On uniqueness of Lamé coefficients from partial Cauchy data in three dimensions

On uniqueness of Lamé coefficients from partial Cauchy data in three dimensions

Oleg Yu Imanuvilov[1], Gunther Uhlmann[2,3] and Masahiro Yamamoto[4]

[1] Department of Mathematics, Colorado State University, 101 Weber Building, Fort Collins, CO 80523, USA
[2] Department of Mathematics, UC Irvine, Irvine, CA 92697, USA
[3] Department of Mathematics, University of Washington, Seattle, WA 98195, USA
[4] Department of Mathematical Sciences, The University of Tokyo, Komaba, Meguro, Tokyo 153, Japan

E-mail: oleg@math.colostate.edu, gunther@math.washington.edu and myama@ms.u-tokyo.ac.jp

Abstract

For the Lamé system, we prove in the three-dimensional case that both Lamé coefficients are uniquely recovered from partial Cauchy data on an arbitrary open subset of the boundary provided that the coefficient μ is a constant.

In a bounded domain $\Omega \subset \mathbf{R}^3$ with smooth boundary, we consider the Lamé system

$$\sum_{j,k,l=1}^{3} \frac{\partial}{\partial x_j}\left(C_{ijkl}\frac{\partial u^k}{\partial x_l}\right) = 0 \quad \text{in } \Omega, \ 1 \leqslant i \leqslant 3, \tag{1}$$

and

$$u|_{\partial\Omega} = f, \tag{2}$$

where

$$C_{ijkl} = \lambda(x)\delta_{ij}\delta_{kl} + \mu(x)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \quad 1 \leqslant i, j, k, l \leqslant 3$$

with the Kronecker delta δ_{ij}. The functions λ and μ are called the Lamé coefficients, and $u(x) = (u^1(x), u^2(x), u^3(x))$ is the displacement. Assume that

$$\mu(x) > 0 \quad \text{on } \overline{\Omega}, \quad (3\lambda + 2\mu)(x) > 0 \quad \text{on } \overline{\Omega}. \tag{3}$$

We set

$$\Lambda_{\lambda,\mu}f = \left(\sum_{j,k,l=1}^{3} \nu_j C_{1jkl}\frac{\partial u^k}{\partial x_l}, \sum_{j,k,l=1}^{3} \nu_j C_{2jkl}\frac{\partial u^k}{\partial x_l}, \sum_{j,k,l=1}^{3} \nu_j C_{3jkl}\frac{\partial u^k}{\partial x_l}\right), \tag{4}$$

where $\nu = (\nu_1, \nu_2, \nu_3)$ is the outward unit normal vector to $\partial\Omega$ and u is the solution to (1) and (2). Denote

$$\mathcal{L}_{\lambda,\mu}(x,D)u = \left(\sum_{j,k,l=1}^{3} \frac{\partial}{\partial x_j}\left(C_{1jkl}\frac{\partial u^k}{\partial x_l}\right), \sum_{j,k,l=1}^{3} \frac{\partial}{\partial x_j}\left(C_{2jkl}\frac{\partial u^k}{\partial x_l}\right), \sum_{j,k,l=1}^{3} \frac{\partial}{\partial x_j}\left(C_{3jkl}\frac{\partial u^k}{\partial x_l}\right)\right).$$

The partial Cauchy data $\mathcal{C}_{\lambda,\mu}$ are defined by

$$\mathcal{C}_{\lambda,\mu} = \{(u, \Lambda_{\lambda,\mu}u)|_{\widetilde{\Gamma}}; \ \mathcal{L}_{\lambda,\mu}(x,D)u = 0 \quad \text{in } \Omega, u|_{\partial\Omega} = f, \quad \text{supp} f \subset \widetilde{\Gamma}, f \in H^{\frac{3}{2}}(\partial\Omega)\}.$$

Here, $\widetilde{\Gamma}$ is an arbitrarily fixed open subset of $\partial\Omega$. We set $\Gamma_0 = \partial\Omega \setminus \overline{\widetilde{\Gamma}}$.

