Depth dependent stability estimate in electrical impedance tomography

Sei Nagayasu* Gunther Uhlmann† Jenn-Nan Wang‡

Abstract

We study the inverse problem of determining an electrical inclusion from boundary measurements. We derive a stability estimate for the linearized map with explicit formulae on generic constants that shows that the problem becomes more ill-posed as the inclusion is farther from the boundary. We also show that this estimate is optimal.

1 Introduction

Electrical Impedance Tomography (EIT) is an inverse method that attempts to determine the conductivity distribution inside a body by making voltage and current measurements at the boundary. The boundary information is encoded in the Dirichlet-to-Neumann map associated to the conductivity equation. More precisely, let Ω be an open bounded domain with smooth boundary in $\mathbb{R}^d$ with $d = 2$ or 3. Assume that $\gamma(x) > 0$ in Ω possesses a suitable regularity. The conductivity equation is described by the following elliptic equation:

$$\nabla \cdot (\gamma(x)\nabla u) = 0 \quad \text{in} \quad \Omega. \tag{1}$$

For an appropriate function f defined on $\partial\Omega$, there exists a unique solution $u(x)$ to the boundary value problem for (1) with Dirichlet condition $u|_{\partial\Omega} = f$. Thus, one can define a map Λ_γ sending the Dirichlet data to the Neumann data by

$$\Lambda_\gamma(f) = \gamma \left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega}.$$

The map Λ_γ is the Dirichlet-to-Neumann map associated with the conductivity equation (1). It is worth to mention that even though the equation (1) is linear, the

*Department of Mathematics, Taida Institute of Mathematical Sciences, NCTS (Taipei), National Taiwan University, Taipei 106, Taiwan. Email:nagayasu@math.ntu.edu.tw

†Department of Mathematics, University of Washington, Box 354305, Seattle, WA 98195-4350, USA. Email:gunther@math.washington.edu

‡Department of Mathematics, Taida Institute of Mathematical Sciences, NCTS (Taipei), National Taiwan University, Taipei 106, Taiwan. Email:jnwang@math.ntu.edu.tw

map Λ_γ depends nonlinearly on γ. The famous Calderón problem [3] is to determine γ from the knowledge of Λ_γ. The EIT problem is notoriously known to be ill-posed. A log-type stability was obtained by Alessandrini [1] and, in fact, this estimate is optimal [8]. A Lipschitz type stability estimate for the values of the conductivity from the Dirichlet-to-Neumann map was proven in [9].

In several practical situations we only need to get partial information on the conductivity. An important example is the determination of electrical inclusions. In this situation, the conductivity function $\gamma(x) = \gamma_0(x) + \gamma_1(x)\chi_D$, where $D \Subset \Omega$ is called an inclusion and χ_D is the characteristic function of D. Here γ_0 is the background medium and γ_1, D are the abnormalities. For this problem, assuming γ_0 is known. We are interested in determining the shape of D by the Dirichlet-to-Neumann map, denoted by Λ_D. Under some natural assumptions on γ_0 and γ_1, uniqueness was shown by Isakov [7]. Numerical methods based on special complex geometrical optics solutions for $\nabla \cdot \gamma_0 \nabla u = 0$ are given in [4], [10] (also see [5], [6], [11] for related results). It has been observed numerically that the deeper the inclusion, the worst the numerical reconstruction. See for instance [4], [10], and [11]. In this paper we give a precise quantitative description of this phenomenon in a model case.

We consider the problem in two dimensions, i.e., $\Omega \subset \mathbb{R}^2$. Let $k > 0$, $k \neq 1$ and define $L_D u := \nabla \cdot ((1 + (k-1)\chi_D)\nabla u)$. For any $f \in H^{1/2}(\partial\Omega)$, there exists a unique weak solution to

$$\begin{cases} L_D u = 0 \text{ in } \Omega, \\ u = f \text{ on } \partial\Omega. \end{cases}$$

The Dirichlet-to-Neumann map is given by $\Lambda_D : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ as

$$\Lambda_D f = \left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega},$$

where ν is the unit outer normal of $\partial\Omega$. The inverse problem is to determine D from Λ_D. As mentioned above, the uniqueness for this problem is known [7]. A log-type stability was obtained in [2]. More precisely, it was proved in [2] that under some minor a priori assumptions on the inclusions, if $\|\Lambda_{D_1} - \Lambda_{D_2}\|_{\mathcal{L}(H^{1/2}, H^{-1/2})} < \epsilon$ with $\epsilon > 0$, then the Hausdorff distance between ∂D_1 and ∂D_2 satisfies

$$d_{\mathcal{H}}(\partial D_1, \partial D_2) < \omega(\epsilon),$$

where $\omega(t)$ is an increasing function in $[0, \infty)$ and satisfies

$$\omega(t) \leq C|\log t|^{-\eta} \quad \text{for } t \in (0, 1).$$

The constants C and $0 < \eta < 1$ depend on the a priori data of the inclusions, but their dependence is not explicitly given in [2]. As a matter of fact, to our best knowledge, we do not know any available stability estimates for inverse problems having explicit descriptions of the data-dependent constants.

Our concern here is to understand how the stability estimate depends on the depth of the inclusion. In this paper, we consider the linearized map of Λ_D around a known inclusion. We believe that, either from numerical or theoretical viewpoint, the stability estimate using Λ_D should behave similarly to the estimate using the linearized map of Λ_D. To set up our problem, we let $\Omega = \{|x| < R\}$ and $B = \{|x| < r\}$, where $0 < r < R$. We introduce a smooth function

$$\psi : \partial B \to \mathbb{R}$$

in order to describe a perturbation B_s of the domain B, namely, the boundary ∂B_s of the domain B_s is described by the image of

$$y = F_s(x) := x + s\psi(x)\nu_x(x), \quad x \in \partial B,$$

where $\nu_x(x)$ is the unit outward normal vector to ∂B at $x \in \partial B$. For $f \in H^{1/2}(\partial\Omega)$, let u_0 be the solution to the problem

$$\begin{cases} L_B u_0 = 0 \text{ in } \Omega, \\ u_0 = f \text{ on } \partial\Omega. \end{cases} \tag{2}$$

Likewise, let u_s be the solution to the problem

$$\begin{cases} L_{B_s} u_s = 0 \text{ in } \Omega, \\ u_s = f \text{ on } \partial\Omega. \end{cases} \tag{3}$$

The linearized map of the Dirichlet-to-Neumann map at the direction $\psi(x)$, denoted by $d\Lambda_B(\psi)$, is formally defined by

$$d\Lambda_B(\psi) = \lim_{s\to 0} \frac{1}{s}(\Lambda_{B_s} - \Lambda_B).$$

We will show that $d\Lambda_B(\psi)$ is legitimately defined in the later section. We now state our main theorem.

