RECONSTRUCTING THE METRIC AND MAGNETIC FIELD FROM THE SCATTERING RELATION

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Abstract. We develop a method for reconstructing the conformal factor of a Riemannian metric and the magnetic field on a surface from the scattering relation associated to the corresponding magnetic flow. The scattering relation maps a starting point and direction of a magnetic geodesic into its end point and direction. The key point in the reconstruction is the interplay between the magnetic ray transform, the fiberwise Hilbert transform on the circle bundle of the surface, and the Laplace–Beltrami operator of the underlying Riemannian metric.

1. Introduction

Let M be an n-dimensional compact manifold with boundary, $n \geq 2$, endowed with a Riemannian metric g and a *magnetic field* Ω, which is a closed 2-form on M. Consider the motion a charged particle of unit mass and unit charge in this magnetic field, which is described by Newton's law of motion

$$\nabla_{\dot\gamma}\dot\gamma = Y(\dot\gamma), \tag{1.1}$$

where ∇ is the Levy–Civita connection of g and $Y : TM \to TM$ is the *Lorentz force* associated with Ω, i.e., the bundle map uniquely determined by

$$\Omega_x(\xi, \eta) = \langle Y_x(\xi), \eta\rangle_g \tag{1.2}$$

for all $x \in M$ and $\xi, \eta \in T_xM$. A path $\gamma : [a, b] \to M$, satisfying (1.1), is referred to as *magnetic geodesic*. In a natural way, equation (1.1) also defines a flow ψ^t on TM, called a *magnetic* (or *twisted geodesic*) *flow*. Magnetic flows were first considered by V. I. Arnold in [2] and by D. V. Anosov and Y. G. Sinai in [1]. As shown in [3, 8, 9, 10, 7, 11]), they are closely related to other problems of classical mechanics, mathematical physics, symplectic geometry, and dynamical systems.

Assuming that g is known on ∂M, consider the inverse problem of determining the metric g and magnetic field Ω on the whole of M from the scattering relation at a fixed energy level. Without loss of generality, we may fix the energy level to be the unit sphere bundle SM of M.

The set of inward unit vector is defined by

$$\partial_+SM = \{(x, \xi) \in SM : x \in \partial M, \langle\xi, \nu(x)\rangle \geq 0\},$$

and the set of outward unit vectors by

$$\partial_-SM = \{(x, \xi) \in SM : x \in \partial M, \langle\xi, \nu(x)\rangle \leq 0\},$$

with ν the unit inner normal to ∂M.

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The *scattering relation* is the map

$$\mathcal{S}_{g,\Omega} : \partial_+ SM \to \partial_- SM$$

that is defined as follows:

$$\mathcal{S}_{g,\Omega}(x,\xi) = \psi^{\ell(x,\xi)}(x,\xi) = (\gamma_{x,\xi}(\ell(x,\xi)), \dot{\gamma}_{x,\xi}(\ell(x,\xi))),$$

where $\gamma_{x,\xi} : [0, \ell(x,\xi)] \to M$ is a magnetic geodesic with $\gamma_{x,\xi}(0) = x$, $\dot{\gamma}_{x,\xi}(0) = \xi$, and $\ell(x,\xi) \geq 0$ is the first time instance at which $\gamma_{x,\xi}$ meets the boundary, $\gamma_{x,\xi}(\ell(x,\xi)) \in \partial M$. Clearly, the scattering relation is well defined whenever there are no trapped unit speed magnetic geodesics.

In the case when the magnetic field is absent, this problem is nothing else than the ordinary lens rigidity problem for a Riemannian manifold. In that case, a natural assumption to impose is simplicity of the metric. In the magnetic case, simplicity is defined as follows, see [5].

Let Λ denote the second fundamental form of ∂M. We say that ∂M is *strictly magnetic convex* if

$$\Lambda(x,\xi) > \langle Y_x(\xi), \nu(x)\rangle \tag{1.3}$$

for all $(x,\xi) \in S(\partial M)$. For $x \in M$, the *magnetic exponential map* at x is the map $\exp^\mu_x : T_x M \to M$ given by

$$\exp^\mu_x(t\xi) = \gamma_{x,\xi}(t), \quad t \geq 0,\ \xi \in S_x M.$$

Definition 1.1. The manifold M is said to be *simple* with respect to (g, Ω) if ∂M is strictly magnetic convex and the magnetic exponential map $\exp^\mu_x : (\exp^\mu_x)^{-1}(M) \to M$ is a diffeomorphism for every $x \in M$.

The inverse problem under consideration is intimately related to the inverse scattering problem for a semiclassical magnetic Schrödinger operator, as well as to the boundary rigidity problem in the presence of a magnetic field [5].

There is a natural gauge transform in this problem. Indeed, if $f : M \to M$ is a diffeomorphism leaving the boundary ∂M pointwise fixed and such that $f^* g|_{\partial M} = g|_{\partial M}$, then the scattering relation of the pair (g, Ω) coincides with that of the pair $(f^* g, f^* \Omega)$. We can exclude this trivial nonuniqueness by fixing the conformal class of the metric. Theorem 6.1 of [5] then gives uniqueness of the conformal factor and the magnetic field from the magnetic scattering relation. The natural question we address in this paper is the *reconstruction* of the conformal factor and the magnetic field from the magnetic scattering relation. Our main result shows that this is in fact possible if $\dim M = 2$. More precisely, we have:

Theorem 1.2. *Let M be a two-dimensional compact manifold with boundary and let g_0 be a fixed Riemannian metric on M. Assume that ρ is a smooth positive function on M, $g = \rho g_0$, and Ω is a smooth 2-form on M such that M is simple with respect to (g, Ω). Then we develop a reconstruction procedure to recover ρ and Ω on M from g_0, the restriction $\rho|_{\partial M}$, and the scattering relation $\mathcal{S}_{g,\Omega}$.*

This result is a generalization of Theorem 1.3 of [13] (see also [14]) that deals with the case when the magnetic field is absent. Our proof of Theorem 1.2 is a modification of the arguments in [13], [14] to deal with the additional difficulty of determining also the magnetic field. The key ingredient is the interplay between the magnetic ray transform, the fiberwise Hilbert transform, and the Laplace–Beltrami operator of the underlying Riemannian metric.

