesic  $\gamma_{x_5, \xi_5} \cap W_1$  and  $a_5 \in S_{cl}^0(W_1; \mathbb{R})$  is a classical symbol, see (62). For  $y = \gamma_{x_5, \xi_5}(t) \in W_1$  we have  $d\varphi(y) = c(t)\dot{\gamma}_{x_5, \xi_5}(t)^b$  with  $c(t) \in \mathbb{R} \setminus \{0\}$ .

Next, we consider the asymptotics of terms  $T_\tau^{(4), \beta}$  and  $\tilde{T}_\tau^{(4), \beta}$  of the type (64) and (65) where  $S_1 = S_2 = \mathbf{Q}$ . In our analysis, we replace the  $\mathcal{B}^L$ -section valued symbols by scalar valued classical symbols  $a_j(x, \theta_j)$  that are of the form  $a_j(z, \theta_j) = \chi(\theta_j)\hat{a}_j(z, \theta_j)$ , where  $\hat{a}_j(z, s\theta_j) = s^{p_j}\hat{a}_j(z, \theta_j)$  for  $s > 0$  and  $\chi(\theta_j) \in C^\infty(\mathbb{R})$  is a function that vanish near  $\theta_j = 0$  and is equal to 1 for  $|\theta_j| \geq 1$ . Here,  $p_j \in \mathbb{Z}_-$ . Then  $a_j(x, \theta_j)$  are in  $C^\infty(\mathbb{R}^4 \times \mathbb{R})$ , vanish near  $\theta_j = 0$ , are positively homogeneous for  $|\theta_j| > 1$  and have principal symbols  $\hat{a}_j(y, \theta_j)$ . Also, the operators  $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$  are replaced by the multiplication operators by  $\Phi_0$ , and  $\mathcal{B}_4$  by a multiplication operators by  $\Phi_1$ . The asymptotics of  $T_\tau^{(4), \beta}$  and  $\tilde{T}_\tau^{(4), \beta}$  for waves  $u_j$  with  $\mathcal{B}^L$ -section valued symbols and general operators  $S_j$  and  $\mathcal{B}_j$  may be obtained in a similar manner with slightly modified computations. Below, we denote  $\kappa(j) = 0$  for  $j = 1, 2, 3$  and  $\kappa(4) = 1$  so that  $U_j$  is supported in  $W_{\kappa(j)}$ . We also denote  $\kappa(5) = 1$ .

Moreover, using a suitable partition of unity we replace the waves  $u_j$  with functions  $U_j \in \mathcal{I}^{p_j}(K_j)$ ,  $j = 1, 2, 3$ , that are supported in  $W_0$  and  $U_4 \in \mathcal{I}^{p_4}(K_4)$  that is supported in  $W_1$ ,

$$(76) \quad U_j(x) = \int_{\mathbb{R}} e^{i\theta_j x^j} a_j(x, \theta_j) d\theta_j, \quad a_j \in S_{cl}^{p_j}(W_{\kappa(j)}; \mathbb{R}),$$

for all  $j = 1, 2, 3, 4$ . Note that in (76) there is no summing over the index  $j$ , and the symbols  $a_j(z, \theta_j)$ ,  $j \leq 3$  and  $a_j(y, \theta_j)$ ,  $j \in \{4, 5\}$  are scalar-valued symbols written in the  $Z$  and  $Y$  coordinates, respectively. Note that  $p_j = n$  correspond to the case when  $U_j \in \mathcal{I}^n(K_j) = \mathcal{I}^{n-1/2}(N^*K_j)$ .

Recall that  $\Lambda_j = N^*K_j$  and denote  $\Lambda_{jk} = N^*(K_j \cap K_k)$ . By [41, Lem. 1.2 and 1.3], the pointwise product  $U_2 \cdot U_1 \in \mathcal{I}(\Lambda_1, \Lambda_{12}) + \mathcal{I}(\Lambda_2, \Lambda_{12})$ . By [41, Prop. 2.2],  $\mathbf{Q}(U_2 \cdot U_1) \in \mathcal{I}(\Lambda_1, \Lambda_{12}) + \mathcal{I}(\Lambda_2, \Lambda_{12})$  and it can be written as

$$(77) \quad G_{12}(z) := \mathbf{Q}(U_2 \cdot U_1) = \int_{\mathbb{R}^2} e^{i(\theta_1 z^1 + \theta_2 z^2)} \sigma_{G_{12}}(z, \theta_1, \theta_2) d\theta_1 d\theta_2,$$

where  $\sigma_{G_{12}}(z, \theta_1, \theta_2)$  is sum of product type symbols, see (33). As  $N^*(K_1 \cap K_2) \setminus (N^*K_1 \cup N^*K_2)$  consist of vectors which are non-characteristic for  $\square_{\hat{g}}$ , the principal symbol  $c_1$  of  $G_{12}$  on  $N^*(K_1 \cap K_2) \setminus (N^*K_1 \cup N^*K_2)$  is given by

$$(78) \quad c_1(z, \theta_1, \theta_2) = s(z, \theta_1, \theta_2)\hat{a}_1(z, \theta_1)\hat{a}_2(z, \theta_2), \\ s(z, \theta_1, \theta_2) = 1/\hat{g}(\theta_1 b^{(1)} + \theta_2 b^{(2)}, \theta_1 b^{(1)} + \theta_2 b^{(2)}) = 1/(2\hat{g}(\theta_1 b^{(1)}, \theta_2 b^{(2)})).$$

Next, we consider  $T_\tau^{(4), \beta} = \sum_{p=1}^2 T_{\tau, p}^{(4), \beta}$  where  $T_{\tau, p}^{(4), \beta}$  is defined as  $T_\tau^{(4), \beta}$ , in (64), by replacing the term  $\mathbf{Q}^*(U_4 \cdot u_\tau)$  by  $\mathbf{Q}_p^*(U_4 \cdot u_\tau)$ . We consider different cases:

**Case 1:** Let us consider  $T_{\tau,p}^{(4),\beta}$  with  $p = 2$ , that is,

$$(79) \quad T_{\tau,2}^{(4),\beta} = \tau^4 \int_{\mathbb{R}^{12}} e^{i\tau\Psi_2(z,y,\theta)} \sigma_{G_{12}}(z, \tau\theta_1, \tau\theta_2) \cdot \\ \cdot a_3(z, \tau\theta_3) \mathbf{Q}_2^*(z, y) a_4(y, \tau\theta_4) a_5(y, \tau) d\theta_1 d\theta_2 d\theta_3 d\theta_4 dy dz, \\ \Psi_2(z, y, \theta) = \theta_1 z^1 + \theta_2 z^2 + \theta_3 z^3 + \theta_4 y^4 + \varphi(y),$$

where  $\mathbf{Q}_2^*(z, y)$  is the Schwartz kernel of  $\mathbf{Q}_2^*$  and  $\theta = (\theta_1, \theta_2, \theta_3, \theta_4) \in \mathbb{R}^4$ .

**Case 1, step 1:** Let first assume that  $(z, y, \theta)$  is a critical point of  $\Psi_2$  satisfying  $\text{Im } \varphi(y) = 0$ . Then we have  $\theta' = (\theta_1, \theta_2, \theta_3) = 0$  and  $z' = (z^1, z^2, z^3) = 0$ ,  $y^4 = 0$ ,  $d_y \varphi(y) = (0, 0, 0, -\theta_4)$ , implying that  $y \in K_4$  and  $(y, d_y \varphi(y)) \in N^* K_4$ . As  $\text{Im } \varphi(y) = 0$ , we have that  $y = \gamma_{x_5, \xi_5}(s)$  with some  $s \in \mathbb{R}_-$ . Since we have  $\dot{\gamma}_{x_5, \xi_5}(s)^b = d_y \varphi(y) \in N_y^* K_4$ , we obtain  $\gamma_{x_5, \xi_5}([s, 0]) \subset K_4$ . However, this is not possible by our assumption that  $x_5 \notin \cup_{j=1}^4 \gamma_{x_j, \xi_j}(\mathbb{R}_+)$  and thus  $x_5 \notin \cup_{j=1}^4 K(x_j, \xi_j; s_0)$  when  $s_0$  is small enough. Thus the phase function  $\Psi_2(z, y, \theta)$  has no critical points satisfying  $\text{Im } \varphi(y) = 0$ .

When the orders  $p_j$  of the symbols  $a_j$  are small enough, the integrals in the  $\theta$  variable in (79) are convergent as Riemann integrals. Next we consider  $\tilde{\mathbf{Q}}_2^*(z, y, \theta) = \mathbf{Q}_2^*(z, y)$  that is constant in the  $\theta$  variable. Below we denote  $\psi_4(y, \theta_4) = \theta_4 y^4$  and  $r = d\varphi(y)$ .

**Case 1, step 2:** Assume that  $((z, y, \theta), d_{z,y,\theta} \Psi_2) \in \text{WF}(\tilde{\mathbf{Q}}_2^*)$  and  $\text{Im } \varphi(y) = 0$ . Then we have  $(z^1, z^2, z^3) = 0$ ,  $y^4 = d_{\theta_4} \psi_4(y, \theta_4) = 0$  and  $y \in \gamma_{x_5, \xi_5}$ . Thus  $z \in K_{123}$  and  $y \in \gamma_{x_5, \xi_5} \cap K_4$ .

Let us use the following notations

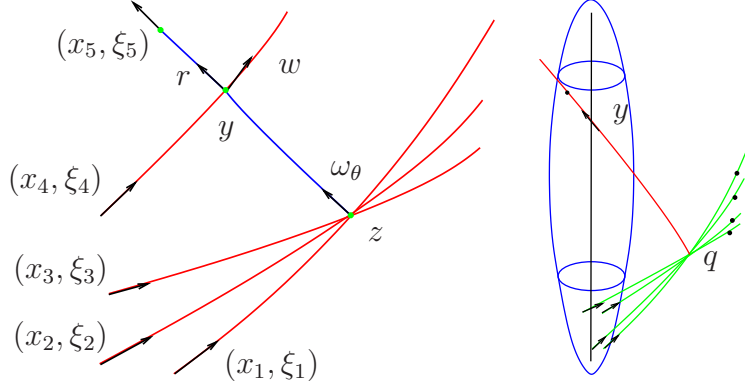
$$(80) \quad z \in K_{123}, \quad \omega_\theta := (\theta_1, \theta_2, \theta_2, 0) = \sum_{j=1}^3 \theta_j dz^j \in T_z^* M_0, \\ y \in K_4 \cap \gamma_{x_5, \xi_5}, \quad (y, w) := (y, d_y \psi_4(y, \theta_4)) \in N^* K_4, \\ r = d\varphi(y) = r_j dy^j \in T_y^* M_0, \quad \kappa := r + w.$$

Then,  $y$  and  $\theta_4$  satisfy  $y^4 = d_{\theta_4} \psi_4(y, \theta_4) = 0$  and  $w = (0, 0, 0, \theta_4)$ . By the definition of the  $Y$  coordinates  $w$  is a light-like covector. Also, by the definition of the  $Z$  coordinates,  $\omega_\theta \in N_z^* K_{123} = N_z^* K_1 + N_z^* K_2 + N_z^* K_3$ .

Let us first consider what happens if  $\kappa = r + w$  is light-like. In this case, all vectors  $\kappa$ ,  $w$ , and  $r$  are light-like and satisfy  $\kappa = r + w$ . This is possible only if  $r \parallel w$ , i.e.,  $r$  and  $w$  are parallel, see [79, Cor. 1.1.5]. Thus  $r + w$  is light-like if and only if  $r$  and  $w$  are parallel.

Recall that we consider the case when  $(z, y, \theta) \in W_1 \times W_0 \times \mathbb{R}^4$  is such that  $((z, y, \theta), d_{z,y,\theta} \Psi_2) \in \text{WF}(\tilde{\mathbf{Q}}_2^*)$  and  $\text{Im } \varphi(y) = 0$ . Using the above notations (80), we obtain  $d_{z,y,\theta} \Psi_2 = (\omega_\theta, r + w; (0, 0, 0, d_{\theta_4} \psi_4(y, \theta_4)))$ , where  $y^4 = d_{\theta_4} \psi_4(y, \theta_4) = 0$  on  $N^* K_4$ , and thus

$$(81) \quad ((z, \omega_\theta), (y, r + w)) \in \text{WF}(\mathbf{Q}_2^*) \subset \Lambda_{\hat{g}} \cup \Delta'_{T^* M_0}.$$



**FIGURE 5. Left:** The figure shows the case A1 when three geodesics intersect at  $z$  and the waves propagating near these geodesics interact and create a wave that hits the fourth geodesic at the point  $y$ . The produced singularities are detected by the gaussian beam source at the point  $x_5$ . Note that  $z$  and  $y$  can be conjugate points on the geodesic connecting them. In the case A2 the points  $y$  and  $z$  are the same. **Right:** Condition I is valid for the point  $y$  with vectors  $(x_j, \xi_j)$ ,  $j = 1, 2, 3, 4$  and with the parameter  $q$ . The black points are the conjugate points of  $\gamma_{x_j, \xi_j}([t_0, \infty))$  and  $\gamma_{q, \zeta}([0, \infty))$ .

Let  $\gamma_0$  be the geodesic with  $\gamma_0(0) = z$ ,  $\dot{\gamma}_0(0) = \omega_\theta^\sharp$ . We see that if  $((z, \omega_\theta), (y, r + w)) \in \Lambda_{\hat{g}}$  then

(A1) There is  $s \in \mathbb{R}$  such that  $(\gamma_0(s), \dot{\gamma}_0(s)^b) = (y, \kappa)$  and the vector  $\kappa$  is light-like.

On the other hand, if  $((z, \omega_\theta), (y, r + w)) \in \Delta'_{T^*M_0}$ , then

(A2)  $z = y$  and  $\kappa = -\omega_\theta$ .

By (81) we see that either (A1) or (A2) has to be valid.

**Case 1, step 2.A1:** (See Fig. 5(Left)) Consider next the case when (A1) is valid. Since  $\kappa$  is light-like,  $r$  and  $w$  are parallel. Then, as  $(\gamma_{x_5, \xi_5}(t_1), \dot{\gamma}_{x_5, \xi_5}(t_1)^b) = (y, r)$ , we see that  $\gamma_0$  is a continuation of the geodesic  $\gamma_{x_5, \xi_5}$ , that is, for some  $t_2$  we have  $(\gamma_{x_5, \xi_5}(t_2), \dot{\gamma}_{x_5, \xi_5}(t_2)) = (z, \omega_\theta) \in N^*K_{123}$ . This implies that  $x_5 \in \mathcal{Y}$  which is not possible by our assumptions. Hence (A1) is not possible.

**Case 1, step 2.A2:** Consider next the case when (A2) is valid. Let us first consider what would happen if  $\kappa$  is light-like. Then, we would have that  $r \parallel w$ . This implies that  $r$  is parallel to  $\kappa = -\omega_\theta \in N^*K_{123}$ , and as  $(\gamma_{x_5, \xi_5}(t_1), \dot{\gamma}_{x_5, \xi_5}(t_1)^b) = (y, r)$ , we would have  $x_5 \in \mathcal{Y}$ . This is not possible by our assumptions. Hence,  $\omega_\theta = -\kappa$  is not light-like.

For any given  $(\vec{x}, \vec{\xi})$  and  $(x_5, \xi_5)$  there exists  $\varepsilon_2 > 0$  so that  $(\{(y, \omega_\theta)\} \times T^*M) \cap \mathcal{W}_2(\varepsilon_2) = \emptyset$ , see (72), and thus

$$(82) \quad (\{(y, \omega_\theta)\} \times T^*M) \cap \text{WF}(\mathbf{Q}_2^*) = \emptyset.$$

Below, we assume that  $\varepsilon_2 > 0$  is chosen so that (82) is valid. Then there are no  $(z, y, \theta)$  such that  $((z, y, \theta), d\Psi_2(z, y, \theta)) \in \text{WF}(\tilde{\mathbf{Q}}_2^*)$  and  $\text{Im } \varphi(y) = 0$ . Thus Corollary 1.4 in [27] yields  $T_{\tau,2}^{(4),\beta} = O(\tau^{-N})$  for all  $N > 0$ . This ends the analysis of the case  $p = 2$ .

**Case 2:** Let us consider  $T_{\tau,p}^{(4),\beta}$  with  $p = 1$ . Using (73), we obtain

$$(83) \quad T_{\tau,1}^{(4),\beta} = \tau^8 \int_{\mathbb{R}^{16}} e^{i\tau(\theta_1 z^1 + \theta_2 z^2 + \theta_3 z^3 + (y-z) \cdot \xi + \theta_4 y^4 + \varphi(y))} \sigma_{G_{12}}(z, \tau\theta_1, \tau\theta_2) \cdot a_3(z, \tau\theta_3) \sigma_{Q_1^*}(z, y, \tau\xi) a_4(y, \tau\theta_4) a_5(y, \tau) d\theta_1 d\theta_2 d\theta_3 d\theta_4 dy dz d\xi.$$

Next, assume that  $(\bar{z}, \bar{\theta}, \bar{y}, \bar{\xi})$  is a critical point of the phase function

$$(84) \quad \Psi_3(z, \theta, y, \xi) = \theta_1 z^1 + \theta_2 z^2 + \theta_3 z^3 + (y - z) \cdot \xi + \theta_4 y^4 + \varphi(y).$$

Denote  $\bar{w} = (0, 0, 0, \bar{\theta}_4)$  and  $\bar{r} = d\varphi(\bar{y}) = \bar{r}_j dy^j$ . Then

$$(85) \quad \begin{aligned} \partial_{\theta_j} \Psi_3 = 0, \quad j = 1, 2, 3 & \quad \text{yields} \quad \bar{z} \in K_{123}, \\ \partial_{\theta_4} \Psi_3 = 0 & \quad \text{yields} \quad \bar{y} \in K_4, \\ \partial_z \Psi_3 = 0 & \quad \text{yields} \quad \bar{\xi} = \omega_{\bar{\theta}} := \sum_{j=1}^3 \bar{\theta}_j dz^j, \\ \partial_\xi \Psi_3 = 0 & \quad \text{yields} \quad \bar{y} = \bar{z}, \\ \partial_y \Psi_3 = 0 & \quad \text{yields} \quad \bar{\xi} = -\bar{r} - \bar{w}. \end{aligned}$$

The critical points we need to consider for the asymptotics satisfy also

$$(86) \quad \text{Im } \varphi(\bar{y}) = 0, \quad \text{so that } \bar{y} \in \gamma_{x_5, \xi_5}, \quad \text{Im } d\varphi(\bar{y}) = 0, \quad \text{Re } d\varphi(\bar{y}) \in L_{\bar{y}}^* M_0.$$

Let us use the notations similar to (80). By (85) and (86),

$$(87) \quad \bar{z} = \bar{y} \in \gamma_{x_5, \xi_5} \cap \bigcap_{j=1}^4 K_j, \quad \bar{\xi} = \sum_{j=1}^3 \bar{\theta}_j dz^j = -\bar{r} - \bar{w}.$$

**Case 2.1:** Consider the case when all the four geodesics  $\gamma_{x_j, \xi_j}$  intersect at the point  $q$ . Then, as  $x_5 \notin \mathcal{Y}$ , we have  $\bar{r} = d\varphi(\bar{y})$  is such that in the  $Y$ -coordinates  $\bar{r} = \bar{r}_j dy^j$  with  $\bar{r}_j \neq 0$  for all  $j = 1, 2, 3, 4$ . Thus the existence of the critical point  $(\bar{z}, \bar{\theta}, \bar{y}, \bar{\xi})$  of  $\Psi_3(z, \theta, y, \xi)$  implies that there exists an intersection point of  $\gamma_{x_5, \xi_5}$  and  $\bigcap_{j=1}^4 K_j$ .

