Wave Phenomena

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Abstract

This chapter discusses imaging methods related to wave phenomena, and in particular inverse problems for the wave equation will be considered. The first part of the chapter explains the boundary control method for determining a wave speed of a medium from the response operator which models boundary measurements. The second part discusses the scattering relation and travel times, which are different types of boundary data contained in the response operator. The third part gives a brief introduction to curvelets in wave imaging for media with nonsmooth wave speeds. The focus will be on theoretical results and methods.

Contents

1 Introduction

This chapter discusses imaging methods related to wave phenomena. Of the different types of waves that exist, we will focus on acoustic waves and problems which can be modelled by the acoustic wave equation. In the simplest case this is the second order linear hyperbolic equation

$$\partial_t^2 u(x,t) - c(x)^2 \Delta u(x,t) = 0$$

for a sound speed $c(x)$. This equation can be considered as a model for other hyperbolic equations, and the methods presented here can in some cases be extended to study wave phenomena in other fields such as electromagnetism or elasticity.

We will mostly be interested in inverse problems for the wave equation. In these problems one has access to certain measurements of waves (the solutions u) on the surface of a medium, and one would like to determine material parameters (the sound speed c) of the interior of the medium from these boundary measurements. A typical field where such problems arise is seismic imaging, where one wishes to determine the interior structure of Earth by making various measurements of waves at the surface. We will not describe seismic imaging applications in more detail here, since they are discussed elsewhere in this volume.

Another feature in this chapter is that we will consistently consider *anisotropic* materials, where the sound speed depends on the direction of propagation. This means that the scalar sound speed $c(x)$, where $x = (x^1, x^2, \ldots, x^n) \in \Omega \subset \mathbb{R}^n$ is replaced by a positive definite symmetric matrix $(g^{jk}(x))_{j,k=1}^n$, and the wave equation becomes

$$\partial_t^2 u(x,t) - \sum_{j,k=1}^n g^{jk}(x) \frac{\partial^2 u}{\partial x^j \partial x^k}(x,t) = 0.$$

Anisotropic materials appear frequently in applications such as in seismic imaging.

It will be convenient to interpret the anisotropic sound speed (g^{jk}) as the inverse of a Riemannian metric, thus modelling the medium as a *Riemannian manifold*. The benefits of such an approach are twofold. First, the well established methods of Riemannian geometry become available to study the problems, and second, this provides an efficient way of dealing with the invariance under changes of coordinates present in many anisotropic wave imaging problems. The second point means that in inverse problems in anisotropic media, one can often only expect to recover the matrix (g^{jk}) up to a change of coordinates given by some diffeomorphism. In practice this ambiguity could be removed by some a priori knowledge of the medium properties (such as the medium being in fact isotropic, see Section 3.1.2).

2 Background

This chapter contains three parts which discuss different topics related to wave imaging. The first part considers the inverse problem of determining a sound speed in a wave equation from the response operator, also known as the hyperbolic Dirichlet-to-Neumann map, by using the boundary control method, see [5, 7, 43]. The second part considers other types of boundary measurements of waves, namely the scattering relation and boundary distance function, and discusses corresponding inverse problems. The third part is somewhat different in nature and does not consider any inverse problems, but rather gives an introduction to the use of curvelet decompositions in wave imaging for nonsmooth sound speeds. We briefly describe these three topics.

2.1 Wave imaging and boundary control method

Let us consider an isotropic wave equation. Let $\Omega \subset \mathbb{R}^n$ be an open, bounded set with smooth boundary $\partial\Omega$ and let $c(x)$ be a scalar-valued positive function in $C^\infty(\overline{\Omega})$ modeling the wave speed in Ω. First, we consider the wave equation

$$\begin{aligned} \partial_t^2 u(x,t) - c(x)^2 \Delta u(x,t) = 0 \qquad &\text{in } \Omega \times \mathbb{R}_+, \\ u|_{t=0} = 0, \quad u_t|_{t=0} = 0,& \\ c(x)^{-n+1}\partial_{\mathbf{n}} u = f(x,t) \qquad &\text{in } \partial\Omega \times \mathbb{R}_+, \end{aligned} \tag{2.1}$$

where $\partial_{\mathbf{n}}$ denotes the Euclidean normal derivative, where n is the unit interior normal. We denote by $u^f = u^f(x,t)$ the solution of (2.1) corresponding to the boundary source term f.

Let us assume that the domain $\Omega \subset \mathbb{R}^n$ is known. The inverse problem is to reconstruct the wave speed $c(x)$ when we are given the set

$$\{(f|_{\partial\Omega\times(0,2T)},\, u^f|_{\partial\Omega\times(0,2T)}) :\ f \in C_0^\infty(\partial\Omega \times \mathbb{R}_+)\},$$

that is, the Cauchy data of solutions corresponding to all possible boundary sources $f \in C_0^\infty(\partial\Omega \times \mathbb{R}_+)$, $T \in (0,\infty]$. This data is equivalent to the *response operator*

$$\Lambda_\Omega : f \mapsto u^f|_{\partial\Omega\times\mathbb{R}_+}, \tag{2.2}$$

which is also called the *non-stationary Neumann-to-Dirichlet map*. Physically, $\Lambda_\Omega f$ describes the measurement of the medium response to any applied boundary source f and it is equivalent to various physical measurements. For instance, measuring how much energy is needed to force the boundary value $c(x)^{-n+1}\partial_{\mathbf{n}} u|_{\partial\Omega\times\mathbb{R}_+}$ to be equal to any given boundary value $f \in C_0^\infty(\partial\Omega \times \mathbb{R}_+)$ is equivalent to measuring the map Λ_Ω on $\partial\Omega \times \mathbb{R}_+$, see [43, 45]. Measuring Λ_Ω is also equivalent to measuring the corresponding Neumann-to-Dirichlet map for the heat or the Schrödinger equations, or measuring the eigenvalues and the boundary values of the normalized eigenfunctions of the elliptic operator $-c(x)^2\Delta$, see [45].

The inverse problems for the wave equation and the equivalent inverse problems for the heat or the Schrödinger equations go back to works of M. Krein at the end of 50's who used the causality principle in dealing with the one-dimensional inverse problem for an inhomogeneous string, $u_{tt} - c^2(x)u_{xx} = 0$, see e.g. [46]. In his works, causality was transformed into analyticity of the Fourier transform of the solution. A more straightforward hyperbolic version of the method was suggested by A. Blagovestchenskii at the end of 60's-70's [12, 13]. The multidimensional case was studied by M. Belishev [4] in late 80's who understood the role of the PDE-control for these problems and developed the boundary control method for hyperbolic inverse problems in domains of Euclidean space. Of crucial importance for the boundary control method was the result of D. Tataru in 1995 [78, 80] concerning a Holmgren-type uniqueness theorem for non-analytic coefficients. The boundary control method was extended to the anisotropic case by M. Belishev and Y. Kurylev [7]. The geometric version of the boundary control method which we consider in this chapter was developed in [7, 42, 47, 43]. We will consider the inverse problem in the more general setting of an anisotropic wave equation in an unbounded domain or on a non-compact manifold. These problems have been studied in detail in [39, 44] also in the case when the measurements are done only on a part of the boundary. In this paper we present a simplified construction method applicable for non-compact manifolds in the case when measurements are done on the whole boundary. We demonstrate these results in the case when we have an isotropic wave speed $c(x)$ in a bounded domain of Euclidean space. For this we use the fact that in the Euclidean space the only conformal deformation of a compact domain fixing the boundary is the identity map. This implies that after the abstract manifold structure (M, g) corresponding to the wave speed $c(x)$ in a given domain Ω is constructed, we can construct in an explicit way the embedding of the manifold M to the domain Ω and determine $c(x)$ at each point $x \in \Omega$. We note on the history of this result that using Tataru's unique continuation result [78], Theorem 3.5 concerning this case can be proven directly using the boundary control method developed for domains in Euclidean space in [4].

The reconstruction of non-compact manifolds has been considered also in [11, 27] with different kind of data, using iterated time reversal for solutions of the wave equation. We note that the boundary control method can be generalized also for Maxwell and Dirac equations under appropriate geometric conditions, [50, 51], and its stability has been analyzed in [1, 41].

2.2 Travel times and scattering relation

The problem considered in the previous section of recovering a sound speed from the response operator is highly overdetermined in dimensions $n \geq 2$. The Schwartz kernel of the response operator depends on $2n$ variables and the sound speed c depends on n variables.

In Section 3.2 we will show that other types of boundary measurements in wave imaging can be directly obtained from the response operator. One such measurement is the *boundary distance function*, a function of $2n-2$ variables, which measures the travel times of shortest geodesics between boundary points. The problem of determining a sound speed from the travel times of shortest geodesics is the *inverse kinematic problem*. The more general problem of determining a Riemannian metric (corresponding to an anisotropic sound speed) up to isometry from the boundary distance function is the *boundary rigidity problem*. The problem is formally determined if $n = 2$ but overdetermined for $n \geq 3$.

This problem arose in geophysics in an attempt to determine the inner structure of the Earth by measuring the travel times of seismic waves. It goes back to Herglotz [37] and Wiechert and Zoeppritz [85] who considered the case of a radial metric conformal to the Euclidean metric. Although the emphasis has been in the case that the medium is isotropic, the anisotropic case has been of interest in geophysics since the Earth is anisotropic. It has been found that even the inner core of the Earth exhibits anisotropic behavior [24].

To give a proper definition of the boundary distance function, we will consider a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary to be equipped with a Riemannian metric g, that is, a family of positive definite symmetric matrices $g(x) = (g_{jk}(x))_{j,k=1}^n$ depending smoothly on $x \in \overline{\Omega}$. The length of a smooth curve $\gamma : [a, b] \to \overline{\Omega}$ is defined to be

$$L_g(\gamma) = \int_a^b \left(\sum_{j,k=1}^n g_{jk}(\gamma(t))\dot{\gamma}^j(t)\dot{\gamma}^k(t) \right)^{1/2} dt.$$

The distance function $d_g(x, y)$ for $x, y \in \overline{\Omega}$ is the infimum of the lengths of all piecewise smooth curves in $\overline{\Omega}$ joining x and y. The boundary distance function is $d_g(x, y)$ for $x, y \in \partial\Omega$.

In the boundary rigidity problem one would like to determine a Riemannian metric g from the boundary distance function d_g. In fact, since $d_g = d_{\psi^* g}$ for any diffeomorphism $\psi : \overline{\Omega} \to \overline{\Omega}$ which fixes each boundary point, we are looking to recover from d_g the metric g up to such a diffeomorphism. Here $\psi^* g(y) = D\psi(y)^t g(\psi(y)) D\psi(y)$ is the pullback of g by ψ.

It is easy to give counterexamples showing that this can not be done in general, consider for instance the closed hemisphere where boundary distances are given by boundary arcs so making the metric larger in the interior does not change d_g. Michel [55] conjectured that a *simple* metric g is uniquely determined, up to an action of a diffeomorphism fixing the boundary, by the boundary distance function $d_g(x, y)$ known for all x and y on $\partial\Omega$. A metric is called simple if for any two points in $\overline{\Omega}$ there is a unique length minimizing geodesic joining them, and if the boundary is strictly convex.

The conjecture of Michel has been proved for two dimensional simple manifolds [61]. In higher dimensions it is open but several partial results are known, including the recent results of Burago and Ivanov for metrics close to Euclidean [15] and close to hyperbolic [16] (see the survey [40]). Earlier and related works include results for simple metrics conformal to each other [26], [10], [56], [57], [59], [8], for flat metrics [34], for locally symmetric spaces of negative curvature [9], for two dimensional simple metrics with negative curvature [25] and [60], a local result [71], a semiglobal solvability result [54], and a result for generic simple metrics [72].

In case the metric is not simple, instead of the boundary distance function one can consider the more general *scattering relation* which encodes, for any geodesic starting and ending at the boundary, the start point and direction, the end point and direction, and the length of the geodesic. We will see in Section 3.2 that also this information can be determined directly from the response operator. If the metric is simple then the scattering relation and boundary distance function are equivalent, and either one is determined by the other.

The *lens rigidity problem* is to determine a metric up to isometry from the scattering relation. There are counterexamples of manifolds which are trapping, and the conjecture is that on a nontrapping manifold the metric is determined by the scattering relation up to isometry. We refer to [73] and the references therein for known results on this problem.

2.3 Curvelets and wave equations

In Section 3.3 we describe an alternative approach to the analysis of solutions of wave equations, based on a decomposition of functions into basic elements called *curvelets* or *wave packets*. This approach also works for wave speeds of limited smoothness unlike some of the approaches presented earlier. Furthermore, the curvelet decomposition yields efficient representations of functions containing sharp wavefronts along curves or surfaces, thus providing a common framework for representing such data and analyzing wave phenomena and imaging operators. Curvelets and related methods have been proposed as computational tools for wave imaging, and the numerical aspects of the theory are a subject of ongoing research.

A curvelet decomposition was introduced by Smith [68] to construct a solution operator for the wave equation with $C^{1,1}$ sound speed, and to prove Strichartz estimates for such equations. This started a body of research on L^p estimates for low regularity wave equations based on curvelet type methods, see for instance Tataru [81]–[83], Smith [69], Smith-Sogge [70]. Curvelet decompositions have their roots in harmonic analysis and the theory of Fourier integral operators, where relevant works include Córdoba-Fefferman [23] and Seeger-Sogge-Stein [66] (see also Stein [74]).

In a rather different direction, curvelet decompositions came up in image analysis as an optimally sparse way of representing images with C^2 edges, see Candés-Donoho [21] (the name "curvelet" was introduced in [20]). The property that curvelets yield sparse representations for wave propagators was studied in Candés-Demanet [17], [18]. Numerical aspects of curvelet type methods in wave computation are discussed in [19], [30]. Finally, both theoretical and practical aspects of curvelet methods related to certain seismic imaging applications are studied in [2], [14], [29], [31], [65].

3 Mathematical modeling and analysis

3.1 Boundary control method

3.1.1 Inverse problems on Riemannian manifolds.

Let $\Omega \subset \mathbb{R}^n$ be an open, bounded set with smooth boundary $\partial\Omega$ and let $c(x)$ be a scalar-valued positive function in $C^\infty(\overline{\Omega})$ modeling the wave speed in Ω. We consider the closure $\overline{\Omega}$ as a differentiable manifold M with a smooth, non-empty boundary. We consider also a more general case, and allow (M,g) to be a possibly non-compact, complete manifold with boundary. This means that the manifold contains its boundary ∂M and M is complete with metric d_g defined below. Moreover, near each point $x \in M$ there are coordinates (U,X) where $U \subset M$ is a neighborhood of x and $X : U \to \mathbb{R}^n$ if x is an interior point, or $X : U \to \mathbb{R}^{n-1} \times [0,\infty)$ is x is a boundary point such that for any coordinate neighborhoods (U,X) and $(\widetilde{U},\widetilde{X})$ the transition functions $X \circ \widetilde{X}^{-1} : \widetilde{X}(U \cap \widetilde{U}) \to X(U \cap \widetilde{U})$ are C^∞-smooth. Note that all compact Riemannian manifolds are complete according to this definition. Usually we denote the components of X by $X(y) = (x^1(y),\dots,x^n(y))$.