Theorem 1. *Let $k > 0$ satisfy $k \neq 1$. Let $m > 0$. Given $M_0, r_0 > 0$ and $X_0 > 1$. Assume that*

$$M \geq M_0, \quad r \leq r_0, \quad \frac{R}{r} \geq X_0.$$

Then for any $\psi \in H^m(\partial B)$ satisfying

$$\|\psi\|_{H^m(\partial B)} \leq M \quad \text{and} \quad \|d\Lambda_B(\psi)\|_{\mathcal{L}} < 1,$$

the following estimate holds:

$$\|\psi\|_{L^2(\partial B)} \leq CM \left[\log\left(\frac{R}{r}\right)\right]^m \left|\log \|d\Lambda_B(\psi)\|_{\mathcal{L}}\right|^{-m}, \tag{4}$$

where a positive constant C depends only on k, m, M_0, r_0, X_0.

Here $\|\cdot\|_{\mathcal{L}}$ denotes the operator norm on the space of bounded linear operators between $H^{1/2}(\partial\Omega)$ and $H^{-1/2}(\partial\Omega)$. Moreover, this estimate (4) is optimal in the sense of Propositions 12 and 13 (See Section 4).

Remark 2. Estimate (4) clearly indicates that the determination of an inclusion by boundary measurements is getting more ill-posed when the inclusion is hidden deeper inside of the conductor, i.e., R/r becomes large.

The paper is organized as follows. In Section 2, we discuss the linearized map $d\Lambda_B(\psi)$ of the Dirichlet-to-Neumann map. In Section 3, we state some technical lemmas which we need and then we prove our main theorem. In Section 4, we discuss the optimality of the stability estimate.

2 The linearized map

In this section, we discuss the linearized map $d\Lambda_B(\psi)$ of the Dirichlet-to-Neumann map Λ_D. We first remark that it is known that the map $\gamma \mapsto \Lambda_\gamma$ is bounded and analytic in the subset of $L^\infty(D)$ consisting of functions which are real and have a positive lower bound (see [3]). We now introduce polar coordinates (ρ, θ), that is, $x = \rho(\cos\theta, \sin\theta) \in \mathbb{R}^2$, in order to express the linearized operator explicitly as the solution to some transmission problem. We put $\widetilde{\psi}(\theta) := \psi(r\cos\theta, r\sin\theta)$ and $\psi_l := \int_0^{2\pi} \widetilde{\psi}(\theta)\, e^{-il\theta}\, d\theta$ for a function $\psi \in L^2(\partial B)$. We remark that $\widetilde{\psi}(\theta) = (2\pi)^{-1} \sum_{l\in\mathbb{Z}} \psi_l e^{il\theta}$,

$$\|\psi\|^2_{L^2(\partial B)} = \frac{r}{2\pi} \sum_{l\in\mathbb{Z}} |\psi_l|^2 \quad \text{and} \quad \|\psi\|^2_{H^m(\partial B)} = \frac{r}{2\pi} \sum_{l\in\mathbb{Z}} (1+l^2)^m |\psi_l|^2. \tag{5}$$

Let $\widetilde{f}(\theta) := f(R\cos\theta, R\sin\theta)$ and $f_l := \int_0^{2\pi} \widetilde{f}(\theta)\, e^{-il\theta}\, d\theta$ for a function f defined on $\partial\Omega$ in the same way. Throughout this paper the subscripts $+$ (respectively $-$) denote the limit from outside (respectively inside) the inclusion.

Lemma 3. *The linearized operator $d\Lambda_B(\psi)$ satisfies*

$$d\Lambda_B(\psi)(f) = \left.\frac{\partial U}{\partial \nu}\right|_{\partial\Omega} \tag{6}$$

for any $f \in H^{1/2}(\partial\Omega)$, where U is the solution to the problem

$$\begin{cases} \Delta U = 0 \ \text{in}\ \Omega \setminus \partial B, \\ U|_+ - U|_- = \dfrac{1-k}{k} \psi \left.\dfrac{\partial u_0}{\partial \nu}\right|_+ \ \text{on}\ \partial B, \\ \left.\dfrac{\partial U}{\partial \nu}\right|_+ - k \left.\dfrac{\partial U}{\partial \nu}\right|_- = r^{-2}(1-k)\, \partial_\theta \Big(\widetilde{\psi}(\theta)\, \partial_\theta u_0|_+ \Big) \ \text{on}\ \partial B, \\ U = 0 \ \text{on}\ \partial\Omega \end{cases} \tag{7}$$

and u_0 is the solution to (2).

Proof. The solutions u_0 and u_s to the problems (2) and (3) satisfy

$$\begin{cases} \Delta u_0 = 0 \text{ in } \Omega, \\ u_0|_+ = u_0|_- \text{ on } \partial B, \\ \left.\dfrac{\partial u_0}{\partial \nu}\right|_+ = k \left.\dfrac{\partial u_0}{\partial \nu}\right|_- \text{ on } \partial B, \\ u_0 = f \text{ on } \partial\Omega \end{cases} \quad \text{and} \quad \begin{cases} \Delta u_s = 0 \text{ in } \Omega, \\ u_s|_+ = u_s|_- \text{ on } \partial B_s, \\ \left.\dfrac{\partial u_s}{\partial \nu}\right|_+ = k \left.\dfrac{\partial u_s}{\partial \nu}\right|_- \text{ on } \partial B_s, \\ u_s = f \text{ on } \partial\Omega, \end{cases}$$