Next we consider the case (87) where  $(\bar{y}, \bar{z}, \bar{\xi}, \bar{\theta})$  is the critical point of  $\Psi_3$ . In particular, then  $\bar{y} = \bar{z}$ . Thus we may assume for a while that  $W_0 = W_1$  and that the  $Y$ -coordinates and  $Z$ -coordinates coincide, that is,  $Y(x) = Z(x)$ . Then  $Y(\bar{y}) = Z(\bar{z}) = 0$  and the covectors  $dz^j = dZ^j$  and  $dy^j = dY^j$  at  $\bar{y}$  coincide for  $j = 1, 2, 3, 4$ . Moreover, (87) imply that

$$\bar{r} = \sum_{j=1}^4 \bar{r}_j dy^j = -\omega_{\bar{\theta}} - \bar{w} = -\sum_{j=1}^3 \bar{\theta}_j dz^j - \bar{\theta}_4 dy^4,$$

so that

$$(88) \quad \bar{r}_j = -\bar{\theta}_j, \quad \text{i.e., } \bar{\theta} := \bar{\theta}_j dz^j = -\bar{r} = \bar{r}_j dy^j \in T_{\bar{y}}^* M_0.$$

Using the method of stationary phase similarly to the proof of [43, Thm. 3.4] and the fact that  $\det(\text{Hess}_{z,y,\theta,\xi}\Psi_3) = 1$  we obtain, in the  $Y$  coordinates where  $Y(\bar{y}) = 0$ ,

$$(89) \quad T_\tau^{(4),\beta} \sim \tau^{-4+\rho} \sum_{k=0}^{\infty} c_k \tau^{-k}, \quad \text{where}$$

$$c_0 = \widehat{C} c_1(0, -\bar{r}_1, -\bar{r}_2) \widehat{a}_3(0, -\bar{r}_3) q_1(0, 0, -(\bar{r}_1, \bar{r}_2, \bar{r}_3, 0)) \widehat{a}_4(0, -\bar{r}_4) \widehat{a}_5(0, 1),$$

and  $\widehat{C} \neq 0$ ,  $\rho = \sum_{j=1}^4 p_j$ , and  $\widehat{a}_j(0, \theta_j)$ ,  $j \leq 4$  and  $\widehat{a}_5(0, 1)$  are the principal symbols of  $U_j$  and  $u_\tau$  at  $Y(\bar{y}) = 0$ , respectively. Above,  $q_1$  is given in (74) and  $c_1$  in (78). Also, observe that  $\bar{r} = d\varphi(\bar{y})$  depends only on  $\widehat{g}$  and  $\mathbf{b}$ , and thus we write below  $\bar{r} = \bar{r}(\mathbf{b})$ . Also,  $(\bar{r}_1, \bar{r}_2, \bar{r}_3, 0)$  is not light-like and hence the nominator of  $q_1$  does not vanish in (89).

**Case 2.2:** If  $\Psi_3(z, \theta, y, \xi)$  has no critical points, that is, there are no intersection points of the five geodesics  $\gamma_{x_j, \xi_j}$ ,  $j = 1, 2, \dots, 5$ , we obtain the asymptotics  $T_{\tau,1}^{(4),\beta} = O(\tau^{-N})$  for all  $N > 0$ .

**Case 3:** Next, we consider the terms  $\widetilde{T}_\tau^{(4),\beta}$  of the type (65). Such term is an integral of the product of  $u_\tau$  and the terms  $\mathbf{Q}(U_2 \cdot U_1)$  and  $\mathbf{Q}(U_4 \cdot U_3)$ . The last two factors can be written in the form (77). Assume that  $\Psi_3$  has a critical point  $(\bar{z}, \bar{\theta}, \bar{y}, \bar{\xi})$ . Then,  $\widetilde{T}_\tau^{(4),\beta}$  has similar asymptotics to  $T_\tau^{(4),\beta}$  as  $\tau \rightarrow \infty$ , with the leading order coefficient  $\widetilde{c}_0 = \widehat{C} c_1(0, -\bar{r}_1, -\bar{r}_2) c_2(0, -\bar{r}_3, -\bar{r}_4)$  where  $\bar{r} = \bar{r}(\mathbf{b})$  and  $c_2$  is given as in (78) with principal symbols  $\widehat{a}_3$  and  $\widehat{a}_4$ .

In the above computations based on method of stationary phase, we obtain the leading order coefficient  $s_m$  in (70) by evaluating the integrated function at the critical point  $(\bar{z}, \bar{\theta}, \bar{y}, \bar{\xi})$ . For example, denoting again  $\bar{r} = d\varphi(\bar{y})$ , the term (83) gives,

$$(90) \quad T_{\tau, id}^{(4),\beta_1} = C \tau^{2-\rho} \frac{q_1(\bar{r}(\mathbf{b}))}{s_1(\bar{r}(\mathbf{b}), \mathbf{b})} v_{nm}^{(5)} v_{(4)}^{rs} v_{(3)}^{ac} v_{(2)}^{ik} b_r^{(1)} b_s^{(1)} b_a^{(1)} b_c^{(1)} b_i^{(1)} b_k^{(1)} v_{(1)}^{nm},$$

where  $s_1(r, \mathbf{b}) = r_1 r_2 \widehat{g}(b^{(1)}, b^{(2)})$ ,  $q_1(r) = (\sum_{j,k=1}^3 \widehat{g}^{jk}(0) r_j r_k)^{-1}$ , and  $v_{(j)}^{ik} = \widehat{g}^{in}(0) \widehat{g}^{km}(0) v_{nm}^{(j)}$ , where  $v_{nm}^{(j)}$  is the metric part of the polarization  $v_{(j)}$ . For the other terms involving different operators  $\mathcal{B}_j^\beta$ ,  $S_j^\beta$ , and  $\sigma$  we obtain similar expressions in terms of  $\mathbf{b}$  and  $\mathbf{w}$ . This shows that  $s_m$  coincides with some real-analytic function  $G(\mathbf{b}, \mathbf{w})$ . This proves (ii).

(iii) Assume that  $\gamma_{x_j, \xi_j}$ ,  $j = 1, 2, 3, 4$  intersect at the point  $q$ . Using formulas (52), (76), and (77) we have that near  $q$  in the  $Y$  coordinates  $\mathcal{M}_1^{(4)} = \mathbf{Q}\mathcal{F}_1^{(4)}$ , where

$$(91) \quad \mathcal{F}_1^{(4)}(y) = \int_{\mathbb{R}^4} e^{iy^j \theta_j} b(y, \theta) d\theta.$$

Recall that here in the  $Y$ -coordinates  $K_j$  is given by  $\{y^j = 0\}$  and  $b(y, \theta)$  is a finite sum of terms that are products of product type symbols.

Let us choose sufficiently small  $\varepsilon_3 > 0$  and  $\chi(\theta) \in C^\infty(\mathbb{R}^4)$  that is positive homogeneous of order zero in the domain  $|\theta| > 1$ , vanishes in a conic  $\varepsilon_3$ -neighborhood (in the  $\widehat{g}^+$  metric) of  $\mathcal{A}_q$ ,

$$(92) \quad \mathcal{A}_q := N_q^* K_{123} \cup N_q^* K_{134} \cup N_q^* K_{124} \cup N_q^* K_{234}$$

and is equal to 1 outside the union of the conic  $(2\varepsilon_3)$ -neighborhood of the set  $\mathcal{A}_q$  and the set where  $|\theta| < 1/2$ .

Let  $\phi \in C_0^\infty(W_1)$  be a function that is one near  $q$ . Then, considering carefully the construction of the symbol  $b(y, \theta)$ , we see that  $b_0(y, \theta) = \phi(y)\chi(\theta)b(y, \theta)$  is a classical symbol of order  $p = \sum_{j=1}^4 p_j$ . Let  $\mathcal{F}^{(4),0}(y) \in \mathcal{I}^{p-4}(\{q\})$  be the conormal distribution that is given by the formula (91) with  $b(y, \theta)$  being replaced by  $b_0(y, \theta)$ .

Using the above computations based on the method of stationary phase and [68], we see that the function  $\mathcal{M}^4 - \mathbf{Q}\mathcal{F}^{(4),0}$  has no wave front set in  $T^*(V)$ , where  $V \subset U_{\widehat{g}} \setminus (\mathcal{Y} \cup \bigcup_{j=1}^4 K_j)$  is a neighborhood of  $x_5$ , and thus it is a  $C^\infty$ -smooth function in  $V$ .

By [41],  $\mathbf{Q} : \mathcal{I}^{p-4}(\{q\}) \rightarrow \mathcal{I}^{p-4-3/2, -1/2}(N^*(\{q\}), \Lambda_q^+)$ , which proves that  $\mathbf{Q}\mathcal{F}^{(4),0}|_V$  and thus  $\mathcal{M}^4|_V$  are in  $\mathcal{I}^{p-4-3/2}(\Lambda_q^+)$ .  $\square$

Next we will show that  $\mathcal{G}(\mathbf{b}, \mathbf{w})$  is not vanishing identically.

### 3.4. Non-vanishing interaction in the Minkowski space.

3.4.1. *WKB computations and the indicator functions in the Minkowski space.* To show that the function  $\mathcal{G}(\mathbf{b}, \mathbf{w})$ , see (71), is not vanishing identically, we will next consider waves in Minkowski space.

In this section,  $x = (x^0, x^1, x^3, x^4)$  are the standard coordinates in Minkowski space and  $\widehat{g}_{jk} = \text{diag}(-1, 1, 1, 1)$  denote the metric in the standard coordinates of the Minkowski space  $\mathbb{R}^4$ . Below we call the principal symbols of the linearized waves the *polarizations* to emphasize their physical meaning. We denote  $\mathbf{w} = (w_{(j)})_{j=1}^5$ . Then for  $j \leq 4$ , the polarizations  $w_j = (v_{(j)}^{met}, v_{(j)}^{scal})$ , represented as a pair of the metric part of the polarization  $v_{(j)}^{met} \in \text{sym}(\mathbb{R}^{4 \times 4}) \cong \mathbb{R}^{10}$  and the scalar field part of the polarization  $v_{(j)}^{scal} \in \mathbb{R}^L$ , are such that for the metric part  $v_{(j)}^{met}$  of the polarization has to satisfy 4 linear conditions (44) with the Minkowski metric (that follow from the linearized harmonicity condition (42)). We study the special case when all polarizations of the scalar fields  $\phi_\ell$ ,  $\ell = 1, 2, \dots, L$ , vanish, that is,  $v_{(j)}^{scal} = 0$  for all  $j$ . To simplify the notations, we denote below  $v_{(j)}^{met} = v_{(j)}$  and  $\mathbf{v} = (v_{(j)})_{j=1}^5$  so that  $\mathbf{w} = (\mathbf{v}, 0)$ . In this case, in Minkowski space the function  $\mathcal{G}(\mathbf{v}, 0, \mathbf{b})$  can be analyzed by assuming that there are no matter fields, which we do next. Later we return to the case of general polarizations.

We assume that the waves  $u_j(x)$ ,  $j = 1, 2, 3, 4$ , solving the linear wave equation in the Minkowski space, are of the form

$$(93) \quad u_j(x) = v_{(j)} \left( b_p^{(j)} x^p \right)_+^a, \quad t_+^a = |t|^a H(t),$$

where  $b_p^{(j)} dx^p$ ,  $p = 1, 2, 3, 4$  are four linearly independent light-like co-vectors of  $\mathbb{R}^4$ ,  $a > 0$  and  $v_{(j)}$  are constant  $4 \times 4$  matrices. We also assume that  $b^{(5)}$  is not in the linear span of any three vectors  $b^{(j)}$ ,  $j = 1, 2, 3, 4$ . In the following, we denote  $b^{(j)} \cdot x := b_p^{(j)} x^p$  and  $\mathbf{b} = (b^{(j)})_{j=1}^5$ .

Let us next consider the wave produced by interaction of two plane wave solutions in the Minkowski space.

Let  $b^{(1)}$  and  $b^{(2)}$  be light like co-vectors. We use the notations

$$u^{a_1, a_2}(x; b^{(1)}, b^{(2)}) = (b^{(1)} \cdot x)_+^{a_1} \cdot (b^{(2)} \cdot x)_+^{a_2}$$

for a product of two plane waves. We define the formal parametrix  $\mathbf{Q}_0$ ,

$$(94) \quad \mathbf{Q}_0(u^{a_1, a_2}(x; b^{(1)}, b^{(2)})) = \frac{u^{a_1+1, a_2+1}(x; b^{(1)}, b^{(2)})}{2(a_1+1)(a_2+1)\widehat{g}(b^{(1)}, b^{(2)})}.$$

Then  $\square_{\widehat{g}}(\mathbf{Q}_0(u^{a_1, a_2}(x; b^{(1)}, b^{(2)}))) = u^{a_1, a_2}(x; b^{(1)}, b^{(2)})$ . Also, let

$$u_\tau^a(x; b^{(4)}, b^{(5)}) = u_4(x) u_\tau(x), \quad u_4(x) = (b^{(4)} \cdot x)_+^a, \quad u_\tau(x) = e^{i\tau b^{(5)} \cdot x},$$

so that  $\square_{\widehat{g}}(u_\tau^a(x; b^{(4)}, b^{(5)})) = 2a\widehat{g}(b^{(4)}, b^{(5)})i\tau u_\tau^{a-1, 0}(x; b^{(4)}, b^{(5)})$ . Let

$$(95) \quad \mathbf{Q}_0(u_\tau^a(x; b^{(4)}, b^{(5)})) = \frac{1}{2i(a+1)\widehat{g}(b^{(4)}, b^{(5)})\tau} u_\tau^{a+1}(x; b^{(4)}, b^{(5)}).$$

We will prove that the indicator function  $\mathcal{G}(\mathbf{v}, 0, \mathbf{b})$  in (71) does not vanish identically by showing that it coincides with the *formal* indicator function  $\mathcal{G}^{(\mathbf{m})}(\mathbf{v}, \mathbf{b})$ , defined below, which is a real-analytic function that does not vanish identically. We define the (Minkowski) indicator function (c.f. (63) and (71)) by

$$\mathcal{G}^{(\mathbf{m})}(\mathbf{v}, \mathbf{b}) = \lim_{\tau \rightarrow \infty} \tau^m \left( \sum_{\beta \leq n_1} \sum_{\sigma \in \Sigma(4)} T_{\tau, \sigma}^{(\mathbf{m}), \beta} + \widetilde{T}_{\tau, \sigma}^{(\mathbf{m}), \beta} \right),$$

where the super-index  $(\mathbf{m})$  refers to the word ‘‘Minkowski’’. Above,  $\sigma$  runs over all permutations of the set  $\{1, 2, 3, 4\}$ . The functions  $T_{\tau, \sigma}^{(\mathbf{m}), \beta}$  and  $\widetilde{T}_{\tau, \sigma}^{(\mathbf{m}), \beta}$  are counterparts of the functions  $T_\tau^{(4), \beta}$  and  $\widetilde{T}_\tau^{(4), \beta}$ , see (64)–(65), obtained by replacing the distorted plane waves and the gaussian beam by the plane waves. We also replace the parametrices  $\mathbf{Q}$  and  $\mathbf{Q}^*$  by a formal parametrix  $\mathbf{Q}_0$ . Also, we include a smooth cut off function  $h \in C_0^\infty(M)$  which is one near the intersection point  $q$  of the  $K_j$ . Thus,

$$(96) \quad T_{\tau, \sigma}^{(\mathbf{m}), \beta} = \langle S_2^0(u_\tau \cdot \mathcal{B}_4 u_{\sigma(4)}), h \cdot \mathcal{B}_3 u_{\sigma(3)} \cdot S_1^0(\mathcal{B}_2 u_{\sigma(2)} \cdot \mathcal{B}_1 u_{\sigma(1)}) \rangle_{L^2(\mathbb{R}^4)},$$

$$(97) \quad \widetilde{T}_{\tau, \sigma}^{(\mathbf{m}), \beta} = \langle u_\tau, h \cdot S_2^0(\mathcal{B}_4 u_{\sigma(4)} \cdot \mathcal{B}_3 u_{\sigma(3)}) \cdot S_1^0(\mathcal{B}_2 u_{\sigma(2)} \cdot \mathcal{B}_1 u_{\sigma(1)}) \rangle_{L^2(\mathbb{R}^4)},$$

where  $u_j$  are given by (93) with  $a = -n - 1$ ,  $j = 1, 2, 3, 4$ . Here, the differential operators  $\mathcal{B}_j = \mathcal{B}_j^\beta$  are in Minkowski space constant coefficient operators and finally,  $S_j^0 = S_{j, \beta}^0 \in \{\mathbf{Q}_0, I\}$ .

Let us now consider the orders of the differential operators appearing above. Recall that  $k_j = \text{ord}(\mathcal{B}_j^\beta)$  are the orders of the differential operators  $\mathcal{B}_j^\beta$ , defined in (51). For  $j = 1, 2$ , we define  $K_{\beta, j} = 1$  when



$S_{\beta,j}^0 = \mathbf{Q}_0$  and  $K_{\beta,j} = 0$  when  $S_{\beta,j}^0 = I$ . Then the allowed values of  $\vec{k} = (k_1, k_2, k_3, k_4)$  depend on  $K_{\beta,1}$  and  $K_{\beta,2}$  as follows: We require that

$$(98) \quad \begin{aligned} k_1 + k_2 + k_3 + k_4 &\leq 2K_{\beta,1} + 2K_{\beta,2} + 2, & k_3 + k_4 &\leq 2K_{\beta,2} + 2, \\ k_4 &\leq 2, \text{ for all terms } T_{\tau,\sigma}^{(\mathbf{m}),\beta}, \text{ and} \\ k_1 + k_2 &\leq 2K_{\beta,1} + 2, \text{ for all terms } \tilde{T}_{\tau,\sigma}^{(\mathbf{m}),\beta} \end{aligned}$$

cf. (53) and (54).

Summarizing the above: Let  $b^{(j)}$ ,  $j = 1, 2, 3, 4$  be linearly independent light-like co-vectors and  $b^{(5)}$  be a light-like co-vector that is not in the linear span of any three vectors  $b^{(j)}$ ,  $j = 1, 2, 3, 4$ . Then, by analyzing the microlocal computations done in the proof of Prop. 3.3, we see that  $\mathcal{G}(\mathbf{w}, \mathbf{b}) = \mathcal{G}^{(\mathbf{m})}(\mathbf{v}, \mathbf{b})$  when  $w_{(j)} = (v_{(j)}, 0) \in \mathbb{R}^{10} \times \mathbb{R}^L$ ,  $\mathbf{w} = (w_{(j)})_{j=1}^5$ , and  $\mathbf{v} = (v_{(j)})_{j=1}^5$ .

**Proposition 3.4.** *Let  $\mathbb{X}$  be the set of  $(\mathbf{b}, v_{(2)}, v_{(3)}, v_{(4)})$ , where  $\mathbf{b}$  is a 5-tuple of light-like covectors  $\mathbf{b} = (b^{(1)}, b^{(2)}, b^{(3)}, b^{(4)}, b^{(5)})$  and  $v_{(j)} \in \mathbb{R}^{10}$ ,  $j = 2, 3, 4$  are the polarizations that satisfy the equation (44) with respect to  $b^{(j)}$ , i.e., the harmonicity condition for the principal symbols. For  $\widehat{b}^{(5)} \in \mathbb{R}^4$ , let  $\mathbb{X}(\widehat{b}^{(5)})$  be the set elements in  $\mathbb{X}$  where  $b^{(5)} = \widehat{b}^{(5)}$ . Then for any light-like  $\widehat{b}^{(5)}$  there is a generic (i.e. open and dense) subset  $\mathbb{X}'(\widehat{b}^{(5)})$  of  $\mathbb{X}(\widehat{b}^{(5)})$  such that for all  $(\mathbf{b}, v_{(2)}, v_{(3)}, v_{(4)}) \in \mathbb{X}'(\widehat{b}^{(5)})$  there exist linearly independent vectors  $v_{(5)}^q$ ,  $q = 1, 2, 3, 4, 5, 6$ , with the following property:*

*If  $v_{(5)} \in \text{span}(\{v_{(5)}^q; q = 1, 2, 3, 4, 5, 6\})$  is non-zero, then there exists a vector  $v_{(1)}$  for which the pair  $(b^{(1)}, v_{(1)})$  satisfies the equation (44) and  $\mathcal{G}^{(\mathbf{m})}(\mathbf{v}, \mathbf{b}) \neq 0$  with  $\mathbf{v} = (v_{(1)}, v_{(2)}, v_{(3)}, v_{(4)}, v_{(5)})$ .*

**Proof.** In the proof below, let  $a \in \mathbb{Z}_+$  be large enough. Consider the light-like vectors of the form

$$(99) \quad b^{(5)} = (1, 1, 0, 0), \quad b^{(j)} = (1, 1 - \frac{1}{2}\rho_j^2, \rho_j + O(\rho_j^3), \rho_j^3),$$

where  $j = 1, 2, 3, 4$ , and  $\rho_j \in (0, 1)$  are small parameters. With an appropriate choice of  $O(\rho_k^3)$  above, the vectors  $b^{(k)}$  are light-like and

$$(100) \quad \widehat{g}(b^{(5)}, b^{(j)}) = -\frac{1}{2}\rho_j^2, \quad \widehat{g}(b^{(k)}, b^{(j)}) = -\frac{1}{2}\rho_k^2 - \frac{1}{2}\rho_j^2 + O(\rho_k\rho_j).$$

Below, we denote  $\omega_{kj} = \widehat{g}(b^{(k)}, b^{(j)})$ . We consider the case when the orders of  $\rho_j$  are

$$(101) \quad \rho_4 = \rho_2^{100}, \quad \rho_2 = \rho_3^{100}, \quad \text{and} \quad \rho_3 = \rho_1^{100}$$

so that  $\rho_4 < \rho_2 < \rho_3 < \rho_1$ . When  $\rho_1$  is small enough,  $b_j$ ,  $j \leq 4$  are linearly independent and  $b^{(5)}$  is not a linear combination of any three vectors  $b^{(j)}$ ,  $j = 1, 2, 3, 4$ .

