THE CALDERÓN PROBLEM WITH PARTIAL DATA IN TWO DIMENSIONS

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Abstract. We prove for a two dimensional bounded domain that the Cauchy data for the Schrödinger equation measured on an arbitrary open subset of the boundary determines uniquely the potential. This implies, for the conductivity equation, that if we measure the current fluxes at the boundary on an arbitrary open subset of the boundary produced by voltage potentials supported in the same subset, we can determine uniquely the conductivity. We use Carleman estimates with degenerate weight functions to construct appropriate complex geometrical optics solutions to prove the results.

1. Introduction

We consider the problem of determining a complex-valued potential q in a bounded two dimensional domain from the Cauchy data measured on an arbitrary open subset of the boundary for the associated Schrödinger equation $\Delta + q$. A motivation comes from the classical inverse problem of electrical impedance tomography. In this inverse problem one attempts to determine the electrical conductivity of a body by measurements of voltage and current on the boundary of the body. This problem was proposed by Calderón [9] and is also known as Calderón's problem. In dimensions $n \geq 3$, the first global uniqueness result for C^2-conductivities was proven in [28]. In [25], [5] the global uniqueness result was extended to less regular conductivities. Also see [14] for the determination of more singular conormal conductivities. In dimension $n \geq 3$ global uniqueness was shown for the Schrödinger equation with bounded potentials in [28]. The case of more singular conormal potentials was studied in [14].

In two dimensions the first global uniqueness result for Calderón's problem was obtained in [24] for C^2-conductivities. Later the regularity assumptions were relaxed in [6], and [2]. In particular, the paper [2] proves uniqueness for L^∞- conductivities. In two dimensions a recent breakthrough result of Bukhgeim [7] gives unique identifiability of the potential from Cauchy data measured on the whole boundary for the associated Schrödinger equation. As for the uniqueness in determining two coefficients, see [10], [18].

In all the above mentioned articles, the measurements are made on the whole boundary. The purpose of this paper is to show global uniqueness in two dimensions, both for the Schrödinger and conductivity equations, by measuring all the Neumann data on an arbitrary open subset $\widetilde{\Gamma}$ of the boundary produced by inputs of Dirichlet data supported on $\widetilde{\Gamma}$. We formulate this inverse problem more precisely below.

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Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary which consists of N smooth closed curves γ_j, $\partial\Omega = \cup_{j=}^N \gamma_j$, and let ν be the unit outward normal vector to $\partial\Omega$. We denote $\frac{\partial u}{\partial \nu} = \nabla u \cdot \nu$. A bounded and non-zero function $\gamma(x)$ (possibly complex-valued) models the electrical conductivity of Ω. Then a potential $u \in H^1(\Omega)$ satisfies the Dirichlet problem

$$\begin{aligned} \operatorname{div}(\gamma \nabla u) &= 0 \text{ in } \Omega, \\ u\big|_{\partial\Omega} &= f, \end{aligned} \tag{1.1}$$

where $f \in H^{\frac{1}{2}}(\partial\Omega)$ is a given boundary voltage potential. The Dirichlet-to-Neumann (DN) map is defined by

$$\Lambda_\gamma(f) = \gamma \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega}. \tag{1.2}$$

The inverse problem is to recover γ from Λ_γ. This problem can be reduced to studying the set of Cauchy data for the Schrödinger equation with the potential q given by:

$$q = -\frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}}. \tag{1.3}$$

More generally we define the set of Cauchy data for a bounded potential q by:

$$\widehat{C_q} = \left\{ \left(u|_{\partial\Omega}, \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} \right) \mid (\Delta + q)u = 0 \text{ on } \Omega, \quad u \in H^1(\Omega) \right\}. \tag{1.4}$$

We have $\widehat{C_q} \subset H^{\frac{1}{2}}(\partial\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$.

Let $\widetilde{\Gamma} \subset \partial\Omega$ be a non-empty open subset of the boundary. Denote $\Gamma_0 = \partial\Omega \setminus \overline{\widetilde{\Gamma}}$.

Our main result gives global uniqueness by measuring the Cauchy data on $\widetilde{\Gamma}$. Let $q_j \in C^{2+\alpha}(\overline{\Omega})$, $j = 1, 2$ for some $\alpha > 0$ and let q_j be complex-valued. Consider the following sets of Cauchy data on $\widetilde{\Gamma}$:

$$\mathcal{C}_{q_j} = \left\{ \left(u|_{\widetilde{\Gamma}}, \frac{\partial u}{\partial \nu}\Big|_{\widetilde{\Gamma}} \right) \mid (\Delta + q_j)u = 0 \,\text{in}\, \Omega,\ u|_{\Gamma_0} = 0,\ u \in H^1(\Omega) \right\}, \quad j = 1, 2. \tag{1.5}$$

Theorem 1.1. *Assume $\mathcal{C}_{q_1} = \mathcal{C}_{q_2}$. Then $q_1 \equiv q_2$.*

Remark. *As far as a regularity of the potentials q_j is concerned we have to assume $C^{2+\alpha}$ regularity only in a neighborhood of the boundary $\partial\Omega$.*

Using Theorem 1.1 one concludes immediately as a corollary the following global identifiability result for the conductivity equation (1.1). This result uses that knowledge of the Dirichlet-to-Neumann map on an open subset of the boundary determines γ and its first derivatives on $\tilde{\Gamma}$ (see [22], [29].)

Corollary 1.1. *With some $\alpha > 0$, let $\gamma_j \in C^{4+\alpha}(\overline{\Omega})$, $j = 1, 2$, be non-vanishing functions. Assume that*

$$\Lambda_{\gamma_1}(f) = \Lambda_{\gamma_2}(f) \text{ on } \widetilde{\Gamma} \text{ for all } f \in H^{\frac{1}{2}}(\Gamma),\ supp\, f \subset \widetilde{\Gamma}.$$

Then $\gamma_1 = \gamma_2$.

It is easy to see that Theorem 1.1 implies the analogous result to [19] in the two dimensional case.

Notice that Theorem 1.1 does not assume that Ω is simply connected. An interesting inverse problem is whether one can determine the potential or conductivity in a region of the plane with holes by measuring the Cauchy data only on the accessible boundary. This is also called the obstacle problem.

Let Ω, D be domains in $\mathbb{R}^2$ with smooth boundaries such that $\overline{D} \subset \Omega$. Let $V \subset \partial\Omega$ be an open set. Let $q_j \in C^{2+\alpha}(\overline{\Omega \setminus D})$, for some $\alpha > 0$ and $j = 1, 2$. Let us consider the following set of partial Cauchy data

$$\tilde{C}_{q_j} = \{(u|_V, \frac{\partial u}{\partial \nu}|_V)|(\Delta + q_j)u = 0 \text{ in } \Omega \setminus \overline{D},\ u|_{\partial D \cup \partial\Omega \setminus V} = 0, u \in H^1(\Omega \setminus \overline{D})\}.$$

Corollary 1.2. *Assume $\tilde{C}_{q_1} = \tilde{C}_{q_2}$. Then $q_1 = q_2$.*

A similar result holds for the conductivity equation.