respectively. Now we put

$$U(x) := \lim_{s\to 0} \frac{u_s(x) - u_0(x)}{s}.$$

and formula (6) is obvious. Moreover, if we write $y = F_s(x)$, we have

$$\frac{1}{s}\big(u_s(y)|_\pm - u_0(x)|_\pm\big) \to U(x)|_\pm + \psi(x) \left.\frac{\partial u_0}{\partial \nu}(x)\right|_\pm \quad \text{and}$$
$$\frac{1}{s}\left(\left.\frac{\partial u_s}{\partial \nu_y}(y)\right|_\pm - \left.\frac{\partial u_s}{\partial \nu_x}(x)\right|_\pm \right) \to \partial_\rho U(x)|_\pm - r^{-2}\widetilde{\psi}'(\theta)\, \partial_\theta u_0(x)|_\pm + \widetilde{\psi}(\theta)\, \partial_\rho^2 u_0(x)|_\pm$$

on ∂B as $s \to 0$. Thus we prove this lemma by using

$$\partial_\rho^2 u_0(x)|_\pm = -\frac{1}{r}\partial_\rho u_0(x)|_\pm - \frac{1}{r^2}\partial_\theta^2 u_0(x)|_\pm$$

on ∂B. □

Using Fourier series, we can write the linearized operator $d\Lambda_B(\psi)$ more explicitly.

Lemma 4. *For $f \in H^{1/2}(\partial\Omega)$ we have*

$$d\Lambda_B(\psi)(f)(R\cos\theta, R\sin\theta) = \sum_{l\in\mathbb{Z}} \lambda_l e^{il\theta},$$

where we put $\lambda_0 := 0$ and

$$\lambda_{-l} := \frac{k-1}{\pi^2}(Rr)^{-1} S_l \sum_{p=1}^{\infty} S_p \left\{ (k+1)\psi_{-l+p} f_{-p} + (k-1)\psi_{-l-p} f_p \right\},$$
$$\lambda_l := \frac{k-1}{\pi^2}(Rr)^{-1} S_l \sum_{p=1}^{\infty} S_p \left\{ (k+1)\psi_{l-p} f_p + (k-1)\psi_{l+p} f_{-p} \right\},$$
$$S_l := \frac{l}{(k-1)R^{-l}r^l - (k+1)R^l r^{-l}}$$

for any positive integer l.

Proof. Note that the solution to the problem (2) is expressed as follows:

$$
\begin{aligned}
&u_0(\rho\cos\theta, \rho\sin\theta) \\
&= \begin{cases}
\dfrac{1}{2\pi}\left[\displaystyle\sum_{l=1}^{\infty}\frac{S_l}{l}\left\{(k-1)r^l\rho^{-l}-(k+1)r^{-l}\rho^l\right\}\left(f_{-l}e^{-il\theta}+f_le^{il\theta}\right)+f_0\right], & r<\rho<R, \\
\dfrac{1}{2\pi}\left[-2\displaystyle\sum_{l=1}^{\infty}\frac{S_l}{l}r^{-l}\rho^l\left(f_{-l}e^{-il\theta}+f_le^{il\theta}\right)+f_0\right], & 0<\rho<r.
\end{cases}
\end{aligned}
$$

In particular, the right-hand sides of the transmission conditions on ∂B in (7) can be written as:

$$
\begin{aligned}
\frac{1-k}{k}\psi\frac{\partial u_0}{\partial\nu}\bigg|_+ &= -\frac{(1-k)r^{-1}}{2\pi^2}\sum_{p\in\mathbb{Z}}\sum_{j=1}^{\infty}S_j(\psi_{p+j}f_{-j}+\psi_{p-j}f_j)e^{ip\theta}, \\
r^{-2}(1-k)\,\partial_\theta\Big(\widetilde{\psi}(\theta)\,\partial_\theta u_0|_+\Big) &= \frac{(1-k)r^{-2}}{2\pi^2}\sum_{p\in\mathbb{Z}}p\sum_{j=1}^{\infty}S_j(-\psi_{p+j}f_{-j}+\psi_{p-j}f_j)e^{ip\theta}.
\end{aligned}
$$

Hence, the solution to the problem (7) is given by

$$
\begin{aligned}
&U(\rho\cos\theta, \rho\sin\theta) \\
&= \begin{cases}
\dfrac{2}{(2\pi)^2}(1-k)r^{-1}\displaystyle\sum_{l=1}^{\infty}\frac{S_l}{l}(R^l\rho^{-l}-R^{-l}\rho^l) \\
\quad\times\displaystyle\sum_{p=1}^{\infty}S_p\big\{(k+1)(\psi_{-l+p}f_{-p}e^{-il\theta}+\psi_{l-p}f_pe^{il\theta}) \\
\qquad\qquad+(k-1)(\psi_{-l-p}f_pe^{-il\theta}+\psi_{l+p}f_{-p}e^{il\theta})\big\}, & r<\rho<R, \\
-\dfrac{4}{(2\pi)^2}(1-k)r^{-1}\displaystyle\sum_{l=1}^{\infty}\frac{S_l}{l}\rho^l \\
\quad\times\displaystyle\sum_{p=1}^{\infty}S_p\big\{R^{-l}(\psi_{-l+p}f_{-p}e^{-il\theta}+\psi_{l-p}f_pe^{il\theta}) \\
\qquad\qquad+R^lr^{-2l}(\psi_{-l-p}f_pe^{-il\theta}+\psi_{l+p}f_{-p}e^{il\theta})\big\} \\
\quad+\dfrac{2}{(2\pi)^2}(1-k)r^{-1}\displaystyle\sum_{p=1}^{\infty}S_p(\psi_pf_{-p}+\psi_{-p}f_p), & 0<\rho<r.
\end{cases}
\end{aligned}
$$

We then finish the proof of the lemma using formula (6). □

With the help of Lemma 4, we can given an estimate of the size of $d\Lambda_B(\psi)$ in terms of r and R.