We will start by analyzing the most important terms  $T_\tau^{(\mathbf{m}),\beta}$  of the type (96) when  $\beta$  is such that  $\vec{S}_\beta = (\mathbf{Q}_0, \mathbf{Q}_0)$ . When  $k_j = k_j^\beta$  is the order of  $\mathcal{B}_j$ , we see that

$$(102) \quad \begin{aligned} T_\tau^{(\mathbf{m}),\beta} &= \langle \mathbf{Q}_0(\mathcal{B}_4 u_4 \cdot u_\tau), h \cdot \mathcal{B}_3 u_3 \cdot \mathbf{Q}_0(\mathcal{B}_2 u_2 \cdot \mathcal{B}_1 u_1) \rangle \\ &= C \frac{\mathcal{P}_\beta}{\omega_{45\tau} \omega_{12}} \int_{\mathbb{R}^4} (b^{(4)} \cdot x)_+^{a-k_4+1} e^{i\tau(b^{(5)} \cdot x)} h(x) (b^{(3)} \cdot x)_+^{a-k_3} \\ &\quad \cdot (b^{(2)} \cdot x)_+^{a-k_2+1} (b^{(1)} \cdot x)_+^{a-k_1+1} dx, \end{aligned}$$

where  $\mathcal{P} = \mathcal{P}_\beta$  is a polarization factor involving the coefficients of  $\mathcal{B}_j$ , the directions  $b^{(j)}$ , and the polarization  $v_{(j)}$ . Moreover,  $C$  is a generic constant which may depend on  $a$  and  $\beta$  but not on  $b^{(j)}$  or  $v_{(j)}$ .

Let us use in  $\mathbb{R}^4$  the coordinates  $y = (y^1, y^2, y^3, y^4)^t$  where  $y^j = b_k^{(j)} x^k$  and let  $A \in \mathbb{R}^{4 \times 4}$  be the matrix for which  $y = A^{-1}x$ . Let  $\mathbf{p} = (A^{-1})^t b^{(5)}$ . Then  $b^{(5)} \cdot x = \mathbf{p} \cdot y$  and  $\det(A) = 2\rho_1^{-3} \rho_2^{-2} \rho_3^{-1} (1 + O(\rho_1))$  and

$$T_\tau^{(\mathbf{m}),\beta} = \frac{C \mathcal{P}_\beta \det(A)}{\omega_{45\tau} \omega_{12}} \int_{(\mathbb{R}_+)^4} e^{i\tau \mathbf{p} \cdot y} h(Ay) y_4^{a-k_4+1} y_3^{a-k_3} y_2^{a-k_2+1} y_1^{a-k_1+1} dy.$$

Using repeated integration by parts we see from (100) that

$$(103) \quad T_\tau^{(\mathbf{m}),\beta} = \frac{C \det(A) \mathcal{P}_\beta (i\tau)^{-(12+4a-|\vec{k}_\beta|)} (1 + O(\tau^{-1}))}{\rho_4^{2(a-k_4+1+2)} \rho_3^{2(a-k_3+1)} \rho_2^{2(a-k_2+2)} \rho_1^{2(a-k_1+1+2)}}.$$

Note that here and below  $O(\tau^{-1})$  may depend also on  $\rho_j$ , that is, we have  $|O(\tau^{-1})| \leq C(\rho_1, \rho_2, \rho_3, \rho_4) \tau^{-1}$ .

Next we consider the polarization term when  $\beta = \beta_1$ , see (66). It appears in the term  $\langle F_\tau, \mathbf{Q}(A[u_4, \mathbf{Q}(A[u_3, \mathbf{Q}(A[u_2, u_1])])]) \rangle$  where all operators  $A[v, w]$  are of the type  $A_1[v, w] = \widehat{g}^{np} \widehat{g}^{mq} v_{nm} \partial_p \partial_q w_{jk}$ , cf. (55). For the term  $\beta = \beta_1$  we have the polarization factor

$$(104) \quad \mathcal{P}_{\beta_1} = (v_{(4)}^{rs} b_r^{(1)} b_s^{(1)}) (v_{(3)}^{pq} b_p^{(1)} b_q^{(1)}) (v_{(2)}^{nm} b_n^{(1)} b_m^{(1)}) \mathcal{D}, \quad \mathcal{D} = \widehat{g}_{nj} \widehat{g}_{mk} v_{(5)}^{nm} v_{(1)}^{jk},$$

where  $v_{(\ell)}^{nm} = \widehat{g}^{nj} \widehat{g}^{mk} v_{jk}^{(\ell)}$ . To show that  $\mathcal{G}^{(\mathbf{m})}(\mathbf{v}, \mathbf{b})$  is non-vanishing we estimate  $\mathcal{P}_{\beta_1}$  from below when we consider a particular choice of polarizations  $v_{(r)}$ , namely

$$(105) \quad v_{mk}^{(r)} = b_m^{(r)} b_k^{(r)}, \quad \text{for } r = 2, 3, 4, \text{ but not for } r = 1, 5$$

so that for  $r = 2, 3, 4$ , we have, as  $b^{(j)}$  are light-like,

$$\widehat{g}^{nm} b_n^{(r)} v_{mk}^{(r)} = 0, \quad \widehat{g}^{mk} v_{mk}^{(r)} = 0, \quad \widehat{g}^{nm} b_n^{(r)} v_{mk}^{(r)} - \frac{1}{2} (\widehat{g}^{mk} v_{mk}^{(r)}) b_k^{(r)} = 0.$$

For this choice of  $v^{(r)}$  the linearized harmonicity conditions (44) hold. Moreover, for this choice of  $v^{(r)}$  we see that for  $\rho_j \leq \rho_r^{100}$

$$(106) \quad v_{(r)}^{ns} b_n^{(j)} b_s^{(j)} = \widehat{g}(b^{(r)}, b^{(j)}) \widehat{g}(b^{(r)}, b^{(j)}) = \frac{1}{4} \rho_r^4 + O(\rho_r^5).$$

In the case  $\beta = \beta_1$ , as  $\vec{k}_{\beta_1} = (6, 0, 0, 0)$  and the polarizations are given by (105), we have  $\mathcal{P}_{\beta_1} = 2^{-6} (\mathcal{D} + O(\rho_1)) \rho_1^4 \cdot \rho_1^4 \cdot \rho_1^4$ , where  $\mathcal{D}$  is the inner

product of  $v^{(1)}$  and  $v^{(5)}$  given in (104). Then the term  $T_\tau^{(\mathbf{m}),\beta_1}$ , which will turn out to have the strongest asymptotics in our considerations, has the asymptotics

$$(107) \quad T_\tau^{(\mathbf{m}),\beta_1} = \mathcal{L}_\tau, \quad \text{where} \\ \mathcal{L}_\tau = C \det(A) (i\tau)^{-(6+4a)} (1 + O(\tau^{-1})) \vec{\rho}^{\vec{d}} \rho_4^{-4} \rho_2^{-2} \rho_3^0 \rho_1^{20} \mathcal{D},$$

where  $\vec{\rho} = (\rho_1, \rho_2, \rho_3, \rho_4)$ ,  $\vec{a} = (a, a, a, a)$ ,  $\vec{d} = (-2a - 2, -2a - 2, -2a - 2, -2a - 2)$ . To compare different terms, we express  $\rho_j$  in powers of  $\rho_1$  as explained in formula (101), that is, we write  $\rho_4^{n_4} \rho_2^{n_2} \rho_3^{n_3} \rho_1^{n_1} = \rho_1^m$  with  $m = 100^3 n_4 + 100^2 n_2 + 100 n_3 + n_1$ . In particular, we will write below

$$\mathcal{L}_\tau = C_{\beta_1}(\vec{\rho}) \tau^{n_0} (1 + O(\tau^{-1})) \quad \text{as } \tau \rightarrow \infty \text{ for each fixed } \vec{\varepsilon}, \text{ and} \\ C_{\beta_1}(\vec{\rho}) = c'_{\beta_1} \rho_1^{m_0} (1 + o(\rho_1)) \quad \text{as } \rho_1 \rightarrow 0,$$

where  $n_0 = -4a - 6$ . Below we will show that  $c'_{\beta_1}$  does not vanish for generic  $(\vec{x}, \vec{\xi})$  and  $(x_5, \xi_5)$  and polarizations  $\mathbf{v}$ . We will consider below  $\beta \neq \beta_1$  and show that all these terms have the asymptotics

$$T_\tau^{(\mathbf{m}),\beta} = C_\beta(\vec{\rho}) \tau^n (1 + O(\tau^{-1})) \quad \text{as } \tau \rightarrow \infty \text{ for each fixed } \vec{\varepsilon}, \text{ and} \\ C_\beta(\vec{\rho}) = c'_\beta \rho_1^m (1 + o(\rho_1)) \quad \text{as } \rho_1 \rightarrow 0.$$

Then we have that either  $n < n_0$  or  $n = n_0$  and  $m < m_0$ . In this case we say that  $T_\tau^{(\mathbf{m}),\beta}$  has weaker asymptotics than  $T_\tau^{(\mathbf{m}),\beta_1}$  and denote  $T_\tau^{(\mathbf{m}),\beta} \prec \mathcal{L}_\tau$ .

Then, we can analyze the terms  $T_{\tau,\sigma}^{(\mathbf{m}),\beta}$  and  $\tilde{T}_{\tau,\sigma}^{(\mathbf{m}),\beta}$ . Note that here the terms, in which the permutation  $\sigma$  is either the identical permutation  $id$  or the permutation  $\sigma_0 = (2, 1, 3, 4)$ , are the same.

**Proposition 3.5.** *Assume that  $v^{(r)}$ ,  $r = 2, 3, 4$  are given by (105). Then for all  $(\beta, \sigma) \notin \{(\beta_1, id), (\beta_1, \sigma_0)\}$  we have  $T_{\tau,\sigma}^{(\mathbf{m}),\beta} \prec T_{\tau,id}^{(\mathbf{m}),\beta_1}$ . Also, for all  $(\beta, \sigma)$  we have  $\tilde{T}_{\tau,\sigma}^{(\mathbf{m}),\beta} \prec T_{\tau,id}^{(\mathbf{m}),\beta_1}$ .*

**Proof.** The claim follows from straightforward computations (whose details are included in [56]) that are similar to the computation of the integral (102) for all other terms except for the term  $T_{\tau,\sigma}^{(\mathbf{m}),\beta}$  where  $\sigma$  is either  $\sigma = (3, 2, 1, 4)$  or  $\sigma = (2, 3, 1, 4)$ . These terms are very similar and thus we analyze the case when  $\sigma = (3, 2, 1, 4)$ . First we consider the case when  $\beta = \beta_2$  is such that  $\vec{S}^{\beta_2} = (\mathbf{Q}_0, \mathbf{Q}_0)$ ,  $\vec{k}_{\beta_2} = (2, 0, 4, 0)$ , which contains the maximal number of derivatives of  $u_1$ , namely 4. By a permutation of the indexes in (102) we obtain the formula

$$(108) \quad T_{\tau,\sigma}^{(\mathbf{m}),\beta_2} = c'_1 \det(A) (i\tau)^{-(6+4a)} (1 + O(\tau^{-1})) \vec{\rho}^{\vec{d}} \\ \cdot (\omega_{45} \omega_{32})^{-1} \rho_4^{2(k_4-1)} \rho_1^{2k_3} \rho_2^{2(k_2-1)} \rho_3^{2(k_1-1)} \mathcal{P}_{\beta_2}, \\ \tilde{\mathcal{P}}_{\beta_2} = (v_{(4)}^{pq} b_p^{(1)} b_q^{(1)}) (v_{(3)}^{rs} b_r^{(1)} b_s^{(1)}) (v_{(2)}^{nm} b_n^{(3)} b_m^{(3)}) \mathcal{D}.$$

Hence, in the case when we use the polarizations (105), we obtain

$$T_{\tau,\sigma}^{(\mathbf{m}),\beta_2} = c_1(i\tau)^{-(6+4a)}(1 + O(\tau^{-1}))\vec{\rho}^{\vec{d}}\rho_4^{-4}\rho_2^{-2}\rho_3^{0+4}\rho_1^{6+8}\mathcal{D}.$$

Comparing the power of  $\rho_3$  in the above expression, we have that in this case  $T_{\tau,\sigma}^{(\mathbf{m}),\beta_2} \prec \mathcal{L}_\tau$ . When  $\sigma = (3, 2, 1, 4)$ , we see in a straightforward way for all other  $\beta$  that  $T_{\tau,\sigma}^{(\mathbf{m}),\beta} \prec \mathcal{L}_\tau$ . Note that taking  $S_j^\beta$  to be  $I$  instead of  $\mathbf{Q}_0$  decreases, by (94) and (102), the total power of  $\tau$  by one. This proves Proposition 3.5.  $\square$

Summarizing; we have analyzed the terms  $T_{\tau,\sigma}^{(\mathbf{m}),\beta}$  corresponding to any  $\beta$  and all  $\sigma$  except  $\sigma = \sigma_0 = (2, 1, 3, 4)$ . Clearly, the sum  $\sum_\beta T_{\tau,\sigma_0}^{(\mathbf{m}),\beta}$  is equal to the sum  $\sum_\beta T_{\tau,id}^{(\mathbf{m}),\beta}$ . Thus, when the asymptotic orders of  $\rho_j$  are given in (101) and the polarizations satisfy (105), we have

$$\begin{aligned} \mathcal{G}^{(\mathbf{m})}(\mathbf{v}, \mathbf{b}) &= \lim_{\tau \rightarrow \infty} \sum_{\beta, \sigma} \frac{T_{\tau,\sigma}^{(\mathbf{m}),\beta}}{(i\tau)^{(6+4a)}} = \lim_{\tau \rightarrow \infty} \frac{2T_{\tau,id}^{(\mathbf{m}),\beta_1}(1 + O(\rho_1))}{(i\tau)^{(6+4a)}} \\ (109) \quad &= 2c_1 \det(A)(1 + O(\rho_1)) \vec{\rho}^{\vec{d}} \rho_4^{-4} \rho_2^{-2} \rho_3^0 \rho_1^{20} \mathcal{D}. \end{aligned}$$

Next we consider general polarizations. Let  $Y = \text{sym}(\mathbb{R}^{4 \times 4})$  and consider the non-degenerate, symmetric, bi-linear form  $B : (v, w) \mapsto \widehat{g}_{nj} \widehat{g}_{mk} v^{nm} w^{jk}$  in  $Y$ . Then  $\mathcal{D} = B(v^{(5)}, v^{(1)})$ .

Let  $L(b^{(j)})$  denote the subspace of dimension 6 of the symmetric matrices  $v \in Y$  that satisfy equation (44) with covector  $\xi = b^{(j)}$ .

Let  $\mathcal{W}(b^{(5)})$  be the real analytic submanifold of  $(\mathbb{R}^4)^5 \times (\mathbb{R}^{4 \times 4})^3 \times (\mathbb{R}^{4 \times 4})^6 \times (\mathbb{R}^{4 \times 4})^6$  consisting of elements  $\eta = (\mathbf{b}, \underline{v}, V^{(1)}, V^{(5)})$ , where  $\mathbf{b} = (b^{(1)}, b^{(2)}, b^{(3)}, b^{(4)}, b^{(5)})$  is a sequence of light-like vectors with the given vector  $b^{(5)}$ , and  $\underline{v} = (v^{(2)}, v^{(3)}, v^{(4)})$  satisfy  $v^{(j)} \in L(b^{(j)})$  for all  $j = 2, 3, 4$ , and  $V^{(1)} = (v_p^{(1)})_{p=1}^6 \in (\mathbb{R}^{4 \times 4})^6$  is a basis of  $L(b^{(1)})$  and  $V^{(5)} = (v_p^{(5)})_{p=1}^6 \in (\mathbb{R}^{4 \times 4})^6$  is sequence of vectors in  $Y$  such that  $B(v_p^{(5)}, v_q^{(1)}) = \delta_{pq}$  for  $p \leq q$ . Note that  $\mathcal{W}(b^{(5)})$  has two components where the orientation of the basis  $V^{(1)}$  is different.

By (96) and (96),  $\mathcal{G}^{(\mathbf{m})}(\mathbf{v}, \mathbf{b})$  is linear in each  $v^{(j)}$ . For  $\eta \in \mathcal{W}(b^{(5)})$ , we define

$$\kappa(\eta) := \det \left( \mathcal{G}^{(\mathbf{m})}(\mathbf{v}_{(p,q)}, \mathbf{b}) \right)_{p,q=1}^6, \text{ where } \mathbf{v}_{(p,q)} = (v_p^{(1)}, v^{(2)}, v^{(3)}, v^{(4)}, v_q^{(5)}).$$

Then  $\kappa(\eta)$  can be written as  $\kappa(\eta) = A_1(\eta)/A_2(\eta)$  where  $A_1(\eta)$  and  $A_2(\eta)$  are real-analytic functions on  $\mathcal{W}(b^{(5)})$ . In fact,  $A_2(\eta)$  is a product of terms  $\widehat{g}(b^{(j)}, b^{(k)})^p$  with some positive integer  $p$ , cf. (100) and (103).

Let us next consider the case when the sequence of the light-like vectors,  $\mathbf{b} = (b^{(1)}, b^{(2)}, b^{(3)}, b^{(4)}, b^{(5)})$  given in (99) with  $\vec{\rho}$  given in (101) with some small  $\rho_1 > 0$  and let the polarizations  $\underline{v} = (v^{(2)}, v^{(3)}, v^{(4)})$  be such that  $v^{(j)} \in L(b^{(j)})$ ,  $j = 2, 3, 4$ , are those given by (105), and  $V^{(1)} = (v_p^{(1)})_{p=1}^6$  be a basis of  $L(b^{(1)})$ . Let  $V^{(5)} = (v_p^{(5)})_{p=1}^6$  be vectors in

$Y$  such that  $B(v_p^{(5)}, v_q^{(1)}) = \delta_{pq}$  for  $p \leq q$ . When  $\rho_1 > 0$  is small enough, formula (109) yields that  $\kappa(\eta) \neq 0$  for  $\eta = (\mathbf{b}, \underline{v}, V^{(1)}, V^{(5)})$ . Since  $\kappa(\eta) = \kappa(\eta) = A_1(\eta)/A_2(\eta)$  where  $A_1(\eta)$  and  $A_2(\eta)$  are real-analytic on  $\mathcal{L}(b^{(5)})$ , we have that  $\kappa(\eta)$  is non-vanishing and finite on an open and dense subset of the component of  $\mathcal{W}(b^{(5)})$  containing  $\eta$ . The fact that  $\kappa(\eta)$  is non-vanishing on a generic subset of the other component of  $\mathcal{W}(b^{(5)})$  can be seen by changing the orientation of  $V^{(1)}$ . This yields that claim.  $\square$

#### 4. OBSERVATIONS IN NORMAL COORDINATES

We have considered above the singularities of the metric  $g$  in the wave gauge coordinates. As the wave gauge coordinates may also be non-smooth, we do not know if the observed singularities are caused by the metric or the coordinates. Because of this, we consider next the metric in normal coordinates.

Let  $v^{\vec{\varepsilon}} = (g^{\vec{\varepsilon}}, \phi^{\vec{\varepsilon}})$  be the solution of the  $\widehat{g}$ -reduced Einstein equations (8) with the source  $\mathbf{f}_{\vec{\varepsilon}}$  given in (48). We emphasize that  $g^{\vec{\varepsilon}}$  is the metric in the  $(g, \widehat{g})$ -wave gauge coordinates.

Let  $(z, \eta) \in \mathcal{U}_{(z_0, \eta_0)}(\widehat{h})$ ,  $\mu_{\vec{\varepsilon}} = \mu_{g^{\vec{\varepsilon}}, z, \eta}$  and  $(Z_{j, \vec{\varepsilon}})_{j=1}^4$  be a frame of vectors obtained by  $g^{\vec{\varepsilon}}$ -parallel continuation of some  $\vec{\varepsilon}$ -independent frame along the geodesic  $\mu_{\vec{\varepsilon}}([-1, 1])$  from  $\mu_{\vec{\varepsilon}}(-1)$  to a point  $p_{\vec{\varepsilon}} = \mu_{\vec{\varepsilon}}(\tilde{r})$ . Recall that  $g$  and  $\widehat{g}$  coincide in the set  $(-\infty, 0) \times N$  that contains the point  $\mu_{\vec{\varepsilon}}(-1)$ . Let  $\Psi_{\vec{\varepsilon}} : W_{\vec{\varepsilon}} \rightarrow \Psi_{\vec{\varepsilon}}(W_{\vec{\varepsilon}}) \subset \mathbb{R}^4$  denote normal coordinates of  $(M_0, g^{\vec{\varepsilon}})$  defined using the center  $p_{\vec{\varepsilon}}$  and the frame  $Z_{j, \vec{\varepsilon}}$ . We say that  $\mu_{\vec{\varepsilon}}([-1, 1])$  are observation geodesics, and that  $\Psi_{\vec{\varepsilon}}$  are the normal coordinates associated to  $\mu_{\vec{\varepsilon}}$  and  $p_{\vec{\varepsilon}} = \mu_{\vec{\varepsilon}}(\tilde{r})$ . Denote  $\Psi_0 = \Psi_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0}$ , and  $W = W_0$ . We also denote  $g_{\vec{\varepsilon}} = g^{\vec{\varepsilon}}$  and  $U_{\vec{\varepsilon}} = U_{g^{\vec{\varepsilon}}}$ .