The needed properties of the magnetic ray transform, I, are established in Section 2. The main result of this section, Theorem 2.1, about surjectivity of the adjoint operator, I^*, is a refined version of Theorem 7.3 in [5] and generalizes Theorem 3.3 of [13] (also see [15, Theorem 4.2] and [14]), giving a more precise statement even in that case. It is also valid in any dimension.

In Section 3 we present a connection between the fiberwise Hilbert transform, H, on the circle bundle of a surface, the operators I and I^*, and harmonic functions. Such a connection was discovered in [12] for Riemannian surfaces and played a crucial role in proving boundary rigidity of simple surfaces, [12, Theorem 1.1]. For the magnetic case, it was developed in [5], and was used to prove rigidity of simple magnetic systems on surfaces. Here we specialize this connection to our problem. We state the needed results in the lemmas: Lemmas 3.1–3.2 and Lemmas 3.3–3.4.

The closing Section 4 presents a reconstruction procedure for the inverse problem under consideration and finishes the proof of Theorem 1.2.

2. Magnetic ray transform

As in the introduction, let M be an n-dimensional compact manifold with boundary, $n \geq 2$, and let g and Ω be a Riemannian metric and a closed 2-form on M, respectively. Throughout this paper we assume that M is simple with respect to (g, Ω) as stated in Definition 1.1.

If $\phi : SM \to \mathbb{R}$ a continuous function on the unit sphere bundle SM of M, the magnetic ray transform of ϕ is defined to be the following function on the space of unit speed magnetic geodesics going from a boundary point to a boundary point:

$$I\phi(\gamma) = \int_\gamma \phi := \int_0^T \phi(\gamma(t), \dot{\gamma}(t))\, dt,$$

where $\gamma : [0, T] \to M$ is any unit speed magnetic geodesic such that $\gamma(0) \in \partial M$ and $\gamma(T) \in \partial M$. Assuming that the magnetic geodesics are parametrized by $\partial_+ SM$, we obtain a map $I : C(SM) \to C(\partial_+ SM)$,

$$\begin{aligned} I\phi(x,\xi) &= \int_0^{\ell(x,\xi)} \phi(\psi^t(x,\xi))\, dt \\ &= \int_0^{\ell(x,\xi)} \phi(\gamma_{x,\xi}(t), \dot{\gamma}_{x,\xi}(t))\, dt, \quad (x,\xi) \in \partial_+ SM, \end{aligned} \tag{2.1}$$

where $\gamma_{x,\xi} : [0, \ell(x,\xi)] \to M$ is a magnetic geodesic with $\gamma_{x,\xi}(0) = x$, $\dot{\gamma}_{x,\xi}(0) = \xi$, and $\gamma(\ell(x,\xi)) \in \partial M$.

We are interested in the magnetic ray transform of functions of the form $\phi(x,\xi) = v_i(x)\xi^i + \varphi(x)$. Such a function can be identified with a pair $\mathbf{f} = [v, \varphi]$, where v is a one-form and φ is a function on M. When we denote F for a function space (C^k, L^p, H^k, etc.), we will denote by $\mathcal{F}(M)$ the corresponding space of such pairs. In particular, $\mathcal{L}^2(M)$ consists of square integrable pairs, and we endow it with the inner product

$$\langle \mathbf{f}, \mathbf{h} \rangle = \int_M (\langle v, w \rangle_g + \varphi\psi)\, d\,\mathrm{Vol} \tag{2.2}$$

for $\mathbf{f} = [v, \varphi]$ and $\mathbf{h} = [w, \psi]$.

The *magnetic ray transform* of a pair $[v,\varphi] \in \mathcal{C}(M)$ is then defined to be

$$I[v,\varphi](x,\xi) = \int_0^{\ell(x,\xi)} v_i(\gamma_{x,\xi}(t))\dot\gamma^i_{x,\xi}(t)\,dt + \int_0^{\ell(x,\xi)} \varphi(\gamma_{x,\xi}(t))\,dt$$
$$= I_1 v(x,\xi) + I_0\varphi(x,\xi), \quad (x,\xi) \in \partial_+ SM.$$

In view of [5, Lemma 3.2], I extends to a bounded operator

$$I : \mathcal{L}^2(M) \to L^2_\mu(\partial_+ SM), \tag{2.3}$$

where $d\mu$ the Liouville measure on $\partial_+ SM$ given by

$$d\mu(x,\xi) = \langle \xi, \nu(x)\rangle\, d\Sigma^{2n-2}(x,\xi),$$

with $d\Sigma^{2n-2}$ being the standard (local product) measure on $\partial_+ SM$, and with $\nu(x)$ the inward unit normal to ∂M at a point x. Let

$$I^* : L^2_\mu(\partial_+ SM) \to \mathcal{L}^2(M) \tag{2.4}$$

be its adjoint. We have (see [5, Section 7.1])

$$I^* w = [I_1^* w, I_0^* w]$$

with

$$I_1^* w = \left(\int_{S_xM} \xi^i w^\sharp(x,\xi)\,d\sigma_x(\xi)\right), \qquad I_0^* w = \int_{S_xM} w^\sharp(x,\xi)\,d\sigma_x(\xi), \tag{2.5}$$

where $w^\sharp(x,\xi)$ is the function on SM that is constant along the orbits of the magnetic flow and that equals $w(x,\xi)$ on $\partial_+ SM$,

$$w^\sharp(\psi^t(x,\xi)) = w(x,\xi), \quad (x,\xi) \in \partial_+ SM,$$

and where $d\sigma_x$ is the volume element on the fiber S_xM above x which is induced by the Riemannian metric g.