Lemma 5. *The operator $d\Lambda_B(\psi) : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ is a bounded linear operator. In particular, we have the following estimate:*

$$\begin{aligned}
&\|d\Lambda_B(\psi)(f)\|_{H^{-1/2}(\partial\Omega)} \\
&\le \frac{2^{3/2}|k-1|}{(k+1)\pi}\frac{1}{(1-(r/R)^2)^2}(Rr)^{-1} \\
&\quad\times\left[\sum_{l=1}^{\infty} l\left(\frac{r}{R}\right)^{2l}\sum_{j=1}^{\infty} j\left(\frac{r}{R}\right)^{2j}\left(|\psi_{-l+j}|^2+|\psi_{-l-j}|^2+|\psi_{l-j}|^2+|\psi_{l+j}|^2\right)\right]^{1/2} \\
&\quad\times\|f\|_{H^{1/2}(\partial\Omega)}.
\end{aligned} \tag{8}$$

Proof. We first remark that

$$|S_l| = \frac{l}{k+1}\left(\frac{r}{R}\right)^l\frac{1}{1-((k-1)/(k+1))(r/R)^{2l}} < \frac{l}{k+1}\frac{1}{1-(r/R)^2}\left(\frac{r}{R}\right)^l.$$

So, it follows from Lemma 4 that

$$\begin{aligned}
|\lambda_{\pm l}| \le &\frac{|k-1|}{(k+1)\pi^2}\frac{1}{(1-(r/R)^2)^2}(Rr)^{-1} \\
&\times l\left(\frac{r}{R}\right)^l\sum_{p=1}^{\infty} p\left(\frac{r}{R}\right)^p\left(|\psi_{\pm l-p}||f_p|+|\psi_{\pm l+p}||f_{-p}|\right)
\end{aligned}$$

for any positive integer l. Hence we have

$$\begin{aligned}
&\|d\Lambda_B(\psi)(f)\|^2_{H^{-1/2}(\partial\Omega)} = 2\pi R\sum_{l\in\mathbb{Z}}(1+l^2)^{-1/2}|\lambda_l|^2 \le 2\pi R\sum_{l=1}^{\infty} l^{-1}(|\lambda_l|+|\lambda_{-l}|)^2 \\
&\le \frac{2|k-1|^2}{(k+1)^2\pi^3}\frac{1}{(1-(r/R)^2)^4}R^{-1}r^{-2}\sum_{l=1}^{\infty} l\left(\frac{r}{R}\right)^{2l} \\
&\quad\times\left[\sum_{p=1}^{\infty} p\left(\frac{r}{R}\right)^p\left(|\psi_{-l-p}||f_p|+|\psi_{-l+p}||f_{-p}|+|\psi_{l-p}||f_p|+|\psi_{l+p}||f_{-p}|\right)\right]^2.
\end{aligned}$$

We immediately obtain this lemma since we have

$$\begin{aligned}
&\left[\sum_{p=1}^{\infty} p\left(\frac{r}{R}\right)^{p}\left(|\psi_{-l-p}||f_p|+|\psi_{-l+p}||f_{-p}|+|\psi_{l-p}||f_p|+|\psi_{l+p}||f_{-p}|\right)\right]^2\\
&\le \left[\sum_{p=1}^{\infty} p\left(\frac{r}{R}\right)^{2p}\left(|\psi_{-l-p}|^2+|\psi_{-l+p}|^2+|\psi_{l-p}|^2+|\psi_{l+p}|^2\right)\right]\\
&\quad\times\left[\sum_{p=1}^{\infty} p\left(|f_p|^2+|f_{-p}|^2+|f_p|^2+|f_{-p}|^2\right)\right]\\
&=\left[\sum_{p=1}^{\infty} p\left(\frac{r}{R}\right)^{2p}\left(|\psi_{-l-p}|^2+|\psi_{-l+p}|^2+|\psi_{l-p}|^2+|\psi_{l+p}|^2\right)\right]\times 2\sum_{p\in\mathbb{Z}}^{\infty}|p||f_p|^2\\
&\le \left[\sum_{p=1}^{\infty} p\left(\frac{r}{R}\right)^{2p}\left(|\psi_{-l-p}|^2+|\psi_{-l+p}|^2+|\psi_{l-p}|^2+|\psi_{l+p}|^2\right)\right]\times \frac{4\pi}{R}\|f\|^2_{H^{1/2}(\partial\Omega)}
\end{aligned}$$

by the Schwarz inequality. □

Remark 6. By changing the index, we can write the term on the right-hand side of (8) as follows:

$$\begin{aligned}
&\sum_{l=1}^{\infty} l\left(\frac{r}{R}\right)^{2l}\sum_{j=1}^{\infty} j\left(\frac{r}{R}\right)^{2j}\left(|\psi_{-l+j}|^2+|\psi_{-l-j}|^2+|\psi_{l-j}|^2+|\psi_{l+j}|^2\right)\\
&=2(1-s^2)^{-3}(1+s^2)s^2|\psi_0|^2\\
&\quad+\sum_{p=1}^{\infty} s^p\left[\frac{p^3-p}{6}+2(1-s^2)^{-3}s^2\left[p(1-s^2)+(1+s^2)\right]\right]\left(|\psi_p|^2+|\psi_{-p}|^2\right),
\end{aligned}$$

where we put $s := (r/R)^2$ for simplicity.

Corollary 7. *We have the estimate*

$$\|d\Lambda_B(\psi)\|_{\mathcal{L}} \le \frac{8|k-1|}{\pi^{1/2}(k+1)}\frac{1}{\{1-(r/R)^2\}^4}r^{1/2}R^{-3}\|\psi\|_{L^2(\partial B)}.$$

Proof. We obtain this corollary by Lemma 5 and the estimate $|\psi_l|^2 \le (2\pi/r)\|\psi\|^2_{L^2(\partial B)}$ since we have $\sum_{j=1}^{\infty} jt^j = (1-t)^{-2}t$ for $|t| < 1$. □

In the following corollary, we consider a particular case, which will be needed in the proof of the optimality of the stability estimate (see Section 4).

Corollary 8. *Let $a > 0$ and μ be a positive integer. Let $\widetilde{\psi}(\theta) = 2a\cos\mu\theta$. Then we have the following estimate:*

$$\|d\Lambda_B(\psi)\|_{\mathcal{L}} \le C_0 a\mu^{3/2}\frac{1}{\{1-(r/R)^2\}^{7/2}}(Rr)^{-1}\left(\frac{r}{R}\right)^{\mu}, \tag{9}$$

where the positive constant C_0 depends only on k.