**Lemma 4.1.** *Let  $v^{\vec{\varepsilon}} = (g^{\vec{\varepsilon}}, \phi^{\vec{\varepsilon}})$  and  $\Psi_{\vec{\varepsilon}}$  be as above. Let  $S \subset U_{\widehat{g}}$  be a smooth 3-dimensional surface such that  $p_0 = p_{\vec{\varepsilon}}|_{\vec{\varepsilon}=0} \in S$  and*

$$(110) \quad g^{(\alpha)} = \partial_{\vec{\varepsilon}}^{\alpha} g^{\vec{\varepsilon}}|_{\vec{\varepsilon}=0}, \quad \phi_{\ell}^{(\alpha)} = \partial_{\vec{\varepsilon}}^{\alpha} \phi_{\ell}^{\vec{\varepsilon}}|_{\vec{\varepsilon}=0}, \quad \text{for } |\alpha| \leq 4, \quad \alpha \in \{0, 1\}^4,$$

*and assume that  $g^{(\alpha)}$  and  $\phi_{\ell}^{(\alpha)}$  are in  $C^{\infty}(W)$  for  $|\alpha| \leq 3$  and  $g_{pq}^{(\alpha_0)}|_W \in \mathcal{I}^{m_0}(W \cap S)$  and  $\phi_{\ell}^{(\alpha_0)}|_W \in \mathcal{I}^{m_0}(W \cap S)$  for  $\alpha_0 = (1, 1, 1, 1)$ .*

*(i) Assume that  $S \cap W$  is empty. Then the tensors  $\partial_{\vec{\varepsilon}}^{\alpha_0} ((\Psi_{\vec{\varepsilon}})_* g_{\vec{\varepsilon}})|_{\vec{\varepsilon}=0}$  and  $\partial_{\vec{\varepsilon}}^{\alpha_0} ((\Psi_{\vec{\varepsilon}})_* \phi_{\ell}^{\vec{\varepsilon}})|_{\vec{\varepsilon}=0}$  are  $C^{\infty}$ -smooth in  $\Psi_0(W)$ .*

*(ii) Assume that  $\mu_0([-1, 1])$  intersects  $S$  transversally at  $p_0$ . Consider the conditions*

*(a) There is a 2-contravariant tensor field  $v$  that is a smooth section of  $TW \otimes TW$  such that  $v(x) \in T_x S \otimes T_x S$  for  $x \in S$  and the principal symbol of  $\langle v, g^{(\alpha_0)} \rangle|_W \in \mathcal{I}^{m_0}(W \cap S)$  is non-vanishing at  $p_0$ .*

*(b) The principal symbol of  $\phi_{\ell}^{(\alpha_0)}|_W \in \mathcal{I}^{m_0}(W \cap S)$  is non-vanishing at  $p_0$  for some  $\ell = 1, 2, \dots, L$ .*

If (a) or (b) holds, then either  $\partial_{\varepsilon}^{\alpha_0}((\Psi_{\varepsilon})_*g_{\varepsilon})|_{\varepsilon=0}$  or  $\partial_{\varepsilon}^{\alpha_0}((\Psi_{\varepsilon})_*\phi_{\ell}^{\varepsilon})|_{\varepsilon=0}$  is not  $C^{\infty}$ -smooth in  $\Psi_0(W)$ .

**Proof.** (i) is obvious.

(ii) Denote  $\gamma_{\varepsilon}(t) = \mu_{\varepsilon}(t + \tilde{r})$ . Let  $X : W_0 \rightarrow V_0 \subset \mathbb{R}^4$ ,  $X(y) = (X^j(y))_{j=1}^4$  be local coordinates in  $W_0$  such that  $X(p_0) = 0$  and  $X(S \cap W_0) = \{(x^1, x^2, x^3, x^4) \in V_0; x^1 = 0\}$  and  $y(t) = X(\gamma_0(t)) = (t, 0, 0, 0)$ . Note that the coordinates  $X$  are independent of  $\varepsilon$ . We assume that the vector fields  $Z_{\varepsilon, j}$  defining the normal coordinates are such that  $Z_{0, j}(p_0) = \partial/\partial X^j$ . To do computations in local coordinates, let us denote

$$\tilde{g}^{(\alpha)} = \partial_{\varepsilon}^{\alpha}(X_*g^{\varepsilon})|_{\varepsilon=0}, \quad \tilde{\phi}_{\ell}^{(\alpha)} = \partial_{\varepsilon}^{\alpha}(X_*\phi_{\ell}^{\varepsilon})|_{\varepsilon=0}, \quad \text{for } |\alpha| \leq 4, \alpha \in \{0, 1\}^4.$$

Let  $v$  be a tensor field given in (a) such that in the  $X$  coordinates  $v(x) = v^{pq}(x)\frac{\partial}{\partial x^p}\frac{\partial}{\partial x^q}$  so that  $v^{pq}(0) = 0$  if  $(p, q) \notin \{2, 3, 4\}^2$  at the point  $0 = X(p_0)$  and the functions  $v^{pq}(x)$  do not depend on  $x$ , that is,  $v^{pq}(x) = \widehat{v}^{mq} \in \mathbb{R}^{4 \times 4}$ . Let  $R^{\varepsilon}$  be the curvature tensor of  $g^{\varepsilon}$  and define the functions

$$h_{mk}^{\varepsilon}(t) = g^{\varepsilon}(R^{\varepsilon}(\dot{\gamma}_{\varepsilon}(t), Z_m^{\varepsilon}(t))\dot{\gamma}_{\varepsilon}(t), Z_k^{\varepsilon}(t)), \quad J_v(t) = \partial_{\varepsilon}^{\alpha_0}(\widehat{v}^{mq} h_{mq}^{\varepsilon}(t))|_{\varepsilon=0}.$$

The function  $J_v(t)$  is an invariantly defined function on the curve  $\gamma_0(t)$  and thus it can be computed in any coordinates. If  $\partial_{\varepsilon}^{\alpha_0}((\Psi_{\varepsilon})_*g_{\varepsilon})|_{\varepsilon=0}$  would be smooth near  $0 \in \mathbb{R}^4$ , then the function  $J_v(t)$  would be smooth near  $t = 0$ . To show that the  $\partial_{\varepsilon}^{\alpha_0}$ -derivatives of the metric tensor in the normal coordinates are not smooth, we need to show that  $J_v(t)$  is non-smooth at  $t = 0$  for some values of  $\widehat{v}^{mq}$ .

We will work in the  $X$  coordinates and denote  $\tilde{R}(x) = \partial_{\varepsilon}^{\alpha_0} X_*(R_{\varepsilon})(x)|_{\varepsilon=0}$ . Moreover,  $\tilde{\gamma}^j$  are the analogous 4th order  $\varepsilon$ -derivatives and we denote  $\tilde{g} = \tilde{g}^{(\alpha_0)}$ . For simplicity we also denote  $X_*\widehat{g}$  and  $X_*\widehat{\phi}_l$  by  $\widehat{g}$  and  $\widehat{\phi}_l$ , respectively.

We analyze the functions of  $t \in I = (-t_1, t_1)$ , e.g.,  $a(t)$ , where  $t_1 > 0$  is small. We say that  $a(t)$  is of order  $n$  if  $a(\cdot) \in \mathcal{I}^n(\{0\})$ . By the assumptions of the theorem,  $(\partial_x^{\beta} \tilde{g}_{jk})(\gamma(t)) \in \mathcal{I}^{m_0 + \beta_1}(\{0\})$  when  $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)$ . It follows from (110) and the linearized equations for the parallel transport that  $(\tilde{\gamma}(t), \partial_t \tilde{\gamma}(t))$  and  $\tilde{Z}_k$  are in  $\mathcal{I}^{m_0}(\{0\})$ . The above analysis shows that  $\tilde{R}|_{\gamma_0(I)} \in \mathcal{I}^{m_0+2}(\{0\})$ . Thus in the  $X$  coordinates  $\partial_{\varepsilon}^{\alpha_0}(h_{mk}^{\varepsilon}(t))|_{\varepsilon=0} \in \mathcal{I}^{m_0+2}(\{0\})$  can be written as

$$\begin{aligned} & \partial_{\varepsilon}^{\alpha_0}(h_{mk}^{\varepsilon}(t))|_{\varepsilon=0} = \widehat{g}(\tilde{R}(\dot{\gamma}_0(t), \widehat{Z}_m(t))\dot{\gamma}_0(t), \widehat{Z}_k(t)) + s.t. \\ & = \frac{1}{2} \left( \frac{\partial}{\partial x^1} \left( \frac{\partial \tilde{g}_{km}}{\partial x^1} + \frac{\partial \tilde{g}_{k1}}{\partial x^m} - \frac{\partial \tilde{g}_{1m}}{\partial x^k} \right) - \frac{\partial}{\partial x^m} \left( \frac{\partial \tilde{g}_{k1}}{\partial x^1} + \frac{\partial \tilde{g}_{k1}}{\partial x^1} - \frac{\partial \tilde{g}_{11}}{\partial x^k} \right) \right) + s.t., \end{aligned}$$

where all *s.t.* = “smoother terms” are in  $\mathcal{I}^{m_0+1}(\{0\})$ .

Consider next the case (a). Assume that for given  $(k, m) \in \{2, 3, 4\}^2$ , the principal symbol of  $\tilde{g}_{km}^{(\alpha_0)}$  is non-vanishing at  $0 = X(p_0)$ . Let  $v$  be such a tensor field that  $v^{mk}(0) = v^{km}(0) \neq 0$  and  $v^{in}(0) = 0$  when

$(i, n) \notin \{(k, m), (m, k)\}$ . Then the above yields (in the formula below, we do not sum over  $k, m$ )

$$J_v(t) = v^{ij} \partial_{\tilde{\varepsilon}}^{\alpha_0} (h_{ij}^{\tilde{\varepsilon}}(t))|_{\tilde{\varepsilon}=0} = \frac{e(k, m)}{2} \tilde{v}^{km} \left( \frac{\partial}{\partial x^1} \frac{\partial \tilde{g}_{km}}{\partial x^1} \right) + \text{s.t.},$$

where  $e(k, m) = 2 - \delta_{km}$ . Thus the principal symbol of  $J_v(t)$  in  $\mathcal{I}^{m_0+2}(\{0\})$  is non-vanishing and  $J_v(t)$  is not a smooth function. Thus in this case  $\partial_{\tilde{\varepsilon}}^{\alpha_0} ((\Psi_{\tilde{\varepsilon}})_* g_{\tilde{\varepsilon}})|_{\tilde{\varepsilon}=0}$  is not smooth.

Next, we consider the case (b). Assume that there is  $\ell$  such that the principal symbol of the field  $\tilde{\phi}_{\ell}^{(\alpha_0)}$  is non-vanishing. As  $\partial_t \tilde{\gamma}(t) \in \mathcal{I}^{m_0}(\{0\})$ , we see that  $\tilde{\gamma}(t) \in \mathcal{I}^{m_0-1}(\{0\})$ . Then as  $\phi_{\ell}^{\tilde{\varepsilon}}$  are scalar fields,

$$j_{\ell}(t) = \partial_{\tilde{\varepsilon}}^{\alpha_0} \left( \phi_{\ell}^{\tilde{\varepsilon}}(\gamma_{\tilde{\varepsilon}}(t)) \right) \Big|_{\tilde{\varepsilon}=0} = \tilde{\phi}_{\ell}(\gamma(t)) + \text{s.t.},$$

where  $j_{\ell} \in \mathcal{I}^{m_0}(\{0\})$  and the smoother terms (*s.t.*) are in  $\mathcal{I}^{m_0-1}(\{0\})$ . Thus in the case (b),  $j_{\ell}(t)$  is not smooth and hence both  $\partial_{\tilde{\varepsilon}}^{\alpha_0} ((\Psi_{\tilde{\varepsilon}})_* g_{\tilde{\varepsilon}})|_{\tilde{\varepsilon}=0}$  and  $\partial_{\tilde{\varepsilon}}^{\alpha_0} ((\Psi_{\tilde{\varepsilon}})_* \phi_{\ell}^{\tilde{\varepsilon}})|_{\tilde{\varepsilon}=0}$  cannot be smooth.  $\square$

We use now the results above to detect singularities in normal coordinates. We say that the *interaction condition* (I) is satisfied for  $y \in U_{\hat{g}}$  with light-like vectors  $(\vec{x}, \vec{\xi}) = ((x_j, \xi_j))_{j=1}^4$  and  $t_0 \geq 0$ , with parameters  $(q, \zeta, t)$ , if

(I) There exist  $q \in \bigcap_{j=1}^4 \gamma_{x_j(t_0), \xi_j(t_0)}((0, \mathbf{t}_j))$ ,  $\mathbf{t}_j = \rho(x_j(t_0), \xi_j(t_0))$ ,  $\zeta \in L^+(M_0, \hat{g})$  and  $t \geq 0$  such that  $y = \gamma_{q, \zeta}(t)$ .

For  $\mathbf{f}_1 \in \mathcal{I}_C^{n+1}(Y((x_1, \xi_1); t_0, s_0))$  the wave  $\mathbf{Qf}_1 = \mathcal{M}^{(1)}$  is a solution of the linear wave equation. Thus, when  $\mathbf{f}_1$  runs through the set  $\mathcal{I}_C^{n+1}(Y((x_1, \xi_1); t_0, s_0))$  the union of the sets  $\text{WF}(\mathbf{Qf}_1)$  is the manifold  $\Lambda((x_1, \xi_1); t_0, s_0)$ . Thus,  $\mathcal{D}(\hat{g}, \hat{\phi}, \varepsilon)$  determines  $\Lambda((x_1, \xi_1); t_0, s_0) \cap T^*U_{\hat{g}}$ . In particular, using these data we can determine the geodesic segments  $\gamma_{x_1, \xi_1}(\mathbb{R}_+) \cap U_{\hat{g}}$  for all  $x_1 \in U_{\hat{g}}$ ,  $\xi_1 \in L^+M_0$ .

Below, in  $TM_0$  we use the Sasaki metric corresponding to  $\hat{g}^+$ . Moreover, let  $s' \in (s_- + r_2, s_+)$ ,  $0 < r_2 < r_1$  and  $B_j \subset U_{\hat{g}}$  be open sets such that, cf. (10) and (49), for some  $r'_0 \in (0, r_0)$  we have

$$(111) \quad B_j \subset \subset I_{\hat{g}}(\mu_{\hat{g}}(s' - r_2), \mu_{\hat{g}}(s')), \text{ and} \\ \text{for all } g' \in \mathcal{V}(r'_0), B_j \cap J_{g'}^+(B_k) = \emptyset \text{ for all } j \neq k.$$

We say that  $y \in U_{\hat{g}}$  satisfies the *singularity detection condition* (D) with light-like directions  $(\vec{x}, \vec{\xi})$  and  $t_0, \hat{s} > 0$  if

(D) For any  $s, s_0 \in (0, \hat{s})$  and  $j = 1, 2, 3, 4$  there are  $(x'_j, \xi'_j)$  in the  $s$ -neighborhood of  $(x_j, \xi_j)$ ,  $(2s)$ -neighborhoods  $B_j$  of  $x_j$ , in  $(U_{\hat{g}}, \hat{g}^+)$ , satisfying (111), and sources  $\mathbf{f}_j \in \mathcal{I}_C^{n+1}(Y((x'_j, \xi'_j); t_0, s_0))$  having the LS property (14) in  $C^{s_1}(M_0)$  with a family  $\mathcal{F}_j(\varepsilon)$ . This family is supported in  $B_j$  and satisfies  $\partial_{\varepsilon} \mathcal{F}_j(\varepsilon)|_{\varepsilon=0} = \mathbf{f}_j$ . Moreover, let  $u_{\tilde{\varepsilon}}$  be the solution of (8) with the source  $\mathcal{F}_{\tilde{\varepsilon}} = \sum_{j=1}^4 \mathcal{F}_j(\varepsilon_j)$  and  $\mu_{\tilde{\varepsilon}}([-1, 1])$  be observation

geodesics with  $y = \mu_0(\tilde{r})$ , and  $\Psi_{\tilde{\varepsilon}}$  be the normal coordinates associated to  $\mu_{\tilde{\varepsilon}}$  at  $\mu_{\tilde{\varepsilon}}(\tilde{r})$ . Then  $\partial_{\tilde{\varepsilon}}^{\alpha_0}((\Psi_{\tilde{\varepsilon}})_*g_{\tilde{\varepsilon}})|_{\tilde{\varepsilon}=0}$  or  $\partial_{\tilde{\varepsilon}}^{\alpha_0}((\Psi_{\tilde{\varepsilon}})_*\phi_{\tilde{\ell}}^{\tilde{\varepsilon}})|_{\tilde{\varepsilon}=0}$  is not  $C^\infty$ -smooth near  $0 = \Psi_0(y)$ .

**Lemma 4.2.** *Let  $(\vec{x}, \vec{\xi})$ , and  $\mathbf{t}_j$  with  $j = 1, 2, 3, 4$  and  $t_0 > 0$  satisfy (68)-(69). Let  $t_0, \hat{s} > 0$  be sufficiently small and assume that  $y \in \mathcal{V}((\vec{x}, \vec{\xi}), t_0) \cap U_{\hat{g}}$  satisfies  $y \notin \mathcal{Y}((\vec{x}, \vec{\xi}); t_0, \hat{s}) \cup \bigcup_{j=1}^4 \gamma_{x_j, \xi_j}(\mathbb{R})$ . Then*

(i) *If  $y$  does not satisfy condition (I) with  $(\vec{x}, \vec{\xi})$  and  $t_0$ , then  $y$  does not satisfy condition (D) with  $(\vec{x}, \vec{\xi})$  and  $t_0, \hat{s} > 0$ .*

(ii) *Assume  $y \in U_{\hat{g}}$  satisfies condition (I) with  $(\vec{x}, \vec{\xi})$  and  $t_0$  and parameters  $q, \zeta$ , and  $0 < t < \rho(q, \zeta)$ . Then  $y$  satisfies condition (D) with  $(\vec{x}, \vec{\xi}), t_0$ , and any sufficiently small  $\hat{s} > 0$ .*

(iii) *Using the data set  $\mathcal{D}(\hat{g}, \hat{\phi}, \varepsilon)$  we can determine whether the condition (D) is valid for the given point  $y \in W_{\hat{g}}$  with the parameters  $(\vec{x}, \vec{\xi}), \hat{s}$ , and  $t_0$  or not.*

**Proof.** (i) If  $y \notin \mathcal{Y}((\vec{x}, \vec{\xi}); t_0, \hat{s}) \cup \bigcup_{j=1}^4 \gamma_{x_j, \xi_j}(\mathbb{R})$ , the same condition holds also for  $(\vec{x}', \vec{\xi}')$  close to  $(\vec{x}, \vec{\xi})$ . Thus Prop. 3.3 and Lemma 4.1 imply that (i) holds.

(ii) Let  $\mu_{\tilde{\varepsilon}}([-1, 1])$  be observation geodesics with  $y = \mu_0(\tilde{r})$  and  $\Psi_{\tilde{\varepsilon}}$  be the normal coordinates associated to  $\mu_{\tilde{\varepsilon}}([-1, 1])$  at  $\mu_{\tilde{\varepsilon}}(\tilde{r})$ . Our aim is to show that there is a source  $\mathbf{f}_{\tilde{\varepsilon}}$  described in (D) such that  $\partial_{\tilde{\varepsilon}}^4((\Psi_{\tilde{\varepsilon}})_*g_{\tilde{\varepsilon}})|_{\tilde{\varepsilon}=0}$  or  $\partial_{\tilde{\varepsilon}}^4((\Psi_{\tilde{\varepsilon}})_*\phi_{\tilde{\varepsilon}})|_{\tilde{\varepsilon}=0}$  is not  $C^\infty$ -smooth at  $y = \gamma_{q, \zeta}(t)$ .

Let  $\eta = \partial_t \gamma_{q, \zeta}(t)$  and denote  $(y, \eta) = (x_5, \xi_5)$ . Let  $t_j > 0$  be such that  $\gamma_{x_j, \xi_j}(t_j) = q$  and denote  $b_j = (\partial_t \gamma_{x_j, \xi_j}(t_j))^b$ ,  $j = 1, 2, 3, 4$ . Also, let us denote  $b_5 = \zeta^b$  and  $t_5 = -t$ , so that  $q = \gamma_{x_5, \xi_5}(t_5)$  and  $b_5 = (\dot{\gamma}_{x_5, \xi_5}(t_5))^b$ .