Corollary 1.3. *Let $\gamma_j \in C^{4+\alpha}(\overline{\Omega \setminus D})$ $j = 1, 2$ be non vanishing functions. Assume*

$$\Lambda_{\gamma_1}(f) = \Lambda_{\gamma_2}(f) \text{ on } V \quad \forall f \in H^{\frac{1}{2}}(\partial(\overline{\Omega} \setminus D)),\ supp\, f \subset V$$

Then $\gamma_1 = \gamma_2$.

Another application of Theorem 1.1 is to the anisotropic conductivity problem. In this case the conductivity depends on direction and is represented by a positive definite symmetric matrix

$$\sigma = \{\sigma^{ij}\} \quad \text{in } \Omega.$$

The conductivity equation with voltage potential g on $\partial\Omega$ is given by

$$\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}(\sigma^{ij}\frac{\partial u}{\partial x_j}) = 0 \quad \text{in } \Omega,$$

$$u|_{\partial\Omega} = g.$$

The Dirichlet-to-Neumann map is defined by

$$\Lambda_\sigma(g) = \sum_{i,j=1}^{2} \sigma^{ij}\nu_i \frac{\partial u}{\partial x_j}|_{\partial\Omega}.$$

It has been known for a long time that Λ_σ does not determine σ uniquely in the anisotropic case [23]. Let $F : \overline{\Omega} \to \overline{\Omega}$ be a diffeomorphism such that $F(x) = x$ for x from $\partial\Omega$. Then

$$\Lambda_{F_*\sigma} = \Lambda_\sigma,$$

where

$$F_*\sigma = \left(\frac{(DF) \circ \sigma \circ (DF)^T}{|detDF|}\right) \circ F^{-1}. \tag{1.6}$$

Here DF denotes the differential of F, $(DF)^T$ its transpose and the composition inside parenthesis (1.6) is matrix composition. The question of whether one can determine the conductivity up to the obstruction (1.6) has been solved in two dimensions for C^2 conductivities in [24], Lipschitz conductivities in [26] and merely L^∞ conductivities in [3]. The method of proof in all these papers is the reduction to the isotropic case performed using isothermal coordinates [27]. Using the same method and Corollary 1.1, we obtain the following result.

Theorem 1.2. *Let $\sigma_k = \{\sigma_k^{ij}\} \in C^{5+\alpha}(\overline{\Omega})$ for $k = 1, 2$ and some positive α. Suppose that σ_k are positive definite symmetric matrices on $\overline{\Omega}$. Let $\widetilde{\Gamma} \subset \partial\Omega$ be some open set. Assume*

$$\Lambda_{\sigma_1}(g)|_{\Gamma} = \Lambda_{\sigma_2}(g)|_{\Gamma} \quad \forall g \in H^{\frac{1}{2}}(\partial\Omega),\ supp\, g \subset \widetilde{\Gamma}.$$

Then there exists a diffeomorphism

$$F : \overline{\Omega} \to \overline{\Omega}, \quad F|_{\partial\Omega} = Identity, \quad F \in C^{4+\alpha}(\bar{\Omega}), \alpha > 0$$

such that

$$F_*\sigma_1 = \sigma_2.$$

We mention that in [3] K. Astala, M. Lassas, and L. Päiväirinta have shown a partial data result in the anisotropic problem in two dimensions for bounded measurable conductivities, similar to Theorem 1.2, assuming that one knows both the Dirichlet to Neumann and Neumann to Dirichlet map on $\widetilde{\Gamma}$. On the other hand, to the authors' knowledge, there are no uniqueness results similar to Theorem 1.1 with Dirichlet data supported and Neumann data measured on the same arbitrary open subset of the boundary, even for smooth potentials or conductivities. In dimension $n \ge 3$ Isakov [17] proved global uniqueness assuming that Γ_0 is a subset of a plane or a sphere. In dimensions $n \ge 3$, [8] proves global uniqueness in determining a bounded potential for the Schrödinger equation with Dirichlet data supported on the whole boundary and Neumann data measured in roughly half the boundary. The proof relies on a Carleman estimate with a linear weight function. This implies a similar result for the conductivity equation with C^2 conductivities. In [20] the regularity assumption on the conductivity was relaxed to $C^{3/2+\alpha}$ with some $\alpha > 0$. The corresponding stability estimates are proved in [15]. In [19], the result in [8] was generalized to show that by measuring all possible pairs of Dirichlet data on a possible very small subsets of the boundary Γ_+ and Neumann data on a slightly larger open domain than $\partial\Omega \setminus \Gamma_+$, one can uniquely determine the potential. The method of the proof uses Carleman estimates with non-linear weights. The case of the magnetic Schrödinger equation was considered in [11] and an improvement on the regularity of the coefficients is done in [21]. Stability estimates for the magnetic Schrödinger equation with partial data were proven in [30].

The two dimensional case has special features since one can construct a much larger set of complex geometrical optics solutions than in higher dimensions. On the other hand, the problem is formally determined in two dimensions and therefore more difficult. The proof of our main result is based on the construction of appropriate complex geometrical optics solutions by Carleman estimates with degenerate weight functions.

This paper is composed of four sections. In Section 2, we establish our key Carleman estimates, and in Section 3, we construct complex geometrical optics solutions. In Section 4, we complete the proof of Theorem 1.1. In the Appendix we consider the solvability of the Cauchy-Riemann equations with Cauchy data on a subset of the boundary. We also establish a Carleman estimate for Laplace's equation with degenerate harmonic weights that we use in the earlier sections.

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2. **Carleman estimates with degenerate weights**

Throughout the paper we use the following notations.

Notations.
$i=\sqrt{-1}$, $x_1,x_2,\xi_1,\xi_2\in\mathbb{R}^1$, $z=x_1+ix_2$, $\zeta=\xi_1+i\xi_2$, $\overline{z}$ denotes the complex conjugate of $z\in\mathbb{C}$. We identify $x=(x_1,x_2)\in\mathbb{R}^2$ with $z=x_1+ix_2\in\mathbb{C}$. $\partial_z=\frac{1}{2}(\partial_{x_1}-i\partial_{x_2})$, $\partial_{\overline{z}}=\frac{1}{2}(\partial_{x_1}+i\partial_{x_2})$, $D=(\frac{1}{i}\frac{\partial}{\partial x_1},\frac{1}{i}\frac{\partial}{\partial x_2})$, $\beta=(\beta_1,\beta_2)$, $|\beta|=\beta_1+\beta_2$, $D^\beta=(\frac{1}{i^{\beta_1}}\frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}},\frac{1}{i^{\beta_2}}\frac{\partial^{\beta_2}}{\partial x_2^{\beta_2}})$. The tangential derivative on the boundary is given by $\partial_{\vec\tau}=\nu_2\frac{\partial}{\partial x_1}-\nu_1\frac{\partial}{\partial x_2}$, with $\nu=(\nu_1,\nu_2)$ the unit outer normal to $\partial\Omega$, $B(\widehat{x},\delta)=\{x\in\mathbb{R}^2||x-\widehat{x}|<\delta\}$, $f:\mathbb{R}^2\to\mathbb{R}^1$, f'' is the Hessian matrix with entries $\frac{\partial^2 f}{\partial x_i\partial x_j}$. $\mathcal{L}(X,Y)$ denotes the Banach space of all bounded linear operators from a Banach space X to another Banach space Y.