Proof. By Lemma 5 and Remark 6, we get that

$$
\begin{aligned}
&\|d\Lambda_B(\psi)\|_{\mathcal{L}} \\
&\le \frac{8a|k-1|}{k+1} \frac{1}{\{1-(r/R)^2\}^2} (Rr)^{-1} \left(\frac{r}{R}\right)^{\mu} \\
&\quad \times \left[\frac{\mu^3-\mu}{6} + 2\left(1-\left(\frac{r}{R}\right)^4\right)^{-3} \left(\frac{r}{R}\right)^4 \left[\mu\left\{1-\left(\frac{r}{R}\right)^4\right\} + \left\{1+\left(\frac{r}{R}\right)^4\right\}\right]\right]^{1/2}
\end{aligned}
$$

because of $\psi_{\pm\mu} = 2\pi a$ and $\psi_l = 0$ for $l \neq \pm\mu$. This corollary follows from

$$
\begin{aligned}
&\frac{\mu^3-\mu}{6} + 2\left(1-\left(\frac{r}{R}\right)^4\right)^{-3} \left(\frac{r}{R}\right)^4 \left[\mu\left\{1-\left(\frac{r}{R}\right)^4\right\} + \left\{1+\left(\frac{r}{R}\right)^4\right\}\right] \\
&\le \left(1-\left(\frac{r}{R}\right)^4\right)^{-3} \left(\frac{\mu^3}{6} + 2(\mu+2)\right) \le \frac{37}{6}\mu^3 \left(1-\left(\frac{r}{R}\right)^2\right)^{-3},
\end{aligned}
$$

where the constant $C_0 = (37/6)^{1/2} \cdot 8|k-1|/(k+1)$. □

3 The proof of the stability estimate

In this section, we prove our main theorem. We first state some useful identities.

Lemma 9. *For $f, g \in H^{1/2}(\partial\Omega)$ we have the identity*

$$
\begin{aligned}
&\int_{\partial\Omega} d\Lambda_B(\psi)(f)\, g\, d\sigma \\
&= -(1-k)\left(r^{-2}\int_{\partial B} \psi\, \partial_\theta u_0|_+ \,\partial_\theta v_0|_+ \, d\sigma + \frac{1}{k}\int_{\partial B} \psi\, \frac{\partial u_0}{\partial \nu}\bigg|_+ \frac{\partial v_0}{\partial \nu}\bigg|_+ d\sigma\right), \qquad (10)
\end{aligned}
$$

where u_0 and v_0 are the solutions to the problem (2) *with the boundary conditions $u_0 = f$ and $v_0 = g$, respectively.*

Proof. Applying Green's formula yields

$$
\begin{aligned}
0 &= \int_{\Omega\setminus B} \Delta U\, v_0\, dx - \int_{\Omega\setminus B} U\, \Delta v_0\, dx \\
&= \int_{\partial\Omega} d\Lambda_B(\psi)(f)\, g\, d\sigma - \int_{\partial B} \frac{\partial U}{\partial \nu}\bigg|_+ v_0|_+\, d\sigma + \int_{\partial B} U|_+ \frac{\partial v_0}{\partial \nu}\bigg|_+ d\sigma \quad \text{and} \\
0 &= \int_B \Delta U\, v_0\, dx - \int_B U\, \Delta v_0\, dx = \int_{\partial B} \frac{\partial U}{\partial \nu}\bigg|_- v_0|_-\, d\sigma - \int_{\partial B} U|_- \frac{\partial v_0}{\partial \nu}\bigg|_- d\sigma,
\end{aligned}
$$

where U is the solution to the problem (7). Using these identities and the transmission conditions for U and v_0, we obtain this lemma. □

Lemma 10. *Let g_j on $\partial\Omega$ be given by $g_j = e^{ij\theta}$ for any integer j, where $i = \sqrt{-1}$. Then we have*

$$\int_{\partial\Omega} d\Lambda_B(\psi)(g_{\pm l})\, g_{\pm p}\, d\sigma = 4(1-k)^2 r^{-1} S_l S_p \psi_{\mp(l+p)}, \tag{11}$$

$$\int_{\partial\Omega} d\Lambda_B(\psi)(g_{\pm l})\, g_{\mp p}\, d\sigma = -4(1-k)(1+k) r^{-1} S_l S_p \psi_{\mp(l-p)} \tag{12}$$

for positive integers l and p.

Proof. We first remark that the solution u_0 to the problem (2) with the boundary condition $u_0 = g_{\pm l}$ is

$$u_0(\rho\sin\theta, \rho\cos\theta) = \begin{cases} \dfrac{S_l}{l}\left\{(k-1)r^l\rho^{-l} - (k+1)r^{-l}\rho^l\right\}e^{\pm il\theta}, & r < \rho < R, \\ -2\dfrac{S_l}{l}r^{-l}\rho^l e^{\pm il\theta}, & \rho < r \end{cases}$$

for any positive integer l and in particular we have

$$\left.\frac{\partial u_0}{\partial \rho}\right|_+ = -2kr^{-1}S_l e^{\pm il\theta} \quad \text{and} \quad \left.\frac{\partial u_0}{\partial \theta}\right|_+ = \mp 2iS_l e^{\pm il\theta}$$

on ∂B. So, by taking $f = g_{\pm l}$ and $g = g_{\pm p}$ (or $g = g_{\mp p}$) and applying Lemma 9, we obtain this lemma. □

Now we denote $X := R/r$. It is important to estimate each ψ_j in view of formula (5).

Lemma 11. *We have that*

$$|\psi_0| \le C_1 r^2 X^3 \|d\Lambda_B(\psi)\|_{\mathcal{L}}, \quad |\psi_{\pm 1}| \le C_1 r^2 X^4 \|d\Lambda_B(\psi)\|_{\mathcal{L}}$$

and

$$|\psi_{\pm l}| \le \frac{C_1}{l} r^2 X^{l+1} \|d\Lambda_B(\psi)\|_{\mathcal{L}}$$

for any integer $l \ge 2$, where the positive constant C_1 depends only on k.