By assuming that  $V \subset U_{\hat{g}}$  is a sufficiently small neighborhood of  $y$ , we have that  $S := \mathcal{L}_{\hat{g}}^+(q) \cap V$  is a smooth 3-submanifold, as  $t < \rho(q, \zeta)$ .

Let  $u_\tau = \mathbf{Q}_{\hat{g}}^* F_\tau$  be a gaussian beam, produced by a source  $F_\tau(x) = F_\tau(x; x_5, \xi_5)$  and function  $h(x)$  given in (60). Then we can use the techniques of [54, 75], see also [3, 53], to obtain a result analogous to Lemma 3.1 for the propagation of singularities along the geodesic  $\gamma_{q, \zeta}([0, t])$ : We have that when  $h(x_5) = H$  and  $w$  is the principal symbol of  $u_\tau$  at  $(q, b_5)$ , then  $w = (R_{(5)})^* H$ , where  $R_{(5)}$  is a bijective linear map.

Let  $s \in (0, \hat{s})$  be sufficiently small and denote  $b'_5 = b_5$ . Using Propositions 3.3 and 3.4, we see that there are  $(b'_j)^\sharp \in L_q^+ M_0$ ,  $j = 1, 2, 3, 4$ , in the  $s$ -neighborhoods of  $b_j^\sharp \in L_q^+ M_0$ , vectors  $v^{(j)} \in \mathbb{R}^{10+L}$ ,  $j \in \{2, 3, 4\}$ , linearly independent vectors  $v_p^{(5)} \in \mathbb{R}^{10+L}$ ,  $p = 1, 2, 3, 4, 5, 6$ , and linearly independent vectors  $v_r^{(1)} \in \mathbb{R}^{10+L}$ ,  $r = 1, 2, 3, 4, 5, 6$ , that have the following properties:

(a) All  $v^{(j)}$ ,  $j = 2, 3, 4$ , and  $v_r^{(1)}$ ,  $r = 1, 2, \dots, 6$  satisfy the harmonicity conditions for the symbols (44) with the covector  $\xi$  being  $b'_j$  and  $b'_1$ , respectively.



(b) Let  $X_1 = \text{span}(\{v_r^{(1)}; r = 1, 2, 3, \dots, 6\})$  and  $X_5 = \text{span}(\{v_p^{(5)}; p = 1, 2, 3, \dots, 6\})$ . If  $v^{(5)} \in X_5 \setminus \{0\}$  then there exists a vector  $v^{(1)} \in X_1$  such that for  $\mathbf{v} = (v^{(1)}, v^{(2)}, v^{(3)}, v^{(4)}, v^{(5)})$  and  $\mathbf{b}' = (b'_j)_{j=1}^5$  we have  $\mathcal{G}(\mathbf{v}, \mathbf{b}') \neq 0$ .

Let  $Y_5 := \text{sym}(T_y S \otimes T_y S) \times \mathbb{R}^L$ . Since the codimension of  $((R_{(5)})^*)^{-1}Y_5$  in  $\text{sym}(T_q M \otimes T_q M) \times \mathbb{R}^L$  is 4 and the dimension of  $X_5$  is 6, we see that dimension of the intersection  $Z_5 = X_5 \cap ((R_{(5)})^*)^{-1}Y_5$  is at least 2. Thus there exist  $v^{(j)} \neq 0$ ,  $j = 1, 2, 3, 4, 5$  that satisfy the above conditions (a) and (b) and  $v^{(5)} \in Z_5$ . Let  $H^{(5)} = ((R_{(5)})^*)^{-1}v^{(5)} \in ((R_{(5)})^*)^{-1}Z_5$ .

Let  $s_0 \in (0, \widehat{s})$  and  $x'_j = \gamma_{q, (b'_j)^\#}(-t_j)$ , and  $\xi'_j = \partial_t \gamma_{q, (b'_j)^\#}(-t_j)$ ,  $j = 1, 2, 3, 4$ . We denote  $(\vec{x}', \vec{\xi}') = ((x'_j, \xi'_j))_{j=1}^4$ .

Moreover, let  $B_j$  be neighborhoods of  $x'_j$  satisfying (111). Then, by condition  $\mu$ -LS there are sources  $\mathbf{f}_j \in \mathcal{I}_C^{n+1}(Y((x'_j, \xi'_j); t_0, s_0))$  having the LS property (14) in  $C^{s_1}(M_0)$  with some family  $\mathcal{F}_j(\varepsilon)$  supported in  $B_j$  such that the principal symbols of  $\mathbf{f}_j$  at  $(x'_j(t_0), (\xi'_j(t_0))^b)$  are equal to  $w^{(j)} = R_j^{-1}v^{(j)}$ , where  $R_j = R_{(1)}(q, b'_j; x'_j(t_0), (\xi'_j(t_0))^b)$  are defined by formula (46). Then the principal symbols of  $\mathbf{Q}_{\widehat{g}}\mathbf{f}_j$  at  $(q, b'_j)$  are equal to  $v^{(j)}$ . Let  $u_\varepsilon = (g_\varepsilon - \widehat{g}, \phi_\varepsilon - \widehat{\phi})$  be the solution of (25) corresponding to  $\mathcal{F}_\varepsilon = \sum_{j=1}^4 \mathcal{F}_j(\varepsilon_j)$  with  $\partial_{\varepsilon_j} \mathcal{F}_j(\varepsilon_j)|_{\varepsilon_j=0} = \mathbf{f}_j$ . When  $s_0$  is small enough,  $\mathcal{M}^{(4)} = \partial_\varepsilon^4 u_\varepsilon|_{\varepsilon=0}$  is a conormal distribution in the neighborhood  $V$  of  $y$  and  $\mathcal{M}^{(4)}|_V \in \mathcal{I}(S)$ . By Propositions 3.3 and 3.4, the inner product  $\langle F_\tau, \mathcal{M}^{(4)} \rangle_{L^2(U_{\widehat{g}})}$  is not of order  $O(\tau^{-N})$  for all  $N > 0$ , so that  $\mathcal{M}^{(4)}$  is not smooth near  $y$ , see [43]. Let  $h(x) \in C_0^\infty(U_{\widehat{g}})$  be such that  $h(x_5) = H^{(5)}$ . The above implies that the principal symbol of the function  $x \mapsto \langle h(x), \mathcal{M}^{(4)}(x) \rangle_{\mathcal{B}^L}$  is not vanishing at  $(y, \eta^b)$ . Thus the principal symbols of the functions (110) are not vanishing and as  $h(x_5) \in ((R_{(5)})^*)^{-1}Z_5$ , conditions (ii) in Lemma 4.1 are satisfied. Thus either  $\partial_\varepsilon^4((\Psi_\varepsilon)_* g_\varepsilon)|_{\varepsilon=0}$  or  $\partial_\varepsilon^4((\Psi_\varepsilon)_* \phi_\varepsilon)|_{\varepsilon=0}$  is not  $C^\infty$ -smooth at  $0 = \Psi_0(y)$ . Thus, condition (D) is valid for  $y$ . This proves (ii).

(iii) Below, we will assume that  $\mathcal{D}(\widehat{g}, \widehat{\phi}, \widehat{\varepsilon})$  is given with  $\widehat{\varepsilon} > 0$ .

To verify (D), we need to consider the solution  $v_\varepsilon = (g_\varepsilon, \phi_\varepsilon)$  in the wave gauge coordinates. For a general element  $[(U_g, g|_{U_g}, \phi|_{U_g}, F|_{U_g})] \in \mathcal{D}(\widehat{g}, \widehat{\phi}, \varepsilon)$  we encounter the difficulty that we do not know the wave gauge coordinates in the set  $(U_g, g)$ . However, we construct the wave map (i.e. the wave gauge) coordinates for sources  $F$  of a special form. Below, we give the proof in several steps.

**Step 1.** Let  $s_- \leq s' \leq s_0$  and  $r_2 \in (0, r_1)$  be so small that  $I_{\widehat{g}}(\mu_{\widehat{g}}(s' - r_2), \mu_{\widehat{g}}(s')) \subset W_{\widehat{g}}$ . Assume that  $\widehat{\varepsilon}$  is small enough and that we are given an element  $[(U_g, g|_{U_g}, \phi|_{U_g}, F|_{U_g})] \in \mathcal{D}(\widehat{g}, \widehat{\phi}, \widehat{\varepsilon})$  such that  $F$  is supported in  $I_g(\mu_g(s' - r_2), \mu_g(s')) \subset W_g$ . As we know  $(U_g, g|_{U_g})$ , assuming that  $\widehat{\varepsilon}$  is small enough, we can find the wave map  $\Psi : I_g(\mu_g(s' - r_2), \mu_g(s')) \rightarrow U_{\widehat{g}}$  by solving (5) in  $I_g^-(\mu_g(s')) \cap U_g$ . Then

$\Psi$  is the restriction of the wave map  $f$  solving (5)-(6). Then we can determine in the wave gauge coordinates  $\Psi$  the source  $\Psi_*F$  and the solution  $(\Psi_*g, \Psi_*\phi)$  in  $\Psi(I_g^-(\mu_g(s')) \cap U_g)$ . Observe that the function  $\Psi_*F$  vanishes on  $U_{\widehat{g}}$  outside the set  $\Psi(I_g^-(\mu_g(s')) \cap U_g)$ . Thus the source  $f_*F$  is determined in the wave gauge coordinates  $f$  in the whole set  $U_{\widehat{g}}$ , where  $f$  solves (5)-(6). This construction can be done for all equivalence classes  $[(U_g, g|_{U_g}, \phi|_{U_g}, F|_{U_g})]$  in  $\mathcal{D}(\widehat{g}, \widehat{\phi}, \widehat{\varepsilon})$  such that  $F$  is supported in  $I_g(\mu_g(s' - r_2), \mu_g(s')) \subset W_g$ . Next, we consider sources on the set  $U_{\widehat{g}}$  endowed with the background metric  $\widehat{g}$ . Let  $\mathcal{F}$  be a source function on the set  $U_{\widehat{g}}$  such that  $J_{\widehat{g}}^-(\text{supp}(\mathcal{F})) \cap J_{\widehat{g}}^+(\text{supp}(\mathcal{F})) \subset I_{\widehat{g}}(\mu_{\widehat{g}}(s' - r_2), \mu_{\widehat{g}}(s'))$  and assume that  $\mathcal{F}$  is sufficiently small in the  $C^{s_1}$ -norm. Then, using the above considerations, we can determine the unique equivalence class  $[(U_g, g|_{U_g}, \phi|_{U_g}, F|_{U_g})]$  in  $\mathcal{D}(\widehat{g}, \widehat{\phi}, \widehat{\varepsilon})$  for which  $f_*F$  is equal to  $\mathcal{F}$ . In this case we say that the equivalence class corresponds to  $\mathcal{F}$  in the wave gauge coordinates and that  $\mathcal{F}$  is an admissible source.

**Step 2.** Let  $k_1 \geq 8$ ,  $s_1 \geq k_1 + 5$ , and  $n \in \mathbb{Z}_+$  be large enough. Let  $(x'_j, \xi'_j)$  be covectors in an  $s$ -neighborhood of  $(x_j, \xi_j)$  and  $B_j$  be neighborhoods of  $x'_j$  satisfying (111). Consider then  $\mathbf{f}_j \in \mathcal{I}_C^{n+1}(Y((x'_j, \xi'_j); t_0, s_0))$ , and a family  $\mathcal{F}_j(\varepsilon_j) \in C^{s_1}(M_0)$ ,  $\varepsilon_j \in [0, \varepsilon_0)$  of functions supported in  $B_j$  that depend smoothly on  $\varepsilon_j$ . Moreover, assume that  $\partial_{\varepsilon_j} \mathcal{F}_j(\varepsilon_j)|_{\varepsilon_j=0} = \mathbf{f}_j$ . Then, we can use step 1 to test if all sources  $\mathbf{f}_j$  and  $\mathcal{F}_j(\varepsilon_j)$ ,  $\varepsilon_j \in [0, \varepsilon_0)$  are admissible.

**Step 3.** Next, assume that  $\mathbf{f}_j$  and  $\mathcal{F}_j(\varepsilon_j)$ ,  $\varepsilon_j \in [0, \varepsilon_0)$ ,  $j = 1, 2, 3, 4$  are admissible and are compactly supported in neighborhoods  $B_j$  of  $x'_j$  satisfying (111).

Let us next consider  $\vec{a} = (a_1, a_2, a_3, a_4) \in (-1, 1)^4$  and define  $\mathcal{F}(\vec{\varepsilon}, \vec{a}) = \sum_{j=1}^4 \mathcal{F}_j(a_j \vec{\varepsilon})$ . Let  $(g_{\vec{\varepsilon}, \vec{a}}, \phi_{\vec{\varepsilon}, \vec{a}})$  be the solution of (8) with the source  $\mathcal{F}(\vec{\varepsilon}, \vec{a})$ . By (111),  $B_j \cap J_{g_{\vec{\varepsilon}, \vec{a}}}^+(B_k) = \emptyset$  for  $j \neq k$  and  $\vec{\varepsilon}$  small enough. Hence we have that for sufficiently small  $\vec{\varepsilon}$  the conservation law (9) is satisfied for  $(g_{\vec{\varepsilon}, \vec{a}}, \phi_{\vec{\varepsilon}, \vec{a}})$  in the set  $Q = (M_0 \setminus J_{g_{\vec{\varepsilon}, \vec{a}}}^+(p^-)) \cup \bigcup_{j=1}^4 J_{g_{\vec{\varepsilon}, \vec{a}}}^-(B_j)$ , see Fig. 6(Left). Solving the Einstein equations (8) in a neighborhood of the closure of the complement of the set  $Q$ , where the source  $\mathcal{F}(\vec{\varepsilon}, \vec{a})$  vanishes, we see that the conservation law (9) is satisfied in the whole set  $M_0$ , see [14, Sec. III.6.4.1]. Hence  $\sum_{j=1}^4 a_j \mathbf{f}_j$  has the LS property (14) in  $C^{s_1}(M_0)$  with the family  $\mathcal{F}(\vec{\varepsilon}, \vec{a})$ . When  $\vec{\varepsilon} > 0$  is small enough, the sources  $\mathcal{F}(\vec{\varepsilon}, \vec{a})$  are admissible.

**Step 4.** Let  $v_{\vec{\varepsilon}, \vec{a}} = (g_{\vec{\varepsilon}, \vec{a}}, \phi_{\vec{\varepsilon}, \vec{a}})$  be the solutions of the Einstein equations (8) with the source  $\mathcal{F}(\vec{\varepsilon}, \vec{a})$ .

Using Step 1, we can find for all  $\vec{a} \in (-1, 1)^4$  and sufficiently small  $\vec{\varepsilon}$  the equivalence classes  $[(U_{\vec{\varepsilon}, \vec{a}}, g_{\vec{\varepsilon}, \vec{a}}|_{U_{\vec{\varepsilon}, \vec{a}}}, \phi_{\vec{\varepsilon}, \vec{a}}|_{U_{\vec{\varepsilon}, \vec{a}}}, F(\vec{\varepsilon}, \vec{a})|_{U_{\vec{\varepsilon}, \vec{a}}})]$  correspond to  $\mathcal{F}(\vec{\varepsilon}, \vec{a})$  in the wave guide coordinates. Then we can determine, using the normal coordinates  $\Psi_{\vec{\varepsilon}, \vec{a}}$  associated to the observation geodesics  $\mu_{\vec{\varepsilon}, \vec{a}}$  and the solution  $v_{\vec{\varepsilon}, \vec{a}}$ , the function  $(\Psi_{\vec{\varepsilon}, \vec{a}})_* v_{\vec{\varepsilon}, \vec{a}}$ . Also, we can compute the derivatives of this function with respect to  $\vec{\varepsilon}$  and  $a_j$ .

Observe that  $\partial_{\vec{\varepsilon}}^4 f(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)|_{\vec{\varepsilon}=0} = \partial_{\vec{a}}^4 (\partial_{\vec{\varepsilon}}^4 f(a_1 \tilde{\varepsilon}, a_2 \tilde{\varepsilon}, a_3 \tilde{\varepsilon}, a_4 \tilde{\varepsilon})|_{\vec{\varepsilon}=0})|_{\vec{a}=0}$  so that  $\mathcal{M}^{(4)} = \partial_{a_1} \partial_{a_2} \partial_{a_3} \partial_{a_4} \partial_{\vec{\varepsilon}}^4 v_{\vec{\varepsilon}, \vec{a}}|_{\vec{\varepsilon}=0, \vec{a}=0}$ , where  $\mathcal{M}^{(4)}$  given in (50). Thus, using the solutions  $v_{\vec{\varepsilon}, \vec{a}}$  we determine if the function  $\partial_{\vec{\varepsilon}}^4 ((\Psi_{\vec{\varepsilon}})_* v_{\vec{\varepsilon}})|_{\vec{\varepsilon}=0}$ , corresponding to the source  $f_{\vec{\varepsilon}} = \sum_{j=1}^4 \varepsilon_j \mathbf{f}_j$  is singular, where  $\Psi_{\vec{\varepsilon}}$  are the the normal coordinates associated to any observation geodesic. Thus we can verify if the condition (D) holds.  $\square$

## 5. DETERMINATION OF EARLIEST LIGHT OBSERVATION SETS

Below, we use only the metric  $\widehat{g}$  and often denote  $\widehat{g} = g$ ,  $U = U_{\widehat{g}}$ .

Our next aim is to consider the global problem of constructing the set of the earliest light observations of all points  $q \in J(p^-, p^+)$ . To this end, we need to handle the technical problem that in the set  $\mathcal{Y}((\vec{x}, \vec{\xi}))$  we have not analyzed if we observe singularities or not. Also, we have not analyzed the waves in the set where singularities caused by caustics or their interactions may appear, see (69). To avoid these difficulties, we define next the sets  $\mathcal{S}_{reg}((\vec{x}, \vec{\xi}), t_0)$  of points near which we observe singularities in a 3-dimensional set.

**Definition 5.1.** *Let  $(\vec{x}, \vec{\xi}) = ((x_j, \xi_j))_{j=1}^4$  be a collection of light-like vectors with  $x_j \in U_{\widehat{g}}$  and  $t_0 > 0$ . We define  $\mathcal{S}((\vec{x}, \vec{\xi}), t_0)$  be the set of those  $y \in U_{\widehat{g}}$  that satisfies the property (D) with  $(\vec{x}, \vec{\xi})$  and  $t_0$  and some  $\widehat{s} > 0$ . Moreover, let  $\mathcal{S}_{reg}((\vec{x}, \vec{\xi}), t_0)$  be the set of the points  $y_0 \in \mathcal{S}((\vec{x}, \vec{\xi}), t_0)$  having a neighborhood  $W \subset U_{\widehat{g}}$  such that the intersection  $W \cap \mathcal{S}((\vec{x}, \vec{\xi}), t_0)$  is a non-empty smooth 3-dimensional submanifold. We denote (see (18) and Def. 2.4)*

$$(112) \quad \mathcal{S}^{cl}((\vec{x}, \vec{\xi}), t_0) = cl(\mathcal{S}_{reg}((\vec{x}, \vec{\xi}), t_0)) \cap U_{\widehat{g}},$$

$$(113) \quad \mathcal{S}_e((\vec{x}, \vec{\xi}), t_0) = \bigcup_{(z, \eta) \in \mathcal{U}_{z_0, \eta_0}} \mathbf{e}_{z, \eta}(\mathcal{S}^{cl}((\vec{x}, \vec{\xi}), t_0)).$$

The data  $\mathcal{D}(\widehat{g}, \widehat{\phi}, \varepsilon)$  determines the set  $\mathcal{S}_e((\vec{x}, \vec{\xi}), t_0)$ . Below, we fix  $t_0$  to be  $t_0 = 4\kappa_1$ , cf. Lemma 2.3.

If the set  $\bigcap_{j=1}^4 \gamma_{x_j, \xi_j}([t_0, \infty))$  is non-empty we denote its earliest point by  $Q((\vec{x}, \vec{\xi}), t_0)$ . If such intersection point does not exists, we define  $Q((\vec{x}, \vec{\xi}), t_0)$  to be the empty set. Next we consider the relation of  $\mathcal{S}_e((\vec{x}, \vec{\xi}), t_0)$  and  $\mathcal{E}_U(q)$ ,  $q = Q((\vec{x}, \vec{\xi}), t_0)$ , see Def. 2.4.

**Lemma 5.2.** *Let  $(\vec{x}, \vec{\xi})$ ,  $j = 1, 2, 3, 4$  and  $t_0 > 0$  satisfy (68)-(69) and assume that  $\vartheta_1$  in (68) and Lemma 2.3 is so small that for all  $j \leq 4$ ,  $x_j \in I(\widehat{\mu}(s_1), \widehat{\mu}(s_2)) \subset W_{\widehat{g}}$  with some  $s_1, s_2 \in (s_-, s_+)$ .*

*Let  $\mathcal{V} = \mathcal{V}((\vec{x}, \vec{\xi}), t_0)$  be the set defined in (69). Then*

*(i) Assume that  $y \in \mathcal{V} \cap U_{\widehat{g}}$  satisfies the condition (I) with  $(\vec{x}, \vec{\xi})$  and  $t_0$  and parameters  $q, \zeta$ , and  $t$  such that  $0 \leq t \leq \rho(q, \zeta)$ . Then  $y \in \mathcal{S}^{cl}((\vec{x}, \vec{\xi}), t_0)$ .*

(ii) Assume  $y \in \mathcal{V} \cap U_{\widehat{g}}$  does not satisfy condition (I) with  $(\vec{x}, \vec{\xi})$  and  $t_0$ . Then  $y \notin \mathcal{S}^{cl}((\vec{x}, \vec{\xi}), t_0)$ .