Let $\Phi(z)=\varphi(x_1,x_2)+i\psi(x_1,x_2)\in C^2(\overline{\Omega})$ be a holomorphic function in Ω with real-valued φ and ψ:

$$\partial_{\overline{z}}\Phi(z)=0\quad\text{in }\Omega. \tag{2.1}$$

Denote by $\mathcal{H}$ the set of critical points of the function Φ

$$\mathcal{H}=\{z\in\overline{\Omega}|\partial_z\Phi(z)=0\}.$$

Assume that Φ has no critical points on $\overline{\widetilde{\Gamma}}$, and that all the critical points are nondegenerate:

$$\mathcal{H}\cap\overline{\partial\Omega\setminus\Gamma_0}=\{\emptyset\},\quad \partial_z^2\Phi(z)\ne 0,\quad \forall z\in\mathcal{H}. \tag{2.2}$$

Then we know that Φ has only a finite number of critical points and we can set:

$$\mathcal{H}=\{\widetilde{x}_1,...,\widetilde{x}_\ell\}. \tag{2.3}$$

Consider the following problem

$$\Delta u+q_0u=f\quad\text{in }\Omega,\quad u|_{\Gamma_0}=g, \tag{2.4}$$

where ν is the unit outward normal vector to $\partial\Omega$.

Assume that Φ satisfies

$$\Gamma_0\subset\{x\in\partial\Omega|(\nu,\nabla\varphi)=0\}. \tag{2.5}$$

We have

Proposition 2.1. *Let $q_0\in L^\infty(\Omega)$. Assume (2.1), (2.2), (2.5). There exists $\tau_0>0$ such that for all $|\tau|>\tau_0$ there exists a solution to problem (2.4) such that*

$$\|ue^{-\tau\varphi}\|_{L^2(\Omega)}\le C(\|fe^{-\tau\varphi}\|_{L^2(\Omega)}/\sqrt{|\tau|}+\|ge^{-\tau\varphi}\|_{L^2(\Gamma_0)}). \tag{2.6}$$

The proof of this proposition given in the appendix.

Let us introduce the operators:

$$\partial_{\overline{z}}^{-1}g = \frac{1}{2\pi i}\int_\Omega \frac{g(\zeta,\overline{\zeta})}{\zeta - z}d\zeta\wedge d\overline{\zeta} = -\frac{1}{\pi}\int_\Omega \frac{g(\zeta,\overline{\zeta})}{\zeta - z}d\xi_2 d\xi_1,$$

$$\partial_z^{-1}g = -\frac{1}{2\pi i}\overline{\int_\Omega \frac{\overline{g}(\zeta,\overline{\zeta})}{\zeta - z}d\zeta\wedge d\overline{\zeta}} = -\frac{1}{\pi}\int_\Omega \frac{g(\zeta,\overline{\zeta})}{\overline{\zeta} - \overline{z}}d\xi_2 d\xi_1 = \overline{\partial_{\overline{z}}^{-1}\overline{g}}.$$

See e.g., pp.28-31 in [32] where $\partial_{\overline{z}}^{-1}$ and ∂_z^{-1} are denoted by T and $\overline{T}$ respectively. Then we have (e.g., p.47 and p.56 in [32]):

Proposition 2.2. A) *Let $m \ge 0$ be an integer number and $\alpha \in (0,1)$. Then $\partial_{\overline{z}}^{-1}, \partial_z^{-1} \in \mathcal{L}(C^{m+\alpha}(\overline{\Omega}), C^{m+\alpha+1}(\overline{\Omega}))$.*
B) *Let $1 \le p \le 2$ and $1 < \gamma < \frac{2p}{2-p}$. Then $\partial_{\overline{z}}^{-1}, \partial_z^{-1} \in \mathcal{L}(L^p(\Omega), L^\gamma(\Omega))$.*

We define two other operators:

$$R_{\Phi,\tau}g = e^{\tau(\overline{\Phi(z)}-\Phi(z))}\partial_{\overline{z}}^{-1}(ge^{\tau(\Phi(z)-\overline{\Phi(z)})}), \quad \widetilde{R}_{\Phi,\tau}g = e^{\tau(\overline{\Phi(z)}-\Phi(z))}\partial_z^{-1}(ge^{\tau(\Phi(z)-\overline{\Phi(z)})}). \tag{2.7}$$

We have

Proposition 2.3. *Let $g \in C^\alpha(\overline{\Omega})$ for some positive α. The function $R_{\Phi,\tau}g$ is a solution to*

$$\partial_{\overline{z}}R_{\Phi,\tau}g - \tau(\overline{\partial_z\Phi(z)})R_{\Phi,\tau}g = g \quad in\ \Omega. \tag{2.8}$$

The function $\widetilde{R}_{\Phi,\tau}g$ solves

$$\partial_z\widetilde{R}_{\Phi,\tau}g + \tau(\partial_z\Phi(z))\widetilde{R}_{\Phi,\tau}g = g \quad in\ \Omega. \tag{2.9}$$

Proof. The proof is by direct computations:

$$\begin{aligned}\partial_z\widetilde{R}_{\Phi,\tau}g + \tau\frac{\partial\Phi(z)}{\partial z}\widetilde{R}_{\Phi,\tau}g &= \partial_z(e^{\tau(\overline{\Phi(z)}-\Phi(z))}\partial_z^{-1}(ge^{\tau(\Phi(z)-\overline{\Phi(z)})}))\\ &+\tau\frac{\partial\Phi(z)}{\partial z}(e^{\tau(\overline{\Phi(z)}-\Phi(z))}\partial_z^{-1}(ge^{\tau(\Phi(z)-\overline{\Phi(z)})})) =\\ -\tau\frac{\partial\Phi(z)}{\partial z}(e^{\tau(\overline{\Phi(z)}-\Phi(z))}\partial_z^{-1}(ge^{\tau(\Phi(z)-\overline{\Phi(z)})})) &+ (e^{\tau(\overline{\Phi(z)}-\Phi(z))}(ge^{\tau(\Phi(z)-\overline{\Phi(z)})}))\\ &+\tau\frac{\partial\Phi(z)}{\partial z}(e^{\tau(\overline{\Phi(z)}-\Phi(z))}\partial_z^{-1}(ge^{\tau(\Phi(z)-\overline{\Phi(z)})})) = g.\end{aligned}$$

□

Using the stationary phase argument we show

Proposition 2.4. *Let $g \in L^1(\Omega)$ and function Φ satisfy (2.1),(2.2). Then*

$$lim_{|\tau|\to+\infty}\int_\Omega ge^{\tau(\Phi(z)-\overline{\Phi(z)})}dx = 0.$$