Let u be the solution the wave equation

$$\begin{aligned} u_{tt}(x,t) + Au(x,t) = 0 \qquad &\text{in} \quad M \times \mathbb{R}_+, \\ u|_{t=0} = 0, \qquad &u_t|_{t=0} = 0, \\ B_{\nu,\eta} u|_{\partial M \times \mathbb{R}_+} = f. & \end{aligned} \tag{3.1}$$

Here, $f \in C_0^\infty(\partial M \times \mathbb{R}_+)$ is a real valued function, $A = A(x,D)$ is an elliptic partial differential operator of the form

$$Av=-\sum_{j,k=1}^{n}\mu(x)^{-1}|g(x)|^{-\frac{1}{2}}\frac{\partial}{\partial x^j}\left(\mu(x)|g(x)|^{\frac{1}{2}}g^{jk}(x)\frac{\partial v}{\partial x^k}(x)\right)+q(x)v(x), \tag{3.2}$$

where $g^{jk}(x)$ is a smooth, symmetric, real, positive definite matrix, $|g|=\det(g^{jk}(x))^{-1}$, and $\mu(x)>0$ and $q(x)$ are smooth real valued functions. On existence and properties of the solutions of the equation (3.1), see [52]. The inverse of the matrix $(g^{jk}(x))_{j,k=1}^n$, denoted $(g_{jk}(x))_{j,k=1}^n$ defines a Riemanian metric on M. The tangent space of M at x is denoted by T_xM and it consist of vectors p which in local coordinates (U,X), $X(y)=(x^1(y),\dots,x^n(y))$ are written as $p=\sum_{k=1}^n p^k\frac{\partial}{\partial x^k}$. Similarly, the cotangent space T_x^*M of M at x consist of covectors which are written in the local coordinates as $\xi=\sum_{k=1}^n \xi_k dx^k$. The inner product which g determines in the cotangent space T_x^*M of M at the point x is denoted by $\langle\xi,\eta\rangle_g=g(\xi,\eta)=\sum_{j,k=1}^n g^{jk}(x)\xi_j\eta_k$ for $\xi,\eta\in T_x^*M$. We use the same notation for the inner product at the tangent space T_xM, that is, $\langle p,q\rangle_g=g(p,q)=\sum_{j,k=1}^n g_{jk}(x)p^jq^k$ for $p,q\in T_xM$.

The metric defines a distance function, which we call also the travel time function,

$$d_g(x,y)=\inf|\mu|,\quad |\mu|=\int_0^1\langle\partial_s\mu(s),\partial_s\mu(s)\rangle_g^{1/2}ds,$$

where $|\mu|$ denotes the length of the path μ and the infimum is taken over all piecewise C^1-smooth paths $\mu:[0,1]\to M$ with $\mu(0)=x$ and $\mu(1)=y$.

We define the space $L^2(M,dV_\mu)$ with inner product

$$\langle u,v\rangle_{L^2(M,dV_\mu)}=\int_M u(x)v(x)\,dV_\mu(x),$$

where $dV_\mu=\mu(x)|g(x)|^{1/2}dx^1dx^2\dots dx^n$. By the above assumptions, A is formally selfadjoint, that is,

$$\langle Au,v\rangle_{L^2(M,dV_\mu)}=\langle u,Av\rangle_{L^2(M,dV_\mu)}\quad\text{for } u,v\in C_0^\infty(M^{\mathrm{int}}).$$

Furthermore, let

$$B_{\nu,\eta}v=-\partial_\nu v+\eta v,$$

where $\eta:\partial M\to\mathbb{R}$ is a smooth function and

$$\partial_\nu v=\sum_{j,k=1}^n\mu(x)g^{jk}(x)\nu_k\frac{\partial}{\partial x^j}v(x),$$

where $\nu(x) = (\nu_1, \nu_2, \dots, \nu_m)$ is the interior conormal vector field of ∂M, satisfying $\sum_{j,k=1}^n g^{jk}\nu_j\xi_k = 0$ for all cotangent vectors of the boundary, $\xi \in T^*(\partial M)$. We assume that ν is normalized so that $\sum_{j,k=1}^n g^{jk}\nu_j\nu_k = 1$. If M is compact, then the operator A in the domain $\mathcal{D}(A) = \{v \in H^2(M) :\ \partial_\nu v|_{\partial M} = 0\}$, where $H^s(M)$ denotes the Sobolev spaces on M, is an unbounded selfadjoint operator in $L^2(M, dV_\mu)$.

An important example is the operator

$$A_0 = -c^2(x)\Delta + q(x) \tag{3.3}$$

on a bounded smooth domain $\Omega \subset \mathbb{R}^n$ with $\partial_\nu v = c(x)^{-n+1}\partial_{\mathbf{n}} v$, where $\partial_{\mathbf{n}} v$ is the Euclidean normal derivative of v.

We denote the solutions of (3.1) by

$$u(x,t) = u^f(x,t).$$

For the initial boundary value problem (3.1) we define the non-stationary Robin-to-Dirichlet map, or the response operator Λ by

$$\Lambda f = u^f|_{\partial M \times \mathbb{R}_+}. \tag{3.4}$$

The finite time response operator Λ^T corresponding to the finite observation time $T > 0$ is given by

$$\Lambda^T f = u^f|_{\partial M \times (0,T)}. \tag{3.5}$$

For any set $B \subset \partial M \times \mathbb{R}_+$, we denote $L^2(B) = \{f \in L^2(\partial M \times \mathbb{R}_+) : \operatorname{supp}(f) \subset B\}$. This means that we identify the functions and their zero continuations.

By [79], the map Λ^T can be extended to bounded linear map $\Lambda^T : L^2(B) \to H^{1/3}(\partial M \times (0,T))$ when $B \subset \partial M \times (0,T)$ is compact. Here, $H^s(\partial M \times (0,T))$ denotes the Sobolev space on $\partial M \times (0,T)$. Below we consider Λ^T also as a linear operator $\Lambda^T : L^2_{cpt}(\partial M \times (0,T)) \to L^2(\partial M \times (0,T))$, where $L^2_{cpt}(\partial M \times (0,T))$ denotes the compactly supported functions in $L^2(\partial M \times (0,T))$.

For $t > 0$ and a relatively compact open set $\Gamma \subset \partial M$, let

$$M(\Gamma, t) = \{x \in M \ :\ d_g(x, \Gamma) < t\}. \tag{3.6}$$

This set is called the domain of influence of Γ at time t.

When $\Gamma \subset \partial M$ is an open relatively compact set and $f \in C_0^\infty(\Gamma \times \mathbb{R}_+)$, it follows from finite speed of wave propagation (see e.g. [38]) that the wave $u^f(t) = u^f(\cdot\,, t)$ is supported in the domain $M(\Gamma, t)$, that is,

$$u^f(t) \in L^2(M(\Gamma, t)) = \{v \in L^2(M) :\ \operatorname{supp}(v) \subset M(\Gamma, t)\}. \tag{3.7}$$

We will consider the boundary of the manifold ∂M with the metric $g_{\partial M} = \iota^* g$ inherited from the embedding $\iota : \partial M \to M$. We assume that we are given the *boundary data*, that is, the collection

$$(\partial M, g_{\partial M}) \text{ and } \Lambda \tag{3.8}$$

where $(\partial M, g_{\partial M})$ is considered as a smooth Riemannian manifold with a known differentiable and metric structure and Λ is the non-stationary Robin-to-Dirichlet map given in (3.4).

Our goal is reconstruct the isometry type of the Riemannian manifold (M, g), that is, a Riemannian manifold which is isometric to the manifold (M, g). This is often stated by saying that we reconstruct (M, g) up to an isometry. Our next goal is to prove the following result:

Theorem 3.1. Let (M, g) to be a smooth, complete Riemannian manifold with a non-empty boundary. Assume that we are given the boundary data (3.8). Then it is possible to determine the isometry type of manifold (M, g).

3.1.2 From boundary distance functions to Riemannian metric

In order to reconstruct (M, g) we use a special representation, the *boundary distance representation*, $R(M)$, of M and later show that the boundary data (3.8) determine $R(M)$. We consider next the (possibly unbounded) continuous functions $h : C(\partial M) \to \mathbb{R}$. Let us choose a spesific point $Q_0 \in \partial M$ and a constant $C_0 > 0$ and using these, endow $C(\partial M)$ with the metric

$$d_C(h_1, h_2) = |h_1(Q_0) - h_2(Q_0)| + \sup_{z \in \partial M} \min(C_0, |h_1(z) - h_2(z)|). \tag{3.9}$$

Consider a map $R : M \to C(\partial M)$,

$$R(x) = r_x(\cdot); \quad r_x(z) = d_g(x, z), \ z \in \partial M, \tag{3.10}$$

i.e., $r_x(\cdot)$ is the *distance function* from $x \in M$ to the points on ∂M. The image $R(M) \subset C(\partial M)$ of R is called the boundary distance representation of M. The set $R(M)$ is a metric space with the distance inherited from $C(\partial M)$ which we denote by d_C, too. The map R, due to the triangular inequality, is Lipschitz,

$$d_C(r_x, r_y) \leq 2 d_g(x, y). \tag{3.11}$$

We note that when M is compact and $C_0 = \text{diam}\,(M)$, the metric $d_C : C(\partial M) \to \mathbb{R}$ is a norm which is equivalent to the standard norm $\|f\|_\infty = \max_{x \in \partial M} |f(x)|$ of $C(\partial M)$.

We will see below that the map $R : M \to R(M) \subset C(\partial M)$ is an embedding. Many results of differential geometry, such as Whitney or Nash embedding theorems, concern the question how an abstract manifold can be embedded to some simple space, such as a higher dimensional Euclidean space. In the inverse problem we need to construct a "copy" of the unknown manifold in some known space, and as we assume that the boundary is given, we do this by embedding the manifold M to the known, although infinite dimensional function space $C(\partial M)$.

Next we recall some basic definitions on Riemannian manifolds, see e.g. [22] for an extensive treatment. A path $\mu : [a,b] \to N$ is called a geodesic if, for any $c \in [a,b]$ there is $\varepsilon > 0$ such that if $s,t \in [a,b]$ such that $c - \varepsilon < s < t < c + \varepsilon$, the path $\mu([s,t])$ is a shortest path between its endpoints, i.e.

$$|\mu([s,t])| = d_g(\mu(s), \mu(t)).$$

In the future, we will denote a geodesic path μ by γ and parameterize γ with its arclength s so that $|\mu([s_1,s_2])| = d_g(\mu(s_1), \mu(s_2))$. Let $x(s)$,

$$x(s) = (x^1(s), \dots, x^n(s))$$

be the representation of the geodesic γ in local coordinates (U, X). In the interior of the manifold, that is, for $U \subset M^{\text{int}}$ the path $x(s)$ satisfies the second-order differential equations

$$\frac{d^2 x^k(s)}{ds^2} = -\sum_{i,j=1}^{n} \Gamma^k_{ij}(x(s)) \frac{dx^i(s)}{ds} \frac{dx^j(s)}{ds}, \tag{3.12}$$

where Γ^k_{ij} are the Christoffel symbols, given in local coordinates by the formula

$$\Gamma^k_{ij}(x) = \sum_{p=1}^{n} \frac{1}{2} g^{kp}(x) \left(\frac{\partial g_{jp}}{\partial x^i}(x) + \frac{\partial g_{ip}}{\partial x^j}(x) - \frac{\partial g_{ij}}{\partial x^p}(x) \right).$$

Let $y \in M$ and $\xi \in T_x M$ be a unit vector satisfying the condition $g(\xi, \nu(y)) > 0$ in the case when $y \in \partial M$. Then, we can consider the solution of the initial value problem for the differential equation (3.12) with the initial data

$$x(0) = y, \quad \frac{dx}{ds}(0) = \xi.$$

This initial value problem has a unique solution $x(s)$ on an interval $[0, s_0(y,\xi))$ such that $s_0(y,\xi) > 0$ is the smallest value $s_0 > 0$ for which $x(s_0) \in \partial M$, or $s_0(y,\xi) = \infty$ in case no such s_0 exists. We will denote $x(s) = \gamma_{y,\xi}(s)$ and say that the geodesic is a normal geodesic starting at y if $y \in \partial M$ and $\xi = \nu(y)$.

Example. In the case when (M,g) is such a compact manifold that all geodesics are the shortest curves between their endpoints and all geodesics can be continued to geodesics that hit the boundary, we can see that the metric spaces (M, d_g) and $(R(M), \|\cdot\|_\infty)$ are isometric. Indeed, for any two points $x, y \in M$ there is a geodesic γ from x to a boundary point z, which is a continuation of the geodesic from x to y. As in the considered case the geodesics are distance minimizing curves, we see that

$$r_x(z) - r_y(z) = d_g(x, z) - d_g(y, z) = d_g(x, y),$$

and thus $\|r_x - r_y\|_\infty \geq d_g(x, y)$. Combining this with the triangular inequality, we see that $\|r_x - r_y\|_\infty = d_g(x, y)$ for $x, y \in M$ and R is isometry of (M, d_g) and $(R(M), \|\cdot\|_\infty)$.

Notice that when even M is a compact manifold, the metric spaces (M, d_g) and $(R(M), \|\cdot\|_\infty)$ are not always isometric. As an example, consider a unit sphere in $\mathbb{R}^3$ with a small circular hole near the South pole of, say, diameter ε. Then, for any x, y on the equator and $z \in \partial M$, $\pi/2 - \varepsilon \leq r_x(z) \leq \pi/2$ and $\pi/2 - \varepsilon \leq r_y(z) \leq \pi/2$. Then $d_C(r_x, r_y) \leq \varepsilon$, while $d_g(x, y)$ may be equal to π.

Next, we introduce the *boundary normal coordinates* on M. For a normal geodesic $\gamma_{z,\nu}(s)$ starting from $z \in \partial M$ consider $d_g(\gamma_{z,\nu}(s), \partial M)$. For small s,

$$d_g(\gamma_{z,\nu}(s), \partial M) = s, \tag{3.13}$$

and z is the unique nearest point to $\gamma_{z,\nu}(s)$ on ∂M. Let $\tau(z) \in (0, \infty]$ be the largest value for which (3.13) is valid for all $s \in [0, \tau(z)]$. Then for $s > \tau(z)$,

$$d_g(\gamma_{z,\nu}(s), \partial M) < s,$$

and z is no more the nearest boundary point for $\gamma_{z,\nu}(s)$. The function $\tau(z) \in C(\partial M)$ is called the cut locus distance function and the set

$$\omega = \{\gamma_{z,\nu}(\tau(z)) \in M : \, z \in \partial M, \text{ and } \tau(z) < \infty\}, \tag{3.14}$$

is the *cut locus of M with respect to ∂M*. The set ω is a closed subset of M having zero measure. In particular, $M \setminus \omega$ is dense in M. In the remaining domain $M \setminus \omega$ we can use the coordinates

$$x \mapsto (z(x), t(x)), \tag{3.15}$$

where $z(x) \in \partial M$ is the unique nearest point to x and $t(x) = d_g(x, \partial M)$. (Strictly speaking, one also has to use some local coordinates of the boundary, $y : z \mapsto (y^1(z), \dots, y^{(n-1)}(z))$ and define that

$$x \mapsto (y(z(x)), t(x)) = (y^1(z(x)), \dots, y^{(n-1)}(z(x)), t(x)) \in \mathbb{R}^n, \tag{3.16}$$

are the boundary normal coordinates.) Using these coordinates we show that $R : M \to C(\partial M)$ is an embedding. The result of Lemma 3.2 is considered in detail for compact manifolds in [43].