Proof. We first note that $\|g_{\pm l}\|_{H^{1/2}(\partial\Omega)} = (1+l^2)^{1/4}R^{1/2}$ for any positive integer l. It is easy to see that

$$\begin{aligned}\left|\int_{\partial\Omega} d\Lambda_B(\psi)(g_j)\, g_{j'}\, d\sigma\right| &\le \|d\Lambda_B(\psi)(g_j)\|_{H^{-1/2}(\partial\Omega)}\|g_{j'}\|_{H^{1/2}(\partial\Omega)} \\ &\le \|d\Lambda_B(\psi)\|_{\mathcal{L}}\|g_j\|_{H^{1/2}(\partial\Omega)}\|g_{j'}\|_{H^{1/2}(\partial\Omega)} \\ &= (1+j^2)^{1/4}(1+(j')^2)^{1/4}R\,\|d\Lambda_B(\psi)\|_{\mathcal{L}} \\ &= (1+j^2)^{1/4}(1+(j')^2)^{1/4}rX\,\|d\Lambda_B(\psi)\|_{\mathcal{L}}\end{aligned}$$

for any integers $j, j' \neq 0$. On the other hand, we have

$$\frac{1}{|S_l|} = \frac{k+1}{l} X^l \left[1 - \frac{k-1}{k+1} \left(\frac{1}{X} \right)^{2l} \right] \leq \frac{2(k+1)}{l} X^l.$$

By taking $l = p = 1$ in the identity (12), we get

$$|\psi_0| = \frac{r}{4|1-k|(1+k)} \frac{1}{S_1^2} \left| \int_{\partial\Omega} d\Lambda_B(\psi)(g_1)\, g_{-1}\, d\sigma \right| \leq \frac{2^{1/2}(k+1)}{|1-k|} r^2 X^3 \|d\Lambda_B(\psi)\|_{\mathcal{L}}.$$

Likewise, taking $l = 2$ and $p = 1$ in the identity (12) gives

$$|\psi_{\pm 1}| \leq \frac{10^{1/4}(k+1)}{2|1-k|} r^2 X^4 \|d\Lambda_B(\psi)\|_{\mathcal{L}}.$$

On the other hand, taking $p = l \geq 1$ in the identity (11), we obtain

$$\begin{aligned} |\psi_{\mp 2l}| &= \frac{r}{4(1-k)^2} \frac{1}{S_l^2} \left| \int_{\partial\Omega} d\Lambda_B(\psi)(g_{\pm l})\, g_{\pm l}\, d\sigma \right| \\ &\leq \frac{(k+1)^2}{(1-k)^2} \frac{(1+l^2)^{1/2}}{l^2} r^2 X^{2l+1} \|d\Lambda_B(\psi)\|_{\mathcal{L}} \\ &\leq \frac{2^{3/2}(k+1)^2}{(1-k)^2} \frac{1}{2l} r^2 X^{2l+1} \|d\Lambda_B(\psi)\|_{\mathcal{L}}. \end{aligned}$$

In the same way, taking $l \geq 1$ and $p = l+1$ in the identity (11), we get

$$|\psi_{\pm(2l+1)}| \leq \frac{2 \cdot 10^{1/4}(k+1)^2}{(1-k)^2} \frac{1}{2l+1} r^2 X^{(2l+1)+1} \|d\Lambda_B(\psi)\|_{\mathcal{L}}.$$

The proof of the lemma is complete. □

We now prove our main theorem.

Proof of Theorem 1. Note that the a priori assumption $\|\psi\|_{H^m(\partial B)} \leq M$ is equivalent to

$$\sum_{l \in \mathbb{Z}} (1+l^2)^m |\psi_l|^2 \leq \frac{2\pi}{r} M^2. \tag{13}$$

We first consider $A := \|d\Lambda_B(\psi)\|_{\mathcal{L}}^2$ sufficiently small. Let $0 < t < 2 \cdot 3^{-2m} \pi M^2 r^{-1}$ be given. We remark that $(2\pi M^2/rt)^{1/2m} > 3$. Let N be the minimum integer satisfying $2\pi M^2 N^{-2m} r^{-1} \leq t$, namely,

$$N - 1 < \left(\frac{2\pi M^2}{rt} \right)^{1/2m} \leq N. \tag{14}$$

One can see that $N \geq 4$. Using Lemma 11, we have

$$\begin{aligned}
&\sum_{|l|\leq N-1} |\psi_l|^2 \\
&\leq C_1^2 r^4 X^6 \|d\Lambda_B(\psi)\|_{\mathcal{L}}^2 + 2C_1^2 r^4 X^8 \|d\Lambda_B(\psi)\|_{\mathcal{L}}^2 + 2\sum_{l=2}^{N-1} \frac{C_1^2}{l^2} r^4 X^{2(l+1)} \|d\Lambda_B(\psi)\|_{\mathcal{L}}^2 \\
&\leq C_1^2 r^4 \left(X^6 + 2X^8 + 2X^{2N} \sum_{l=2}^{N-1} \frac{1}{l^2} \right) A \leq 5C_1^2 r^4 X^{2N} A.
\end{aligned}$$

On the other hand, we can estimate

$$\sum_{|l|\geq N} |\psi_l|^2 \leq (1+N^2)^{-m} \sum_{|l|\geq N} (1+l^2)^m |\psi_l|^2 \leq (1+N^2)^{-m} \frac{2\pi}{r} M^2 \leq N^{-2m} \frac{2\pi}{r} M^2 \leq t$$

by estimate (13). Combining the estimates above and (14), we get that

$$\sum_{l\in\mathbb{Z}} |\psi_l|^2 \leq F(t),$$

where

$$F(t) := 5C_1^2 r^4 X^{2\left\{ \left(2\pi M^2/r\right)^{1/2m} t^{-1/2m} + 1 \right\}} A + t.$$