Proof. Let $\{g_k\}_{k=1}^\infty \in C_0^\infty(\Omega)$ be a sequence of functions such that $g_k \to g$ in $L^1(\Omega)$. Let $\epsilon > 0$ be an arbitrary number. Suppose that $\widehat{j}$ is large enough such that $\|g - g_{\widehat{j}}\|_{L^1(\Omega)} \le \frac{\epsilon}{2}$. Then

$$|\int_\Omega g e^{\tau(\Phi(z)-\overline{\Phi(z)})}dx| \le |\int_\Omega (g - g_{\widehat{j}})e^{\tau(\Phi(z)-\overline{\Phi(z)})}dx| + |\int_\Omega g_{\widehat{j}} e^{\tau(\Phi(z)-\overline{\Phi(z)})}dx|.$$

The first term on the right hand side of this inequality is less then $\epsilon/2$ and the second goes to zero as $|\tau|$ approaches to infinity by the stationary phase argument. □

Denote

$$\mathcal{O}_\epsilon = \{x \in \Omega | dist(x, \partial\Omega) \le \epsilon\}.$$

We have

Proposition 2.5. *Let $\alpha > 0$, $g \in C^{1+\alpha}(\Omega)$ and $g|_{\mathcal{O}_\epsilon} = 0$. Then*

$$|R_{\Phi,\tau}g(x)| + |\widetilde{R}_{\Phi,\tau}g(x)| \le C\|g\|_{C^{1+\alpha}(\overline{\Omega})}/|\tau| \quad \forall x \in \mathcal{O}_{\epsilon/2}. \tag{2.10}$$

If $g \in C^{2+\alpha}(\overline{\Omega})$, $g|_{\mathcal{O}_\epsilon} = 0$ and $g|_{\mathcal{H}} = 0$, then

$$\|R_{\Phi,\tau}g\|_{C^0(\overline{\mathcal{O}_{\frac{\epsilon}{2}}})} + \|\widetilde{R}_{\Phi,\tau}g\|_{C^0(\overline{\mathcal{O}_{\frac{\epsilon}{2}}})} = o(\frac{1}{\tau}) \tag{2.11}$$

for all $x \in \mathcal{O}_{\epsilon/2}$.

Proof. Denote $\tilde{g}(x, \xi_1, \xi_2) = -\frac{1}{\pi}\frac{g(\xi_1,\xi_2)}{\zeta - z}$. Let $x = (x_1, x_2)$ be an arbitrary point in $\overline{\mathcal{O}_{\frac{\epsilon}{2}}}$. We set $z = x_1 + ix_2$. We prove (2.10) and (2.11) for the function $R_{\Phi,\tau}g$. Proof of the estimates for the function $\widetilde{R}_{\Phi,\tau}g$ is exactly the same. Let us prove (2.10) first. Let $\delta > 0$ be sufficiently small and $e_k \in C_0^\infty(B(\widetilde{x}_k, \delta))$ such that $e_k|_{B(\widetilde{x}_k,\delta/2)} = 1$. We decompose

$$I(\tau) = \int_\Omega \tilde{g}e^{\tau(\Phi-\overline{\Phi})}d\xi_1 d\xi_2 = \sum_{k=1}^{\ell}\int_{B(\widetilde{x}_k,\delta)} e_k\tilde{g}e^{\tau(\Phi-\overline{\Phi})}d\xi_1 d\xi_2 + \int_\Omega (1 - \sum_{k=1}^{\ell} e_k)\tilde{g}e^{\tau(\Phi-\overline{\Phi})}d\xi_1 d\xi_2. \tag{2.12}$$

By the stationary phase argument we can estimate the second integral on the right hand side of in (2.12) as

$$\|\int_\Omega (1 - \sum_{k=1}^{\ell} e_k)\tilde{g}e^{\tau(\Phi-\overline{\Phi})}d\xi_1 d\xi_2\|_{C^0(\overline{\mathcal{O}_{\frac{\epsilon}{2}}})} \le \frac{C\|g\|_{C^{1+\alpha}(\overline{\Omega})}}{|\tau|}. \tag{2.13}$$

In order to estimate the first term on the right hand side of (2.12) we use that

$$\sum_{k=1}^{\ell}\int_{B(\widetilde{x}_k,\delta)} e_k\tilde{g}e^{\tau(\Phi-\overline{\Phi})}d\xi_1 d\xi_2 = \tag{2.14}$$

$$\sum_{k=1}^{\ell}\left\{\int_{B(\widetilde{x}_k,\delta)} e_k\left(\tilde{g} + \frac{1}{\pi}\frac{g(\tilde{x}_k)}{\zeta - z}\right)e^{\tau(\Phi-\overline{\Phi})}d\xi_1 d\xi_2 - \int_{B(\widetilde{x}_k,\delta)} e_k\frac{1}{\pi}\frac{g(\tilde{x}_k)}{\zeta - z}e^{\tau(\Phi-\overline{\Phi})}d\xi_1 d\xi_2\right\}.$$

Applying the stationary phase argument to the second term in (2.14) again we get

$$\|\int_{B(\widetilde{x}_k,\delta)} e_k\frac{1}{\pi}\frac{g(\tilde{x}_k)}{\zeta - z}e^{\tau(\Phi-\overline{\Phi})}d\xi_1 d\xi_2\|_{C^1(\overline{\mathcal{O}_{\frac{\epsilon}{2}}})} \le \frac{C\|g\|_{C^0(\overline{\Omega})}}{|\tau|}. \tag{2.15}$$

In order to estimate the first term on the right hand side of (2.14) we observe

$$\sum_{k=1}^{\ell}\int_{B(\tilde x_k,\delta)} e_k\left(\tilde g+\frac{1}{\pi}\frac{g(\tilde x_k)}{\zeta-z}\right)e^{\tau(\Phi-\overline{\Phi})}d\xi_1 d\xi_2=$$

$$\sum_{k=1}^{\ell}\lim_{\delta'\to+0}\int_{B(\tilde x_k,\delta)\setminus B(\tilde x_k,\delta')} e_k\left(\tilde g+\frac{1}{\pi}\frac{g(\tilde x_k)}{\zeta-z}\right)e^{\tau(\Phi-\overline{\Phi})}d\xi_1 d\xi_2=$$

$$\sum_{k=1}^{\ell}\lim_{\delta'\to+0}\int_{B(\tilde x_k,\delta)\setminus B(\tilde x_k,\delta')} e_k\left(\tilde g+\frac{1}{\pi}\frac{g(\tilde x_k)}{\zeta-z}\right)\frac{1}{\tau\partial_z\Phi}\partial_z e^{\tau(\Phi-\overline{\Phi})}d\xi_1 d\xi_2=$$

$$-\sum_{k=1}^{\ell}\lim_{\delta'\to+0}\int_{B(\tilde x_k,\delta)\setminus B(\tilde x_k,\delta')} \partial_z\left(e_k(\tilde g+\frac{1}{\pi}\frac{g(\tilde x_k)}{\zeta-z})\frac{1}{\tau\partial_z\Phi}\right)e^{\tau(\Phi-\overline{\Phi})}d\xi_1 d\xi_2$$

$$-\sum_{k=1}^{\ell}\lim_{\delta'\to+0}\int_{S(\tilde x_k,\delta')}\frac{1}{2\delta'}(\xi_1-i\xi_2)e_k\left(\tilde g+\frac{1}{\pi}\frac{g(\tilde x_k)}{\zeta-z}\right)\frac{1}{\tau\partial_z\Phi}e^{\tau(\Phi-\overline{\Phi})}d\xi_1 d\xi_2. \tag{2.16}$$