Lemma 3.2. Let (M, d_g) be the metric space corresponding to a complete Riemannian manifold (M, g) with a non-empty boundary. The map $R : (M, d_g) \to (R(M), d_C)$ is a homeomorphism. Moreover, given $R(M)$ as a subset of $C(\partial M)$ it is possible to construct a distance function d_R on $R(M)$ that makes the metric space $(R(M), d_R)$ isometric to (M, d_g).

Proof. We start by proving that R is a homeomorphism. Recall the following simple result from topology:

Assume that X and Y are Hausdorff spaces, X is compact and $F : X \to Y$ is a continuous, bijective map from X to Y. Then $F : X \to Y$ is a homeomorphism.

Let us next extend this principle. Assume that (X, d_X) and (Y, d_Y) are metric spaces and let $X_j \subset X$, $j \in \mathbb{Z}_+$ be compact sets such that $\bigcup_{j\in\mathbb{Z}_+} X_j = X$. Assume that $F : X \to Y$ is a continuous, bijective map. Moreover, let $Y_j = F(X_j)$ and assume that there is a point $p \in Y$ such that

$$a_j = \inf_{y\in Y\setminus Y_j} d_Y(y, p) \to \infty \text{ as } j \to \infty. \tag{3.17}$$

Then by the above, the maps $F : \cup_{j=1}^n X_j \to \cup_{j=1}^n Y_j$ are homeomorphisms for all $n \in \mathbb{Z}_+$. Next, consider a sequence $y_k \in Y$ such that $y_k \to y$ in Y as $k \to \infty$. By removing first elements of the sequence $(y_k)_{k=1}^\infty$ if needed, we can assume that $d_Y(y_k, y) \le 1$. Let now $N \in \mathbb{Z}_+$ be such that for $j > N$ we have $a_j > b := d_Y(y, p) + 1$. Then $y_k \in \cup_{j=1}^N Y_j$ and as the map $F : \cup_{j=1}^N X_j \to \cup_{j=1}^N Y_j$ is a homeomorphism, we see that $F^{-1}(y_k) \to F^{-1}(y)$ in X as $k \to \infty$. This shows that $F^{-1} : Y \to X$ is continuous and thus $F : X \to Y$ is a homeomorphism.

By definition, $R : M \to R(M)$ is surjective and, by (3.11), continuous. In order to prove the injectivity, assume the contrary, i.e. $r_x(\cdot) = r_y(\cdot)$ but $x \neq y$. Denote by z_0 any point where

$$\min_{z\in\partial M} r_x(z) = r_x(z_0).$$

Then

$$\begin{aligned} d_g(x, \partial M) &= \min_{z\in\partial M} r_x(z) = r_x(z_0) \\ &= r_y(z_0) = \min_{z\in\partial M} r_y(z) = d_g(y, \partial M), \end{aligned} \tag{3.18}$$

and $z_0 \in \partial M$ is a nearest boundary point to x. Let μ_x be the shortest path from z_0 to x. Then, the path μ_x is a geodesic from x to z_0 which intersects ∂M first time at z_0. By using the first variation on length formula, we see that μ_x has to hit to z_0 normally, see [22]. The same considerations are true for the point y with the same point z_0. Thus, both x and y lie on the normal geodesic $\gamma_{z_0,\nu}(s)$ to ∂M. As the geodesics are unique solutions of a system of ordinary differential equations (the Hamilton-Jacobi equation (3.12)), they are uniquely determined by their initial points and directions, that is, the geodesics are non-branching. Thus we see that

$$x = \gamma_{z_0}(s_0) = y,$$

where $s_0 = r_x(z_0) = r_y(z_0)$. Hence $R : M \to C(\partial M)$ is injective.

Next, we consider the condition (3.17) for $R : M \to R(M)$. Let $z \in M$ and consider closed sets $X_j = \{x \in M : \ d_C(R(x), R(z)) \leq j\}$, $j \in \mathbb{Z}_+$. Then for $x \in X_j$ we have by definition (3.9) of the metric d_C that

$$d_g(x, Q_0) \leq j + d_g(z, Q_0),$$

implying that the sets X_j, $j \in \mathbb{Z}_+$ are compact. Clearly, $\bigcup_{j\in\mathbb{Z}_+} X_j = X$. Let next $Y_j = R(X_j) \subset Y = R(M)$ and $p = R(Q_0) \in R(M)$. Then for $r_x \in Y \setminus Y_j$ we have

$$\begin{aligned} d_C(r_x, p) &\geq r_x(Q_0) - p(Q_0) = d_g(x, Q_0) \\ &\geq j - d_g(z, Q_0) - C_0 \to \infty \text{ as } j \to \infty \end{aligned}$$

and thus the condition (3.17) is satisfied. As $R : M \to R(M)$ is a continuous, bijective map, this implies that $R : M \to R(M)$ is a homeomorphism.

Next we introduce a differentiable structure and a metric tensor, g_R, on $R(M)$ to have an isometric diffeomorpism

$$R : (M, g) \to (R(M), g_R). \tag{3.19}$$

Such structures clearly exists – the map R pushes the differentiable structure of M and the metric g to some differentiable structure on $R(M)$ and the metric $g_R := R_* g$ which makes the map (3.19) an isometric diffeomorphism. Next we construct these coordinates and the metric tensor in those on $R(M)$ using the fact that $R(M)$ is known as a subset of $C(\partial M)$.

We will start by construction of the differentiable and metric structures on $R(M) \setminus R(\omega)$, where ω is the cut locus of M with respect to ∂M. First, we show that we can identify in the set $R(M)$ all the elements of the form $r = r_x \in R(M)$ where $x \in M \setminus \omega$. To do this, we observe that $r = r_x$ with $x = \gamma_{z,\nu}(s)$, $s < \tau(z)$ if and only if

i. $r(\cdot)$ has a unique global minimum at some point $z \in \partial M$;

ii. there is $\widetilde{r} \in R(M)$ having a unique global minimum at the same z and $r(z) < \widetilde{r}(z)$. This is equivalent to saying that there is y with $r_y(\cdot)$ having a unique global minimum at the same z and $r_x(z) < r_y(z)$.

Thus we can find $R(M \setminus \omega)$ by choosing all those $r \in R(M)$ for which the above conditions i and ii are valid.

Next, we choose a differentiable structure on $R(M \setminus \omega)$ which makes the map $R : M \setminus \omega \to R(M \setminus \omega)$ a diffeomorphism. This can be done by introducing coordinates near each $r^0 \in R(M \setminus \omega)$. In a sufficiently small neighborhood $W \subset R(M)$ of r^0 the coordinates

$$r \mapsto (Y(r), T(r)) = (y(\mathrm{argmin}_{z\in\partial M} r), \min_{z\in\partial M} r)$$

are well defined. These coordinates have the property that the map $x \mapsto (Y(r_x), T(r_x))$ coincides with the boundary normal coordinates (3.15), (3.16). When we choose the differential structure on $R(M \setminus \omega)$ that corresponds to these coordinates, the map

$$R : M \setminus \omega \to R(M \setminus \omega)$$

is a diffeomorphism.

Next we construct the metric g_R on $R(M)$. Let $r^0 \in R(M \setminus \omega)$. As above, in a sufficiently small neighborhood $W \subset R(M)$ of r^0 there are coordinates $r \mapsto X(r) := (Y(r), T(r))$ that correspond to the boundary normal coordinates. Let $(y^0, t^0) = X(r^0)$. We consider next the evaluation function

$$K_w : W \to \mathbb{R}, \quad K_w(r) = r(w),$$

where $w \in \partial M$. The inverse of $X : W \to \mathbb{R}^n$ is well defined in a neighborhood $U \subset \mathbb{R}^n$ of (y^0, t^0) and thus we can define the function

$$E_w = K_w \circ X^{-1} : U \to \mathbb{R}$$

that satisfies

$$E_w(y, t) := d_g(w, \gamma_{z(y),\nu(y)}(t)), \quad (y, t) \in U, \tag{3.20}$$

where $\gamma_{z(y),\nu(y)}(t)$ is the normal geodesic starting from the boundary point $z(y)$ with coordinates $y = (y^1, \dots, y^{n-1})$ and $\nu(y)$ is the interior unit normal vector at y.

Let now $g_R = R_* g$ be the push-forward of g to $R(M \setminus \omega)$. We denote its representation in X-coordinates by $g_{jk}(y, t)$. Since X corresponds to the boundary normal coordinates, the metric tensor satisfies

$$g_{mm} = 1, \quad g_{\alpha m} = 0, \quad \alpha = 1, \dots, n-1.$$

Consider the function $E_w(y, t)$ as a function of (y, t) with a fixed w. Then its differential, dE_w at point (y, t) defines a covector in $T^*_{(y,t)}(U) = \mathbb{R}^n$. Since the gradient of a distance function is a unit vector field, we see from (3.20) that

$$\|dE_w(y,t)\|^2_{(g_{jk})} := (\frac{\partial}{\partial t} E_w(y,t))^2 + \sum_{\alpha,\beta=1}^{n-1} (g_R)^{\alpha\beta}(y,t) \frac{\partial E_w}{\partial y^\alpha}(y,t) \frac{\partial E_w}{\partial y^\beta}(y,t) = 1.$$

Let us next fix a point $(y^0,t^0)\in U$. Varying the point $w\in\partial M$ we obtain a set of covectors $dE_w(y^0,t^0)$ in the unit ball of $(T^*_{(y^0,t^0)}U,g_{jk})$ which contains an open neighborhood of $(0,\dots,0,1)$. This determines uniquely the tensor $g^{jk}(y^0,t^0)$. Thus we can construct the metric tensor in the boundary normal coordinates at arbitrary $r\in R(M\setminus\omega)$. This means that we can find the metric g_R on $R(M\setminus\omega)$ when $R(M)$ is given.

To complete the reconstruction, we need to find the differentiable structure and the metric tensor near $R(\omega)$. Let $r^{(0)}\in R(\omega)$ and $x^{(0)}\in M^{\rm int}$ be such a point that $r^{(0)}=r_{x^{(0)}}=R(x^{(0)})$. Let z_0 be some of the closest points of ∂M to the point $x^{(0)}$. Then there are points $z_1,\dots,z_{n-1}$ on ∂M, given by $z_j=\mu_{z_0,\theta_j}(s_0)$, where $\mu_{z_0,\theta_j}(s)$ are geodesics of $(\partial M,g_{\partial M})$ and $\theta_1,\dots,\theta_{n-1}$ are orthonormal vectors of $T_{z_0}(\partial M)$ with respect to metric $g_{\partial M}$ and $s_0>0$ is sufficiently small, so that the distance functions $y\mapsto d_g(z_i,y)$, $i=0,1,2,\dots,n-1$ form local coordinates $y\mapsto(d_g(z_i,y))_{i=0}^{n-1}$ on M in some neighborhood of the point $x^{(0)}$ (we omit here the proof which can be found in [43, Lemma 2.14]).

Let now $W\subset R(M)$ be a neighborhood of $r^{(0)}$ and let $\widetilde r\in W$. Moreover, let $V=R^{-1}(W)\subset M$ and $\widetilde x=R^{-1}(\widetilde r)\in V$. Let us next consider arbitrary points $z_1,\dots,z_{n-1}$ on ∂M. Our aim is to verify whether the functions $x\mapsto X^i(x)=d_g(x,z_i)$, $i=0,1,\dots,n-1$ form smooth coordinates in V. As $M\setminus\omega$ is dense on M and we have found topological structure of $R(M)$ and constructed the metric g_R on $R(M\setminus\omega)$, we can choose $r^{(j)}\in R(M\setminus\omega)$ such that $\lim_{j\to\infty}r^{(j)}=\widetilde r$ in $R(M)$. Let $x^{(j)}\in M\setminus\omega$ be the points for which $r^{(j)}=R(x^{(j)})$. Now the function $x\mapsto(X^i(x))_{i=0}^{n-1}$ defines smooth coordinates near $\widetilde x$ if and only if for functions $Z^i(r)=K_{z_i}(r)$ we have

$$\begin{aligned}&\lim_{j\to\infty}\det((g_R(dZ^i(r),dZ^l(r)))_{i,l=0}^{n-1})|_{r=r^{(j)}}\\ =\ &\lim_{j\to\infty}\det((g(dX^i(x),dX^l(x)))_{i,l=0}^{n-1})|_{x=x^{(j)}}\neq 0.\end{aligned}\tag{3.21}$$

Thus for all $\widetilde r\in W$ we can verify for any points $z_1,\dots,z_{n-1}\in\partial M$ whether the condition (3.21) is valid or not and this condition is valid for all $\widetilde r\in W$ if and only if the functions $x\mapsto X^i(x)=d_g(x,z_i)$, $i=0,1,\dots,n-1$ form smooth coordinates in V. Moreover, by the above reasoning we know that any $r^{(0)}\in R(\omega)$ has some neighborhood W and some points $z_1,\dots,z_{n-1}\in\partial M$ for which the condition (3.21) is valid for all $\widetilde r\in W$. By choosing such points, we find also near $r^{(0)}\in(\omega)$ smooth coordinates $r\mapsto(Z^i(r))_{i=0}^{n-1}$ which make the map $R:M\to R(M)$ a diffeomorphism near $x^{(0)}$.

Summarizing, we have constructed differentiable structure (i.e. local coordinates) on the whole set $R(M)$, and this differentiable structure makes the map $R:M\to R(M)$ a diffeomorphism. Moreover, since the metric $g_R=R_*g$ is a smooth tensor, and we have found it in a dense subset $R(M\setminus\omega)$ of $R(M)$, we can continue it in the local coordinates. This gives us the metric g_R on the whole $R(M)$ which makes the map $R: M \to R(M)$ an isometric diffeomorphism. $\square$

In the above proof, the reconstruction of the metric tensor in the boundary normal coordinates can be considered as finding the image of the metric in the travel time coordinates.

Let us next consider the case when we have an unknown isotropic wave speed $c(x)$ in a bounded domain $\Omega \subset \mathbb{R}^n$. We will assume that we are given the set Ω and an abstract Riemannian manifold (M, g) which is isometric to Ω endowed with its travel time metric corresponding to the wave speed $c(x)$. Also, we assume that we are given a map $\psi : \partial\Omega \to \partial M$ which gives the correspondence between the boundary points of Ω and M. Next we show that it is then possible to find an embedding from the manifold M to Ω which gives us the wave speed $c(x)$ at each point $x \in \Omega$. This construction is presented in detail in [43].

For this end, we need first to reconstruct a function σ on M which corresponds to the function $c(x)^2$ on Ω. This is done on the following lemma.

Lemma 3.3. Assume we are given a Riemannian manifold (M, g) such that there exists an open set $\Omega \subset \mathbb{R}^n$ and an isometry $\Psi : (\Omega, (\sigma(x))^{-1}\delta_{ij}) \to (M, g)$ and a function α on M such that $\alpha(\Psi(x)) = \sigma(x)$. Then knowing the Riemannian manifold (M, g), the restriction $\psi = \Psi|_{\partial\Omega} : \partial\Omega \to \partial M$, and the boundary value $\sigma|_{\partial\Omega}$, we can determine the function α.