In this paper, we consider the following inverse problem. *Suppose that the partial Cauchy data $\mathcal{C}_{\lambda,\mu}$ are given. Can we determine the Lamé coefficients λ and μ?*

This inverse problem has been studied since the 1990s. A linearized version of this inverse problem for full data was studied by Ikehata [7]. In two dimensions, Akamatsu *et al* [1] proved that for the case of full Cauchy data ($\widetilde{\Gamma} = \partial\Omega$) one can recover the Lamé coefficients and their normal derivatives of arbitrary orders on the boundary provided that the Lamé coefficients are C^∞ functions. This boundary determination result was extended by Nakamura and Uhlmann [17] to higher dimensions. In [15], Nakamura and Uhlmann for the case of full Cauchy data established that in two dimensions the Lamé coefficients are uniquely determined, assuming that they are sufficiently close to a pair of positive constants. Recently, Imanuvilov and Yamamoto in [13] proved for the two-dimensional case that the Lamé coefficient λ can be recovered from partial Cauchy data if the coefficient μ is some positive constant. In two dimensions, there is another possible way to prove a local uniqueness result for λ and μ which are close to constants: the proof is a combination of [1] and a local uniqueness result for the Love–Kirchhoff plate equation in [8], and the equivalence of the two problems is proved in [9].

For the three-dimensional case, the uniqueness for both Lamé coefficients is proved provided that μ is close to a positive constant [6, 16] and the proof relies on construction of complex geometric optics solution [5, 18]. Our result relies on these uniqueness results for the case of full data.

All the above works are concerned with the full Cauchy data (i.e. $\widetilde{\Gamma} = \partial\Omega$). The recovery of Lamé coefficients by partial Cauchy data on an arbitrary sub-boundary is useful from the practical point of view, because one can limit input and measurement subsets of $\partial\Omega$ as much as possible. In the case of partial Cauchy data for the Lamé system, unlike the case of the Schrödinger operator, the construction of complex geometric optics solutions seems to be possible only for a dense set of Lamé coefficients. To the best of our knowledge, there are no results on the unique recovery of the Lamé coefficients from the partial Cauchy data in the three-dimensional case. The purpose of this paper is to prove such uniqueness in three dimensions.

Finally, we mention that this inverse problem is closely related to the method known as electrical impedance tomography (EIT). EIT is used in prospection of oil and minerals and in medical imaging in detecting breast cancer, pulmonary edema, etc. For the mathematical treatments of this problem, we refer, e.g., to [2–4, 11, 12, 14, 19] and the review paper [20].

Our result is the following theorem.

Theorem 0.1. *Let μ_1, μ_2 be some positive constants and λ_1, $\lambda_2 \in C^\infty(\overline{\Omega})$ be some functions satisfying (3) and $\lambda_1 = \lambda_2$ on Γ_0. If $\mathcal{C}_{\lambda_1,\mu_1} = \mathcal{C}_{\lambda_2,\mu_2}$, then $(\lambda_1, \mu_1) = (\lambda_2, \mu_2)$.*

Proof. The proof consists in showing that from partial Cauchy data one can recover the full data. First, following [17], we prove that

$$(\lambda_1, \mu_1) = (\lambda_2, \mu_2) \quad \text{on } \widetilde{\Gamma} \quad \text{and} \quad \frac{\partial\lambda_1}{\partial\nu} = \frac{\partial\lambda_2}{\partial\nu} \quad \text{on } \widetilde{\Gamma}. \tag{5}$$

Let $u_j \in H^2(\Omega)$, $j = 1, 2$, be functions such that

$$\mathcal{L}_{\lambda_j,\mu_j}(x,D)u_j = 0 \quad \text{in } \Omega, \quad u_j|_{\partial\Omega} = f, \tag{6}$$

where supp$f \subset \widetilde{\Gamma}$. Since the partial Cauchy data are the same, we obtain

$$\Lambda_{\lambda_1,\mu_1} u_1 = \Lambda_{\lambda_2,\mu_2} u_2 \quad \text{on} \quad \widetilde{\Gamma}, \tag{7}$$

where $\Lambda_{\lambda_i,\mu_i}$, $i = 1, 2$, are defined in (4).