Note that for each fixed x from $\mathcal{O}_{\frac{\epsilon}{2}}$ function $e_k(\xi_1,\xi_2)(\tilde g+\frac{1}{\pi}\frac{g(\tilde x_k)}{\zeta-z})\in C^{1+\alpha}(\overline{\Omega})$ and $(\tilde g+\frac{1}{\pi}\frac{g(\tilde x_k)}{\zeta-z})(x,\tilde x_k)=0$. Thus

$$\lim_{\delta'\to+0}\int_{S(\tilde x_k,\delta')}\frac{1}{2\delta'}(\xi_1-i\xi_2)e_k(\tilde g+\frac{1}{\pi}\frac{g(\tilde x_k)}{\zeta-z})\frac{1}{\partial_z\Phi}e^{\tau(\Phi-\overline{\Phi})}d\xi_1 d\xi_2=0.$$

By (2.2) there exists a constant C such that

$$|\partial_z\left(\frac{e_k}{\partial_z\Phi}(\tilde g+\frac{1}{\pi}\frac{g(\tilde x_k)}{\zeta-z})\right)|\le C\sum_{k=1}^{\ell}\frac{\|g\|_{C^{1+\alpha}(\overline{\Omega})}}{|x-\tilde x_k|^{2-\alpha}}.$$

Using these inequalities we pass to the limit in (2.16) and we obtain

$$\sum_{k=1}^{\ell}\int_{\Omega} e_k\left(\tilde g+\frac{1}{\pi}\frac{g(\tilde x_k)}{\zeta-z}\right)e^{\tau(\Phi-\overline{\Phi})}d\xi_1 d\xi_2=\frac{1}{\tau}\sum_{k=1}^{\ell}\int_{B(\tilde x_k,\delta)}\partial_z\left(e_k\left(\tilde g+\frac{1}{\pi}\frac{g(\tilde x_k)}{\zeta-z}\right)\frac{1}{\partial_z\Phi}\right)e^{\tau(\Phi-\overline{\Phi})}d\xi_1 d\xi_2.$$

This inequality and (2.13),(2.15) imply (2.10).

Now we prove (2.11). Thanks to the improved regularity of the function g similarly to (2.16) we have

$$\|\int_{\Omega}(1-\sum_{k=1}^{\ell}e_k)\tilde g e^{\tau(\Phi-\overline{\Phi})}d\xi_1 d\xi_2\|_{C^0(\overline{\mathcal{O}_{\frac{\epsilon}{2}}})}\le\frac{C}{|\tau|^2}. \tag{2.17}$$

By (2.17) and the assumption that $g|_{\mathcal{H}}=0$ we get

$$I(\tau)=\sum_{k=1}^{\ell}\int_{B(\tilde x_k,\delta)}\partial_z(\frac{e_k\tilde g}{\tau\partial_z\Phi})e^{\tau(\Phi-\overline{\Phi})}d\xi_1 d\xi_2+o(\frac{1}{\tau}). \tag{2.18}$$

Consider the radial cut off function $\chi\in C_0^\infty(B(0,1))$ such that

$$\chi\ge 0,\quad \chi|_{B(0,\frac{1}{2})}=1.$$

Then by (2.18)

$$I(\tau)=\sum_{k=1}^{\ell}\int_{B(\widetilde{x}_k,\delta)}\partial_z(\frac{e_k\tilde g}{\tau\partial_z\Phi})\chi(|x-\tilde x_k|\ln|\tau|)e^{\tau(\Phi-\overline{\Phi})}d\xi_1d\xi_2+$$

$$\sum_{k=1}^{\ell}\int_{B(\widetilde{x}_k,\delta)}\partial_z(\frac{e_k\tilde g}{\tau\partial_z\Phi})(1-\chi(|x-\tilde x_k|\ln|\tau|))e^{\tau(\Phi-\overline{\Phi})}d\xi_1d\xi_2+o(\frac{1}{\tau})=$$

$$-\sum_{k=1}^{\ell}\int_{B(\widetilde{x}_k,\delta)}\partial_z\left(\frac{1}{\tau\partial_z\Phi}\partial_z(\frac{e_k\tilde g}{\tau\partial_z\Phi})(1-\chi(|x-\tilde x_k|\ln|\tau|))\right)e^{\tau(\Phi-\overline{\Phi})}d\xi_1d\xi_2+$$

$$\sum_{k=1}^{\ell}\int_{B(\widetilde{x}_k,\delta)}\partial_z(\frac{e_k\tilde g}{\tau\partial_z\Phi})\chi(|x-\tilde x_k|\ln|\tau|)e^{\tau(\Phi-\overline{\Phi})}d\xi_1d\xi_2+o(\frac{1}{\tau}).\tag{2.19}$$

Using the inequalities

$$\sum_{k=1}^{\ell}|\int_{B(\widetilde{x}_k,\delta)}\partial_z\left(\frac{1}{\tau\partial_z\Phi}\partial_z(\frac{e_k\tilde g}{\tau\partial_z\Phi})(1-\chi(|x-\tilde x_k|\ln|\tau|))\right)e^{\tau(\Phi-\overline{\Phi})}d\xi_1d\xi_2|\le\frac{C}{\tau^{\frac32}}$$

and

$$\sum_{k=1}^{\ell}|\int_{B(\widetilde{x}_k,\delta)}\partial_z(\frac{e_k\tilde g}{\tau\partial_z\Phi})\chi(|x-\tilde x_k|\ln|\tau|)e^{\tau(\Phi-\overline{\Phi})}d\xi_1d\xi_2|$$

$$\le\frac{C}{\tau}\sum_{k=1}^{\ell}|\int_{B(\widetilde{x}_k,\delta)}\frac{1}{|x-\tilde x|^{2-\alpha}}\chi(|x-\tilde x_k|\ln|\tau|)e^{\tau(\Phi-\overline{\Phi})}d\xi_1d\xi_2|=o(\frac{1}{\tau})$$

we get (2.11).