(iii) If  $Q((\vec{x}, \vec{\xi}), t_0) \neq \emptyset$  and  $q = Q((\vec{x}, \vec{\xi}), t_0) \in \mathcal{V}$ , we have

$$\mathcal{S}_e((\vec{x}, \vec{\xi}), t_0) = \mathcal{E}_U(q) \subset \mathcal{V}.$$

Otherwise, if  $Q((\vec{x}, \vec{\xi}), t_0) \cap \mathcal{V} = \emptyset$ , then  $\mathcal{S}_e((\vec{x}, \vec{\xi}), t_0) \cap \mathcal{V} = \emptyset$ .

**Proof.** (i) Assume first that  $y$  is not in  $\mathcal{Y}((\vec{x}, \vec{\xi}))$  and  $t < \rho(q, \zeta)$ . Then the assumptions in (i) and Lemma 4.2 (ii) and (iii) imply that  $y \in \mathcal{S}^{cl}((\vec{x}, \vec{\xi}), t_0)$ . Consider next a general point  $y$  satisfying the assumptions in (i) and let  $q = Q((\vec{x}, \vec{\xi}), t_0)$ . Then  $y \in \mathcal{E}_U(q)$ . Since  $\rho(x, \xi)$  is lower semi-continuous and the set  $\mathcal{E}_U(q) \setminus \mathcal{Y}((\vec{x}, \vec{\xi}))$  is dense in  $\mathcal{E}_U(q)$ , and  $y$  is a limit point of points  $y_n \notin \mathcal{Y}((\vec{x}, \vec{\xi}))$  that satisfy the assumptions in (i). Hence  $y \in \mathcal{S}^{cl}((\vec{x}, \vec{\xi}), t_0)$ .

The claim (ii) follows from Lemma 4.2 (i).

(iii) Suppose  $q = Q((\vec{x}, \vec{\xi}), t_0) \in \mathcal{V}$  and  $y \in \mathcal{E}_U(q) \setminus \bigcup_{j=1}^4 \gamma_{x_j, \xi_j}([t_0, \infty))$ . Let  $\gamma_{q, \eta}([0, l])$  be a light-like geodesic that is the longest causal geodesic from  $q$  to  $y$  with  $l \leq \rho(q, \eta)$ , and let  $p_j = \gamma_{x_j, \xi_j}(t_0 + \mathbf{t}_j)$ ,  $\mathbf{t}_j = \rho(x_j(t_0), \xi_j(t_0))$ , be the first cut point on the geodesic  $\gamma_{x_j, \xi_j}([t_0, \infty))$ . To show that  $y$  is in  $\mathcal{V}$ , we assume the opposite,  $y \notin \mathcal{V}$ . Then for some  $j$  there is a causal geodesic  $\gamma_{p_j, \theta_j}([0, l_j])$  from  $p_j$  to  $y$ . Now we can use a short-cut argument: Let  $q' = \gamma_{x_j, \xi_j}(t')$ . As  $q \in \mathcal{V}$ , we have  $t' < t_0 + \mathbf{t}_j$ . Moreover, as  $y \notin \gamma_{x_j, \xi_j}([t_0, \infty))$ , the union of the geodesic  $\gamma_{x_j, \xi_j}([t', t_0 + \mathbf{t}_j])$  from  $q'$  to  $p_j$  and  $\gamma_{p_j, \theta_j}([0, l_j])$  from  $p_j$  to  $y$  does not form a light-like geodesic and thus  $\tau(q', y) > 0$ . As  $y \in \mathcal{E}_U(q)$ , this is not possible. Hence  $y \in \mathcal{V}$ . Thus by (i),  $y \in \mathcal{S}^{cl}((\vec{x}, \vec{\xi}), t_0)$  and hence  $\mathcal{E}_U(q) \setminus (\bigcup_{j=1}^4 \gamma_{x_j, \xi_j}([t_0, \infty))) \subset \mathcal{S}^{cl}((\vec{x}, \vec{\xi}), t_0)$ . Since the set  $\mathcal{E}_U(q) \setminus (\bigcup_{j=1}^4 \gamma_{x_j, \xi_j}([t_0, \infty)))$  is dense in the closed set  $\mathcal{E}_U(q)$ , the above shows that  $\mathcal{E}_U(q) \subset \mathcal{S}^{cl}((\vec{x}, \vec{\xi}), t_0)$ . Also by (ii),  $\mathcal{S}^{cl}((\vec{x}, \vec{\xi}), t_0) \subset \mathcal{L}^+(q)$ . Using Definition 2.4 and (113), we see that  $\mathcal{S}_e((\vec{x}, \vec{\xi}), t_0) = \mathcal{E}_U(q)$ .

On the other hand, if  $Q((\vec{x}, \vec{\xi}), t_0) \cap \mathcal{V} = \emptyset$ , we can apply (ii) for all  $y \in \mathcal{V} \cap U$  and see that  $\mathcal{S}^{cl}((\vec{x}, \vec{\xi}), t_0) \cap \mathcal{V} = \emptyset$ . This proves (iii).  $\square$

Let

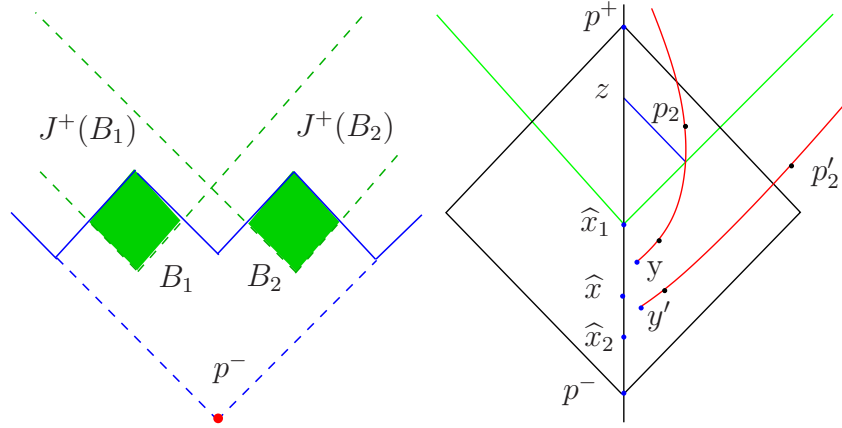
$$(114) \quad \mathcal{K}_{t_0} = \{x \in U_{\widehat{g}} \ ; \ x = \gamma_{\widehat{x}, \widehat{\xi}}(r), \ \widehat{x} = \widehat{\mu}(s), \ s \in [s^-, s^+], \\ \xi \in L_{\widehat{x}}^+ M_0, \ \|\xi\|_{\widehat{g}^+} = 1, \ r \in [0, 2t_0)\}.$$

Recall that we are given the set  $U_{\widehat{g}}$  that is determined via the parameter  $\widehat{h} > 0$ . By using compactness of  $\widehat{\mu}([s_-, s_+])$ , the continuity of  $\tau(x, y)$ , and the existence of convex neighborhoods [74, Prop. 5.7], we can determine the sets  $\mathcal{E}_U(\mathcal{K}_{t_0} \cap J^+(\widehat{\mu}(s)))$  for some  $t'_0 > 0$  and all  $s \in [s_-, s_+]$ . Here, recall that  $\mathcal{E}_U(V) = \{\mathcal{E}_U(q); q \in V\} \subset 2^U$ . Thus, by making  $\widehat{h}$  and  $t_0 = 4\kappa_1$  smaller, we may assume below that we are given

the sets  $\mathcal{E}_U(\mathcal{K}_{t_0} \cap J^+(\widehat{\mu}(s)))$  for  $s \in [s_-, s_+]$ . Below, we may assume that  $\vartheta_1$  is so small that  $\gamma_{y,\zeta}([0, t_0]) \cap J(p^-, p^+) \subset \mathcal{K}_{t_0}$  when  $y \in J(p^-, p^+)$ ,  $d_{g^+}(y, \widehat{\mu}) < \vartheta_1$  and  $\zeta \in L_y^+ M$ ,  $\|\zeta\|_{g^+} \leq 1 + \vartheta_1$ .

Let  $\kappa_1, \kappa_2$  be constants given as in Lemma 2.3. Let  $s_0 \in [s_-, s_+]$  be so close to  $s_+$  that  $J^+(\widehat{\mu}(s_0)) \cap J^-(p^+) \subset \mathcal{K}_{t_0}$ . Then the given data  $\mathcal{D}(\widehat{g}, \widehat{\phi}, \varepsilon)$  determines  $\mathcal{E}_U(J^+(\widehat{\mu}(s_0)) \cap J^-(p^+))$ .

Next we use a step-by-step construction: We consider  $s_1 \in (s_-, s_+)$  and assume that we are given  $\mathcal{E}_U(J^+(\widehat{x}_1) \cap J^-(p^+))$  with  $\widehat{x}_1 = \widehat{\mu}(s_1)$ . Then, let  $s_2 \in (s_1 - \kappa_2, s_1)$ . Our next aim is to find the earliest light observation sets  $\mathcal{E}_U(J^+(\widehat{x}_2) \cap J^-(p^+))$  with  $\widehat{x}_2 = \widehat{\mu}(s_2)$ . To this end we need to make the following definitions (see Fig. 6).



**FIGURE 6. Left:** The setting in the proof of Lemma 4.2. : The blue points on  $\widehat{\mu}$  are  $\widehat{x}_1 = \widehat{\mu}(s_1)$ ,  $\widehat{x}_2 = \widehat{\mu}(s_2)$ , and  $\widehat{x} = \widehat{\mu}(s)$ . The blue points  $y$  and  $y'$  are close to  $\widehat{x}$ . The set with the green boundary is  $J^+(\widehat{x}_1)$ . We consider the geodesics  $\gamma_{y,\zeta}([0, \infty))$  and  $\gamma_{y',\zeta'}([0, \infty))$ . These geodesics corresponding to the cases when the geodesic  $\gamma_{y,\zeta}([0, \infty))$  enters in  $J^-(p^+) \cap J^+(\widehat{x}_1)$ , and the case when the geodesic  $\gamma_{y',\zeta'}([0, \infty))$  does not enter this set. The point  $p_2$  is the cut point of  $\gamma_{y,\zeta}([t_0, \infty))$  and  $p'_2$  is the cut point of  $\gamma_{y',\zeta'}([t_0, \infty))$ . At the point  $z = \widehat{\mu}(\mathbb{S}(y, \zeta, s_1))$  we observe for the first time on the geodesic  $\widehat{\mu}$  that the geodesic  $\gamma_{y,\zeta}([0, \infty))$  has entered  $J^+(\widehat{x}_1)$ .

**Definition 5.3.** Let  $s_- \leq s_2 \leq s < s_1 \leq s_+$  satisfy  $s_1 < s_2 + \kappa_2$ ,  $\widehat{x}_j = \widehat{\mu}(s_j)$ ,  $j = 1, 2$ , and  $\widehat{x} = \widehat{\mu}(s)$ ,  $\widehat{\zeta} \in L_{\widehat{x}}^+ U$ ,  $\|\widehat{\zeta}\|_{g^+} = 1$ . Let  $(y, \zeta) \in L^+ U$  be in  $\vartheta_1$ -neighborhood of  $(\widehat{x}, \widehat{\zeta})$  such that  $y \in J^+(\widehat{x}_2)$  and the geodesic  $\gamma_{y,\zeta}(\mathbb{R}_+)$  does not intersect  $\widehat{\mu}$ .

Let  $r_1(y, \zeta, s_1) = \inf\{r > 0; \gamma_{y,\zeta}(r) \in J^+(\widehat{\mu}(s_1))\}$ . Also, define  $r_2(y, \zeta) = \inf\{r > 0; \gamma_{y,\zeta}(r) \in M \setminus I^-(\widehat{\mu}(s_{+2}))\}$  and  $r_0(y, \zeta, s_1) = \min(r_2(y, \zeta), r_1(y, \zeta, s_1))$ .

When  $\gamma_{y,\zeta}(\mathbb{R}_+)$  intersects  $J^+(\widehat{\mu}(s_1)) \cap J^-(p^+)$  we define

$$(115) \quad \mathbb{S}(y, \zeta, s_1) = f_{\widehat{\mu}}^+(q_0),$$

where  $q_0 = \gamma_{y,\zeta}(r_0)$  and  $r_0 = r_0(y, \zeta, s_1)$ . In the case when  $\gamma_{y,\zeta}(\mathbb{R}_+)$  does not intersect  $J^+(\widehat{\mu}(s_1)) \cap J^-(p^+)$ , we define  $\mathbb{S}(y, \zeta, s_1) = s^+$ .

We note that above  $r_2(y, \zeta)$  is finite by [74, Lem. 14.13]. Below we use the notations used in Def. 5.3. We saw in Sec. 4 that using solutions of the linearized Einstein equations we can find  $\gamma_{x,\xi} \cap U_{\widehat{g}}$  for any  $(x, \xi) \in L^+U$ . Thus we can check for given  $(x, \xi)$  if  $\gamma_{x,\xi} \cap \widehat{\mu} = \emptyset$ .

**Definition 5.4.** Let  $0 < \vartheta < \vartheta_1$  and  $\mathcal{R}_\vartheta(y, \zeta)$  be the set of  $(\vec{x}, \vec{\xi}) = ((x_j, \xi_j))_{j=1}^4$  that satisfy (i) and (ii) in formula (68) with  $\vartheta_1$  replaced by  $\vartheta$  and  $(x_1, \xi_1) = (y, \zeta)$ . We say that the set  $S \subset U_{\widehat{g}}$  is a genuine observation associated to the geodesic  $\gamma_{y,\zeta}$  if there is  $\widehat{\vartheta} > 0$  such that for all  $\vartheta \in (0, \widehat{\vartheta})$  there are  $(\vec{x}, \vec{\xi}) \in \mathcal{R}_\vartheta(y, \zeta)$  such that  $S = \mathcal{S}_e((\vec{x}, \vec{\xi}), t_0)$ .

**Lemma 5.5.** Suppose  $\max(s_-, s_1 - \kappa_2) \leq s < s_1 < s^+$  and let  $\widehat{x} = \widehat{\mu}(s)$ ,  $\widehat{x}_1 = \widehat{\mu}(s_1)$ , and  $\widehat{\zeta} \in L_{\widehat{x}}^+M$ ,  $\|\widehat{\zeta}\|_{g^+} = 1$ . Moreover, let  $(y, \zeta)$  be in a  $\vartheta_1$ -neighborhood of  $(\widehat{x}, \widehat{\zeta})$ . Assume that the geodesic  $\gamma_{y,\zeta}(\mathbb{R}_+)$  does not intersect  $\widehat{\mu}$ .

(A) Then the cut point  $p_0 = \gamma_{y(t_0), \zeta(t_0)}(\mathbf{t}_*)$ ,  $\mathbf{t}_* = \rho(y(t_0), \zeta(t_0))$  of the geodesics  $\gamma_{y(t_0), \zeta(t_0)}([0, \infty))$ , if it exists, satisfies either

$$(i) p_0 \notin J^-(\widehat{\mu}(s_{+2})),$$

or

$$(ii) r_0 = r_0(y, \zeta, s_1) < r_2(y, \zeta) \text{ and } p_0 \in I^+(\widehat{x}_1).$$

(B) There is  $\vartheta_2(y, \zeta, s_1) \in (0, \vartheta_1)$  such that if  $0 < \vartheta < \vartheta_2(y, \zeta, s_1)$ ,  $(\vec{x}, \vec{\xi}) \in \mathcal{R}_\vartheta(y, \zeta)$ , and the geodesics  $\gamma_{x_j(t_0), \xi_j(t_0)}([0, \infty))$ ,  $j \in \{1, 2, 3, 4\}$ , has a cut point  $p_j = \gamma_{x_j(t_0), \xi_j(t_0)}(\mathbf{t}_j)$ , then the following holds:

If either the point  $p_0$  does not exist or it exists and (i) holds then  $p_j \notin J^-(p^+)$ . On the other hand, if  $p_0$  exists and (ii) holds, then  $f_{\widehat{\mu}}^+(p_j) > f_{\widehat{\mu}}^+(q_0)$ , where  $q_0 = \gamma_{y,\zeta}(r_0(y, \zeta, s_1))$ .

Note that  $f_{\widehat{\mu}}^+(q_0) = \mathbb{S}(y, \zeta, s_1)$ .

**Proof.** (A) Assume that (i) does not hold, that is,  $p_0 = \gamma_{y,\zeta}(t_0 + \mathbf{t}_*) \in J^-(\widehat{\mu}(s_{+2}))$ . By Lemma 2.3 (ii) we have  $f_{\widehat{\mu}}^-(p_0) > s + 2\kappa_2 \geq s_1$  that yields  $p_0 \in I^+(\widehat{x}_1)$ . Thus, the geodesic  $\gamma_{y(t_0), \zeta(t_0)}([0, \rho(y(t_0), \zeta(t_0))])$  intersects  $J^+(\widehat{x}_1) \cap J^-(\widehat{\mu}(s_{+2}))$ . Hence the alternative (ii) holds with  $0 < r_0 < r_2(y, \zeta)$  and moreover,  $r_0 < t_0 + \rho(y(t_0), \zeta(t_0))$ .

(B) If (i) holds, the claim follows since the function  $(x, \xi) \mapsto \rho(x, \xi)$  is lower semi-continuous and  $(x, \xi, t) \mapsto \gamma_{x,\xi}(t)$  is continuous.

In the case (ii), we saw above that  $r_0 < t_0 + \rho(y(t_0), \zeta(t_0))$ . Let  $q_0 = \gamma_{y,\zeta}(r_0)$ . Then by using a short cut argument and the fact that  $\gamma_{y,\zeta}(\mathbb{R}_+)$  does not intersect  $\widehat{\mu}$  we see similarly to the above that  $f_{\widehat{\mu}}^+(p_0) > f_{\widehat{\mu}}^+(q_0) = \mathbb{S}(y, \zeta, s_1)$ . Since the function  $(x, \xi, t) \mapsto f_{\widehat{\mu}}^+(\gamma_{x,\xi}(t))$  is continuous and  $t \mapsto f_{\widehat{\mu}}^+(\gamma_{x,\xi}(t))$  is non-decreasing, and the function  $(x, \xi) \mapsto \rho(x, \xi)$  is lower semi-continuous, we have that the function

$(x, \xi) \mapsto f_{\hat{\mu}}^+(\gamma_{x(t_0), \xi(t_0)}(\rho(x(t_0), \xi(t_0))))$  is lower semi-continuous, and the claim follows.  $\square$

**Definition 5.6.** Let  $s_- \leq s_2 \leq s < s_1 \leq s_+$  satisfy  $s_1 < s_2 + \kappa_2$ ,  $\hat{x}_j = \hat{\mu}(s_j)$ ,  $j = 1, 2$ , and  $\hat{x} = \hat{\mu}(s)$ ,  $\hat{\zeta} \in L_{\hat{x}}^+U$ ,  $\|\hat{\zeta}\|_{g^+} = 1$ . Also, let  $(y, \zeta) \in L^+U$  be in  $\vartheta_1$ -neighborhood of  $(\hat{x}, \hat{\zeta})$  and  $\mathcal{G}(y, \zeta, s_1)$  be the set of the genuine observations  $S \subset U_{\hat{g}}$  associated to the geodesic  $\gamma_{y, \zeta}$  such that  $S \in \mathcal{E}_U(J^+(\hat{x}_1) \cap J^-(p^+))$ . Moreover, define  $T(y, \zeta, s_1)$  to be the infimum of  $s' \in [-1, s_+]$  such that  $\hat{\mu}(s') \in S \cap \hat{\mu}$  with some  $S \in \mathcal{G}(y, \zeta, s_1)$ . If no such  $s'$  exists, we define  $T(y, \zeta, s_1) = s^+$ .