Now we would like to show the estimate

$$F(t_0) \leq C_2 M^2 r^{-1} (\log X)^{2m} (-\log A)^{-2m} \quad \text{for some} \quad 0 < t_0 < 2 \cdot 3^{-2m} \pi M^2 r^{-1}, \quad (15)$$

where the positive constant C_2 depends only on k and m. We choose t_0 such that

$$r^4 X^{2\left\{ \left(2\pi M^2/r\right)^{1/2m} t_0^{-1/2m} + 1 \right\}} A = M^2 r^{-1} (\log X)^{2m} (-\log A)^{-2m},$$

i.e., we pick

$$t_0 = 2^{2m+1} \pi M^2 r^{-1} (\log X)^{2m} \left(\log G(A, M, r, X) \right)^{-2m},$$

where $G(A, M, r, X) := A^{-1} (-\log A)^{-2m} M^2 r^{-5} (\log X)^{2m} X^{-2}$. Then we have

$$F(t_0) = 5C_1^2 M^2 r^{-1} (\log X)^{2m} (-\log A)^{-2m} + t_0.$$

Therefore, it is enough to estimate t_0. Now we fix $\eta, \eta' \in (0, 1)$ small enough such that $\eta + 2m\eta' < 1$ and put

$$A_0 := \left[(e\eta')^{2m} M^2 r^{-5} (\log X)^{2m} X^{-8} \right]^{1/(1-\eta-2m\eta')}.$$

Let $0 < A < \min\{A_0, 1\}$, then we have

$$A^{1-\eta-2m\eta'} \leq A_0^{1-\eta-2m\eta'} = (e\eta')^{2m} M^2 r^{-5} (\log X)^{2m} X^{-8}.$$

Consequently, we obtain

$$G(A,M,r,X) \geq A^{-\eta}(-\log A)^{-2m}\big\{(e\eta')^{-1}A^{-\eta'}\big\}^{2m}X^6 \geq A^{-\eta}X^6 \geq \begin{cases} X^6 \\ A^{-\eta} \end{cases}$$

since $0 < -\log t \leq (e\eta')^{-1}t^{-\eta'}$ for all $0 < t < 1$. Thus we deduce that

$$\begin{aligned} t_0 &\leq 2^{2m+1}\pi M^2 r^{-1}(\log X)^{2m}\left(\log A^{-\eta}\right)^{-2m} \\ &= 2^{2m+1}\pi\eta^{-2m}M^2 r^{-1}(\log X)^{2m}\left(-\log A\right)^{-2m} \end{aligned}$$

and

$$t_0 \leq 2^{2m+1}\pi M^2 r^{-1}(\log X)^{2m}\left(\log X^6\right)^{-2m} = 2\cdot 3^{-2m}\pi M^2 r^{-1}.$$

Summing up, we have proved that if $0 < A < \min\{A_0, 1\}$ then the estimate (15) holds with $C_2 = 5C_1^2 + 2^{2m+1}\pi\eta^{-2m}$. In other words, we obtain

$$\|\psi\|_{L^2(\partial B)} = \left(\frac{r}{2\pi}\sum_{l\in\mathbb{Z}}|\psi_l|^2\right)^{1/2} \leq \left(\frac{r}{2\pi}F(t_0)\right)^{1/2} \leq \left(\frac{C_2}{2\pi}\right)^{1/2} M(\log X)^m(-\log A)^{-m}$$

for $0 < A < \min\{A_0, 1\}$.

Next we consider the case where $A_0 \leq A < 1$. Note that

$$A_0 = \left[(e\eta')^{2m}M^2 r^{-5}(\log X)^{2m}X^{-8}\right]^{1/(1-\eta-2m\eta')} \geq cX^{-8/(1-\eta-2m\eta')},$$

where $c := \min\big\{\left[(e\eta')^{2m}M_0^2 r_0^{-5}(\log X_0)^{2m}\right]^{1/(1-\eta-2m\eta')}, 1/2\big\}$. We remark that $-\log c > 0$. So we can estimate

$$\begin{aligned} \|\psi\|_{L^2(\partial B)} &\leq \|\psi\|_{H^m(\partial B)} \leq M \\ &\leq (-\log A_0)^m M(-\log A)^{-m} \leq C_3^m M(\log X)^m(-\log A)^{-m} \end{aligned}$$

for $A_0 \leq A < 1$ since

$$-\log A_0 \leq -\log(cX^{-8/(1-\eta-2m\eta')}) = -\log c + \frac{8}{1-\eta-2m\eta'}\log X \leq C_3\log X,$$

where $C_3 := (-\log c)/(\log X_0) + 8/(1-\eta-2m\eta')$. Thus we obtain estimate (4) with $C = 2^{-m}\min\{(C_2/2\pi)^{1/2}, C_3^m\}$. □

4 Optimality of the stability estimate

In this section, we discuss the optimality of the stability estimate in the sense that the polylogarithmic order m in estimate (4) can not be improved. We divide our discussion into two parts. For the first part, we fix the constants $k, m, R, r, M > 0$. In Theorem 1, we derived the estimate

$$\|\psi\|_{L^2(\partial B)} \leq C_*\big|\log\|d\Lambda_B(\psi)\|_{\mathcal{L}}\big|^{-m} \tag{16}$$

for any $\psi \in H^m(\partial B)$ satisfying $\|\psi\|_{H^m(\partial B)} \leq M$ and $\|d\Lambda_B(\psi)\|_{\mathcal{L}} < 1$, where C_* is independent of ψ. We now prove that the polylogarithmic order in (16) is optimal.

Proposition 12. *Let $k, m, R, r, M, \varepsilon > 0$ be fixed. Assume $k \neq 1$ and $R > r$. Then there exists no positive constant C' which is independent of ψ such that the following estimate holds:*

$$\|\psi\|_{L^2(\partial B)} \leq C' \big|\log \|d\Lambda_B(\psi)\|_{\mathcal{L}}\big|^{-m-\varepsilon} \tag{17}$$

for any $\psi \in H^m(\partial B)$ satisfying

$$\|\psi\|_{H^m(\partial B)} \leq M \tag{18}$$

and

$$\|d\Lambda_B(\psi)\|_{\mathcal{L}} < 1. \tag{19}$$

Proof. We prove this proposition by contradiction. That is, we assume that there exists C' which is independent of ψ such that (17) holds for all $\psi \in H^m(\partial B)$ satisfying (18) and (19). Let μ be a positive integer. Put $a_\mu := 2^{-1}\pi^{-1/2}r^{-1/2}(1+\mu^2)^{-m/2}M$. Define a function ψ on ∂B by $\widetilde{\psi}(\theta) = 2a_\mu \cos \mu\theta$. Then we have

$$\|\psi\|_{H^m(\partial B)} = M \quad \text{and} \quad \|\psi\|_{L^2(\partial B)} = (1+\mu^2)^{-m/2}M. \tag{20}$$