Proof. First, observe that we are given the boundary value $\alpha|_{\partial M}$ of $\alpha(\Psi(x)) = \sigma(x)$. By assumption the metric g on M is conformally Euclidean, that is, the metric tensor, in the some coordinates, has the form $g_{jk}(x) = \sigma(x)^{-1}\delta_{jk}$ where $\sigma(x) > 0$. Hence the function $\beta = \frac{1}{2}\ln(\alpha)$, when $m = 2$, and $\beta = \alpha^{(n-2)/4}$, when $n \geq 3$, satisfies the so-called scalar curvature equation

$$\Delta_g \beta - k_g = 0 \quad (n = 2), \tag{3.22}$$

$$\frac{4(n-1)}{n-2}\Delta_g \beta - k_g \beta = 0 \quad (n \geq 3), \tag{3.23}$$

where k_g is the scalar curvature of (M, g),

$$k_g(x) = \sum_{k,j,l=1}^{n} g^{jl}(x) R^k_{jkl}(x)$$

where R^i_{jkl} is the curvature tensor given in terms of the Christoffel symbols as

$$R^i_{jkl}(x) = \frac{\partial}{\partial x^k}\Gamma^i_{lj}(x) - \frac{\partial}{\partial x^l}\Gamma^i_{kj}(x) + \sum_{r=1}^{n}(\Gamma^r_{lj}(x)\Gamma^i_{kr}(x) - \Gamma^r_{kj}(x)\Gamma^i_{lr}(x)).$$

The idea of these equations is that if β satisfies, e.g., equation (3.23) in the case $m \geq 3$, then the metric $\beta^{4/(n-2)}g$ has zero scalar curvature. Together with boundary data (3.8) being given, we obtain Dirichlet boundary value problem for β in M.

Clearly, Dirichlet problem for equation (3.22) has a unique solution that gives α when $n = 2$. In the case $n \geq 3$, to show that this boundary value problem has a unique solution, it is necessary to check that 0 is not an eigenvalue of the operator $\frac{4(n-1)}{n-2}\Delta_g - k_g$ with Dirichlet boundary condition. Now, the function $\beta = \alpha^{(n-2)/4}$ is a positive solution of the Dirichlet problem for equation (3.23) with boundary condition $\beta|_{\partial M} = \alpha^{(n-2)/4}\big|_{\partial M}$. Assume that there is another possible solution of this problem,

$$\widetilde{\beta} = v\beta, \quad v > 0, \quad v|_{\partial M} = 1. \tag{3.24}$$

Then both $(M, \beta^{4/(n-2)}g)$ and $(M, \widetilde{\beta}^{4/(n-2)}g)$ have zero scalar curvatures. Denoting $g_1 = \beta^{4/(n-2)}g$, $g_2 = \widetilde{\beta}^{4/(n-2)}g$, we obtain that v should satisfy the scalar curvature equation

$$\frac{4(n-1)}{n-2}\Delta_{g_1}v - k_{g_1}v = 0.$$

Here, we have $k_{g_1} = 0$ as g_1 has vanishing scalar curvature. Together with boundary condition (3.24), this equation implies that $v \equiv 1$, i.e. $\beta = \widetilde{\beta}$. This immediately yields that 0 is not the eigenvalue of the Dirichlet operator (3.23) because, otherwise, we could obtain a positive solution $\widetilde{\beta} = \beta + c_0\psi_0$, where ψ_0 is the Dirichlet eigenfunction, corresponding to zero eigenvalue, and $|c_0|$ is sufficiently small. Thus β, and henceforth α, can be uniquely determined by solving Dirichlet boundary value problems for (3.22) and (3.23). $\square$

Our next goal is to embed the abstract manifold (M, g) with conformally Euclidean metric into Ω with metric $(\sigma(x))^{-1}\delta_{ij}$. To achieve this goal, we use the *a priori* knowledge that such embedding exists and the fact that we have already constructed α corresponding to $\sigma(x)$ on M.

Lemma 3.4. Let (M, g) be a compact Riemannian manifold, $\alpha(x)$ a positive smooth function on M, and $\psi: \partial\Omega \to \partial M$ a diffeomorphism. Assume also that there is a diffeomorphism $\Psi : \overline{\Omega} \to M$ such that

$$\Psi|_{\partial\Omega} = \psi, \quad \Psi^* g = (\alpha(\Psi(x)))^{-1}\delta_{ij}.$$

Then, if Ω, (M, g), α, and ψ are known, it is possible to construct the diffeomorphism Ψ by solving ordinary differential equations.

Proof. Let $\zeta = (z,\tau)$ be the boundary normal coordinates on $M \setminus \omega$. Our goal is to construct the coordinate representation for $\Psi^{-1} = X$,

$$\begin{aligned} X \;\; &: \;\; M \setminus \omega \to \Omega, \\ &X(z,\tau) = (x^1(z,\tau), \ldots, x^n(z,\tau)). \end{aligned}$$

Denote by $h_{ij}(x) = \alpha(\Psi(x))^{-1}\delta_{ij}$ the metric tensor in Ω. Let $\Gamma_{i,jk} = \sum_p g_{ip}\Gamma^p_{jk}$ be the Christoffel symbols of (Ω, h_{ij}) in the Euclidean coordinates and let $\widetilde{\Gamma}_{\sigma,\mu\nu}$ be Christoffel symbols of (M,g), in ζ-coordinates. Next, we consider functions h_{ij}, $\Gamma_{k,ij}$, etc. as functions on $M \setminus \omega$ in (z,τ)-coordinates evaluated at the point $x = x(z,\tau)$, e.g., $\Gamma_{k,ij}(z,\tau) = \Gamma_{k,ij}(x(z,\tau))$. Then, since Ψ is an isometry, the transformation rule of Christoffel symbols with respect to the change of coordinates implies

$$\widetilde{\Gamma}_{\sigma,\mu\nu} = \sum_{i,j,k=1}^{n} \Gamma_{k,ij} \frac{\partial x^i}{\partial \zeta^\mu}\frac{\partial x^j}{\partial \zeta^\nu}\frac{\partial x^k}{\partial \zeta^\sigma} + \sum_{i,j=1}^{n} h_{ij}\frac{\partial x^i}{\partial \zeta^\sigma}\frac{\partial^2 x^j}{\partial \zeta^\mu \partial \zeta^\nu}, \tag{3.25}$$

where

$$h_{ij}(z,\tau) = \frac{1}{\alpha(\Psi(z,\tau))}\delta_{ij}. \tag{3.26}$$

Using equations (3.25) and (3.26) we can write $\frac{\partial^2 x^j}{\partial \zeta^\mu \partial \zeta^\nu}$ in the form

$$\frac{\partial^2 x^j}{\partial \zeta^\mu \partial \zeta^\nu}(\zeta) = \sum_{p,\sigma,\mu,\nu=1}^{n} \alpha(\zeta)\delta^{jp}\left(\widetilde{\Gamma}_{\sigma,\mu\nu}\frac{\partial \zeta^\sigma}{\partial x^p}\right.$$

$$\left. - \sum_{n=1}^{n} \frac{1}{2}\frac{\partial \alpha^{-1}}{\partial \zeta^\sigma}\left[\frac{\partial \zeta^\sigma}{\partial x^n}\delta_{pi} + \frac{\partial \zeta^\sigma}{\partial x^i}\delta_{pn} - \frac{\partial \zeta^\sigma}{\partial x^p}\delta_{ni}\right]\frac{\partial x^i}{\partial \zeta^\mu}\frac{\partial x^n}{\partial \zeta^\nu}\right). \tag{3.27}$$

As α and $\widetilde{\Gamma}_{\sigma,\mu\nu}$ are known as a function of ζ, the right-hand side of (3.27) can be written in the form

$$\frac{\partial^2 x^j}{\partial \zeta^\mu \partial \zeta^\nu} = F^j_{\mu,\nu}\left(\zeta, \frac{\partial x}{\partial \zeta}\right), \tag{3.28}$$

where $F^j_{\mu,\nu}$ are known functions. Choose $\nu = m$, so that

$$\frac{\partial^2 x^j}{\partial \zeta^\mu \partial \zeta^n} = \frac{d}{d\tau}\left(\frac{\partial x^j}{\partial \zeta^\mu}\right).$$

Then, equation (3.28) becomes a system of ordinary differential equations along normal geodesics for the matrix $(\frac{\partial x^j}{\partial \zeta^\mu}(\tau))^n_{j,\mu=1}$. Moreover, since diffeomorphism $\Psi : \partial\Omega \to \partial M$ is given, the boundary derivatives $\frac{\partial x^j}{\partial \zeta^\mu}$, $\mu = 1, \ldots,$ $n-1$, are known for $\zeta^n = \tau = 0$. By relation (3.26),

$$\frac{\partial x^j}{\partial \zeta^n} = \frac{\partial x^j}{\partial \tau} = \alpha^{-1}\frac{\partial x^j}{\partial \mathbf{n}} = -\alpha^{-1}\mathbf{n}^j$$

for $\zeta^n = \tau = 0$ where $\mathbf{n} = (\mathbf{n}^1, \dots, \mathbf{n}^n)$ is the Euclidean unit exterior normal vector. Thus, $\frac{\partial x^j}{\partial \tau}(z, 0)$ are also known. Solving a system of ordinary differential equations (3.28) with these initial conditions at $\tau = 0$, we can construct $\frac{\partial x^j}{\partial \zeta^\mu}(z, \tau)$ everywhere on $M \setminus \omega$. In particular, taking $\mu = n$, we find $\frac{dx^j}{d\tau}(z, \tau)$. Using again the fact that $(x^1(z, 0), \dots, x^n(z, 0)) = \psi(z)$ are known, we obtain the functions $x^j(z, \tau)$, z fixed, $0 \le \tau \le \tau_{\partial M}(z)$, i.e., reconstruct all normal geodesics on Ω with respect to metric h_{ij}. Clearly, this gives us the embedding of (M, g) onto (Ω, h_{ij}). $\square$

Combining the above results we get the following result for the isotropic wave equation.

Theorem 3.5. Let $\Omega \subset \mathbb{R}^n$ to be a bounded open set with smooth boundary and $c(x) \in C^\infty(\overline{\Omega})$ be a strictly positive function. Assume that we know Ω, $c|_{\partial\Omega}$, and the non-stationary Robin-to-Neumann map $\Lambda_{\partial\Omega}$. Then it is possible to determine the function $c(x)$.

We note that in Theorem 3.5 the boundary value $c|_{\partial\Omega}$ of the wave speed $c(x)$ can be determined using the finite velocity of wave propagation (3.7) and the knowledge of Ω and $\Lambda_{\partial\Omega}$, but we will not consider this fact in this chapter.

3.1.3 From boundary data to inner products of waves

Let $u^f(x, t)$ denote the solutions of the hyperbolic equation (3.1), Λ^{2T} be the finite time Robin-to-Dirichlet map for the equation (3.1) and let dS_g denote the Riemannian volume form on the manifold $(\partial M, g_{\partial M})$. We start with the Blagovestchenskii identity.

Lemma 3.6. Let $f, h \in C_0^\infty(\partial M \times \mathbb{R}_+)$. Then

$$\int_M u^f(x, T) u^h(x, T) \, dV_\mu(x) = \tag{3.29}$$

$$= \frac{1}{2} \int_L \int_{\partial M} (f(x, t)(\Lambda^{2T} h)(x, s) - (\Lambda^{2T} f)(x, t) h(x, s)) \, dS_g(x) dt ds,$$

where

$$L = \{(s, t) : \ 0 \le t + s \le 2T, \ t < s, \ t, s > 0\}.$$

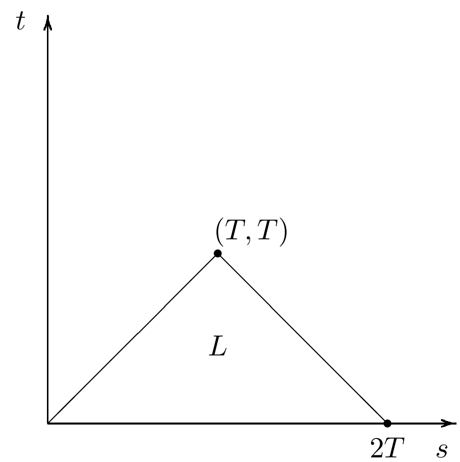
Proof. Let

$$w(t, s) = \int_M u^f(x, t) u^h(x, s) \, dV_\mu(x).$$

Then, by integration by parts, we see that

$$(\partial_t^2 - \partial_s^2) w(t, s) = \int_M [\partial_t^2 u^f(x, t) u^h(x, s) - u^f(x, t) \partial_s^2 u^h(x, s)] \, dV_\mu(x) =$$

Figure 1: Domain of integration in the Blagovestchenskii identity



$$= -\int_M [Au^f(x,t)u^h(x,s) - u^f(x,t)Au^h(x,s)]\, dV_\mu(x) =$$

$$= -\int_{\partial M} [B_{\nu,\eta}u^f(t)u^h(s) - u^f(t)B_{\nu,\eta}u^h(s)]\, dS_g(x) =$$

$$= \int_{\partial M} [\Lambda^{2T}u^f(x,t)u^h(x,s) - u^f(x,t)\Lambda^{2T}u^h(x,s)]\, dS_g(x).$$

Moreover,

$$w|_{t=0} = w|_{s=0} = 0,$$

$$\partial_t w|_{t=0} = \partial_s w|_{s=0} = 0.$$

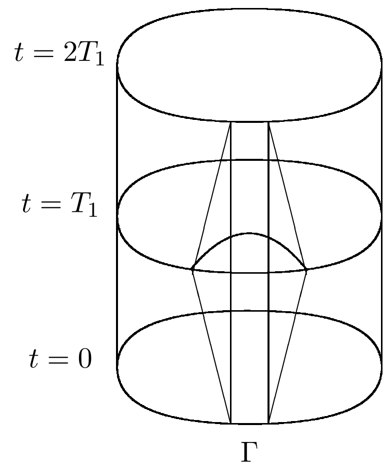
Thus, w is the solution of the initial-boundary value problem for the one-dimensional wave equation in the domain $(t,s) \in [0,2T]\times[0,2T]$ with known source and zero initial and boundary data (3.8). Solving this problem, we determine $w(t,s)$ in the domain where $t+s \leq 2T$ and $t < s$ (see Fig. 1). In particular, $w(T,T)$ gives the assertion. □

The other result is based on the following fundamental theorem by D. Tataru [78, 80].

Theorem 3.7. Let $u(x,t)$ solve the wave equation $u_{tt} + Au = 0$ in $M \times \mathbb{R}$ and $u|_{\Gamma\times(0,2T_1)} = \partial_\nu u|_{\Gamma\times(0,2T_1)} = 0$, where $\emptyset \neq \Gamma \subset \partial M$ is open. Then

$$u = 0 \text{ in } K_{\Gamma,T_1}, \tag{3.30}$$

Figure 2: Double-cone of influence



where

$$K_{\Gamma,T_1} = \{(x,t) \in M \times \mathbb{R} : d_g(x,\Gamma) < T_1 - |t - T_1|\}$$

is the double cone of influence (see Fig. 2).

(The proof of this theorem, in full generality, is in [78]. A simplified proof for the considered case is in [43].)

The observability Theorem 3.7 gives rise to the following approximate controllability:

Corollary 3.8. For any open $\Gamma \subset \partial M$ and $T_1 > 0$,

$$\mathrm{cl}_{L^2(M)}\{u^f(\cdot, T_1) : f \in C_0^\infty(\Gamma \times (0,T_1))\} = L^2(M(\Gamma, T_1)).$$

Here

$$M(\Gamma, T_1) = \{x \in M : d_g(x,\Gamma) < T_1\} = K_{\Gamma,T_1} \cap \{t = T_1\}$$

is the domain of influence of Γ at time T_1 and $L^2(M(\Gamma,T_1)) = \{a \in L^2(M) : \mathrm{supp}(a) \subset M(\Gamma, T_1)\}$.