Let us next consider  $(\vec{x}, \vec{\xi}) \in \mathcal{R}_{\vartheta}(y, \zeta)$  where  $0 < \vartheta < \vartheta_2(y, \zeta, s_1)$ . Here,  $\vartheta_2(y, \zeta, s_1)$  is defined in Lemma 5.5. Assume that for some  $j = 1, 2, 3, 4$  we have that  $\rho(x_j(t_0), \xi_j(t_0)) < \mathcal{T}(x_j(t_0), \xi_j(t_0))$  and consider the cut point  $p_j = \gamma_{x_j(t_0), \xi_j(t_0)}(\rho(x_j(t_0), \xi_j(t_0)))$ . Then either the case (i) or (ii) of Lemma 5.5, (B) holds. If (i) holds,  $p_j$  satisfies  $p_j \notin J^-(p^+)$  and thus  $f_{\hat{\mu}}^+(p_j) > s_+ \geq \mathbb{S}(y, \zeta, s_1)$ . If (ii) holds, there exists  $r_0 = r_0(y, \zeta, s_1) < r_1(y, \zeta)$  such that  $q_0 = \gamma_{y, \zeta}(r_0) \in J^+(\hat{x}_1)$  and  $f_{\hat{\mu}}^+(p_j) > f_{\hat{\mu}}^+(q_0) = \mathbb{S}(y, \zeta, s_1)$ . Thus both in case (i) and (ii) we have

$$(116) \quad f_{\hat{\mu}}^+(p_j) > \mathbb{S}(y, \zeta, s_1).$$

Next, consider a point  $q = \gamma_{y, \zeta}(r) \in J^-(p^+)$ , where  $t_0 < r \leq r_0 = r_0(y, \zeta, s_1)$ . Let  $\theta = -\dot{\gamma}_{y, \zeta}(r) \in T_qM$ . By Lemma 2.3 (ii), the geodesic  $\gamma_{y, \zeta}([t_0, r]) = \gamma_{q, \theta}([0, r - t_0])$  has no cut points. Denote  $\tilde{x} = \hat{\mu}(\mathbb{S}(y, \zeta, s_1))$ . Note that then  $q \in J^-(\tilde{x})$ . Then, consider four geodesics that emanate from  $q$  to the past, in the light-like direction  $\eta_1 = \theta$  and in the light-like directions  $\eta_j \in T_qM$ ,  $j = 2, 3, 4$  that are sufficiently close to the direction  $\theta$ . Let  $\gamma_{q, \eta_j}(r_j)$  be the intersection points of  $\gamma_{q, \eta_j}$  with the surface  $\{x \in M; \mathbf{t}(x) = c_0\}$ , on which the time function  $\mathbf{t}(x)$  has the constant value  $c_0 := \mathbf{t}(\gamma_{y, \zeta}(t_0))$ . Note that such  $r_j = r_j(\eta_j)$  exists by the inverse function theorem when  $\eta_j$  is sufficiently close to  $\eta_1$ . Choosing  $x_j = \gamma_{q, \eta_j}(r_j + t_0)$  and  $\xi_j = -\dot{\gamma}_{q, \eta_j}(r_j + t_0)$  we see that when  $\vartheta_3 \in (0, \vartheta_2(y, \zeta, s_1))$  is small enough, for all  $\vartheta \in (0, \vartheta_3)$ , there is  $(\vec{x}, \vec{\xi}) = ((x_j, \xi_j))_{j=1}^4 \in \mathcal{R}_{\vartheta}(y, \zeta)$  such that the geodesics corresponding to  $(\vec{x}, \vec{\xi})$  intersect at  $q$ . As the set  $(U_{\hat{g}}, \hat{g})$  is known, that for sufficiently small  $\vartheta$  one can verify if given vectors  $(\vec{x}, \vec{\xi})$  satisfy  $(\vec{x}, \vec{\xi}) = ((x_j, \xi_j))_{j=1}^4 \in \mathcal{R}_{\vartheta}(y, \zeta)$ , cf. [74, Prop. 5.7]. Also, note that as then  $\vartheta < \vartheta_2(y, \zeta, s_1)$ , the inequality (116) yields  $\tilde{x} \in \mathcal{V}((\vec{x}, \vec{\xi}), t_0)$  and thus  $q \in J^-(\tilde{x}) \subset \mathcal{V}((\vec{x}, \vec{\xi}), t_0)$ . Then Lemma 5.2 (iii) yields  $\mathcal{S}_e((\vec{x}, \vec{\xi}), t_0) = \mathcal{E}_U(q)$ . As  $\vartheta \in (0, \vartheta_3)$  above can be arbitrarily small, we have that for any  $q = \gamma_{y, \zeta}(r) \in J^-(p^+)$  where  $t_0 < r \leq r_0 = r_0(y, \zeta, s_1)$ ,

we obtain

$$(117) \quad S = \mathcal{E}_U(q) \text{ is a genuine observation associated to } \gamma_{y,\zeta} \text{ and} \\ S \cap \widehat{\mu} = \{\widehat{\mu}(\widehat{s})\}, \widehat{s} := f_{\widehat{\mu}}^+(q) \leq \mathbb{S}(y, \zeta, s_1).$$

**Lemma 5.7.** *Assume that  $\gamma_{y,\zeta}(\mathbb{R}_+)$  does not intersect  $\widehat{\mu}$ . Then we have  $T(y, \zeta, s_1) = \mathbb{S}(y, \zeta, s_1)$ .*

**Proof.** Let us first prove that  $T(y, \zeta, s_1) \geq \mathbb{S}(y, \zeta, s_1)$ . To this end, let  $s' < \mathbb{S}(y, \zeta, s_1)$  and  $x' = \widehat{\mu}(s')$ . Assume  $S \in \mathcal{E}_U(J^+(\widehat{x}_1) \cap J^-(p^+))$  is a genuine observation associated to the geodesic  $\gamma_{y,\zeta}$  and  $S \cap \widehat{\mu} = \{\widehat{\mu}(s')\}$ . Let  $q \in J^+(\widehat{x}_1) \cap J^-(p^+)$  be such that  $S = \mathcal{E}_U(q)$ .

Then for arbitrarily small  $0 < \vartheta < \vartheta_2(y, \zeta, s_1)$  there is  $(\vec{x}, \vec{\xi}) \in \mathcal{R}_\vartheta(y, \zeta)$  satisfying  $\mathcal{S}_e((\vec{x}, \vec{\xi}), t_0) = S$ . Let  $\mathcal{V} = \mathcal{V}((\vec{x}, \vec{\xi}), t_0)$ . Then by (116), we have  $x' \in \mathcal{V}$ .

If the geodesics corresponding to  $(\vec{x}, \vec{\xi})$  intersect at some point  $q' \in J^-(x')$ , then by Lemma 5.2 (i), (ii) we have  $S = \mathcal{E}_U(q')$ . Then  $\widehat{\mu}(s') = \widehat{\mu}(f_{\widehat{\mu}}^+(q'))$  implying  $f_{\widehat{\mu}}^+(q') = s'$ . Moreover, we have then that  $\mathcal{E}_U(q) = \mathcal{E}_U(q')$  and Theorem 2.5 (i) yields  $q' = q$ . Since  $(\vec{x}, \vec{\xi}) \in \mathcal{R}_\vartheta(y, \zeta)$  implies  $(x_1, \xi_1) = (y, \zeta)$ , we see that  $q' \in \gamma_{y,\zeta}([t_0, \infty))$ . As  $q \in J^+(\widehat{x}_1)$ , we see that  $q = q' \in \gamma_{y,\zeta}([t_0, \infty)) \cap J^+(\widehat{x}_1) = \gamma_{y,\zeta}([r_0(y, \zeta, s_1), \infty))$ . However, then  $f_{\widehat{\mu}}^+(q') \geq \mathbb{S}(y, \zeta, s_1) > s'$  and thus  $S \cap \widehat{\mu} = \mathcal{E}_U(q) \cap \widehat{\mu}$  can not be equal to  $\{\widehat{\mu}(s')\}$ .

On the other hand, if the geodesics corresponding to  $(\vec{x}, \vec{\xi})$  do not intersect at any point in  $J^-(x') \subset \mathcal{V}$ , then either they intersect in some  $q'_1 \in (M \setminus J^-(x')) \cap \mathcal{V}$ , do not intersect at all, or intersect at  $q'_2 \in M \setminus \mathcal{V}$ . In the first case,  $S = \mathcal{E}_U(q'_1)$  do not satisfy  $S \cap \widehat{\mu} \in \widehat{\mu}((-1, s'))$ . In the other cases, Lemma 5.2 (iii) yields  $\mathcal{S}_e((\vec{x}, \vec{\xi}), t_0) \cap \mathcal{V} = \emptyset$ . As  $x' = \widehat{\mu}(s') \in \mathcal{V}$ , we see that  $S \cap \widehat{\mu}$  can not be equal to  $\{\widehat{\mu}(s')\}$ . Since above  $s' < \mathbb{S}(y, \zeta, s_1)$  is arbitrary, this shows that  $T(y, \zeta, s_1) \geq \mathbb{S}(y, \zeta, s_1)$ .

Let us next show that  $T(y, \zeta, s_1) \leq \mathbb{S}(y, \zeta, s_1)$ . Assume the opposite. Then, if  $\mathbb{S}(y, \zeta, s_1) = s_+$ , we see by Def. 5.6 that  $T(y, \zeta, s_1) = \mathbb{S}(y, \zeta, s_1)$  which leads to a contradiction. However, if  $\mathbb{S}(y, \zeta, s_1) < s_+$ , by Def. 5.3, we have (116). This implies the existence of  $q_0 = \gamma_{y,\zeta}(r_0)$ ,  $r_0 = r_0(y, \zeta, s_1)$  such that  $q_0 \in J^+(\widehat{x}_1) \cap J^-(p^+)$  and by (117),  $S = \mathcal{E}_U(q_0)$  is a genuine observation associated to the geodesic  $\gamma_{y,\zeta}$ . By Lemma 5.5 (ii),  $\mathbb{S}(y, \zeta, s_1) = f_{\widehat{\mu}}^-(q_0)$  which implies, by Def. 5.6 that  $T(y, \zeta, s_1) \leq \mathbb{S}(y, \zeta, s_1)$ . Thus,  $T(y, \zeta, s_1) = \mathbb{S}(y, \zeta, s_1)$ .  $\square$

Next we reconstruct  $\mathcal{E}_U(q)$  when  $q$  runs over a geodesic segment.

**Lemma 5.8.** *Let  $s_- \leq s_2 \leq s < s_1 \leq s_+$  with  $s_1 < s_2 + \kappa_2$ , let  $\widehat{x}_j = \widehat{\mu}(s_j)$ ,  $j = 1, 2$ , and  $\widehat{x} = \widehat{\mu}(s)$ ,  $\widehat{\zeta} \in L_{\widehat{x}}^+U$ ,  $\|\widehat{\zeta}\|_{g^+} = 1$ . Let  $(y, \zeta) \in L^+U$  be in the  $\vartheta_1$ -neighborhood of  $(\widehat{x}, \widehat{\zeta})$  such that  $y \in J^+(\widehat{x}_2)$ . Assume that  $\gamma_{y,\zeta}(\mathbb{R}_+)$  does not intersect  $\widehat{\mu}$ . Then, if we are given the data set*



$\mathcal{D}(\widehat{g}, \widehat{\phi}, \varepsilon)$ , we can determine the collection  $\{\mathcal{E}_U(q); q \in G_0(y, \zeta, s_1)\}$ , where  $G_0(y, \zeta, s_1) = \{q \in \gamma_{y, \zeta}([t_0, \infty)) \cap (I^-(p^+) \setminus J^+(\widehat{x}_1))\}$ .

**Proof.** Let  $s' = \mathbb{S}(y, \zeta, s_1)$ ,  $x' = \widehat{\mu}(s')$ , and  $\Sigma$  be the set of all genuine observations  $S$  associated to the geodesic  $\gamma_{y, \zeta}$  such that  $S$  intersects  $\widehat{\mu}([-1, s'])$ .

Let  $q = \gamma_{y, \zeta}(r) \in G_0(y, \zeta, s_1)$ . Since  $\gamma_{y, \zeta}(\mathbb{R}_+)$  does not intersect  $\widehat{\mu}$ , using a short cut argument for the geodesics from  $q$  to  $q_0 = \gamma_{y, \zeta}(r_0(y, \zeta, s_1))$  and from  $q_0$  to  $x'$ , we see that  $f_{\widehat{\mu}}^-(q) < s'$ . Then,  $q \in I^-(p^+) \setminus J^+(\widehat{x}_1)$  and  $r < r_0(y, \zeta, s_1)$ , and we have using (117) that  $S = \mathcal{E}_U(q)$  is a genuine observation associated to the geodesic  $\gamma_{y, \zeta}$  and  $S \cap \widehat{\mu} = \{\widehat{\mu}(f_{\widehat{\mu}}^-(q))\}$  with  $f_{\widehat{\mu}}^-(q) < s'$ . Thus  $\mathcal{E}_U(q) \in \Sigma$  and we conclude that  $\mathcal{E}_U(G_0(y, \zeta, s_1)) \subset \Sigma$ .

Next, suppose  $S \in \Sigma$ . Then there is  $\widehat{\vartheta} \in (0, \vartheta_2(y, \zeta, s_1))$  such that for all  $\vartheta \in (0, \widehat{\vartheta})$  there is  $(\vec{x}, \vec{\xi}) \in \mathcal{R}_\vartheta(y, \zeta)$  so that  $\mathcal{S}_e((\vec{x}, \vec{\xi}), t_0) = S$ . Observe that by (116) we have  $\widehat{\mu}([-1, s']) \subset J^-(x') \subset \mathcal{V}((\vec{x}, \vec{\xi}), t_0)$ .

First, consider the case when the geodesics corresponding to  $(\vec{x}, \vec{\xi})$  do not intersect at any point in  $I^-(x')$ . Then Lemma 5.2 (iii) yields that  $\mathcal{S}_e((\vec{x}, \vec{\xi}), t_0)$  is either empty or does not intersect  $I^-(x')$ . Thus  $S \cap I^-(x') = \mathcal{S}_e((\vec{x}, \vec{\xi}), t_0) \cap I^-(x')$  is empty and  $S$  does not intersect  $\widehat{\mu}([-1, s'])$ . Hence  $S$  cannot be in  $\Sigma$ .

Second, consider the case when the geodesics corresponding to  $(\vec{x}, \vec{\xi})$  intersect at some point  $q \in I^-(x')$ . Then, Lemma 5.2 (iii) yields  $S = \mathcal{E}_U(q)$ . Since  $(\vec{x}, \vec{\xi}) \in \mathcal{R}_\vartheta(y, \zeta)$  implies  $(x_1, \xi_1) = (y, \zeta)$ , the intersection point  $q$  has a representation  $q = \gamma_{x_1, \xi_1}(r)$ . As  $q \in I^-(x')$ , this yields  $q \in G_0(y, \zeta, s_1)$  and  $S \in \mathcal{E}_U(G_0(y, \zeta, s_1))$ . Hence  $\Sigma \subset \mathcal{E}_U(G_0(y, \zeta, s_1))$ .

Combining the above arguments, we see that  $\Sigma = \mathcal{E}_U(G_0(y, \zeta, s_1))$ . As  $\Sigma$  is determined by the data set, the claim follows.  $\square$

Let  $B(s_2, s_1)$  be the set of all  $(y, \zeta, t)$  such that there are  $\widehat{x} = \widehat{\mu}(s)$ ,  $s \in [s_2, s_1)$  and  $\widehat{\zeta} \in L_{\widehat{x}}^+ U$ ,  $\|\widehat{\zeta}\|_{g^+} = 1$  so that  $(y, \zeta) \in L^+ U$  in  $\vartheta_1$ -neighborhood of  $(\widehat{x}, \widehat{\zeta})$ ,  $y \in J^+(\widehat{x}_2)$ , and  $t \in [t_0, r_0(y, \zeta, s_1)]$ . Moreover, let  $B_0(s_2, s_1)$  be the set of all  $(y, \zeta, t) \in B(s_2, s_1)$  such that  $t < r_0(y, \zeta, s_1)$  and  $\gamma_{y, \zeta}(\mathbb{R}_+) \cap \widehat{\mu} = \emptyset$ . Lemma 5.8 and the fact that  $\mathcal{D}(\widehat{g}, \widehat{\phi}, \varepsilon)$  determines  $\gamma_{y, \zeta} \cap U_{\widehat{g}}$  show that, when we are given the data set  $\mathcal{D}(\widehat{g}, \widehat{\phi}, \varepsilon)$ , we can determine the collection  $\Sigma_0(s_2, s_1) := \{\mathcal{E}_U(q); q = \gamma_{y, \zeta}(t), (y, \zeta, t) \in B_0(s_2, s_1)\}$ . We denote also  $\Sigma(s_2, s_1) := \{\mathcal{E}_U(q); q = \gamma_{y, \zeta}(t), (y, \zeta, t) \in B(s_2, s_1)\}$ .

Note that the sets  $\mathcal{E}_U(q) \subset U$ , where  $q \in J := J^-(p^+) \cap J^+(p^-)$ , can be identified with the function,  $F_q : \mathcal{U}_{z_0, \eta_0} \rightarrow \mathbb{R}$ ,  $F_q(z, \eta) = f_{\mu(z, \eta)}^+(q)$ , c.f. (1). Let  $\mathcal{U} = \mathcal{U}_{z_0, \eta_0}$ . When we endow the set  $\mathbb{R}^{\mathcal{U}}$  of maps  $\mathcal{U} \rightarrow \mathbb{R}$  with the topology of pointwise convergence, Lemma 2.2 yields that  $F : q \mapsto F_q$  is continuous map  $F : J \rightarrow \mathbb{R}^{\mathcal{U}}$ . By Theorem 2.5,  $F$  is one-to-one, and since  $J$  is compact and  $\mathbb{R}^{\mathcal{U}}$  is Hausdorff, we have that  $F : J \rightarrow F(J)$  is homeomorphism. Next, we identify  $\mathcal{E}_U(q)$  and  $F_q$ .

Using standard results of differential topology, we see that any neighborhood of  $(y, \zeta) \in L^+U$  contains  $(y', \zeta') \in L^+U$  such that the geodesic  $\gamma_{y', \zeta'}([0, \infty))$  does not intersect  $\widehat{\mu}$ . Since  $(y, \zeta) \mapsto r_0(y, \zeta, s_1)$  is lower semicontinuous, this implies that  $\Sigma_0(s_2, s_1)$  is dense in  $\Sigma(s_2, s_1)$ . Hence we obtain the closure  $\overline{\Sigma}(s_2, s_1)$  of  $\Sigma(s_2, s_1)$  as the limits points of  $\Sigma_0(s_2, s_1)$ .

Then, we obtain the set  $\mathcal{E}_U(J^+(\widehat{\mu}(s_2)) \cap J^-(p^+))$  as the union  $\overline{\Sigma}(s_2, s_1) \cup \mathcal{E}_U(J^+(\widehat{\mu}(s_1)) \cap J^-(p^+)) \cup \mathcal{E}_U(\mathcal{K}_{t_0} \cap J^+(\widehat{\mu}(s_2)))$ , see (114).

Let  $s_0, \dots, s_K \in [s_-, s_+]$  be such that  $s_j > s_{j+1} > s_j - \kappa_2$  and  $s_K = s_-$ . Then, by iterating the above construction so that the values of the parameters  $s_1$  and  $s_2$  are replaced by  $s_j$  and  $s_{j+1}$ , respectively, we can construct the set  $\mathcal{E}_U(J^+(\widehat{\mu}(s_-)) \cap J^-(\widehat{\mu}(s_+)))$ .

Moreover, similarly to the above construction, we can find the sets  $\mathcal{E}_U(J^+(\widehat{\mu}(s')) \cap J^-(\widehat{\mu}(s''))) for all  $s_- < s' < s'' < s_+$ , and taking their union, we find the set  $\mathcal{E}_U(I(\widehat{\mu}(s_-), \widehat{\mu}(s_+)))$ . By Theorem 2.5 we can reconstruct the manifold  $I(\widehat{\mu}(s_-), \widehat{\mu}(s_+))$  and the conformal structure on it. This proves Theorem 1.1.  $\square$$

**Remark 5.1.** The proof of Theorem 1.1 can be used to analyze approximative reconstruction of the set  $I_{\widehat{g}}(p^-, p^+) = I_{\widehat{g}}^+(p^-) \cap I_{\widehat{g}}^-(p^+)$  and its conformal structure with only one measurement: We choose one suitably constructed source  $\mathcal{F}$ , supported in  $W_{\widehat{g}}$ , measure the fields  $(f, \phi)$  produced by this source in  $U_{\widehat{g}}$  and aim to construct an approximation of the conformal class of the metric  $g$  in  $I_{\widehat{g}}(p^-, p^+)$ . In the proof above we showed that it is possible to use the non-linearity to create an artificial point source at a point  $q \in J_{\widehat{g}}(p^-, p^+)$ . Using the same method we see that it is possible to create with a single source  $\mathcal{F}$  an arbitrary number of artificial point sources. To see this, let  $\delta > 0$  and  $P, Q \in \mathbb{Z}_+$  and consider points  $x_p \in U_{\widehat{g}} \cap J_{\widehat{g}}(p^-, p^+)$ ,  $p = 1, 2, \dots, P$ . Let  $\xi_{p,k} \in \Sigma_p = \{\xi \in T_{x_p}^*M; \|\xi\|_{\widehat{g}} = 1\}$ ,  $k = 1, 2, \dots, n_Q$  be a maximal  $1/Q$  net on the set  $\Sigma_p$  and  $\Sigma_{p,k}(R)$  be an  $R$ -neighborhood of  $\xi_{p,k}$  in  $\Sigma_p$ . Let  $R_1 = 1/Q$  and  $R_2 = 2/Q$ , and consider real numbers  $a(p, k) \in (-31, -30)$ , chosen to be  $a(p, k) = -31 + 1/j(p, k)$ , where  $j : \mathbb{Z}_+^2 \rightarrow \mathcal{P}$  is a bijection from  $\mathbb{Z}_+^2$  to the set  $\mathcal{P}$  of the prime numbers. Let  $F_{p,k} \in \mathcal{I}^{a(p,k)}(\Sigma_{p,k}(R_2))$  be Lagrangian distributions whose principal symbols are non-vanishing on  $\Sigma_{p,k}(R_1)$ .