□

Denote

$$r(z)=\Pi_{k=1}^{\ell}(z-\widetilde z_k)\ \text{ where } \mathcal{H}=\{\widetilde x_1,\dots,\widetilde x_\ell\},\ \widetilde z_k=\widetilde x_{1,k}+i\widetilde x_{2,k}.$$

We have

Proposition 2.6. *Let α be some positive number $g\in C^{1+\alpha}(\overline{\Omega})$ and $g|_{\mathcal{O}_\epsilon}=0$. Then for each $\delta\in(0,1)$, there exists a constant $C(\delta)>0$ such that*

$$\|\widetilde R_{\Phi,\tau}(\overline{r(z)}g)\|_{L^2(\Omega)}\le C(\delta)\|g\|_{C^{1+\alpha}(\overline{\Omega})}/|\tau|^{1-\delta},\quad \|R_{\Phi,\tau}(r(z)g)\|_{L^2(\Omega)}\le C(\delta)\|g\|_{C^{1+\alpha}(\overline{\Omega})}/|\tau|^{1-\delta}.\tag{2.20}$$

Proof. Denote $v=\widetilde R_{\Phi,\tau}(\overline{r(z)}g)$. By Proposition 2.5

$$\|v\|_{L^2(\mathcal{O}_{\epsilon/2})}\le C\|g\|_{C^{1+\alpha}(\overline{\Omega})}/|\tau|.\tag{2.21}$$

Then by Proposition 2.3 we have

$$\frac{\partial v}{\partial z}+\tau\frac{\partial\Phi}{\partial z}v=\overline{r(z)}g\quad\text{in }\Omega.$$

There exists a function p such that

$$-\frac{\partial p}{\partial\overline{z}}+\tau\frac{\overline{\partial\Phi(z)}}{\partial z}p=v\quad\text{in }\Omega$$

and there exists a constant $C > 0$ independent of τ such that

$$\|p\|_{L^2(\Omega)} \le C\|v\|_{L^2(\Omega)}. \tag{2.22}$$

Let χ be a nonnegative function such that $\chi \equiv 0$ on $\mathcal{O}_{\frac{\epsilon}{16}}$ and $\chi \equiv 1$ on $\Omega \setminus \mathcal{O}_{\frac{\epsilon}{8}}$. Setting $\widetilde{p} = \chi p$ and using $g|_{\mathcal{O}_\epsilon} \equiv 0$, we have that

$$\int_\Omega r(z)\overline{g}pdx = \int_{\Omega\setminus\mathcal{O}_\epsilon} r(z)\overline{g}pdx = \int_\Omega r(z)\overline{g}\widetilde{p}dx$$

and

$$-\frac{\partial\widetilde{p}}{\partial\overline{z}} + \tau\overline{\frac{\partial\Phi(z)}{\partial z}}\widetilde{p} = \chi v - p\frac{\partial\chi}{\partial\overline{z}} \quad \text{in } \Omega. \tag{2.23}$$

Then

$$\|\chi^{\frac{1}{2}}v\|^2_{L^2(\Omega)} = \int_\Omega r(z)\overline{g}pdx + \int_\Omega p\frac{\partial\chi}{\partial\overline{z}}\overline{v}dx. \tag{2.24}$$

Applying to equation (2.23) the operator $\frac{\partial}{\partial z}$ we have

$$-\frac{\partial}{\partial z}\frac{\partial\widetilde{p}}{\partial\overline{z}} = \frac{\partial}{\partial z}(-\tau\overline{\frac{\partial\Phi(z)}{\partial z}}\widetilde{p} + \chi v - p\frac{\partial\chi}{\partial\overline{z}}) \quad \text{in } \Omega \quad \widetilde{p}|_{\partial\Omega} = 0.$$

The classical a-priori estimate for the Laplace operator yields

$$\|\widetilde{p}\|_{H^1(\Omega)} \le C\|\tau\overline{\frac{\partial\Phi(z)}{\partial z}}\widetilde{p} - \chi v + p\frac{\partial\chi}{\partial\overline{z}}\|_{L^2(\Omega)}.$$

Then by (2.22)

$$\|\widetilde{p}\|_{H^1(\Omega)} \le C(|\tau|\|p\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)}) \le C|\tau|\|v\|_{L^2(\Omega)}. \tag{2.25}$$

Taking the scalar product of (2.23) and $\frac{r(z)}{\partial_z\Phi(z)}\overline{g}$ we get

$$\int_\Omega \frac{r(z)}{\partial_z\Phi(z)}\overline{g}\left(-\frac{\partial\widetilde{p}}{\partial\overline{z}} + \tau\overline{\frac{\partial\Phi(z)}{\partial z}}\widetilde{p}\right)dx = \int_\Omega \frac{r(z)}{\partial_z\Phi(z)}\overline{g}\left(\chi v - p\frac{\partial\chi}{\partial\overline{z}}\right)dx.$$

Then

$$\tau\int_\Omega \overline{g}r(z)\widetilde{p}dx = \int_\Omega \frac{r(z)}{\partial_z\Phi(z)}\overline{g}\left(\chi v - p\frac{\partial\chi}{\partial\overline{z}}\right)dx - \int_\Omega \frac{\partial}{\partial\overline{z}}\left(\frac{r(z)}{\partial_z\Phi(z)}\overline{g}\right)\widetilde{p}dx.$$

By (2.25) and the Sobolev embedding theorem, for each $\tilde{\epsilon} \in (0, \frac{1}{2})$, we have

$$\left|\int_\Omega \frac{\partial}{\partial\overline{z}}\left(\frac{r(z)}{\partial_z\Phi(z)}\overline{g}\right)\widetilde{p}dx\right| \le \left|\int_\Omega \frac{r(z)\overline{\partial_z^2\Phi(z)}}{(\partial_z\Phi(z))^2}\overline{g}\widetilde{p}dx\right| + \left|\int_\Omega \frac{r(z)}{\partial_z\Phi(z)}\frac{\partial\overline{g}}{\partial\overline{z}}\widetilde{p}dx\right| \tag{2.26}$$

$$\le C\|g\|_{C^{1+\alpha}(\overline{\Omega})}\left\|\frac{1}{\partial_z\Phi(z)}\right\|_{L^{2-\tilde{\epsilon}}(\Omega)}\|\widetilde{p}\|_{L^{\frac{2-\tilde{\epsilon}}{1-\tilde{\epsilon}}}(\Omega)} \le C\|g\|_{C^{1+\alpha}(\overline{\Omega})}\|\widetilde{p}\|_{H^{\delta_3(\tilde{\epsilon})}(\Omega)} \le C\|g\|_{C^{1+\alpha}(\overline{\Omega})}|\tau|^{\delta_3(\tilde{\epsilon})}\|v\|_{L^2(\Omega)}.$$

Here we choose $\delta_3(\tilde{\epsilon}) > 0$ such that $\delta_3(\tilde{\epsilon}) \to +0$ as $\tilde{\epsilon} \to +0$ and $H^{\delta_3(\tilde{\epsilon})}(\Omega) \subset L^{\frac{2-\tilde{\epsilon}}{1-\tilde{\epsilon}}}(\Omega)$. Therefore

$$\left|\int_\Omega \overline{g}r(z)\widetilde{p}dx\right| \le C\|g\|_{C^{1+\alpha}(\overline{\Omega})}|\tau|^{-1+\delta_3(\tilde{\epsilon})}\|v\|_{L^2(\Omega)} \quad \text{as } \delta_3(\tilde{\epsilon}) \to +0. \tag{2.27}$$