Proof. Let us assume that $a \in L^2(M(\Gamma,T_1))$ is orthogonal to all $u^f(\cdot, T_1),\ f \in C_0^\infty(\Gamma \times (0,T_1))\}$. Denote by v the solution of the wave equation

$$\begin{aligned} &(\partial_t^2 + A)v = 0; \quad v|_{t=T_1} = 0, \quad \text{in } M \times \mathbb{R}, \\ &\partial_t v|_{t=T_1} = a; \quad B_{\nu,\eta} v|_{\partial M \times \mathbb{R}} = 0. \end{aligned}$$

Using integration by parts we obtain for all $f \in C_0^\infty(\Gamma \times (0,T_1))$

$$\int_0^{T_1} \int_{\partial M} f(x,s) v(x,s)\, dS_g(x)\, ds = \int_M a(x) u^f(x,T_1) dV_\mu(x) = 0,$$

due to the orthogonality of a and the solutions $u^f(t)$. Thus $v|_{\Gamma\times(0,T_1)} = 0$. Moreover, as v is odd with respect to $t = T_1$, that is, $v(x, T_1 + s) = -v(x, T_1 - s)$, we see that $v|_{\Gamma\times(T_1,2T_1)} = 0$. As u satisfies the wave equation, standard energy estimates yield that $u \in C(\mathbb{R}; H^1(M))$, and hence $u|_{\partial M\times\mathbb{R}} \in C(\mathbb{R}; H^{1/2}(\partial M))$. Combining the above, we see that $v|_{\Gamma\times(0,2T_1)} = 0$, and as $B_{\nu,\eta}v|_{\Gamma\times(0,2T_1)} = 0$, we see using Theorem 3.7 that $a = 0$. $\square$

Recall that we denote $u^f(t) = u^f(\cdot\,, t)$.

Lemma 3.9. Let $T > 0$ and $\Gamma_j \subset \partial M$, $j = 1, \dots, J$, be non-empty, relatively compact open sets, $0 \leq T_j^- < T_j^+ \leq T$. Assume we are given $(\partial M, g_{\partial M})$ and the response operator Λ^{2T}. This data determines the inner product

$$J_N^T(f_1, f_2) = \int_N u^{f_1}(x,t) u^{f_2}(x,t)\, dV_\mu(x)$$

for given $t > 0$ and $f_1, f_2 \in C_0^\infty(\partial M \times \mathbb{R}_+)$, where

$$N = \bigcap_{j=1}^{J} (M(\Gamma_j, T_j^+) \setminus M(\Gamma_j, T_j^-)) \subset M.$$

Proof. Let us start with the case when $f_1 = f_2 = f$ and $T_j^- = 0$ for all $j = 1, 2, \dots, J$.

Let $B = \bigcup_{j=1}^{J} (\Gamma_j \times [T - T_j, T])$. For all $h \in C_0^\infty(B)$ it holds by (3.7) that $\mathrm{supp}(u^h(\cdot\,, T)) \subset N$, and thus

$$\begin{aligned} &\|u^f(T) - u^h(T)\|^2_{L^2(M,dV_\mu)} \\ &= \int_N (u^f(x,T) - u^h(x,T))^2 dV_\mu(x) + \int_{M\setminus N} (u^f(x,T))^2 dV_\mu(x). \end{aligned}$$

Let $\chi_N(x)$ be the characteristic function of the set N. By Corollary 3.8, there is $h \in C_0^\infty(B)$ such that the norm $\|\chi_N u^f(T) - u^h(T)\|_{L^2(M,dV_\mu)}$ is arbitrarily small. This shows that $J_N^T(f_1, f_2)$ can be found by

$$J_N^T(f,f) = \|u^f(T)\|^2_{L^2(M,dV_\mu)} - \inf_{h\in C_0^\infty(B)} F(h), \tag{3.31}$$

where

$$F(h) = \|u^f(T) - u^h(T)\|^2_{L^2(M,dV_\mu)}.$$

As $F(h)$ can be computed with the given data (3.8) by Lemma 3.6, it follows that we can determine $J_N^T(f,f)$ for any $f \in C_0^\infty(\partial M \times \mathbb{R}_+)$. Now, since

$$J_N^T(f_1, f_2) = \frac{1}{4}(J_N^T(f_1 + f_2, f_1 + f_2) - J_N^T(f_1 - f_2, f_1 - f_2)),$$

the claim follows in the case when $T_j^- = 0$ for all $j = 1, 2, \dots, J$.

Let us consider the general case when T_j^- may be non-zero. We observe that we can write the characteristic function $\chi_N(x)$ of the set $N = \bigcap_{j=1}^J (M(\Gamma_j, T_j^+) \setminus M(\Gamma_j, T_j^-))$ as

$$\chi_N(x) = \sum_{k=1}^{K_1} c_k \chi_{N_k}(x) - \sum_{k=K_1+1}^{K_2} c_k \chi_{N_k}(x)$$

where $c_k \in \mathbb{R}$ are constants which can be determined by solving a simple linear system of equations and the sets N_k are of the form

$$N_k = \bigcup_{j \in I_k} M(\Gamma_j, t_j)$$

where $I_k \subset \{1, 2, \dots, J\}$ and $t_j \in \{T_j^+ : \ j = 1, 2, \dots, J\} \cup \{T_j^- : \ j = 1, 2, \dots, J\}$. Thus

$$J_N^T(f_1, f_2) = \sum_{k=1}^{K_1} c_k J_{N_k}^T(f_1, f_2) - \sum_{k=K_1+1}^{K_2} c_k J_{N_k}^T(f_1, f_2),$$

where all the terms $J_{N_k}^T(f_1, f_2)$ can be computed using the boundary data (3.8). □

3.1.4 From inner products of waves to boundary distance functions

Let us consider open sets $\Gamma_j \subset \partial M$, $j = 1, 2, \dots, J$ and numbers $T_j^+ > T_j^- \geq 0$. For a collection $\{(\Gamma_j, T_j^+, T_j^-) : \ j = 1, \dots, J\}$ we define the number

$$P(\{(\Gamma_j, T_j^+, T_j^-) : \ j = 1, \dots, J\}) = \sup_f J_N^T(f, f),$$

where $T = (\max T_j^+) + 1$,

$$N = \bigcap_{j=1}^J (M(\Gamma_j, T_j^+) \setminus M(\Gamma_j, T_j^-))$$

and the supremum is taken over functions $f \in C_0^\infty(\partial M \times (0, T))$ satisfying $\|u^f(T)\|_{L^2(M)} \leq 1$. When $\Gamma_j^q \subset \partial M$ are open sets so that $\Gamma_j^q \to \{z_j\}$ as $q \to \infty$, that is, $\{z_j\} \subset \Gamma_j^q \subset \Gamma_j^{q-1}$ for all q and $\bigcap_{q=1}^\infty \overline{\Gamma}_j^q = \{z_j\}$, we denote

$$P(\{(z_j, T_j^+, T_j^-) : \ j = 1, \dots, J\}) = \lim_{q \to \infty} P(\{(\Gamma_j^q, T_j^+, T_j^-) : \ j = 1, \dots, J\}).$$

Theorem 3.10. Let $\{z_n\}_{n=1}^{\infty}$ be a dense set on ∂M and $r(\cdot) \in C(\partial M)$ be an arbitrary continuous function. Then $r \in R(M)$ if and only if for all $N > 0$ it holds that

$$P(\{(z_j, r(z_n) + \frac{1}{N}, r(z_n) - \frac{1}{N}) : \ j = 1, \dots, N\}) > 0. \tag{3.32}$$

Moreover, condition (3.32) can be verified using the boundary data (3.8). Hence the boundary data determine uniquely the boundary distance representation $R(M)$ of (M, g) and therefore determines the isometry type of (M, g).

Proof. *"If"–part.* Let $x \in M$ and denote for simplicity $r(\cdot) = r_x(\cdot)$. Consider a ball $B_{1/N}(x) \subset M$ of radius $1/N$ and center x in (M, g). Then, for $z \in \partial M$

$$B_{1/N}(x) \subset M(z, r(z) + \frac{1}{N}) \setminus M(z, r(z) - \frac{1}{N}).$$

By Corollary 3.8, for any $T > r(z)$ there is $f \in C_0^\infty(\partial M \times (0, T))$ such that the function $u^f(\cdot\,, T)$ does not vanish a.e. in $B_{1/N}(x)$. Thus for any $N \in \mathbb{Z}_+$ and $T = \max\{r(z_n) : \ n = 1, 2, \dots, N\}$ we have

$$\begin{aligned} &P(\{(z_j, r(z_n) + \frac{1}{N}, r(z_n) - \frac{1}{N}) : \ j = 1, \dots, N\}) \\ &\geq \int_{B_{1/N}(x)} |u^f(x, T)|^2 \, dV_\mu(x) > 0 \end{aligned}$$

"Only if"–part. Let (3.32) be valid. Then for all $N > 0$ there are points

$$x_N \in A_N = \bigcap_{n=1}^{N} \left(M(z_n, r(z_n) + \frac{1}{N}) \setminus M(z_n, r(z_n) - \frac{1}{N}) \right) \tag{3.33}$$

as the set A_N has to have a non-zero measure. By choosing a suitable subsequence of x_N (denoted also by x_N), there exists a limit $x = \lim_{N\to\infty} x_N$.

Let $j \in \mathbb{Z}_+$. It follows from (3.33) that

$$r(z_j) - \frac{1}{N} \leq d_g(x_N, z_j) \leq r(z_j) + \frac{1}{N} \quad \text{for all } N \geq j.$$

As the distance function d_g on M is continuous, we see by taking limit $N \to \infty$ that

$$d_g(x, z_j) = r(z_j), \quad j = 1, 2, \dots.$$

Since $\{z_j\}_{j=1}^{\infty}$ are dense in ∂M, we see that $r(z) = d_g(x, z)$ for all $z \in \partial M$, that is, $r = r_x$. □

Note that this proof provides an algorithm for construction of an isometric copy of (M, g) when the boundary data (3.8) are given.

3.1.5 Alternative reconstruction of metric via Gaussian beams

Next we consider an alternative construction of the boundary distance representation $R(M)$, developed in [6, 42, 43]. In the previous considerations, we used in Lemma 3.9 the sets of type $N=\bigcap_{j=1}^{J}(M(\Gamma_j,T_j^+)\setminus M(\Gamma_j,T_j^-))\subset M$ and studied the norms $\|\chi_N u^f(\cdot\,,T)\|_{L^2(M)}$. In the alternative construction considered below we need to consider only the sets N of the form $N=M(\Gamma_0,T_0)$. For this end, we consider solutions $u^f(x,t)$ with special sources f which produce wave packets, called the Gaussian beams [3, 64]. For simplicity, we consider just the case when

$$A=-\Delta_g+q,$$

and give a very short exposition on the construction of the Gaussian beam solutions. Details can be found in e.g. in [43, Ch. 2.4] where the properties of Gaussian beams are discussed in detail. In this section, we consider complex valued solutions $u^f(x,t)$.

Gaussian beams, called also "quasiphotons", are a special class of solutions of the wave equation depending on a parameter $\varepsilon>0$ which propagate in a neighborhood of a geodesic $\gamma=\gamma_{y,\xi}([0,L])$, $g(\xi,\xi)=1$. Below, we consider first the construction in the case when γ is in the interior of M.

To construct Gaussian beams we start by considering an asymptotic sum, called formal Gaussian beam,

$$U_\varepsilon(x,t)=M_\varepsilon\exp\{-(i\varepsilon)^{-1}\theta(x,t)\}\sum_{k=0}^{N}u_k(x,t)(i\varepsilon)^k, \tag{3.34}$$

where $x\in M$, $t\in[t_-,t_+]$, and $M_\varepsilon=(\pi\varepsilon)^{-n/4}$ is the normalization constant. The function $\theta(x,t)$ is called the phase function and $u_k(x,t)$, $k=0,1,\dots,N$ are the amplitude functions. A phase function $\theta(x,t)$ is associated with a geodesic $t\mapsto\gamma(t)\in M$ if

$$\operatorname{Im}\theta(\gamma(t),t)=0, \tag{3.35}$$

$$\operatorname{Im}\theta(x,t)\geq C_0 d_g(x,\gamma(t))^2, \tag{3.36}$$

for $t\in[t_-,t_+]$. These conditions guarantee that for any t the absolute value of $U_\varepsilon(x,t)$ looks like a Gaussian function in the x variable which is centered at $\gamma(t)$. Thus the formal Gaussian beam can be considered to move in time along the geodesic $\gamma(t)$. The phase function can be constructed so that it satisfies the eikonal equation

$$(\frac{\partial}{\partial t}\theta(x,t))^2-g^{jl}(x)\frac{\partial}{\partial x^j}\theta(x,t)\,\frac{\partial}{\partial x^l}\theta(x,t)\asymp 0, \tag{3.37}$$

where $\asymp$ means the coincidence of the Taylor coefficients of both sides considered as functions of x at the points $\gamma(t), t \in [t_-, t_+]$, that is,

$$v(x,t) \asymp 0 \quad \text{if } \partial_x^\alpha v(x,t)|_{x=\gamma(t)} = 0 \text{ for all } \alpha \in \mathbb{N}^n \text{ and } t \in [t_-, t_+].$$

The amplitude functions u_k, $k = 0, \dots, N$ can be constructed as solutions of the transport equations

$$\mathcal{L}_\theta u_k \asymp (\partial_t^2 - \Delta_g + q) u_{k-1}, \quad \text{with } u_{-1} = 0. \tag{3.38}$$

Here $\mathcal{L}_\theta$ is the transport operator

$$\mathcal{L}_\theta u = 2\partial_t \theta \partial_t u - 2\langle \nabla\theta, \nabla u\rangle_g + (\partial_t^2 - \Delta_g)\theta \cdot u, \tag{3.39}$$

where $\nabla u(x,t) = \sum_j g^{jk}(x) \frac{\partial u}{\partial x^k}(x,t) \frac{\partial}{\partial x^k}$ is the gradient on (M,g). The following existence result is proven e.g. in [3, 64, 43]:

Theorem 3.11. Let $y \in M^{\text{int}}$, $\xi \in T_x M$ be a unit vector and $\gamma = \gamma_{y,\xi}(t)$, $t \in [t_-, t_+] \subset \mathbb{R}$ be a geodesic lying in M^{int} when $t \in (t_-, t_+)$.

Then there are functions $\theta(x,t)$ and $u_k(x,t)$ satisfying (3.36)–(3.38) and a solution $u_\varepsilon(x,t)$ of equation

$$(\partial_t^2 - \Delta_g + q) u_\varepsilon(x,t) = 0, \quad (x,t) \in M \times [t_-, t_+], \tag{3.40}$$

such that

$$|u_\varepsilon(x,t) - \phi(x,t) U_\varepsilon(x,t)| \le C_N \varepsilon^{\widetilde{N}(N)}, \tag{3.41}$$

where $\widetilde{N}(N) \to \infty$ when $N \to \infty$. Here $\phi \in C_0^\infty(M \times \mathbb{R})$ is a smooth cut-off function satisfying $\phi = 1$ near the trajectory $\{(\gamma(t), t) : \ t \in [t_-, t_+]\} \subset M \times \mathbb{R}$.

In the other words, for an arbitrary geodesic in the interior of M there is a Gaussian beam that propagates along this geodesic.