Let d and δ stand for the differential on functions and the codifferential on one-forms. Define

$$\mathbf{d}h = [dh, 0], \quad \boldsymbol{\delta}[v,\varphi] = \delta v$$

for a function h on M and for a pair $[v,\varphi]$ of a one-form and a function. A pair $[v,\varphi]$ is said to be *potential* if $[v,\varphi] = \mathbf{d}h$ for some $h \in H^1_0(M)$, the latter being the Sobolev space of functions with square integrable derivatives and with zero trace on ∂M. A pair $[v,\varphi]$ is said to be *solenoidal* if $\boldsymbol{\delta}[v,\varphi] = 0$. If F denotes for a function space, then $\mathcal{F}_{\mathcal{P}}(M)$ will denote the subspace of potential pairs in $\mathcal{F}(M)$ and $\mathcal{F}_{\rm sol}(M)$, the subspace of solenoidal pairs in $\mathcal{F}(M)$.

Note that (see [5, (7.3)])

$$\operatorname{Im} I^* \subset \mathcal{L}^2_{\rm sol}(M). \tag{2.6}$$

Consider

$$C^\infty_\alpha(\partial_+ SM) = \{w \in C^\infty(\partial_+ SM) : w^\sharp \in C^\infty(SM)\}.$$

By [5, Lemma 7.6], this space can be described via the scattering relation $\mathcal{S}_{g,\Omega}$ as follows:

$$C^\infty_\alpha(\partial_+ SM) = \{w \in C^\infty(\partial_+ SM) : Aw \in C^\infty(\partial(SM))\}, \tag{2.7}$$

where

$$Aw(x,\xi) = \begin{cases} w(x,\xi), & (x,\xi) \in \partial_+ SM, \\ w \circ \mathcal{S}^{-1}_{g,\Omega}(x,\xi), & (x,\xi) \in \partial_- SM. \end{cases}$$

The following theorem is the main result of this section.

Theorem 2.1. *Let M be simple with respect to (g, Ω). Then for every $\mathbf{f} \in \mathcal{C}^\infty_{\mathrm{sol}}(M)$, there exists $w \in C^\infty_\alpha(\partial_+ SM)$ such that*

$$\mathbf{f} = I^* w.$$

Proof. Let $\widetilde{M}$ be a an n-dimensional compact manifold with boundary whose interior contains M. Extend g and Ω to a Riemannian metric and a closed 2-form on $\widetilde{M}$, preserving the former notations for the extensions. Note that $\widetilde{M}$ is also simple with respect to (g, Ω), if $\widetilde{M}$ is sufficiently close to M, and we take this simplicity for granted.

Denote by r_M the restriction operator from $\widetilde{M}$ to M, denote the magnetic ray transform on $\widetilde{M}$ by $\tilde{I}$, and define

$$N = \tilde{I}^* \tilde{I}. \tag{2.8}$$

Note that, in view of (2.6),

$$\boldsymbol{\delta} N = 0. \tag{2.9}$$

The following holds:

Lemma 2.2. *The operator*

$$r_M N : \mathcal{C}^\infty_0(\widetilde{M}^{\mathrm{int}}) \to \mathcal{C}^\infty_{\mathrm{sol}}(M) \tag{2.10}$$

is surjective.

Postponing the proof of Lemma 2.2 to the end of the section, we finish the proof of the theorem using it. Given $\mathbf{f} \in \mathcal{C}^\infty_{\mathrm{sol}}(M)$, Lemma 2.2 provides us with an $\mathbf{h} = [u, \psi] \in \mathcal{C}^\infty_0(\widetilde{M}^{\mathrm{int}})$ such that

$$\mathbf{f} = r_M N \mathbf{h} = r_M \tilde{I}^* \tilde{I}[u, \psi]. \tag{2.11}$$

Define

$$w^\pm(x, \xi) = \pm \int_0^{\tilde{\ell}^\pm(x,\xi)} \left\{ u_i(\gamma_{x,\xi}(t)) \dot{\gamma}^i_{x,\xi}(t) + \psi(\gamma_{x,\xi}(t)) \right\} dt,$$

where $\tilde{\ell}^\pm(x, \xi)$ is the value of t, $\pm t > 0$, at which $\gamma_{x,\xi}(t)$ meets the boundary of $\widetilde{M}$, $\gamma_{x,\xi}(\tilde{\ell}^\pm(x, \xi)) \in \partial\widetilde{M}$, $\pm\tilde{\ell}^\pm(x, \xi) > 0$.