Assume for the moment that $u_1 \in C^{2+\alpha}(\bar{\Omega})$ for some α from $(0, 1)$. The regularity results for the Lamé system imply immediately that $u_2 \in C^{2+\alpha}(\bar{\Omega})$.

By (5) and $3\lambda_j + 2\mu_j > 0$ on $\overline{\Omega}$, $j = 1, 2$, we can prove

$$\left(u_1, \frac{\partial u_1}{\partial \nu} \right) = \left(u_2, \frac{\partial u_2}{\partial \nu} \right) \quad \text{on} \quad \widetilde{\Gamma}. \tag{8}$$

Moreover, from (5), (8) and equation (6), we conclude

$$\left(u_1, \frac{\partial u_1}{\partial \nu}, \frac{\partial^2 u_1}{\partial x_i \partial x_k} \right) = \left(u_2, \frac{\partial u_2}{\partial \nu}, \frac{\partial^2 u_2}{\partial x_i \partial x_k} \right) \quad \text{on} \quad \widetilde{\Gamma}, \quad \forall i, k \in \{1, 2, 3\}. \tag{9}$$

Hence,

$$\left(\text{rot}\, u_1, \frac{\partial \text{rot}\, u_1}{\partial \nu} \right) = \left(\text{rot}\, u_2, \frac{\partial \text{rot}\, u_2}{\partial \nu} \right) \quad \text{on} \quad \widetilde{\Gamma}. \tag{10}$$

As for the proof of (9) and (8), we refer, for instance, to [10].

Since the functions μ_j are assumed to be constants, from (5) we conclude that

$$\mu := \mu_1 = \mu_2 \quad \text{in } \Omega. \tag{11}$$

For the constant μ, we note that

$$\mathcal{L}_{\lambda,\mu}(x, D)u = \mu \Delta u + (\mu + \lambda)\nabla \text{div}\, u + (\text{div}\, u)\nabla \lambda.$$

Applying to equation (6) the operator rot and using the fact that μ_j is constant, we obtain

$$\mu_j \Delta \text{rot}\, u_j = 0 \quad \text{in } \Omega, \; j = 1, 2. \tag{12}$$

Equality (10) and the uniqueness of the solution for the Cauchy problem for the Laplace equation imply

$$\text{rot}\, u_1 = \text{rot}\, u_2 \quad \text{in } \Omega. \tag{13}$$

The Lamé operator, with the coefficient $\mu = \text{const}$, can be written in the form $\mathcal{L}(x, D)u = \nabla((\lambda + 2\mu)\text{div}\, u) - \mu \text{rot rot} u$. Then, using (11) and (13), we obtain

$$\nabla((\lambda_1 + 2\mu)\text{div}\, u_1) = \nabla((\lambda_2 + 2\mu)\text{div}\, u_2) \quad \text{in } \Omega. \tag{14}$$

Hence, $(\lambda_1 + 2\mu)\text{div}\, u_1 - (\lambda_2 + 2\mu)\text{div}\, u_2$ is a constant function in Ω. Since $(\lambda_1 + 2\mu)\text{div}\, u_1 = (\lambda_2 + 2\mu)\text{div}\, u_2$ on $\widetilde{\Gamma}$ by (5) and (8), equation (14) implies

$$(\lambda_1 + 2\mu)\text{div}\, u_1 = (\lambda_2 + 2\mu)\text{div}\, u_2 \quad \text{in } \Omega. \tag{15}$$

From (11), (15), (13) and the assumption $(\lambda_1 - \lambda_2)|_{\Gamma_0} = 0$, we conclude

$$\frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu} \quad \text{on } \Gamma_0. \tag{16}$$

Indeed in order to obtain equality (16) observe that rotations preserve the isotropicity of the Lamé system. Therefore, after choosing an arbitrary point $x \in \Gamma_0$, we rotate the coordinates to have $\nu(x) = (0, 0, 1)$. Then, we obtain

$$\frac{\partial u_{1,j}}{\partial x_k}(x) = \frac{\partial u_{2,j}}{\partial x_k}(x), \quad \forall j \in \{1, 2, 3\}, \quad \forall k \in \{1, 2\}.$$

Hence, equality (13) implies that $\frac{\partial u_{1,j}}{\partial \nu}(x) = \frac{\partial u_{2,j}}{\partial \nu}(x)$ for $j = 1, 2$ and (15) implies that $\frac{\partial u_{1,3}}{\partial \nu}(x) = \frac{\partial u_{2,3}}{\partial \nu}(x)$.