Next we consider a class of boundary sources in (3.1) which generate Gaussian beams. Let $z_0 \in \partial M$, $t_0 > 0$, and let $x \mapsto z(x) = (z^1(x), \dots, z^{n-1}(x))$ be a local system of coordinates on $W \subset \partial M$ near z_0. For simplicity, we denote these coordinates as $z = (z^1, \dots, z^{n-1})$ and make computations without reference to the point x. Consider a class of functions $f_\varepsilon = f_{\varepsilon, z_0, t_0}(z,t)$ on the boundary cylinder $\partial M \times \mathbb{R}$, where

$$f_\varepsilon(z,t) = B_{\nu,\eta}((\pi\varepsilon)^{-n/4} \phi(z,t) \exp\{i\varepsilon^{-1}\Theta(z,t)\} V(z,t)). \tag{3.42}$$

Here $\phi \in C_0^\infty(\partial M \times \mathbb{R})$ is one near near (z_0, t_0) and

$$\Theta(z,t) = -(t - t_0) + \frac{1}{2}\langle H_0(z - z_0), (z - z_0)\rangle + \frac{i}{2}(t - t_0)^2, \tag{3.43}$$

where $\langle \cdot\,,\cdot \rangle$ is the complexified Euclidean inner product, $\langle a, b \rangle = \sum a_j b_j$, and $H_0 \in \mathbb{C}^{n\times n}$ is a symmetric matrix with a positive definite imaginary part, i.e., $(H_0)_{jk} = (H_0)_{kj}$ and $\mathrm{Im}\, H_0 > 0$, where $(\mathrm{Im}\, H_0)_{jk} = \mathrm{Im}\,(H_0)_{jk}$. Finally, $V(z,t)$ is a smooth function supported in $W \times \mathbb{R}_+$, having non-zero value at (z_0, t_0). The solution $u^{f_\varepsilon}(x,t)$ of the wave equation

$$\begin{aligned} &\partial_t^2 u - \Delta_g u + qu = 0, \quad \text{in } M \times \mathbb{R}_+, \\ &u|_{t=0} = \partial_t u|_{t=0} = 0, \\ &B_{\nu,\eta} u|_{\partial M \times \mathbb{R}_+} = f_\varepsilon(z,t) \end{aligned} \tag{3.44}$$

is a Gaussian beam propagating along the normal normal geodesic $\gamma_{z_0,\nu}$. Let $S(z_0) \in (0,\infty]$ be the smallest values $s > 0$ so that $\gamma_{z_0,\nu}(s) \in \partial M$, that is, the first time when the geodesic $\gamma_{z_0,\nu}$ hits to ∂M, or $S(z_0) = \infty$ if no such value $s > 0$ exists. Then the following result in valid (see e.g. [43]).

Lemma 3.12. For any function $V \in C_0^\infty(W \times \mathbb{R}_+)$ being one near (z_0, t_0), $t_0 > 0$, and $0 < t_1 < S(z_0)$ and $N \in \mathbb{Z}_+$ there are C_N so that the solution $u^{f_\varepsilon}(x,t)$ of problem (3.44) satisfies estimates

$$|u^{f_\varepsilon}(x,t) - \phi(x,t) U_\varepsilon(x,t)| \le C_N \varepsilon^{\widetilde{N}(N)}, \quad 0 \le t < t_0 + t_1 \tag{3.45}$$

where $U_\varepsilon(x,t)$ is of the form (3.34), for all $0 < \varepsilon < 1$, where $\widetilde{N}(N) \to \infty$ when $N \to \infty$ and $\phi \in C_0^\infty(M \times \mathbb{R})$ is ϕ one near the trajectory $\{(\gamma_{z_0,\nu}(t), t + t_0) :\ t \in [0, t_1]\} \subset M \times \mathbb{R}$.

Let us denote

$$P_{y,\tau} v(x) = \chi_{M(y,\tau)}(x) v(x).$$

Then, the boundary data $(\partial M, g_{\partial M})$ and the operator Λ uniquely determine the values $\|P_{y,\tau} u^f(t)\|_{L^2(M)}$ for any $f \in C_0^\infty(\partial M \times \mathbb{R}_+)$, $y \in \partial M$ and $t, \tau > 0$. Let f_ε be of form (3.42)–(3.43) and $u_\varepsilon(x,t) = u^f(x,t),\ f = f_\varepsilon$ be a Gaussian beam propagating along $\gamma_{z_0,\nu}$ described in Lemma 3.12. The asymptotic expansion (3.34) of a Gaussian beam implies that for $s < S(z_0)$ and $\tau > 0$,

$$\lim_{\varepsilon \to 0} \|P_{y,\tau} u_\varepsilon(\cdot\,, s + t_0)\|_{L^2(M)} = \begin{cases} h(s), & \text{for } d_g(\gamma_{z_0,\nu}(s), y) < \tau, \\ 0, & \text{for } d_g(\gamma_{z_0,\nu}(s), y) > \tau, \end{cases} \tag{3.46}$$

where $h(s)$ is a strictly positive function. By varying $\tau > 0$, we can find $d_g(\gamma_{z_0,\nu}(s), y) = r_x(y)$, where $x = \gamma_{z_0,\nu}(t)$. Moreover, we see that $S(z_0)$ can be determined using the boundary data and (3.46) by observing that $S(z_0)$ is the smallest number $S > 0$ such that if $t_k \to S$ is an increasing sequence, then

$$d_g(\gamma_{z_0,\nu}(s_k), \partial M) = \inf_{y \in \partial M} d_g(\gamma_{z_0,\nu}(s_k), y) \to 0 \quad \text{as } k \to \infty.$$

Summarizing, for any $z_0 \in \partial M$ we can find $S(z_0)$ and furthermore, for any $0 \leq t < S(z_0)$ we can find the boundary distance function $r_x(y)$ with $x = \gamma_{z_0,\nu}(t)$. As any point $x \in M$ can be represented in this form, we see that the boundary distance representation $R(M)$ can be constructed from the boundary data using the Gaussian beams.

3.2 Travel times and scattering relation

We will show in this section that if $(\overline{\Omega}, g)$ is a simple Riemannian manifold then by looking at the singularities of the response operator we can determine the boundary distance function $d_g(x, y)$, $x, y \in \partial\Omega$, that is, the travel times of geodesics going through the domain. The boundary distance function is a function of $2n - 2$ variables. Thus the inverse problem of determining the Riemannian metric from the boundary distance function is formally determined in two dimensions and formally overdetermined in dimensions $n \geq 3$.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. If the response operators for the two manifolds $(\overline{\Omega}, g_1)$ and $(\overline{\Omega}, g_2)$ are the same then we can assume, after a change of variables which is the identity at the boundary, the two metrics g_1 and g_2 have the same Taylor series at the boundary [77]. Therefore we can extend both metrics smoothly to be equal outside outside Ω and Euclidean outside a ball of radius R. We denote the extensions to $\mathbb{R}^n$ $g_j, j = 1, 2$, as before. Let $u_j(t, x, \omega)$, be the solution of the continuation problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta_{g_j} u_j & = & 0, \text{ in } \mathbb{R}^n \times \mathbb{R} \\ \\ u_j(x, t) & = & \delta(t - x \cdot \omega), t < -R, \end{cases} \tag{3.47}$$

where $\omega \in \mathbb{S}^{n-1} = \{x \in \mathbb{R}^n; |x| = 1\}$.

It was shown in [77] that if the response operators for $(\overline{\Omega}, g_1)$ and $(\overline{\Omega}, g_2)$ are equal then the two solutions coincide outside Ω, namely

$$u_1(t, x, \omega) = u_2(t, x, \omega), \quad x \in \mathbb{R}^n \setminus \Omega. \tag{3.48}$$

In the case that the manifold $(\Omega, g_j), j = 1, 2$ is simple, we will use methods of geometrical optics to construct solutions of (3.47) to show that if the response operators of g_1 and g_2 are the same then the boundary distance functions of the metrics g_1 and g_2 coincide.

3.2.1 Geometrical optics

Let g denote a smooth Riemannian metric which is Euclidean outside a ball of radius R.

We will construct solutions to the continuation problem for the metric g (which is either g_1 or g_2). We fix ω. Let us assume that there is a solution to equation (3.47) of the form

$$u(x,t,\omega) = a(x,\omega)\delta(t-\phi(x,\omega)) + v(x,\omega), \quad u = 0, t < -R, \tag{3.49}$$

where a, ϕ are functions to be determined and $v \in L^2_{loc}$ Notice that in order to satisfy the initial conditions in (3.47), we require that

$$a = 1, \quad \phi(x,\omega) = x\cdot\omega \text{ for } x\cdot\omega < -R. \tag{3.50}$$

By replacing equation (3.49) in equation (3.47), it follows that

$$\frac{\partial^2 u}{\partial t^2} - \Delta_g u = \tag{3.51}$$
$$A\delta''(t-\phi(x,\omega)) + B\delta'(t-\phi(x,\omega)) - (\Delta_g a)\delta(t-\phi(x,\omega)) + \frac{\partial^2 v}{\partial t^2} - \Delta_g v,$$

where

$$A = a(x,\omega)\left(1 - \sum_{i,j=1}^{n} g^{ij}\frac{\partial\phi}{\partial x^i}\frac{\partial\phi}{\partial x^j}\right) \tag{3.52}$$
$$B = 2\sum_{j,k=1}^{n} g^{jk}\frac{\partial a}{\partial x^k}\frac{\partial\phi}{\partial x^j} + a\Delta_g\phi \tag{3.53}$$
$$\tag{3.54}$$

We choose the functions ϕ, a in the expansion (3.51) to eliminate the singularities δ'' and δ' and then construct v so that

$$\frac{\partial^2 v}{\partial t^2} - \Delta_g v = (\Delta_g a)\delta(t-\phi(x,\omega)), \quad v = 0, t < -R. \tag{3.55}$$

The eikonal equation

In order to solve the equation $A = 0$, it is sufficient to solve the equation

$$\sum_{i,j=1}^{n} g^{ij}\frac{\partial\phi}{\partial x^i}\frac{\partial\phi}{\partial x^j} = 1, \quad \phi(x,\omega) = x\cdot\omega, x\cdot\omega < -R. \tag{3.56}$$

Equation (3.56) is known as the *eikonal equation*. Here we will describe a method, using symplectic geometry, to solve this equation.

Let $H_g(x,\xi) = \frac{1}{2}(\sum_{i,j=1}^{n} g^{ij}(x)\xi_i\xi_j - 1)$ the Hamiltonian associated to the metric g. Note that the metric induced by g in the cotangent space $T^*\mathbb{R}^n$ is given by the principal symbol of the Laplace-Beltrami operator

$g^{-1}(x,\xi) = \sum_{i,j=1}^{n} g^{ij}(x)\xi_i\xi_j$. The equation (3.56) together with the initial condition can be rewritten as

$$H_g(x, d\phi) = 0, \quad \phi(x,\omega) = x\cdot\omega, x\cdot\omega < -R$$

where $d\phi = \sum_{i=1}^{n} \frac{\partial\phi}{\partial x^i}\, dx^i$ is the differential of ϕ.

Let $S = \{(x,\xi) : H_g(x,\xi) = 0\}$, and let $M_\phi = \{(x, \nabla\phi(x)) : x \in \mathbb{R}^n\}$, then solving equation (3.56), is equivalent to finding ϕ such that

$$M_\phi \subset S, \text{ with } M_\phi = \{(x,\omega); x\cdot\omega < -R\}. \tag{3.57}$$

In oder to find ϕ so that (3.57) is valid we need to find a Lagrangian submanifold L so that $L \subset S$, $L = \{(x,\omega); x\cdot\omega < -R\}$ and the projection of $T^*\mathbb{R}^n$ to $\mathbb{R}^n$ is a diffeomorphism [32]. We will construct such a Lagrangian manifold by flowing out from $N = \{(x,\omega) : x\cdot\omega = s \text{ and } s < -R\}$ by the geodesic flow associated to the metric g. We recall the definition of geodesic flow.

We define the Hamiltonian vector field associated to H_g

$$V_g = (\frac{\partial H_g}{\partial \xi}, -\frac{\partial H_g}{\partial x}). \tag{3.58}$$

The bicharacteristics are the integral curves of H_g

$$\frac{d}{ds}x^m = \sum_{j=1}^{n} g^{mj}\xi_j, \quad \frac{d}{ds}\xi_m = -\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial g^{ij}}{\partial x^m}\xi_i\xi_j, m = 1, ..., n. \tag{3.59}$$

The projections of the bicharacteristics in the x variable are the geodesics of the metric g and the parameter s denotes arc length. We denote the associated geodesic flow by

$$X_g(s) = (x_g(s), \xi_g(s)).$$

If we impose the condition that the bicharacteristics are in S initially, then they belong to S for all time, since the Hamiltonian vector field V_g is tangent to S. The Hamiltonian vector field is transverse to N then the resulting manifold obtained by flowing N along the integral curves of V_g will be a Lagrangian manifold L contained in S. We shall write $L = X_g(N)$.

Now the projection of N into the base space is a diffeomorphism so that $L = \{(x, d_x\phi)\}$ locally near a point of N. We can construct a global solution of (3.57) near Ω if the manifold is simple. We recall:

Definition. Let Ω be a bounded domain of Euclidean space with smooth boundary and g a Riemannian metric on $\overline{\Omega}$. We say that $(\overline{\Omega}, g)$ is simple if given two points on the boundary there is a unique minimizing geodesic joining the two points on the boundary and, moreover, $\partial\Omega$ is geodesically convex.

If $(\overline{\Omega}, g)$ is simple then we extend the metric smoothly in a small neighborhood so that the metric g is still simple. In this case we can solve the eikonal equation globally in a neighborhood of Ω.

The transport equation

The equation $B = 0$ is equivalent to solving the following equation:

$$\sum_{i,j=1}^{n} g^{ij} \frac{\partial \phi}{\partial x^j} \frac{\partial a}{\partial x^i} + \frac{a}{2} \Delta_g \phi = 0. \tag{3.60}$$

Equation (3.60) is called the *transport equation*. It is a vector field equation for $a(x, \omega)$, which is solved by integrating along the integral curves of the vector field $v = \sum_{i,j=1}^{n} g^{ij} \frac{\partial \phi}{\partial x^j} \frac{\partial}{\partial x^i}$. It is an easy computation to prove that v has length 1 and that the integral curves of v are the geodesics of the metric g.

The solution of the transport equation (3.60) is then given by:

$$a(x, \omega) = \exp\left(-\frac{1}{2} \int_\gamma \Delta_g \phi \right), \tag{3.61}$$

where γ is the unique geodesic such that $\gamma(0) = y$, $\dot{\gamma}(0) = \omega$, $y \cdot \omega = 0$ and γ passes through x. If (Ω, g) is a simple manifold then $a \in C^\infty(\mathbb{R}^n)$.

To end the construction of the geometrical optics solutions we observe that the function $v(t, x, \omega) \in L^2_{\text{loc}}$ by using standard regularity results for hyperbolic equations.

Now we state the main result of this section:

Theorem 3.13. Let $(\overline{\Omega}, g_i), i = 1, 2$ be simple manifolds, and assume that the response operators for $(\overline{\Omega}, g_1)$ and $(\overline{\Omega}, g_2)$ are equal. Then $d_{g_1} = d_{g_2}$.

Sketch of proof. Assume that we have two metrics g_1, g_2 with the same response operator. Then by (3.48) the solutions of (3.47) are the same outside Ω. Therefore the main singularity of the solutions in the geometrical optics expansion must be the same outside Ω. Thus we conclude that

$$\phi_1(x, \omega) = \phi_2(x, \omega), \quad x \in \mathbb{R}^n \setminus \Omega. \tag{3.62}$$

Now $\phi_j(x, \omega)$ measures the geodesic distance to the hyperplane $x \cdot \omega = -R$ in the metric g. From this we can easily conclude that the geodesic distance between two points in the boundary for the two metrics is the same, that is $d_{g_1}(x, y) = d_{g_2}(x, y), x, y \in \partial\Omega$.