Clearly, $w^\pm \in C^\infty(S\widetilde{M}^{\mathrm{int}})$. We let $w = (w^+ + w^-)|_{\partial_+ SM}$. It is easy to see that $(w^+ + w^-)$ is constant on the orbits of the magnetic flow; therefore, $w^\sharp = (w^+ + w^-)|_{SM}$ and $w^\sharp \in C^\infty(SM)$. Since $\tilde{I}[u, \psi] = (w^+ + w^-)|_{\partial_+ S\widetilde{M}}$, we see from (2.11) that $I^* w = \mathbf{f}$, which completes the proof of the theorem. □

Now, we prove Lemma 2.2. This lemma should be compared to [5, Lemma 7.4], [15, Theorem 4.3] and the corresponding result of [**?**]. Besides, some ideas in its proof are borrowed from [4] and [16].

Proof of Lemma 2.2. We first prove that the operator $r_M N$ in (2.10) has closed range.

Recall from [5, Proposition 7.2] that N is a ΨDO of order -1 on $\widetilde{M}^{\mathrm{int}}$ and that the principal symbol of N is

$$c_n \operatorname{diag}\left(\frac{\delta^j_i}{|\xi|} - \frac{\xi^j \xi_i}{|\xi|^3}, \frac{1}{|\xi|} \right). \tag{2.12}$$

Let Λ be a proper ΨDO on $\widetilde{M}^{\mathrm{int}}$ with principal symbol $-c_n(1/|\xi|^3)$. In view of (2.12), we see that

$$C := N + \mathbf{d}\Lambda\boldsymbol{\delta} \tag{2.13}$$

is a ΨDO of order -1 with principal symbol

$$c_n \operatorname{diag}\left(\delta_i^j |\xi|^{-1}, |\xi|^{-1}\right),$$

which means that C is an elliptic ΨDO of order -1.

Let T be a proper parametrix of C, i.e.

$$CT \equiv E \tag{2.14}$$

and

$$TC \equiv E, \tag{2.15}$$

where $\equiv$ designates equivalence up to a smoothing operator and E is the identity operator.

From (2.13)–(2.15) we find that

$$NT + \mathbf{d}\Lambda\boldsymbol{\delta}T \equiv E \tag{2.16}$$

and

$$TN + T\mathbf{d}\Lambda\boldsymbol{\delta} \equiv E. \tag{2.17}$$

Premultiplying (2.13) by $\boldsymbol{\delta}$ and using (2.9), we get

$$\boldsymbol{\delta}C = \boldsymbol{\delta}N + \boldsymbol{\delta}\mathbf{d}\Lambda\boldsymbol{\delta} = \boldsymbol{\delta}\mathbf{d}\Lambda\boldsymbol{\delta}.$$

Since $\boldsymbol{\delta}\mathbf{d} = \delta d$ is nothing but $-\Delta$, it has a proper parametrix $(-\Delta)^{-1}$. Then

$$\Lambda\boldsymbol{\delta} \equiv (-\Delta)^{-1}\boldsymbol{\delta}C.$$

Combined with (2.16) and (2.14), this yields

$$NT \equiv E - \mathbf{d}(-\Delta)^{-1}\boldsymbol{\delta}. \tag{2.18}$$

Restricted to $\mathcal{C}^\infty_{0,\mathrm{sol}}(\widetilde{M}^{\mathrm{int}})$, this implies

$$NT\mathbf{f} = \mathbf{f} + K\mathbf{f} \tag{2.19}$$

for all $\mathbf{f} \in \mathcal{C}^\infty_{0,\mathrm{sol}}(\widetilde{M}^{\mathrm{int}})$, where K is a smoothing operator on $\widetilde{M}^{\mathrm{int}}$.

Let U be a neighborhood of M, compactly embedded in $\widetilde{M}^{\mathrm{int}}$. By the results of [6], there exists a bounded extension operator $\mathcal{E} : \mathcal{L}^2_{\mathrm{sol}}(M) \to \mathcal{L}^2_{U,\mathrm{sol}}(\widetilde{M}^{\mathrm{int}})$ whose restriction to $\mathcal{C}^\infty_{\mathrm{sol}}(M)$ is a bounded linear operator from $\mathcal{C}^\infty_{\mathrm{sol}}(M)$ to $\mathcal{C}^\infty_{U,\mathrm{sol}}(\widetilde{M}^{\mathrm{int}})$. Here $\mathcal{L}^2_{U,\mathrm{sol}}(\widetilde{M}^{\mathrm{int}})$ ($\mathcal{C}^\infty_{U,\mathrm{sol}}(\widetilde{M}^{\mathrm{int}})$) denotes the subspace of $\mathcal{L}^2_{\mathrm{sol}}(\widetilde{M}^{\mathrm{int}})$ ($\mathcal{C}^\infty_{\mathrm{sol}}(\widetilde{M}^{\mathrm{int}})$), consisting of pairs supported in U. (The results of [6] are stated for domains in the Euclidean $\mathbb{R}^n$, but, as noted at the end of that article, they are valid for subdomains of a smooth manifold equipped with a sufficiently smooth Riemannian metric.) Then we have on $\mathcal{L}^2_{\mathrm{sol}}(M)$

$$r_M NT\mathcal{E} = E + r_M K\mathcal{E}. \tag{2.20}$$

It is easy to see that $r_M K\mathcal{E}$ is a compact operator on $\mathcal{L}^2_{\mathrm{sol}}(M)$, because K is smoothing on $\widetilde{M}^{\mathrm{int}}$. Therefore, the range of the operator $E + r_M K\mathcal{E} : \mathcal{L}^2_{\mathrm{sol}}(M) \to \mathcal{L}^2_{\mathrm{sol}}(M)$ is closed and has finite codimension. Since K is smoothing, this easily implies that the range of the operator $E + r_M K\mathcal{E} : \mathcal{C}^\infty_{\mathrm{sol}}(M) \to \mathcal{C}^\infty_{\mathrm{sol}}(M)$ is closed and also has finite codimension. We conclude that $r_M NT\mathcal{E}(\mathcal{C}^\infty_{\mathrm{sol}}(M))$ is closed and has finite codimension in $\mathcal{C}^\infty_{\mathrm{sol}}(M)$. Since

$$r_M NT\mathcal{E}(\mathcal{C}^\infty_{\mathrm{sol}}(M)) \subset r_M N(\mathcal{C}^\infty_0(\widetilde{M}^{\mathrm{int}})) \subset \mathcal{C}^\infty_{\mathrm{sol}}(M),$$

a well-known fact from functional analysis then implies that the intermediate space $r_M N(\mathcal{C}^\infty_0(\widetilde{M}^{\mathrm{int}}))$ is closed in $\mathcal{C}^\infty_{\mathrm{sol}}(M)$.