So, the function ψ satisfies the condition (18) in particular. Moreover, using Corollary 8, we can see that

$$\begin{aligned} \|d\Lambda_B(\psi)\|_{\mathcal{L}} &\leq C_0 a_\mu \mu^{3/2} \frac{1}{\{1-(r/R)^2\}^{7/2}} (Rr)^{-1} \left(\frac{r}{R}\right)^{\mu} \\ &= C_0'(1+\mu^2)^{-m/2}\mu^{3/2}\left(\frac{r}{R}\right)^{\mu} < C_0' \mu^{-m+3/2}\left(\frac{r}{R}\right)^{\mu}, \end{aligned} \tag{21}$$

where $C_0' := 2^{-1}\pi^{-1/2}C_0 M R^{-1} r^{-3/2}\left\{1-(r/R)^2\right\}^{-7/2}$. Note that the constant C_0' is independent of μ. Hence the function ψ satisfies the condition (19) when μ is large enough. Consequently, the estimate (17) holds for μ sufficiently large. By (20), (21) and (17), we then obtain

$$\begin{aligned} (1+\mu^2)^{-m/2}M = \|\psi\|_{L^2(\partial B)} &\leq C' \big|\log \|d\Lambda_B(\psi)\|_{\mathcal{L}}\big|^{-m-\varepsilon} \\ &< C' \left(-\log\left\{C_0'\mu^{-m+3/2}\left(\frac{r}{R}\right)^{\mu}\right\}\right)^{-m-\varepsilon} \\ &= C'\left(-\log C_0' + \left(m - \frac{3}{2}\right)\log\mu + \mu \log X\right)^{-m-\varepsilon}, \end{aligned}$$

i.e.,

$$M \leq C'(1+\mu^2)^{m/2}\left(-\log C_0' + \left(m-\frac{3}{2}\right)\log\mu + \mu\log X\right)^{-m-\varepsilon} \tag{22}$$

for $\mu \gg 1$. Recall that $X := R/r$. However, the right-hand side of (22) tends to zero as $\mu \to +\infty$. This is a contradiction. □

In the second part, we discuss the dependency of the constant C_* in (16) on R and r. Fix $r_0 > 0$ and $X_0 > 1$. We have shown in Theorem 1 that C_* in (16) satisfies

$$C_* \leq C_\sharp \left[\log\left(\frac{R}{r}\right) \right]^m \tag{23}$$

for $R, r > 0$ with $r \leq r_0$ and $R/r \geq X_0$, where $C_\sharp$ depends only on k, m, r_0, X_0, M. Similar to Proposition 12, we can prove that the polylogarithmic order in (23) is optimal, at least, when the constant R and the ratio R/r are large.

Proposition 13. *Let $k > 0$ satisfy $k \neq 1$, $m > 0$, and $M > 0$. Given $R_0 > 0$ and $X_0 > 1$. Let $\varepsilon > 0$. Then there exists no positive constant C'', depending only on k, m, R_0, X_0, M and ε, such that for any*

$$R \geq R_0, \quad \frac{R}{r} \geq X_0 \tag{24}$$

and for any $\psi \in H^m(\partial B)$ satisfying

$$\|\psi\|_{H^m(\partial B)} \leq M \quad \text{and} \quad \|d\Lambda_B(\psi)\|_{\mathcal{L}} < 1, \tag{25}$$

the following estimate holds:

$$\|\psi\|_{L^2(\partial B)} \leq C'' \left[\log\left(\frac{R}{r}\right) \right]^{m-\varepsilon} \left|\log \|d\Lambda_B(\psi)\|_{\mathcal{L}}\right|^{-m}. \tag{26}$$

Proof. We also prove this proposition by contradiction. We assume that there exists a positive constant C'' which depends only on k, m, R_0, X_0, M and ε such that for any $R, r > 0$ satisfying (24) and for any $\psi \in H^m(\partial B)$ satisfying (25) the estimate (26) holds.

Define a function ψ on ∂B by $\widetilde{\psi}(\theta) = 2a_2 \cos 2\theta$, where $a_2 := 2^{-1}5^{-m/2} \times \pi^{-1/2} r^{-1/2} M$. Then we have that

$$\|\psi\|_{L^2(\partial B)} = 5^{-m/2} M, \quad \|\psi\|_{H^m(\partial B)} = M$$

and

$$\|d\Lambda_B(\psi)\|_{\mathcal{L}} \leq \frac{2^{1/2} 5^{-m/2} C_0}{\pi^{1/2}} M R^{-5/2} \left\{ 1 - \left(\frac{r}{R}\right)^2 \right\}^{-7/2} \left(\frac{r}{R}\right)^{1/2} \leq C_0'' \left(\frac{r}{R}\right)^{1/2}$$

for $r, R > 0$ satisfying (24) as in the proof of Proposition 12, where

$$C_0'' := 2^{1/2} \pi^{-1/2} 5^{-m/2} C_0 M R_0^{-5/2} (1 - X_0^{-2})^{-7/2}.$$

We remark that the constant C_0 is independent of $r, R > 0$. Thus, the conditions in (25) hold whenever R/r is large enough. By the assumptions, the estimate (26) holds for $R/r \gg 1$. It follows that

$$
\begin{aligned}
5^{-m/2}M = \|\psi\|_{L^2(\partial B)} &\le C'' \left[\log\left(\frac{R}{r}\right)\right]^{m-\varepsilon} \left|\log \|d\Lambda_B(\psi)\|_{\mathcal{L}}\right|^{-m} \\
&\le C'' \left[\log\left(\frac{R}{r}\right)\right]^{m-\varepsilon} \left(-\log\left\{C_0''\left(\frac{r}{R}\right)^{1/2}\right\}\right)^{-m} \\
&= C'' \left[\log\left(\frac{R}{r}\right)\right]^{m-\varepsilon} \left[\frac{1}{2}\log\left(\frac{R}{r}\right) - \log C_0''\right]^{-m} \to 0 \text{ as } \frac{R}{r} \to +\infty,
\end{aligned}
$$

which leads to a contradiction. The proposition is now proved. □

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