This type of argument was used in [62] to study a similar inverse problem for the more complicated system of elastodynamics. In particular it is proven in [62] that from the response operator associated to the equations of isotropic elastodynamics one can determine, under the assumption of simplicity of the metrics, the lengths of geodesics of the metrics defined by

$$ds^2 = c_p(x)ds_e^2, \quad ds^2 = c_s(x)^2 ds_e^2, \tag{3.63}$$

where ds_e is the length element corresponding to the Euclidian metric, and $c_p(x) = \sqrt{\frac{(\lambda+2\mu)}{\rho}}$, $c_s(x) = \sqrt{\frac{\mu}{\rho}}$ denote the speed of compressional waves and shear waves respectively. Here λ, μ are the Lamé parameters and ρ the density.

Using Mukhometov's result [57], [58] we can then recover both speeds from the response operator. This shows in particular that if we know the density one can determine the Lamé parameters from the response operator. By using the transport equation of geometrical optics, similar to (3.60), and the results on the ray transform (see for instance [67]), Rachele shows that under certain a-priori conditions one can also determine the density ρ [63].

3.2.2 Scattering relation

In the presence of caustics (i.e. the exponential map is not a diffeomorphism) the expansion (3.49) is not valid since we cannot solve the eikonal equation globally in Ω. The solution of (3.48) is globally a Lagrangian distribution (see for instance [38]). These distributions can locally be written in the form

$$u(t,x,\omega) = \int_{\mathbb{R}^m} e^{i\phi(t,x,\omega,\theta)} a(t,x,\omega,\theta)\, d\theta \tag{3.64}$$

where ϕ is a phase function and $a(t,x,\omega)$ is a classical symbol.

Every Lagrangian distribution is determined (up to smoother terms) by a Lagrangian manifold and its symbol. The Lagrangian manifold associated to $u(t,x,\omega)$ is the flow out from $t = x\cdot\omega, t < -R$ by the Hamilton vector field of $p_g(t,x,\tau,\xi) = \tau^2 - \sum_{j,k=1}^{n} g_{jk}(x)\xi^j\xi^k$. Here (τ,ξ) are the dual variables to (t,x) respectively. The projection in the (x,ξ) variables of the flow is given by the flow out from N by geodesic flow, that is the Lagrangian submanifold L described above.

The **scattering relation** (also called lens map), $C_g \subset \big(T^*(\mathbb{R}\times\partial\Omega)\setminus 0\big)\times \big(T^*(\mathbb{R}\times\partial\Omega)\setminus 0\big)$ of a metric $g = (g^{ij})$ on $\overline{\Omega}$ with dual metric $g^{-1} = (g_{ij})$ is defined as follows. Consider bicharacteristic curves, $\gamma : [a,b] \to T^*(\overline{\Omega}\times\mathbb{R})$, of the Hamilton function $p_g(t,x,\tau,\xi)$ which satisfy the following: $\gamma(]a,b[)$ lies in the interior, γ intersects the boundary non-tangentially at $\gamma(a)$ and $\gamma(b)$, and time increases along γ. Then the canonical projection from $\big(T^*_{\mathbb{R}\times\partial\Omega}\big(\mathbb{R}\times\Omega\big)\setminus 0\big)\times\big(T^*_{\mathbb{R}\times\partial\Omega}\big(\mathbb{R}\times\Omega\big)\setminus 0\big)$ onto $\big(T^*(\mathbb{R}\times\partial\Omega)\setminus 0\big)\times T^*(\mathbb{R}\times\partial\Omega)\setminus 0\big)$ maps the endpoint pair $(\gamma(b),\gamma(a))$ to a point in C_g. In other words C_g gives the geodesic distance between points in the boundary and also the points of exit and direction of exit of the geodesic if we know the point of entrance and direction of entrance.

It is well-known that C_g is a homogeneous canonical relation on $\big(\big(T^*(\mathbb{R}\times \partial\Omega)\setminus 0\big)\times\big(T^*(\mathbb{R}\times\partial\Omega)\setminus 0\big)$. (See [35] for the concept of a scattering relation.) C_g is, in fact, a diffeomorphism between open subsets of $T^*(\mathbb{R}\times\partial\Omega)\setminus 0$.

In analogy with Theorem 3.13 we have

Theorem 3.14. Let $g_i, i=1,2$ be Riemannian metrics on $\overline{\Omega}$ such that the response operators for $(\overline{\Omega}, g_1)$ and $(\overline{\Omega}, g_2)$ are equal. Then

$$C_{g_1} = C_{g_2}.$$

Sketch of proof. Since by (3.48) we know the solutions of (3.47) outside Ω. Therefore the associated Lagrangian manifolds to the Lagrangian distributions u_j must be the same outside Ω. By taking the projection of these Lagrangians onto the boundary we get the desired claim.

In the case that $(\overline{\Omega}, g)$ is simple then the scattering relation doesn't give any new information. In fact $((t_1, x_1, \tau, \xi_1), (t_0, x_0, \tau, \xi_0)) \in C_g$ if $t_1 - t_0 = d_g(x_1, x_0)$ and $\xi_j = -\tau\frac{\partial d_g(x_1,x_0)}{\partial x^j}, j = 0, 1$. In other words d_g is the generating function of the scattering relation.

This result was generalized in [36] to the case of the equations of elastodynamics with residual stress. It is shown that knowing the response operator we can recover the scattering relations associated to P and S waves. For this one uses Lagrangian distributions with appropriate polarization.

The scattering relation contains all travel time data; not just information about minimizing geodesics as is the case of the boundary distance function. The natural conjecture is that on a nontrapping manifold this is enough to determine the metric up to isometry. We refer to [73] and the references therein for results on this problem.

3.3 Curvelets and wave imaging

In this section we will discuss in more detail the use of curvelets in wave imaging. We begin by explaining the curvelet decomposition of functions, using the standard second dyadic decomposition of phase space. The curvelets provide tight frames of $L^2(\mathbb{R}^n)$ and give efficient representations of sharp wave fronts. We then discuss why curvelets are useful for solving the wave equation. This is best illustrated in terms of the half-wave equation (a first order hyperbolic equation), where a good approximation to the solution is obtained by decomposing the initial data in curvelets and then by translating each curvelet along the Hamilton flow for the equation. Then we explain how one deals with wave speeds of limited smoothness, and how one can convert the approximate solution operator into an exact one by doing a Volterra iteration.

The treatment below follows the original approach of Smith [68] and focuses on explaining the theoretical aspects of curvelet methods for solving

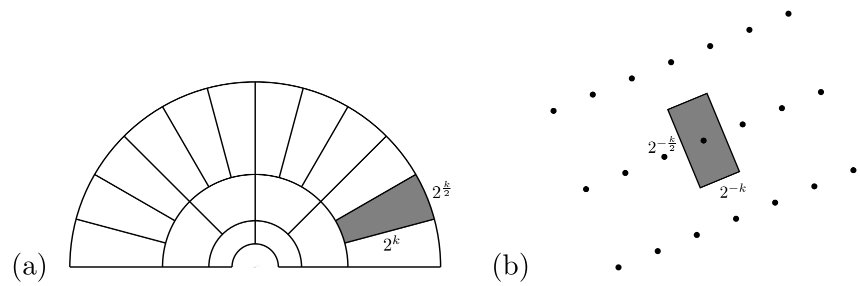


Figure 3: A curvelet φ_γ with $\gamma = (k, \omega, x)$ is concentrated (a) in the frequency domain near a box of length $\sim 2^k$ and width $\sim 2^{k/2}$, and (b) in the spatial side near a box of length $\sim 2^{-k}$ and width $\sim 2^{-k/2}$.

wave equations. We refer to the works mentioned in the introduction for applications and more practical considerations.

3.3.1 Curvelet decomposition

We will explain the curvelet decomposition in its most standard form, as given in [68]. In a nutshell, curvelets are functions which are frequency localized in certain frequency shells and certain directions, according to the second dyadic decomposition and parabolic scaling. On the space side curvelets are concentrated near lattice points which correspond to the frequency localization.

To make this more precise, we recall the *dyadic decomposition* of the frequency space $\{\xi \in \mathbb{R}^n\}$ into the ball $\{|\xi| \leq 1\}$ and dyadic shells $\{2^k \leq |\xi| \leq 2^{k+1}\}$. The *second dyadic decomposition* further subdivides each frequency shell $\{2^k \leq |\xi| \leq 2^{k+1}\}$ into slightly overlapping "boxes" of width roughly $2^{k/2}$ (thus each box resembles a rectangle whose major axis has length $\sim 2^k$ and all other axes have length $\sim 2^{k/2}$). See Figure 3(a) for an illustration. The convention that the width ($2^{k/2}$) of the boxes is the square root of the length (2^k) is called *parabolic scaling*; this scaling is crucial for the wave equation as will be explained later.

In the end, the second dyadic decomposition amounts to having a collection of nonnegative functions $h_0, h_k^\omega \in C_c^\infty(\mathbb{R}^n)$ which form a partition of unity in the sense that

$$1 = h_0(\xi)^2 + \sum_{k=0}^{\infty} \sum_{\omega} h_k^\omega(\xi)^2.$$

Here, for each k, ω runs over roughly $2^{(n-1)k/2}$ unit vectors uniformly distributed over the unit sphere, and h_k^ω is supported in the set

$$2^{k-1/2} \leq |\xi| \leq 2^{k+3/2}, \qquad \left| \frac{\xi}{|\xi|} - \omega \right| \leq 2^{-k/2}.$$

We also require a technical estimate for the derivatives

$$|\langle \omega, \partial_\xi \rangle^j \partial_\xi^\alpha h_k^\omega(\xi)| \leq C_{j,\alpha} 2^{-k(j+|\alpha|/2)},$$

with $C_{j,\alpha}$ independent of k and ω. Such a partition of unity is not hard to construct, we refer to [74, Section 9.4] for the details.

On the frequency side, a curvelet at frequency level 2^k with direction ω will be supported in a rectangle with side length $\sim 2^k$ in direction ω and side lengths $\sim 2^{k/2}$ in the orthogonal directions. By the uncertainty principle, on the spatial side one expects a curvelet to be concentrated in a rectangle with side length $\sim 2^{-k}$ in direction ω and $\sim 2^{-k/2}$ in the other directions. Motivated by this, we define a rectangular lattice Ξ_k^ω in $\mathbb{R}^n$ which has spacing 2^{-k} in direction ω and spacing $2^{-k/2}$ in the orthogonal directions, thus

$$\Xi_k^\omega = \{x \in \mathbb{R}^n \,;\, x = a 2^{-k}\omega + \sum_{j=2}^n b_j 2^{-k/2}\omega_j \text{ where } a, b_j \in \mathbb{Z}\}$$

and $\{\omega, \omega_2, \ldots, \omega_n\}$ is a fixed orthonormal basis of $\mathbb{R}^n$. See Figure 3(b).

We are now ready to give a definition of the curvelet frame.

Definition. For a triplet $\gamma = (k, \omega, x)$ with ω as described above and for $x \in \Xi_k^\omega$, we define the corresponding *fine scale curvelet* φ_γ in terms of its Fourier transform by

$$\hat{\varphi}_\gamma(\xi) = (2\pi)^{-n/2} 2^{-k(n+1)/4} e^{-ix\cdot\xi} h_k^\omega(\xi).$$

The *coarse scale curvelets* for $\gamma = (0, x)$ with $x \in \mathbb{Z}^n$ are given by

$$\hat{\varphi}_\gamma(\xi) = (2\pi)^{-n/2} e^{-ix\cdot\xi} h_0(\xi).$$

The distinction between coarse and fine scale curvelets is analogous to the case of wavelets. The coarse scale curvelets are used to represent data at low frequencies $\{|\xi| \leq 1\}$ and they are direction independent, whereas the fine scale curvelets depend on the direction ω.

The next list collects some properties of the (fine scale) curvelets φ_γ.

- *Frequency localization.* The Fourier transform $\hat{\varphi}_\gamma(\xi)$ is supported in the shell $\{2^{k-1/2} < |\xi| < 2^{k+3/2}\}$ and in a rectangle with side length $\sim 2^k$ in the ω direction and side length $\sim 2^{k/2}$ in directions orthogonal to ω.

- *Spatial localization.* The function $\varphi_\gamma(y)$ is concentrated in (that is, decays away from) a rectangle centered at $x \in \Xi_k^\omega$, having side length 2^{-k} in the ω direction and side lengths $2^{-k/2}$ in directions orthogonal to ω.

- *Tight frame.* Any function $f \in L^2(\mathbb{R}^n)$ may be written in terms of curvelets as
$$f(y) = \sum_\gamma c_\gamma \varphi_\gamma(y)$$
where c_γ are the curvelet coefficients of f:
$$c_\gamma = \int_{\mathbb{R}^n} f(y)\overline{\varphi_\gamma(y)}\,dy.$$
One has the Plancherel identity
$$\int_{\mathbb{R}^n} |f(y)|^2\,dy = \sum_\gamma |c_\gamma|^2.$$

The last statement about how to represent a function $f \in L^2(\mathbb{R}^n)$ in terms of curvelets can be proved by writing

$$\hat{f}(\xi) = h_0(\xi)^2\hat{f}(\xi) + \sum_{k=0}^\infty \sum_\omega h_k^\omega(\xi)^2\hat{f}(\xi)$$

and then by expanding the functions $h_k^\omega(\xi)\hat{f}(\xi)$ in Fourier series in suitable rectangles, and finally by taking the inverse Fourier transform. Note that any L^2 function can be represented as a superposition of curvelets φ_γ, but that the φ_γ are not orthogonal and the representation is not unique.

3.3.2 Curvelets and wave equations

Next we explain, in a purely formal way, how curvelets can be used to solve the Cauchy problem for the wave equation

$$\begin{aligned}(\partial_t^2 + A(x, D_x))u(t, x) &= F(t, x) \quad \text{in } \mathbb{R} \times \mathbb{R}^n,\\ u(0, x) &= u_0(x),\\ \partial_t u(0, x) &= u_1(x).\end{aligned}$$

Further details and references are given in the next section. Here $A(x, D_x) = \sum_{j,k=1}^n g^{jk}(x)D_{x_j}D_{x_k}$ is a uniformly elliptic operator, meaning that $g^{jk} = g^{kj}$ and $0 < \lambda \leq \sum_{j,k=1}^n g^{jk}(x)\xi_j\xi_k \leq \Lambda < \infty$ uniformly over $x \in \mathbb{R}^n$ and $\xi \in S^{n-1}$. We assume that g^{jk} are smooth and have uniformly bounded derivatives of all orders.