To finish the proof of the lemma, it now suffices to show that the adjoint operator has trivial kernel.

We have the direct sum decomposition

$$\mathcal{C}^\infty(M) = \mathcal{C}^\infty_{\mathcal{P}}(M) \oplus \mathcal{C}^\infty_{\mathrm{sol}}(M). \tag{2.21}$$

Therefore, each functional on $\mathcal{C}^\infty_{\mathrm{sol}}(M)$ yields a functional on $\mathcal{C}^\infty(M)$ that vanishes on $\mathcal{C}^\infty_{\mathcal{P}}(M)$ and vice versa, a functional on $\mathcal{C}^\infty(M)$ that vanishes on $\mathcal{C}^\infty_{\mathcal{P}}(M)$ yields a functional on $\mathcal{C}^\infty_{\mathrm{sol}}(M)$. Hence the dual of $\mathcal{C}^\infty_{\mathrm{sol}}(M)$ can be identified with

$$\mathcal{D}'_{M,\delta}(\widetilde{M}^{\mathrm{int}}) := \{\mathbf{f} \in \mathcal{D}'(\widetilde{M}^{\mathrm{int}}) : \operatorname{supp}\mathbf{f} \subset M,\ \langle \mathbf{f} \,|\, \tilde{\mathbf{h}}\rangle = 0 \text{ for all } \mathbf{h} \in \mathcal{C}^\infty_{\mathcal{P}}(M)\}, \tag{2.22}$$

where $\tilde{\mathbf{h}} \in \mathcal{C}^\infty(\widetilde{M}^{\mathrm{int}})$ is any extension of $\mathbf{h}$ from M to $\widetilde{M}^{\mathrm{int}}$.

Now, the dual of the operator in (2.10) is an operator

$$(r_M N)^* : \mathcal{D}'_{M,\delta}(\widetilde{M}^{\mathrm{int}}) \to \mathcal{D}'(\widetilde{M}^{\mathrm{int}}). \tag{2.23}$$

It is easy to see that N is selfadjoint in the sense that

$$\langle N\mathbf{f} \,|\, \mathbf{h}\rangle = \langle \mathbf{f} \,|\, N\mathbf{h}\rangle$$

for all $\mathbf{f} \in \mathcal{E}'(\widetilde{M}^{\mathrm{int}})$ and $\mathbf{h} \in \mathcal{C}^\infty_0(\widetilde{M}^{\mathrm{int}})$. Using this, we find that for all $\mathbf{f} \in \mathcal{D}'_{M,\delta}(\widetilde{M}^{\mathrm{int}})$ and $\mathbf{h} \in \mathcal{C}^\infty(\widetilde{M}^{\mathrm{int}})$

$$\langle (r_M N)^*\mathbf{f} \,|\, \mathbf{h}\rangle = \langle \mathbf{f} \,|\, (r_M N\mathbf{h})^\sim\rangle = \langle \mathbf{f} \,|\, N\mathbf{h}\rangle = \langle N\mathbf{f} \,|\, \mathbf{h}\rangle.$$

In other words,

$$(r_M N)^* = N\big|_{\mathcal{D}'_{M,\delta}(\widetilde{M}^{\mathrm{int}})}.$$

Assume $\mathbf{f} \in \mathcal{D}'_{M,\delta}(\widetilde{M})$ is in the kernel of N,

$$N\mathbf{f} = 0. \tag{2.24}$$

Observe that it follows from (2.22) that

$$\operatorname{sing\,supp} \boldsymbol{\delta}\mathbf{f} \subset \partial M. \tag{2.25}$$

Now, decompose

$$\mathbf{f} = \mathbf{d}p + \mathbf{g}, \tag{2.26}$$

with

$$\boldsymbol{\delta}\mathbf{g} = 0. \tag{2.27}$$

We have

$$\boldsymbol{\delta}\mathbf{d}p = \boldsymbol{\delta}\mathbf{f}$$

and, since $\boldsymbol{\delta}\mathbf{d} = \delta d = -\Delta$ is an elliptic operator, from (2.25) we get

$$\operatorname{sing\,supp} p \subset \partial M. \tag{2.28}$$

Since $\mathbf{f}$ is supported in M, from (2.26) and (2.28) we deduce

$$\operatorname{sing\,supp} \mathbf{g} \subset M. \tag{2.29}$$

Let $\tilde{p}$ be a smooth function on $\widetilde{M}$ equal to p in a neighborhood of $\partial\widetilde{M}$. It is easy to see that

$$N\mathbf{d}\tilde{p} = N\mathbf{d}p. \tag{2.30}$$

Therefore,

$$N\mathbf{g} = N\mathbf{f} - N\mathbf{d}p = -N\mathbf{d}\tilde{p}, \tag{2.31}$$

which implies that $N\mathbf{g}$ is smooth in $\widetilde{M}^{\mathrm{int}}$. Now, using (2.17) and (2.27), we deduce that $\mathbf{g}$ is smooth in $\widetilde{M}^{\mathrm{int}}$. Taking (2.29) into account, we conclude that $\mathbf{g}$ is smooth on $\widetilde{M}$.