Assume next that  $(M, g)$  is in generic manifold (i.e., it is in the intersection of countably many open and dense sets in a suitable space of smooth manifolds), points  $x_p$  have generic positions, and let  $\delta > 0$ . Note that the sets  $\Sigma_{p,k}(R_1)$  are a covering of the unit sphere  $\Sigma_p$  and the linearized waves  $u_{p,k} = \mathbf{Q}_{\widehat{g}}(F_{p,k})$  are singular on a subset of the light-cones  $\mathcal{L}_{\widehat{g}}^+(x_p)$ . Let  $\varepsilon > 0$  be small enough and consider a suitable source  $F_\varepsilon$ , with  $\partial_\varepsilon F_\varepsilon|_{\varepsilon=0} = \sum_{p,k} F_{p,k}$ , that produces the perturbation  $u_\varepsilon(x) = \sum_{n=1}^4 \varepsilon^n u_n(x) + O(\varepsilon^5)$  for  $(\widehat{g}, \widehat{\phi})$ . Assume that we measure the singular supports of the waves  $u_n$ ,  $n = 1, 2, 3, 4$  produced by  $n$ -th

order interaction of the waves. When  $\varepsilon$  is small enough, this could be done e.g. by using thresholding of the curvelet coefficients, of a suitable order, of the solution [13, 26]. Then in  $I_{\widehat{g}}(p^-, p^+)$ , outside the singular support of the wave  $u_3$ , the wave  $u_4$  is a sum of a smooth wave and the waves produced by artificial point sources  $\mathcal{F}_\ell$  located at  $q_\ell$ ,  $\ell = 1, 2, \dots, L$ . Here  $q_\ell \in J_{\widehat{g}}(p^-, p^+)$  are the intersection points of any four light cones  $\mathcal{L}_{\widehat{g}}^+(x_{p_1(\ell)}), p_1(\ell), p_2(\ell), p_3(\ell), p_4(\ell) \leq N$ . When  $Q$  and  $P$  are large enough, so that  $N$  is large, the points  $q_\ell$  are a  $\delta$ -dense subset of  $J_{\widehat{g}}(p^-, p^+)$ . Moreover, for the chosen orders  $a(p, q)$ , the orders of the sources  $\mathcal{F}_\ell$  at points  $q_\ell$  are all different and thus the waves  $\mathbf{Q}_{\widehat{g}}(\mathcal{F}_\ell)$  have different orders. As a very rough analogy, we can produce in  $J_{\widehat{g}}(p^-, p^+)$  an arbitrarily dense collection of artificial point sources having all different colors. Using this observation one can show, using similar methods to those in [2], that in a suitable compact class of Lorentzian manifolds having no conjugate points the measurement with the source  $F_\varepsilon$ , defined using sufficiently large  $P$  and  $Q$  and generic values of principal symbols on  $\Sigma_{p,q}$ , determines a  $\delta$ -approximation (in a suitable sense) of the conformal class of the manifold  $(J_{\widehat{g}}(p^-, p^+), \widehat{g})$ . The details of this construction will be considered elsewhere.

**Appendix A: Model with adaptive source functions.** Let us consider an abstract model, of active measurements. By an active measurement we mean a model where we can control some of the physical fields and the other physical fields adapt to the changes of all fields so that the conservation law holds. Roughly speaking, we can consider measurement devices as a complicated process that changes one energy form to other forms of energy, like a system of explosives that transform some potential energy to kinetic energy. This process creates a perturbation of the metric and the matter fields that we observe in a subset of the spacetime. In this paper, our aim has been to consider a mathematical model that can be rigorously analyzed.

In [58], see also [56], we consider a model direct problem for the Einstein-scalar field equations, with  $V(s) = \frac{1}{2}m^2s^2$ , based on adaptive sources that is an example of abstract models satisfying the assumptions studied in this paper: Let  $g$  and  $\phi$  satisfy

$$(118) \quad \begin{aligned} \text{Ein}_{\widehat{g}}(g) &= P_{jk} + Zg_{jk} + \mathbf{T}_{jk}(g, \phi), & Z &= -\sum_{\ell=1}^L (S_\ell \phi_\ell + \frac{1}{2m^2} S_\ell^2), \\ \square_g \phi_\ell - m^2 \phi_\ell &= S_\ell & \text{in } M_0, & \ell = 1, 2, 3, \dots, L, \\ S_\ell &= Q_\ell + \mathcal{S}_\ell^{2nd}(g, \phi, \nabla^g \phi, Q, \nabla Q, P, \nabla^g P), & \text{in } M_0, \\ g &= \widehat{g}, \quad \phi_\ell = \widehat{\phi}_\ell, & \text{in } M_0 \setminus J_g^+(p^-). \end{aligned}$$

We assume that the background fields  $\widehat{g}$ ,  $\widehat{\phi}$ ,  $\widehat{Q}$ , and  $\widehat{P}$  satisfy these equations and call  $Z$  the stress energy density caused by the sources  $S_\ell$ . Here, the functions  $\mathcal{S}_\ell^{2nd}(g, \phi, \nabla^g \phi, Q, \nabla Q, P, \nabla^g P)$  model the devices that we use to perform active measurements. The precise construction

of  $\mathcal{S}_\ell^{2nd}(g, \phi, \nabla^g \phi, Q, \nabla Q, P, \nabla^g P)$  is quite technical, but these functions can be viewed as the instructions on how to build a device that can be used to measure the structure of the spacetime far away. When  $\widehat{P} = 0$ ,  $\widehat{Q} = 0$  and Condition A is satisfied, one can show that there are adaptive source functions so that (118) satisfies Assumption  $\mu$ -SL.

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#### REFERENCES

- [1] P. Alinhac: Non-Unicité du Problème de Cauchy, *Annals of Mathematics*, 117 (1983), 77–108.
- [2] M. Anderson, A. Katsuda, Y. Kurylev, M. Lassas, M. Taylor: Boundary regularity for the Ricci equation, Geometric Convergence, and Gel'fand's Inverse Boundary Problem, *Inventiones Mathematicae* 158 (2004), 261-321.
- [3] V. Babich, V. Ulin: The complex space-time ray method and "quasiphotons." (Russian) *Zap. Nauchn. Sem. LOMI* 117 (1981), 5–12.
- [4] C. Bär, N. Ginoux, F. Pfäffle: *Wave equations on Lorentzian manifolds and quantization*, ESI Lectures in Mathematics and Physics, 2007, 202 pp.
- [5] J. Beem, P. Ehrlich, K. Easley: *Global Lorentzian geometry*, Pure and Applied Mathematics, vol. 67, Dekker, 1981, p. 270
- [6] M. Belishev: An approach to multidimensional inverse problems for the wave equation. (Russian) *Dokl. Akad. Nauk SSSR* 297 (1987), 524–527.
- [7] M. Belishev, Y. Kurylev: To the reconstruction of a Riemannian manifold via its spectral data (BC-method). *Comm. PDE* 17 (1992), 767–804.
- [8] Y. Berezanskii: The uniqueness theorem in the inverse problem of spectral analysis for the Schrödinger equation. *Trudy Moskov Mat. Obs.* 7 (1958), 1-62.
- [9] A. Bernal, M. Sanchez: Globally hyperbolic spacetimes can be defined as "causal" instead of "strongly causal", *Class. Quant. Grav.* 24 (2007) 745-750.
- [10] A. Bernal, M. Sanchez: Smoothness of time functions and the metric splitting of globally hyperbolic spacetimes. *Comm. Math. Phys.* 257 (2005), 43–50.
- [11] D. Brill, S. Deser: Instability of closed spaces in general relativity. *Comm. Math. Phys.* 32 (1973), 291-304.
- [12] D. Brill, O. Reula, B. Schmidt: Local linearization stability. *J. Math. Phys.* 28 (1987), 1844-1847.
- [13] V. Brytik, M. de Hoop, M. Salo: Sensitivity analysis of wave-equation tomography: a multi-scale approach. *J. Fourier Anal. Appl.* 16 (2010), 544-589.
- [14] Y. Choquet-Bruhat, *General relativity and the Einstein equations*. Oxford Univ. Press, 2009. xxvi+785 pp.
- [15] Y. Choquet-Bruhat, S. Deser: On the stability of flat space. *Ann. Physics* 81 (1973), 165-178.

- [16] Y. Choquet-Bruhat, J. Isenberg, D. Pollack: Applications of theorems of Jean Leray to the Einstein-scalar field equations. *J. Fixed Point Theory Appl.* 1 (2007), 31-46.
- [17] Y. Choquet-Bruhat, J. Isenberg, D. Pollack: The Einstein-scalar field constraints on asymptotically Euclidean manifolds, *Chi. Ann. of Math B*, 27(2006) 31–52.
- [18] Y. Choquet-Bruhat, J. Isenberg, D. Pollack: The constraint equations for the Einstein-scalar field system on compact manifolds, *Classical and Quantum Gravity* 24 (2007), 809.
- [19] M. Choulli, P. Stefanov: Inverse scattering and inverse boundary value problems for the linear Boltzmann equation. *Comm. PDE* 21 (1996), 763-785.
- [20] D. Christodoulou: Global existence of generalized solutions of the spherically symmetric Einstein-scalar equations in the large. *Comm. Math. Phys.* 106 (1986), 587-621.
- [21] D. Christodoulou, S. Klainerman: *The global nonlinear stability of the Minkowski space*, Princeton University Press, Princeton, U.S.A., 1993.
- [22] P. Chrusciel, J. Isenberg D. Pollack: A brief review of initial data engineering *Comm. Math. Phys.* 257 (2005), 29-42.
- [23] P. Chrusciel, R. Mazzeo: On “many black hole” vacuum spacetimes. *Classical and Quantum Gravity* 20 (2003), 1-26.
- [24] T. Collins, A. Greenleaf, M. Pramanik: A multi-dimensional resolution of singularities with applications. *Amer. J. Math.* 135 (2013), 1179-1252.
- [25] M. Dafermos: *The evolution problem in general relativity*. Current developments in mathematics, 2008, 1–66, Int. Press, Somerville, MA, 2009.
- [26] M. de Hoop, S. Holman, H. Smith, G. Uhlmann: Regularity and multi-scale discretization of the solution construction of hyperbolic evolution equations of limited smoothness. *Appl. Comput. Harmon. Anal.* **33** (2012), 330-353.
- [27] J.-M. Delort: *F.B.I. transformation. Second microlocalization and semilinear caustics*. Lecture Notes in Mathematics, 1522. Springer, 1992. vi+101 pp.
- [28] N. Dencker: On the propagation of polarization sets for systems of real principal type, *Journal of Functional Analysis* 46 (1982), 351-372
- [29] D. Dos Santos Ferreira, C. Kenig, M. Salo: Determining an unbounded potential from Cauchy data in admissible geometries *Comm. PDE*, 38(2013), 50-68.
- [30] J. Duistermaat: *Fourier Integral Operators*, Birkhäuser, 1996. x+142 pp.
- [31] G. Eskin: Inverse hyperbolic problems and optical black holes. *Comm. Math. Phys.* **297** (2010), 817–839.
- [32] R. Felea, A. Greenleaf: Fourier integral operators with open umbrellas and seismic inversion for cusp caustics, *Math. Res. Lett.* 17 (2010), 867-886.
- [33] M. Fridman, et al: Demonstration of temporal cloaking *Nature* 481(2012), 62.
- [34] A. Fischer, J. Marsden: The Einstein evolution equations as a first-order quasilinear symmetric hyperbolic system. I. *Comm. Math. Phys.* 28 (1972), 1–38
- [35] A. Fischer and J. Marsden: Linearization stability of the Einstein equations. *Bull. Amer. Math. Soc. Volume* **79** (1973), 997-1003.
- [36] R. Geroch: Domain of dependence. *J. Mathematical Phys.* 11 (1970), 437–449.
- [37] J. Girbau, K. Bruna: *Stability by linearization of Einstein’s field equation*. Birkhäuser, 2010. xvi+208 pp.
- [38] C. Graham, M. Zworski: Scattering matrix in conformal geometry. *Invent. Math.*, 152, 89-118, 2003.
- [39] C. Guillarmou, A. Sa Barreto: Inverse Problems for Einstein manifolds. *Inverse Prob. Imag.* 3 (2009), 1-15.

- [40] C. Guillarmou, M. Salo, L. Tzou: Inverse scattering at fixed energy for surfaces with Euclidean ends. *Comm. Math. Phys.* 303 (2011), 761-784.
- [41] A. Greenleaf, G., Uhlmann: Recovering singularities of a potential from singularities of scattering data. *Comm. Math. Phys.* **157** (1993), 549-572.
- [42] A. Greenleaf, Y. Kurylev, M. Lassas, G. Uhlmann: Full-wave invisibility of active devices at all frequencies. *Comm. Math. Phys.* 275 (2007), 749-789.
- [43] A. Grigis, J. Sjöstrand: *Microlocal analysis for differential operators: an introduction*, LMS, 1999.
- [44] V. Guillemin, G. Uhlmann: Oscillatory integrals with singular symbols. *Duke Math. J.* 48 (1981), 251-267.
- [45] T. Helin, M. Lassas, L. Oksanen: An inverse problem for the wave equation with one measurement and the pseudorandom source. *Analysis and PDE* **5** (2012), 887-912.
- [46] P. Hoskins: Principles of ultrasound elastography, *Ultrasound* (20) 2012, 8-15.
- [47] T. Hughes, T. Kato; J. Marsden: Well-posed quasi-linear second-order hyperbolic systems with applications to nonlinear elastodynamics and general relativity. *Arch. Rational Mech. Anal.* 63 (1976), 273-294
- [48] S. Hawking, G. Ellis: *The Large Scale Structure of Space-Time*, Cambridge Univ. press, 1973.
- [49] L. Hörmander: *The analysis of linear partial differential operators IV*. Springer-Verlag, 1985, vii+352 pp.
- [50] V. Isakov: On uniqueness in inverse problems for semilinear parabolic equations. *Arch. Rational Mech. Anal.* 124 (1993), 1-12.
- [51] V. Isakov and A. Nachman: Global uniqueness in a two-dimensional semilinear elliptic inverse problem, *Trans. Amer. Math. Soc.* 347 (1995), 3375-3390.
- [52] H. Kang, G. Nakamura: Identification of nonlinearity in a conductivity equation via the Dirichlet-to-Neumann map. *Inverse Prob.* 18 (2002), 1079-1088.
- [53] A. Katchalov, A system of coordinate for describing the "quasiphoton". (Russian) *Zap. Nauch. Semin. LOMI*, 140 (1984), 73-76.
- [54] A. Katchalov, Y. Kurylev, M. Lassas: *Inverse boundary spectral problems*. Chapman-Hall/CRC, Boca Raton, FL, 2001. xx+290 pp
- [55] T. Kato: Quasi-linear equations of evolution, with applications to partial differential equations. *Spectral theory and differential equations*, pp. 25-70. Lecture Notes in Math., Vol. 448, Springer, 1975.
- [56] Y. Kurylev, M. Lassas, G. Uhlmann: Inverse problems in spacetime I: Inverse problems for Einstein equations – Extended preprint version. Arxiv preprint arXiv:1405.4503v1.
- [57] Y. Kurylev, M. Lassas, G. Uhlmann: Inverse problems in spacetime II: Reconstruction of a Lorentzian manifold from light observation sets. Preprint arXiv:1405.3386.
- [58] Y. Kurylev, M. Lassas, G. Uhlmann: Linearization stability results and active measurements for the Einstein-scalar field equations, Preprint arXiv:1405.3384.
- [59] S. Klainerman, I. Rodnianski: Improved local well-posedness for quasi-linear wave equations in dimension three, *Duke Math. J.*, 117 (2003), no 1 , 1-124.
- [60] S. Klainerman, I. Rodnianski: A Kirchoff-Sobolev parametrix for wave equations in a curved space-time. *J. Hyperb. Diff. Eq.* 4, (2007), 401-433.
- [61] S. Klainerman, I. Rodnianski: Rough solutions to the Einstein vacuum equations, *Annals of Mathematics*, 161 (2005), 1143-1193.
- [62] J. Lee, G. Uhlmann: Determining anisotropic real-analytic conductivities by boundary measurements. *Comm. Pure Appl. Math.*, **42** (1989), 1097-1112.
- [63] H. Lindblad, I. Rodnianski: Global existence for the Einstein vacuum equations in wave coordinates. *Comm. Math. Phys.* 256 (2005), 43-110.

- [64] H. Lindblad: Global solutions of quasilinear wave equations. *Amer. J. Math.* 130 (2008), 115-157.
- [65] J. Luk, I. Rodnianski: *Nonlinear interaction of impulsive gravitational waves for the vacuum Einstein equations*, Preprint arXiv:1301.1072.
- [66] M. McCall et al: A spacetime cloak, or a history editor, *Journal of Optics* 13 (2011), 024003.
- [67] J. McLaughlin, D. Renzi: Shear Wave Speed Recovery in Transient Elastography And Supersonic Imaging Using Propagating Fronts, *Inverse Problems* 22 (2006), 681-706.
- [68] A. Melin, J. Sjöstrand: Fourier integral operators with complex-valued phase functions, in *Fourier Integral Operators and Partial Differential Equations*, Lecture Notes in Math. 459, Springer, 1975.
- [69] R. Melrose, A. Sa Barreto, A. Vasy: Asymptotics of solutions of the wave equation on de Sitter-Schwarzschild space. To appear in *Comm. PDE*.
- [70] R. Melrose, G. Uhlmann: Lagrangian intersection and the Cauchy problem. *Comm. Pure Appl. Math.* **32** (1979), 483-519.
- [71] R. Melrose, N. Ritter: Interaction of nonlinear progressing waves for semilinear wave equations. *Ann. of Math.* **121** (1985), 187-213.
- [72] R. Melrose, N. Ritter: Interaction of progressing waves for semilinear wave equations. II. *Ark. Mat.* **25** (1987), 91-114.
- [73] G. Nakamura, M. Watanabe: An inverse boundary value problem for a nonlinear wave equation. *Inverse Probl. Imaging* 2 (2008), 121-131.
- [74] B. O'Neill, *Semi-Riemannian geometry*. With applications to relativity. Pure and Applied Mathematics, 103. Academic Press, Inc., 1983. xiii+468 pp.
- [75] J. Ralston: Gaussian beams and the propagation of singularities. *Studies in partial differential equations*, 206-248, MAA Stud. Math., 23, 1982
- [76] J. Rauch, M. Reed: Singularities produced by the nonlinear interaction of three progressing waves; examples. *Comm. PDE* **7** (1982), 1117-1133.
- [77] A. Rendall: *Partial differential equations in general relativity*. Oxford University Press, 2008. xvi+279 pp.
- [78] H. Ringström: *The Cauchy Problem in General Relativity*, EMS, 2009, 307 pp.
- [79] R. Sachs, H. Wu: *General relativity for mathematicians*, Springer 1977, 291 p.
- [80] M. Salo, X. Zhong: An inverse problem for the p-Laplacian: boundary determination. *SIAM J. Math. Anal.* 44 (2012), 2474-2495.
- [81] Z. Sun: On a quasilinear inverse boundary value problem. *Math. Z.* 221 (1996), 293-305.
- [82] Z. Sun, G. Uhlmann: Inverse problems in quasilinear anisotropic media. *Amer. J. Math.* 119 (1997), 771-797.
- [83] D. Tataru: Unique continuation for solutions to PDE's; between Hörmander's theorem and Holmgren's theorem. *Comm. PDE* 20 (1995), 855-884.
- [84] D. Tataru: Unique continuation for operators with partially analytic coefficients. *J. Math. Pures Appl.* 78 (1999), 505-521.
- [85] M. Taylor: *Partial differential equations. III*. Springer 1997, 608 p.
- [86] A. Vasy: Diffraction by edges. *Modern Phys. Lett. B* 22 (2008), 2287-2328.
- [87] A. Vasy: Propagation of singularities for the wave equation on manifolds with corners. *Ann. of Math.* (2) 168 (2008), 749-812.
- [88] A supplementary video on colliding waves. Download from the page <http://www.rni.helsinki.fi/~mjl/Relativity-IP-video.html>

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