It is enough to construct an operator $S(t) : u_1 \mapsto u(t, \cdot)$ such that $u(t,x) = (S(t)u_1)(x)$ solves the above wave equation with $F \equiv 0$ and $u_0 \equiv 0$. Then, by Duhamel's principle, the general solution of the above equation will be

$$u(t,x) = \int_0^t S(t-s)F(s,x)\,ds + (\partial_t S(t)u_0)(x) + (S(t)u_1)(x).$$

To construct $S(t)$, we begin by factoring the wave operator $\partial_t^2 + A(x, D_x)$ into two first order hyperbolic operators, known as *half wave operators*. Let $P(x, D_x) = \sqrt{A(x, D_x)}$ be a formal square root of the elliptic operator $A(x, D_x)$. Then we have

$$\partial_t^2 + A(x, D_x) = (\partial_t - iP)(\partial_t + iP)$$

and the Cauchy problem for the wave equation with data $F \equiv 0$, $u_0 \equiv 0$, $u_1 = f$ is reduced to solving the two first order equations

$$\begin{aligned} (\partial_t - iP)v = 0, \quad v(0) = f, \\ (\partial_t + iP)u = v, \quad u(0) = 0. \end{aligned}$$

If one can solve the first equation, then solvability of the second equation will follow from Duhamel's principle (the sign in front of P is immaterial).

Therefore, we only need to solve

$$\begin{aligned} (\partial_t - iP)v(t,x) &= 0, \\ v(0,x) &= f(x). \end{aligned}$$

For the moment, let us simplify even further and assume that $A(x, D_x)$ is the Laplacian $-\Delta$, so that P will be the operator given by

$$\widehat{Pf}(\xi) = |\xi|\hat{f}(\xi).$$

Taking the spatial Fourier transform of the equation for v and solving the resulting ordinary differential equation gives the full solution

$$v(t,y) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(y\cdot\xi + t|\xi|)} \hat{f}(\xi)\,d\xi.$$

Thus, the solution is given by a Fourier integral operator acting on f:

$$v(t,y) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\Phi(t,y,\xi)} a(t,y,\xi)\hat{f}(\xi)\,d\xi.$$

In this particular case the phase function is $\Phi(t,y,\xi) = y \cdot \xi + t|\xi|$ and the symbol is $a(t,y,\xi) \equiv 1$.

So far we have not used any special properties of f. Here comes the key point. *If f is a curvelet, then the phase function is well approximated on* $\mathrm{supp}(f)$ *by its linearization in ξ:*

$$\Phi(t,y,\xi) \approx \nabla_\xi \Phi(t,y,\omega)\cdot\xi \quad \text{for } \xi \in \mathrm{supp}(f).^1$$

Thus, if $f = \varphi_\gamma$ then the solution v with this initial data is approximately given by

$$v(t,y) \approx (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(y+t\omega)\cdot\xi} \hat{\varphi}_\gamma(\xi)\,d\xi = \varphi_\gamma(y+t\omega).$$

Thus the half wave equation for $P = \sqrt{-\Delta}$, whose initial data is a curvelet in direction ω, is approximately solved by translating the curvelet along a straight line in direction ω.

We now return to the general case where $A(x,\xi)$ is a general elliptic symbol $\sum_{j,k=1}^n g^{jk}(x)\xi_j\xi_k$. We define

$$p(x,\xi) = \sqrt{A(x,\xi)}.$$

Then p is homogeneous of order 1 in ξ, and it generates a *Hamilton flow* $(x(t),\xi(t))$ in the phase space $T^*\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$, determined by the ordinary differential equations

$$\begin{aligned} \dot{x}(t) &= \nabla_\xi p(x(t),\xi(t)), \\ \dot{\xi}(t) &= -\nabla_x p(x(t),\xi(t)). \end{aligned}$$

If $A(x,\xi)$ is smooth then the curves $(x(t),\xi(t))$ starting at some point $(x(0),\xi(0)) = (x,\omega)$ are smooth and exist for all time. Note that if $p(x,\xi) = |\xi|$ then one has straight lines $(x(t),\xi(t)) = (x+t\omega,\omega)$.

Similarly as above, the half wave equation

$$\begin{aligned} (\partial_t - iP)v(t,x) &= 0, \\ v(0,x) &= f(x), \end{aligned}$$

can be approximately solved as follows:

[1]This statement may seem somewhat mysterious, but it really is one reason why curvelets are useful for wave imaging. A slightly more precise statement is as follows: if $\Psi(t,y,\xi)$ is smooth for $\xi \neq 0$, homogeneous of order 1 in ξ, and its derivatives are uniformly bounded over $t \in [-T,T]$ and $y \in \mathbb{R}^n$ and $\xi \in S^{n-1}$, then

$$|\Psi(t,y,\xi) - \nabla_\xi \Psi(t,y,\omega)\cdot\xi| \lesssim 1$$

whenever $\xi\cdot\omega \sim 2^k$ and $|\xi - (\xi\cdot\omega)\omega| \lesssim 2^{k/2}$. Also the derivatives of $\Psi(t,y,\xi) - \nabla_\xi\Psi(t,y,\omega)\cdot\xi$ satisfy suitable symbol bounds. Parabolic scaling is crucial here, we refer to [18, Section 3.2] for more on this point.

1. Write the initial data f in terms of curvelets as $f(y) = \sum_\gamma c_\gamma \varphi_\gamma(y)$.
2. For a curvelet $\varphi_\gamma(y)$ centered at x pointing in direction ω, let $\varphi_\gamma(t, y)$ be another curvelet centered at $x(t)$ pointing in direction $\xi(t)$. That is, translate each curvelet φ_γ for time t along the Hamilton flow for P.
3. Let $v(t, y) = \sum_\gamma c_\gamma \varphi_\gamma(t, y)$ be the approximate solution.

Thus the wave equation can be approximately solved by decomposing the initial data into curvelets and then by translating each curvelet along the Hamilton flow.

3.3.3 Low regularity wave speeds and Volterra iteration

Here we give some further details related to the formal discussion in the previous section, following the arguments in [68]. The precise assumption on the coefficients will be

$$g^{jk}(x) \in C^{1,1}(\mathbb{R}^n).$$

This means that $\partial^\alpha g^{jk} \in L^\infty(\mathbb{R}^n)$ for $|\alpha| \leq 2$, which is a minimal assumption which guarantees a well defined Hamilton flow.

As discussed in Section 3.3.2, by Duhamel's formula it is sufficient to consider the Cauchy problem

$$\begin{aligned} (\partial_t^2 + A(x, D_x))u(t, x) &= 0 \quad \text{in } \mathbb{R} \times \mathbb{R}^n, \\ u(0, x) &= 0, \\ \partial_t u(0, x) &= f. \end{aligned}$$

Here $A(x, D_x) = \sum_{j,k=1}^n g^{jk}(x) D_{x_j} D_{x_k}$ and $g^{jk} \in C^{1,1}(\mathbb{R}^n)$, $g^{jk} = g^{kj}$, and $0 < \lambda \leq \sum_{j,k=1}^n g^{jk}(x)\xi_j\xi_k \leq \Lambda < \infty$ uniformly over $x \in \mathbb{R}^n$ and $\xi \in S^{n-1}$.

To deal with the nonsmooth coefficients, we introduce the smooth approximations

$$A_k(x, \xi) = \sum_{i,j=1}^n g_k^{ij}(x)\xi_i\xi_j, \qquad g_k^{ij} = \chi(2^{-k/2} D_x) g^{ij}$$

where $\chi \in C_c^\infty(\mathbb{R}^n)$ satisfies $0 \leq \chi \leq 1$, $\chi(\xi) = 1$ for $|\xi| \leq 1/2$, and $\chi(\xi) = 0$ for $|\xi| \geq 1$. We have written $(\chi(2^{-k/2}D_x)g)\hat{}(\xi) = \chi(2^{-k/2}\xi)\hat{g}(\xi)$. Thus g_k^{ij} are smooth truncations of g^{ij} to frequencies $\leq 2^{k/2}$. We will use the smooth approximation A_k in the construction of the solution operator at frequency level 2^k, which is in keeping with paradifferential calculus.

Given a curvelet $\varphi_\gamma(y)$ where $\gamma = (k, \omega_\gamma, x_\gamma)$, we wish to consider a curvelet $\varphi_\gamma(t, y)$ which corresponds to a translation of φ_γ for time t along the Hamilton flow for $H_k(x, \xi) = \sqrt{A_k(x, \xi)}$. In fact, we shall define

$$\varphi_\gamma(t, y) = \varphi_\gamma(\Theta_\gamma(t)(y - x_\gamma(t)) + x_\gamma)$$

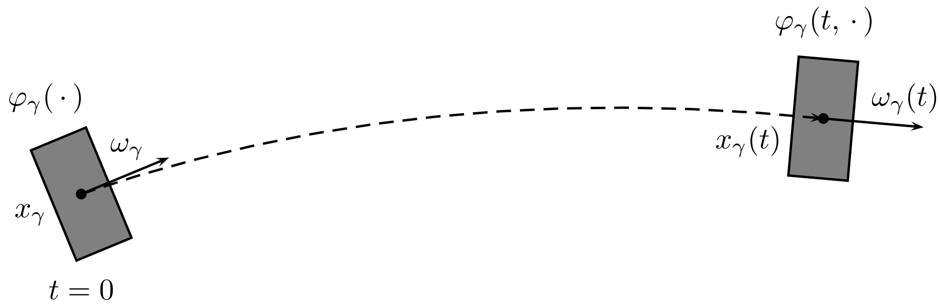


Figure 4: The translation of a curvelet φ_γ for time t along the Hamilton flow..

where $x_\gamma(t)$ and the $n\times n$ matrix $\Theta_\gamma(t)$ arise as the solution of the equations

$$\begin{aligned}\dot{x} &= \nabla_\xi H_k(x,\omega),\\ \dot{\omega} &= -\nabla_x H_k(x,\omega) + (\omega\cdot\nabla_x H_k(x,\omega))\omega,\\ \dot{\Theta} &= -\Theta(\omega\otimes\nabla_x H_k(x,\omega) - \nabla_x H_k(x,\omega)\otimes\omega)\end{aligned}$$

with initial condition $(x_\gamma(0),\omega_\gamma(0),\Theta_\gamma(0)) = (x_\gamma,\omega_\gamma,I)$. Here $v\otimes w$ is the matrix with $(v\otimes w)x = (w\cdot x)v$. The idea is that $(x_\gamma(t),\omega_\gamma(t))$ is the Hamilton flow for H_k restricted to the unit cosphere bundle $S^*\mathbb{R}^n = \{(x,\xi)\in T^*\mathbb{R}^n\,;\,|\xi|=1\}$, and $\Theta_\gamma(t)$ is a matrix which tracks the rotation of ω_γ along the flow and satisfies $\Theta_\gamma(t)\omega_\gamma(t) = \omega_\gamma$ for all t. See Figure 4 for an illustration.

We define an approximate solution operator at frequency level 2^k by

$$E_k(t)f(y) = \sum_{\gamma':k'=k}(f,\varphi_{\gamma'})_{L^2(\mathbb{R}^n)}\varphi_{\gamma'}(t,y).$$

Summing over all frequencies, we consider the operator

$$E(t)f = \sum_{k=0}^{\infty} E_k(t)f.$$

This operator essentially takes a function f, decomposes it into curvelets, and then translates each curvelet at frequency level 2^k for time t along the Hamilton flow for H_k.

It is proved in [68, Theorem 3.2] that $E(t)$ is an operator of order 0, mapping $H^\alpha(\mathbb{R}^n)$to $H^\alpha(\mathbb{R}^n)$ for any α. The fact that $E(t)$is an approximate solution operator is encoded in the result that the wave operator applied to $E(t)$,

$$T(t) = (\partial_t^2 + A(x,D_x))E(t),$$

which is a priori a second order operator, is in fact an operator of order 1 and maps $H^{\alpha+1}(\mathbb{R}^n)$to $H^{\alpha}(\mathbb{R}^n)$ for $-1 \leq \alpha \leq 2$. This is proved in [68, Theorem 4.5], and is due to the two facts. The first one is that when A is replaced by the smooth approximation A_k, the corresponding operator

$$\sum_k (\partial_t^2 + A_k(x, D_x))E_k(t)$$

is of order 1 because the second order terms cancel. Here one uses that translation along Hamilton flow approximately solves the wave equation. The second fact is that the part involving the nonsmooth coefficients,

$$\sum_k (A_k(x, D_x) - A(x, D_x))E_k(t)$$

is also of order 1 using that A_k is truncated to frequencies $\leq 2^{k/2}$ and using estimates for $A - A_k$ obtained from the $C^{1,1}$ regularity of the coefficients.

To obtain the full parametrix one needs to consider the Hamilton flows both for $\sqrt{A_k}$ and $-\sqrt{A_k}$, corresponding to the two half-wave equations appearing in the factorization of the wave operator, and one also needs to introduce corrections to ensure that the initial values of the approximate solution are the given functions. For simplicity we will not consider these details here and only refer to [68, Section 4]. The outcome of this argument is an operator $\mathbf{s}(t, s)$ which is strongly continuous in t and s as a bounded operator $H^{\alpha}(\mathbb{R}^n) \to H^{\alpha+1}(\mathbb{R}^n)$, satisfies $\mathbf{s}(t, s)f|_{t=s} = 0$ and $\partial_t \mathbf{s}(t, s)f|_{t=s} = f$, and further the operator

$$T(t, s) = (\partial_t^2 + A(x, D_x))\mathbf{s}(t, s)$$

is bounded $H^{\alpha}(\mathbb{R}^n) \to H^{\alpha}(\mathbb{R}^n)$ for $-1 \leq \alpha \leq 2$.

We conclude this discussion by explaining the Volterra iteration scheme which is used for converting the approximate solution operator to an exact one, as in [68, Theorem 4.6]. We look for a solution in the form

$$u(t) = \mathbf{s}(t, 0)f + \int_0^t \mathbf{s}(t, s)G(s)\, ds$$

for some $G \in L^1([-t_0, t_0]\,; H^{\alpha}(\mathbb{R}^n))$. From the properties of $\mathbf{s}(t, s)$, we see that u satisfies

$$(\partial_t^2 + A(x, D_x))u = T(t, 0)f + G(t) + \int_0^t T(t, s)G(s)\, ds.$$

Thus, u is a solution if G is such that

$$G(t) + \int_0^t T(t, s)G(s)\, ds = -T(t, 0)f.$$

Since $T(t, s)$ is bounded on $H^{\alpha}(\mathbb{R}^n)$ for $-1 \leq \alpha \leq 2$, with norm bounded by a uniform constant when $|t|, |s| \leq t_0$, the last Volterra equation can be solved by iteration. This yields the required solution u.

4 Conclusion

In this chapter inverse problems for the wave equation were considered with different types of data. All considered data correspond to measurements made on the boundary of a body in which the wave speed is unknown and possibly anisotropic. The case of the complete data, that is, with measurements of amplitudes and phases of waves corresponding to all possible sources on the boundary, was considered using the boundary control method. We showed that the wave speed can be reconstructed from the boundary measurements up to a diffeomorphism of the domain. This corresponds to the determination of the wave speed in the local travel-time coordinates. Next, the inverse problem with less data, the scattering relation, was considered. The scattering relation consists of the travel times and the exit directions of the wave fronts produced by the point sources located on the boundary of the body. Such data can be considered to be obtained by measuring the waves up to smooth errors, or measuring only the singularities of the waves. The scattering relation is a generalization of the travel time data, that is, the travel times of the waves through the body. Finally, we considered the use of wavelets and curvelets in the analysis of the waves. Using the curvelet representation of the waves, the singularities of the waves can be efficiently analyzed. In particular, the curvelets are suitable for the simulation of the scattering relation, even when the wave speed is non-smooth. Summarizing, in this chapter modern approaches to study inverse problems for wave equations, based on control theory, geometry, and microlocal analysis, were presented.

5 Cross-references

Inverse scattering, Mathematics of photoacoustic and thermoacoustic tomography, Photoacoustic and thermoacoustic tomography: image formation principles.

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