3

By (16), if $f \in C^{2+\alpha}(\partial\Omega)$, supp $f \subset \widetilde{\Gamma}$ in (5), then $\frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu}$ on $\partial\Omega$. By a density argument, the above relation holds for the slightly relaxed regularity assumption on the function f. Namely, the function f belongs to $H^{\frac{3}{2}}(\partial\Omega)$. Hence, we removed the assumption that $u_1 \in C^{2+\alpha}(\bar{\Omega})$.

Next, let $f \in H^{\frac{3}{2}}(\partial\Omega)$ and the functions $v_j \in H^2(\Omega)$ be the solutions of the following boundary value problems:

$$\mathcal{L}_{\lambda_j,\mu_j}(x,D)v_j = 0 \quad \text{in } \Omega, \quad v_j|_{\partial\Omega} = f, \quad j \in \{1,2\}. \tag{17}$$

We claim that

$$\frac{\partial v_1}{\partial \nu} = \frac{\partial v_2}{\partial \nu} \quad \text{on} \quad \widetilde{\Gamma}. \tag{18}$$

Indeed, let $w_j \in H^2(\Omega)$ be a solution to the Lamé system

$$\mathcal{L}_{\lambda_j,\mu_j}(x,D)w_j = 0 \quad \text{in} \quad \Omega, \quad w_j|_{\partial\Omega} = g, \quad j \in \{1,2\}, \tag{19}$$

where $g \in H^{\frac{3}{2}}(\partial\Omega)$ and supp $g \subset \widetilde{\Gamma}$ is an arbitrary function. Taking the scalar product of equation (17) with w_j and integrating by parts, we have

$$
\begin{aligned}
0 &= \int_\Omega (\mathcal{L}_{\lambda_j,\mu_j}(x,D)v_j, w_j)\,\mathrm{d}x = \int_\Omega (v_j, \mathcal{L}_{\lambda_j,\mu_j}(x,D)w_j)\,\mathrm{d}x \\
&\quad + \int_{\partial\Omega} ((\Lambda_{\lambda_j,\mu_j}v_j, w_j) - (\Lambda_{\lambda_j,\mu_j}w_j, v_j))\,\mathrm{d}\sigma \\
&= \int_{\partial\Omega} ((\Lambda_{\lambda_j,\mu_j}v_j, g) - (\Lambda_{\lambda_j,\mu_j}w_j, f))\,\mathrm{d}\sigma = \int_{\widetilde{\Gamma}} (\Lambda_{\lambda_j,\mu_j}v_j, g)\,\mathrm{d}\sigma - \int_{\partial\Omega} (\Lambda_{\lambda_j,\mu_j}w_j, f)\,\mathrm{d}\sigma \\
&= \int_{\widetilde{\Gamma}} (\Lambda_{\lambda_j,\mu_j}v_j, g)\,\mathrm{d}\sigma - \int_{\partial\Omega} (\Lambda_{\lambda_1,\mu_1}w_1, f)\,\mathrm{d}\sigma,
\end{aligned}
$$

where $\mathrm{d}\sigma$ denotes the surface measure.