By (2.21)

$$\left|\int_\Omega p\frac{\partial\chi}{\partial\overline{z}}\overline{v}dx\right| \le C\|p\|_{L^2(\Omega)}\|v\|_{L^2(\mathcal{O}_{\frac{\epsilon}{8}})} \le C\|g\|_{C^{1+\alpha}(\overline{\Omega})}\|p\|_{L^2(\Omega)}/|\tau|. \tag{2.28}$$

By (2.22), (2.27) and (2.28) we obtain from (2.24)

$$\|v\|^2_{L^2(\Omega)} \le C\|g\|_{C^{1+\alpha}(\overline{\Omega})}(|\tau|^{-1+\delta_3(\tilde\epsilon)}\|v\|_{L^2(\Omega)} + \|p\|_{L^2(\Omega)}/|\tau|) \le C|\tau|^{-1+\delta_3(\tilde\epsilon)}\|g\|_{C^{1+\alpha}(\overline{\Omega})}\|v\|_{L^2(\Omega)}.$$

The proof of the proposition is complete. □

We have

Proposition 2.7. *Let $\alpha>0$, $g\in C^{2+\alpha}(\Omega)$, $g|_{\mathcal{O}_\epsilon}=0$ and $g|_{\mathcal{H}}=0$. Then*

$$\left\|R_{\Phi,\tau}g+\frac{g}{\tau\overline{\partial_z\Phi}}\right\|_{L^2(\Omega)}+\left\|\widetilde{R}_{\Phi,\tau}g-\frac{g}{\tau\partial_z\Phi}\right\|_{L^2(\Omega)} = o\left(\frac{1}{\tau}\right) \quad \textit{as } |\tau|\to+\infty. \tag{2.29}$$

Proof. By (2.2) and Proposition 2.5

$$\|\widetilde{R}_{\Phi,\tau}g\|_{C^0(\overline{\mathcal{O}_{\frac{\epsilon}{2}}})}+\|R_{\Phi,\tau}g\|_{C^0(\overline{\mathcal{O}_{\frac{\epsilon}{2}}})} = o\left(\frac{1}{\tau}\right). \tag{2.30}$$

Therefore instead of (2.29) it suffices to prove

$$\left\|\chi_1R_{\Phi,\tau}g+\frac{g}{\tau\overline{\partial_z\Phi}}\right\|_{L^2(\Omega)}+\left\|\chi_1\widetilde{R}_{\Phi,\tau}g-\frac{g}{\tau\partial_z\Phi}\right\|_{L^2(\Omega)} = o\left(\frac{1}{\tau}\right) \quad \text{as } |\tau|\to+\infty, \tag{2.31}$$

where $\chi_1\in C_0^\infty(\Omega)$ and $\chi_1|_{\Omega\setminus\mathcal{O}_{\epsilon/2}}=1$. Denote $w=\chi_1\widetilde{R}_{\Phi,\tau}g-\frac{g}{\tau\partial_z\Phi}$. Here we note that $\frac{g}{\partial_z\Phi}\in L^\infty(\Omega)$. This follows from (2.2), $g\in C^{1+\alpha}(\overline{\Omega})$ and $g|_{\mathcal{H}}=0$. Then (2.9) and $g|_{\mathcal{O}_\epsilon}=0$ yield

$$\partial_z w+\tau(\partial_z\Phi)w = -\partial_z\left(\frac{g}{\tau\partial_z\Phi}\right)+(\partial_z\chi_1)\widetilde{R}_{\Phi,\tau}g \quad \text{in } \Omega, \quad w|_{\partial\Omega}=0. \tag{2.32}$$

Note that by (2.2) and the fact that $g|_{\mathcal{H}}=0$, we obtain

$$\left|\partial_z\left(\frac{g}{\partial_z\Phi}\right)\right| = \left|\frac{\partial_z g}{\partial_z\Phi}-\frac{g}{\partial_z\Phi}\frac{\partial_z^2\Phi}{\partial_z\Phi}\right| \le \frac{C}{\Pi_{k=1}^{\ell}|x-\widetilde{x}_k|}. \tag{2.33}$$

Consider the radial cut off function $\chi\in C_0^\infty(B(0,1))$ such that

$$\chi\ge 0, \quad \chi|_{B(0,\frac{1}{2})}=1.$$

By (2.33) and Proposition 2.2 B),

$$\widetilde{R}_{\Phi,\tau}\left(\sum_{k=1}^{\ell}\chi(|x-\widetilde{x}_k|\ln|\tau|)\partial_z\left(\frac{g}{\partial_z\Phi}\right)\right)\to 0 \quad \text{in} \quad L^2(\Omega) \text{ as } |\tau|\to+\infty. \tag{2.34}$$

In fact, fixing large $|\tau|$, small $\delta > 0$ and $p > 1$ such that $p-1$ is sufficiently small, we apply Proposition 2.2 B) and (2.33) to conclude

$$\begin{aligned}
&\left\|\widetilde{R}_{\Phi,\tau}\left(\sum_{k=1}^{\ell}\chi(|x-\widetilde{x}_k|\ln|\tau|)\partial_z\left(\frac{g}{\partial_z\Phi}\right)\right)\right\|_{L^2(\Omega)}\\
\le\ & C\sum_{k=1}^{\ell}\left(\int_{B(\widetilde{x}_k,\delta)}|\chi(|x-\widetilde{x}_k|\ln|\tau|)|^p\left|\partial_z\left(\frac{g}{\partial_z\Phi}\right)\right|^p dx\right)^{\frac{1}{p}}\\
\le\ & C'\|g\|_{C^{1+\alpha}(\overline{\Omega})}\sum_{k=1}^{\ell}\left(\int_{B(\widetilde{x}_k,\delta)}|\chi(|x-\widetilde{x}_k|\ln|\tau|)|^p\frac{1}{|x-\widetilde{x}_k|^p}dx\right)^{\frac{1}{p}}\\
\le\ & C''\|g\|_{C^{1+\alpha}(\overline{\Omega})}(\int_0^{\delta}|\chi(\rho\ln|\tau|)|^p\rho^{1-p}d\rho)^{\frac{1}{p}}.
\end{aligned}$$

Thus we obtain (2.34) by the Riemann-Lebesgue's lemma.