We have

$$N(\mathbf{d}\tilde{p} + \mathbf{g}) = 0$$

with $\tilde{p}$ and $\mathbf{g}$ smooth on $\widetilde{M}$. This implies $\tilde{I}(\mathbf{d}\tilde{p}+\mathbf{g}) = 0$ and from [5, Theorem 5.3] we then obtain

$$\mathbf{d}\tilde{p} + \mathbf{g} = \mathbf{d}\hat{p} \tag{2.32}$$

for some smooth function $\hat{p}$ on $\widetilde{M}$, vanishing on $\partial\widetilde{M}$.

Combining (2.26) and (2.32) yields

$$\mathbf{f} = \mathbf{d}p + \mathbf{d}\hat{p} - \mathbf{d}\tilde{p}. \tag{2.33}$$

For every $\mathbf{h} \in \mathcal{C}^\infty_{\mathrm{sol}}(M)$ we have

$$\langle \mathbf{f} \,|\, \tilde{\mathbf{h}} \rangle = \langle \mathbf{d}(p + \hat{p} - \tilde{p}) \,|\, \tilde{\mathbf{h}} \rangle = \langle p + \hat{p} - \tilde{p} \,|\, \boldsymbol{\delta}\tilde{\mathbf{h}} \rangle$$

for any extension $\tilde{\mathbf{h}} \in \mathcal{C}^\infty_0(\widetilde{M}^{\mathrm{int}})$ of $\mathbf{h}$. By the above-mentioned result of [6], we can take $\tilde{\mathbf{h}}$ to be solenoidal, $\boldsymbol{\delta}\tilde{\mathbf{h}} = 0$. Therefore, we see that $\mathbf{f}$ annihilates $\mathcal{C}^\infty_{\mathrm{sol}}(M)$. Taking it into account that $\mathbf{f}$ also annihilates $\mathcal{C}^\infty_{\mathcal{P}}(M)$ (see (2.22)), we find that $\mathbf{f} = 0$, concluding the proof of the lemma.

□

3. Hilbert transform and harmonic functions

From now on, we assume that $\dim M = 2$. The simplicity assumption implies that M is topologically a disk and we fix some orientation on it. Given $v \in T_xM$, we then denote by $v_\perp$ the vector obtained by rotating v by $\pi/2$, with respect to g, according to the orientation of M. In coordinates $(v_\perp)_i = \varepsilon_{ij}v^j$, where

$$\varepsilon = \sqrt{\det g}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The fiberwise Hilbert transform is defined by (see [13, (1.4)])

$$Hu(x,\xi) = \frac{1}{2\pi}\int_{S_xM} \frac{1 + \langle \xi, \eta\rangle}{\langle \xi_\perp, \eta\rangle} u(x,\eta)\, d\sigma_x(\eta), \quad \xi \in S_xM,$$

where $d\sigma_x(\eta)$ is the line element on S_xM with respect to g. If we fix $x \in M$ and a reference point $a \in S_xM$, a function on S_x can be treated as a function of the angular variable, and

$$Hu(x,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \cot\left(\frac{\varphi - \theta}{2}\right) u(x,\varphi)\, d\varphi.$$

The following identity is a key point in our consideration (see [5, (7.13)]):

$$BHAw = -\frac{1}{2\pi} I[\nabla_\perp I_0^* w, \delta_\perp I_1^* w]. \tag{3.1}$$

Here the operators

$$A : C(\partial_+ SM) \to C(SM), \quad B : C(SM) \to C(\partial_+ SM)$$

are defined by

$$Aw(x,\xi) = \begin{cases} w(x,\xi), & (x,\xi) \in \partial_+ SM, \\ w \circ \mathcal{S}^{-1}(x,\xi), & (x,\xi) \in \partial_- SM, \end{cases} \tag{3.2}$$

$$Bu(x,\xi) = u(x,\xi) - u \circ \mathcal{S}(x,\xi), \quad (x,\xi) \in \partial_+ SM, \tag{3.3}$$

with $\mathcal{S}$ being the scattering relation $\mathcal{S} = \mathcal{S}_{g,\Omega}$. The operators I_0^* and I_1^* are defined by (2.5), and

$$\nabla_\perp = \varepsilon \nabla, \quad \delta_\perp v = -\delta v_\perp .$$

Let h, h_* be a pair of conjugate harmonic functions, i.e.,

$$\nabla h = \nabla_\perp h_*, \quad \nabla h_* = -\nabla_\perp h.$$

By Theorem 2.1, there is $w \in C^\infty_\alpha(\partial_+ SM)$ such that

$$I_0^* w = h, \tag{3.4}$$

$$I_1^* w = 0. \tag{3.5}$$

From (3.4), (3.5), and (3.1) we then have:

$$BHAw = -\frac{1}{2} B h^0_*, \tag{3.6}$$

where h^0_* is the trace of h_* on ∂M.

Recall that

$$I_1^* w(x) = \left(\int_{S_x M} \xi^i w^\sharp(x,\xi)\, d\sigma_x(\xi) \right),$$

where $w^\sharp(x,\xi)$ is defined as the function that is constant along the orbits of the magnetic flow and that equals $w(x,\xi)$ on $\partial_+ SM$.