This integral identity implies

$$\Lambda_{\lambda_1,\mu_1}v_1 = \Lambda_{\lambda_2,\mu_2}v_2 \quad \text{on} \quad \widetilde{\Gamma}.$$

Repeating the arguments in (12)–(16), we conclude

$$\frac{\partial v_1}{\partial \nu} = \frac{\partial v_2}{\partial \nu} \quad \text{on } \Gamma_0. \tag{20}$$

Hence, by (16) and (20), the following full Cauchy data are equal:

$$\widetilde{\mathcal{C}}_{\lambda_1,\mu_1} = \widetilde{\mathcal{C}}_{\lambda_2,\mu_2},$$

where

$$\widetilde{\mathcal{C}}_{\lambda,\mu} = \{(u, \Lambda_{\lambda,\mu}u)|_{\partial\Omega}; \mathcal{L}_{\lambda,\mu}(x,D)u = 0 \quad \text{in } \Omega, u|_{\partial\Omega} = f, f \in H^{\frac{3}{2}}(\partial\Omega)\}.$$

Applying the result of [6], we obtain that $\lambda_1 = \lambda_2$.   □

Acknowledgments

References

[1] Akamatsu M, Nakamura G and Steinberg S 1991 Identification of the Lamé coefficients from boundary observations *Inverse Problems* **7** 335–54

[2] Astala K and Päivärinta L 2006 Calderón's inverse conductivity problem in the plane *Ann. Math.* **163** 265–99

[3] Bukhgeim A 2008 Recovering the potential from Cauchy data in two dimensions *J. Inverse Ill-Posed Problems* **16** 19–34

[4] Calderón A P 1980 On an inverse boundary value problem *Seminar on Numerical Analysis and Its Applications to Continuum Physics* (Río de Janeiro: Sociedade Brasileira de Matemática) pp 65–73

[5] Eskin G 2001 Global uniqueness in the inverse scattering problem for the Schrödinger operator with external Yang–Mills potentials *Commun. Math. Phys.* **222** 503–31

[6] Eskin G and Ralston J 2002 On the inverse boundary value problem for linear isotropic elasticity *Inverse Problems* **18** 907–21

[7] Ikehata M 1990 Inversion formulas for the linearized problem for an inverse boundary value problem in elastic prospection *SIAM J. Appl. Math.* **50** 1635–44

[8] Ikehata M 1993 An inverse problem for the plate in the Love–Kirchhoff theory *SIAM J. Appl. Math.* **53** 942–70

[9] Ikehata M 1996 A relationship between two Dirichlet to Neumann maps in anisotropic elastic plate theory *J. Inverse Ill-Posed Problems* **4** 233–43

[10] Ikehata M, Nakamura G and Yamamoto M 1998 Uniqueness in inverse problems for the isotropic Lamé system *J. Math. Sci. Univ. Tokyo* **5** 627–92

[11] Imanuvilov O, Uhlmann G and Yamamoto M 2010 The Calderón problem with partial data in two dimensions *J. Am. Math. Soc.* **23** 655–91

[12] Imanuvilov O, Uhlmann G and Yamamoto M 2011 Determination of second-order elliptic operators in two dimensions from partial Cauchy data *Proc. Natl Acad. Sci. USA* **1008** 467–72

[13] Imanuvilov O and Yamamoto M 2011 Reconstruction of the Lamé parameters from partial Cauchy data *J. Inverse Ill-Posed Problems* **19** 881–91

[14] Nachman A 1996 Global uniqueness for a two-dimensional inverse boundary value problem *Ann. Math.* **143** 71–96

[15] Nakamura G and Uhlmann G 1993 Identification of Lamé parameters by boundary measurements *Am. J. Math.* **115** 1161–87

[16] Nakamura G and Uhlmann G 1994 Global uniqueness for an inverse boundary value problem arising in elasticity *Invent. Math.* **118** 457–74
 Nakamura G and Uhlmann G 2003 *Invent. Math.* **152** 205–7 (erratum)

[17] Nakamura G and Uhlmann G 1995 Inverse boundary problems at the boundary for an elastic system *SIAM J. Math. Anal.* **26** 263–79

[18] Nakamura G and Uhlmann G 2002 Complex geometrical optics solutions and pseudoanalytic matrices *Ill-posed and Inverse Problems* ed S I Kabanikhin and V G Romanov (Utrecht: VSP) pp 305–38

[19] Sylvester J and Uhlmann G 1986 A uniqueness theorem for an inverse boundary value problem in electrical prospection *Commun. Pure Appl. Math.* **39** 91–112

[20] Uhlmann G 2009 Calderón's problem and electrical impedance tomography *Inverse Problems* **25** 123011