Therefore, from (3.5) we have

$$\left(\int_{S_x M} \xi^i Aw(x,\xi)\, d\sigma_x(\xi) \right) = 0 \quad \text{for all } x \in \partial M. \tag{3.7}$$

Thus, we derive the following:

Lemma 3.1 (cf. [5, Lemma 7.8])**.** *If h, h_* is a pair of smooth conjugate harmonic functions on M, then there is $w \in C^\infty_\alpha(\partial_+ SM)$ such that $I_0^* w = h$, $I_1^* w = 0$, and the equations* (3.6) *and* (3.7) *hold with h^0_* the trace of h_* on ∂M.*

Now, consider (3.6) and (3.7) as simultaneous equations in w. Note that they are completely determined by the metric on ∂M and the scattering relation.

Lemma 3.2 (cf. [5, Lemma 7.9])**.** *Suppose $h^0_* \in C^\infty(\partial M)$ and $w \in C^\infty_\alpha(\partial_+ SM)$ satisfy equation* (3.6) *and* (3.7). *Define $h := I_0^* w$ and let h_* be the harmonic continuation of h^0_* to M. Then h and h_* are conjugate harmonic functions and $I_1^* w = 0$.*

Proof. In view of [5, Lemma 7.9], we must only prove that $I_1^* w = 0$. Observe that in the proof of the mentioned lemma in [5], it was shown that

$$\delta_\perp I_1^* w = 0.$$

This means that

$$I_1^* w = \nabla f$$

for some smooth function f on M. On the other hand, we always have

$$\delta I_1^* = 0.$$

Therefore, f is a harmonic function on M.

Note that (3.7) means $I_1^* w|_{\partial M} = 0$. Hence, f is harmonic and $\nabla f|_{\partial M} = 0$, whence we conclude that $f \equiv const$, so that $I_1^* w = \nabla f = 0$. □

Let φ be a harmonic function on M. By Theorem 2.1, then there is $w \in C_\alpha^\infty(\partial_+ M)$ such that

$$I_0^* w = 0, \tag{3.8}$$

$$I_1^* w = \nabla\varphi. \tag{3.9}$$

Since $\delta_\perp \nabla\varphi = 0$, from (3.8), (3.9), and (3.1) we get

$$BHAw = 0. \tag{3.10}$$

Also, (3.9) implies

$$\left(\int_{S_x M} \xi^i Aw(x,\xi)\, d\sigma_x(\xi) \right) = \nabla\varphi(x) \quad \text{for all } x \in \partial M. \tag{3.11}$$

Thus, we derive the following:

Lemma 3.3. *If φ is a harmonic function on M, then there is $w \in C_\alpha^\infty(\partial_+ SM)$ such that $I_0^* w = 0$, $I_1^* w = \nabla\varphi$, and the equations* (3.10) *and* (3.11) *hold.*

Now, consider (3.10) and (3.11) as simultaneous equations in w. Note that they are completely determined by the metric on ∂M and the scattering relation.

Lemma 3.4. *Let φ be a harmonic function on M. Suppose $w \in C_\alpha^\infty(\partial_+ SM)$ satisfies* (3.10) *and* (3.11). *Then we have $I_0^* w \equiv const$ and $I_1^* w = \nabla\varphi$.*

Proof. Applying [5, Lemma 7.9], we see that $I_0^* w \equiv const$. For the same arguments as in the proof of Lemma 3.2, we have $I_1^* w = \nabla f$ for some harmonic function f. From (3.11) we get $\nabla f|_{\partial M} = \nabla\varphi|_{\partial M}$. Since f and φ are harmonic functions, we conclude that $f - \varphi \equiv const$, which yields $I_1^* w = \nabla\varphi$. □

4. Reconstruction procedure and the proof of Theorem 1.2

As in the previous section, M is assumed to be an oriented surface. Let dA be the area form associated with g. Then we can write down the magnetic 2-form Ω as

$$\Omega = \lambda dA$$

for some function λ on M. The generator of the magnetic flow on SM for (g, Ω) is

$$\mathbf{G}_\mu = \mathbf{G} + \lambda V,$$

where $\mathbf{G}$ is the generator of the geodesic flow and V is the derivation in the vertical variable on SM.

The reconstruction procedure in the proof of Theorem 1.2 is based on the integral identity of the next lemma.

Lemma 4.1. *For every conformal Killing vector field X on M and every $w \in C^\infty(\partial_+ SM)$, the following identity holds:*

$$\int_M \langle X, \nabla I_0^* w\rangle \, dA - 2\int_M \lambda \langle X_\perp, I_1^* w\rangle \, dA$$
$$= 2\pi \int_{\partial M} \langle X, \nu\rangle (Aw)_0 \, ds + 2\int_{\partial_+ SM} B(\langle X, \xi\rangle) w \, d\mu, \quad (4.1)$$

where the operators A and B are defined by (3.2) *and* (3.3)*, and $(Aw)_0$ is the average value of Aw on a fiber,*

$$(Aw)_0(x) = \frac{1}{2\pi}\int_{S_x M} (Aw)(x,\xi)\, d\sigma_x(\xi).$$

Proof. Let X be a conformal Killing vector field; i.e.,

$$\frac{1}{2}(\nabla_i X_j + \nabla_j X_i) = g_{ij}\frac{\delta X}{2}. \quad (4.2)$$

Then straightforward calculations give

$$\mathbf{G}_\mu \langle X, \xi\rangle = \frac{1}{2}\delta X + \lambda\langle X_\perp, \xi\rangle.$$

Applying I yields

$$-B(\langle X, \xi\rangle) = I[\lambda X_\perp, \frac{1}{2}\delta X]. \quad (4.3)$$

Multiplying (4.3) in $L^2_\mu(\partial_+ SM)$ by $w \in C^\infty(\partial_+ SM)$, we get

$$\int_{\partial_+ SM} wI[\lambda X_\perp, \frac{1}{2}\delta X]\, d\mu = -\int_{\partial_+ SM} wB(\langle X, \xi\rangle)\, d\mu. \quad (4.4)$$

Observe that

$$\int_{\partial_+ SM} wI[\lambda X_\perp, \frac{1}{2}\delta X]\, d\mu = \int_M \langle \lambda X_\perp, I_1^* w\rangle \, dA + \frac{1}{2}\int_M \delta X I_0^* w \, dA$$
$$= \int_M \lambda\langle X_\perp, I_1^* w\rangle \, dA - \frac{1}{2}\int_M \langle X, \nabla I_0^* w\rangle \, dA + \frac{1}{2}\int_{\partial M} \langle X, \nu\rangle I_0^* w \, ds. \quad (4.5)$$

Since $I_0^* w|_{\partial M} = 2\pi (Aw)_0$, identity (4.1) follows now from (4.4) and (4.5). $\square$

Proof of Theorem 1.2. First of all, the simplicity assumption implies that M is topologically a disk, and we can find a conformal diffeomorphism $f : (M, g_0) \to (D, e)$, where D is the unit disk in the plane and e is the Euclidean metric. So, there is no loss of generality in assuming that M is a subset of $\mathbb{R}^2$ and that

$$g_{ij} = \rho\delta_{ij}. \quad (4.6)$$

In this case, (4.2) takes the form of the Cauchy-Riemann equations:

$$\frac{\partial X^1}{\partial x^1} = \frac{\partial X^2}{\partial x^2}, \quad \frac{\partial X^1}{\partial x^2} = -\frac{\partial X^2}{\partial x^1}.$$

Let h be a harmonic function. By Lemmas 3.1 and 3.2 we can find a solution $w \in C^\infty_\alpha(\partial_+ SM)$ of equations (3.6) and (3.7):

$$BHAw = -\frac{1}{2}Bh^0_*,$$

$$\left(\int_{S_x M} \xi^i Aw(x,\xi)\, d\sigma_x(\xi)\right) = 0, \quad x \in \partial M,$$

where h_*^0 is the trace of the conjugate harmonic function. These lemmas also give $h = I_0^* w$ and $I_1^* w = 0$, so that (4.1) takes the form:

$$\int_M \langle X, \nabla h\rangle \, dA = \int_{\partial M} \langle X, \nu\rangle h \, ds + 2\int_{\partial_+ SM} (B\langle X, \xi\rangle) w \, d\mu. \tag{4.7}$$

For the metric (4.6), we have

$$\int_M \langle X, \nabla h\rangle \, dA = \int_M \langle X, \nabla h\rangle \rho(x) \, dx. \tag{4.8}$$

Observe that the above equations for w and the integrals on the right-hand side of (4.7) involve only $g|_{\partial M}$ and the scattering relation $\mathcal{S}_{g,\Omega}$ (via B). Therefore, (4.7) and (4.8) imply that, knowing $g|_{\partial M}$ and $\mathcal{S}_{g,\Omega}$, we can determine the quantity

$$S_{X,h}[\rho] = \int_M \langle X, \nabla h\rangle \rho(x) \, dx$$

for any holomorphic vector field X and any harmonic function h. We thus come to the same situation as in [13, Section 4]. Repeating the arguments of [13], [14], we recover the Fourier transform of ρ:

$$\int_M \rho(x) e^{2i\langle x,k\rangle} \, dx = \lim_{\sigma\to-\bar{\zeta}} \frac{S_{X,h[\rho]}}{(\zeta_2\sigma_1 + \zeta_1\sigma_2)},$$

where ζ, σ are complex vectors in $\mathbb{C}^2$ such that $\zeta\cdot\zeta = \sigma\cdot\sigma = 0$, and $\zeta = \eta + ik$, $\eta, k \in \mathbb{R}^2$. This completes the procedure for reconstructing the conformal factor.

Now, we proceed to restoring the magnetic field. Now, we may assume that the conformal factor ρ is known, so that we know the metric g. The reconstruction of Ω is then reduced to reconstructing the function λ, since we have $\Omega = \lambda\rho \, dx$.

Let φ be a harmonic function on M. By Lemmas 3.3 and 3.4 we can find a solution $w \in C_\alpha^\infty(\partial_+ SM)$ of equations (3.10) and (3.11):

$$BHAw = 0,$$

$$\left(\int_{S_x M} \xi^i Aw(x,\xi) \, d\sigma_x(\xi)\right) = \nabla\varphi(x), \quad x \in \partial M.$$

These lemmas also give $I_0^* w \equiv const$ and $I_1^* w = \nabla\varphi$. Therefore, (4.1) acquires in this case the form:

$$\int_M \lambda\langle X_\perp, \nabla\varphi\rangle \, dA = -\pi\int_{\partial M} \langle X, \nu\rangle (Aw)_0 \, ds - \int_{\partial_+ SM} B(\langle X, \xi\rangle) w \, d\mu.$$

The same arguments as before show that, knowing g and $S_{g,\Omega}$, we can determine the quantity

$$\widetilde{S}_{X,h}[\lambda] := \int_M \lambda\langle X_\perp, \nabla\varphi\rangle \, dA$$

for any holomorphic vector field X and any harmonic function h. This is much the same situation as we had before for ρ. Using a similar procedure we reconstruct λ, finishing the proof of Theorem 1.2.

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