By Proposition 2.6, we obtain

$$\widetilde{R}_{\Phi,\tau}\left(\left(1-\sum_{k=1}^{\ell}\chi(|x-\widetilde{x}_k|\ln|\tau|)\right)\partial_z\left(\frac{g}{\partial_z\Phi}\right)\right)\to 0 \quad \text{in} \quad L^2(\Omega) \text{ as } |\tau|\to+\infty. \tag{2.35}$$

In fact the function $\left(\left(1-\sum_{k=1}^{\ell}\chi(|x-\widetilde{x}_k|\ln|\tau|)\right)\partial_z\left(\frac{g}{\partial_z\Phi}\right)\right)\frac{1}{r(z)}\in C^{1+\alpha}(\bar{\Omega})$ for any nonzero τ. Short calculations give the estimate

$$\|\left(\left(1-\sum_{k=1}^{\ell}\chi(|x-\widetilde{x}_k|\ln|\tau|)\right)\partial_z\left(\frac{g}{\partial_z\Phi}\right)\right)\frac{1}{r(z)}\|_{C^{1+\alpha}(\overline{\Omega})}\le C|\tau|^{\frac{1}{2}}.$$

So by Proposition 2.6

$$\|\widetilde{R}_{\Phi,\tau}\left((\left(1-\sum_{k=1}^{\ell}\chi(|x-\widetilde{x}_k|\ln|\tau|)\right)\partial_z\left(\frac{g}{\partial_z\Phi}\right)\right\|_{L^2(\Omega)}\le\frac{C}{|\tau|^{\frac{3}{2}}}.$$

Therefore (2.34) and (2.35) yield

$$\left\|\widetilde{R}_{\Phi,\tau}\left(\partial_z\left(\frac{g}{\partial_z\Phi}\right)\right)\right\|_{L^2(\Omega)}=o(1)\quad\text{as } |\tau|\to+\infty. \tag{2.36}$$

Denote $\widetilde{w}=w+\frac{1}{\tau}\chi_1\widetilde{R}_{\Phi,\tau}(\partial_z(\frac{g}{\partial_z\Phi}))$.

By (2.36), it suffices to prove

$$\|\widetilde{w}\|_{L^2(\Omega)}=o\left(\frac{1}{\tau}\right)\quad\text{as } |\tau|\to+\infty. \tag{2.37}$$

In terms of (2.32) and (2.9), observe that

$$\partial_z\widetilde{w}+\tau(\partial_z\Phi)\widetilde{w}=f\quad\text{in }\Omega,\quad \widetilde{w}|_{\partial\Omega}=0, \tag{2.38}$$

where $f=\frac{1}{\tau}(\partial_z\chi_1)\widetilde{R}_{\Phi,\tau}(\partial_z(\frac{g}{\partial_z\Phi}))+(\partial_z\chi_1)\widetilde{R}_{\Phi,\tau}g$. By (2.36) and (2.30) we have

$$\|f\|_{L^2(\Omega)}=o\left(\frac{1}{\tau}\right)\quad\text{as } |\tau|\to+\infty. \tag{2.39}$$

Applying Proposition 5.2 to equation (2.38) we get

$$\|\partial_{x_1}(e^{i\tau\psi}\widetilde{w})\|^2_{L^2(\Omega)} + \tau\int_{\partial\Omega}(\nabla\varphi,\nu)|\widetilde{w}|^2 d\sigma$$

$$+\mathrm{Re}\int_{\partial\Omega} i\left(\left(\nu_2\frac{\partial}{\partial x_1} - \nu_1\frac{\partial}{\partial x_2}\right)\widetilde{w}\right)\overline{\widetilde{w}}d\sigma + \|\partial_{x_2}(e^{i\tau\psi}\widetilde{w})\|^2_{L^2(\Omega)} = \|f\|^2_{L^2(\Omega).}$$

Thanks to the zero Dirichlet boundary conditions for the function $\tilde{w}$ we obtain

$$\|\partial_{x_1}(e^{i\tau\psi}\widetilde{w})\|^2_{L^2(\Omega)} + \|\partial_{x_2}(e^{i\tau\psi}\widetilde{w})\|^2_{L^2(\Omega)} = \|f\|^2_{L^2(\Omega).}$$

Poincaré's inequality implies

$$\|\widetilde{w}\|_{H^1(\Omega)} \le C\|f\|_{L^2(\Omega).}$$

From this and using (2.39), we obtain (2.37). As for the first term in (2.29), we can argue similarly. The proof of the proposition is completed. □

3. **Complex geometrical optics solutions**

In this section, we construct complex geometrical optics solutions for the Schrödinger equation $\Delta + q_1$ with q_1 satisfying the conditions of Theorem 1.1. Consider

$$L_1 u = \Delta u + q_1 u = 0 \quad \text{in } \Omega. \tag{3.1}$$

We will construct solutions to (3.1) of the form

$$u_1(x) = e^{\tau\Phi(z)}(a(z) + a_0(z)/\tau) + e^{\tau\overline{\Phi(z)}}\overline{(a(z) + a_1(z)/\tau)} + e^{\tau\varphi}u_{11} + e^{\tau\varphi}u_{12}, \quad u_1|_{\Gamma_0} = 0. \tag{3.2}$$

The function Φ satisfies (2.1), (2.2) and

$$\mathrm{Im}\,\Phi|_{\Gamma_0} = 0. \tag{3.3}$$

The amplitude function $a(z)$ is not identically zero on $\overline{\Omega}$ and has the following properties:

$$a \in C^2(\overline{\Omega}), \quad \partial_{\overline{z}} a \equiv 0,\ \mathrm{Re}\, a|_{\Gamma_0} = 0,\ a(z)|_{\mathcal{H}\cap\partial\Omega} = \partial_z a(z)|_{\mathcal{H}\cap\partial\Omega} = 0. \tag{3.4}$$

The function u_{11} is given by

$$\begin{aligned} u_{11} = & -\frac{1}{4}e^{i\tau\psi}\widetilde{R}_{\Phi,\tau}(e_1(\partial_{\overline{z}}^{-1}(aq_1) - M_1(z))) - \frac{1}{4}e^{-i\tau\psi}R_{\Phi,-\tau}(e_1(\partial_z^{-1}(\overline{a(z)}q_1) - M_3(\overline{z}))) \\ & - \frac{e^{i\tau\psi}}{\tau}\frac{e_2(\partial_{\overline{z}}^{-1}(aq_1) - M_1(z))}{4\partial_z\Phi} - \frac{e^{-i\tau\psi}}{\tau}\frac{e_2(\partial_z^{-1}(\overline{a(z)}q_1) - M_3(\overline{z}))}{4\overline{\partial_z\Phi}} \\ = & \ w_1 e^{-\tau\varphi} + w_2 e^{-\tau\varphi}, \end{aligned} \tag{3.5}$$

where the polynomials $M_1(z)$ and $M_3(\bar{z})$ satisfy

$$\partial_z^j(\partial_{\overline{z}}^{-1}(aq_1) - M_1(z)) = 0, \quad x \in \mathcal{H},\ j = 0,1,2, \tag{3.6}$$

$$\partial_{\overline{z}}^j(\partial_z^{-1}(\overline{a}q_1)(z) - M_3(\overline{z})) = 0, \quad x \in \mathcal{H},\ j = 0,1,2. \tag{3.7}$$

Note that by (3.4)

$$\partial_{\overline{z}}^k\partial_z^j(\partial_{\overline{z}}^{-1}(aq_1) - M_1(z)) = 0, \quad x \in \mathcal{H}\cap\partial\Omega,\ j,k \in \{0,1,2\}, \text{ and } j+k \le 2, \tag{3.8}$$

$$\partial_{\overline{z}}^j\partial_z^k(\partial_z^{-1}(\overline{a}q_1)(z)-M_3(\overline{z}))=0,\quad x\in\mathcal{H}\cap\partial\Omega,\ j,k\in\{0,1,2\},\text{ and } j+k\le 2. \tag{3.9}$$

The functions $e_1, e_2 \in C^\infty(\Omega)$ are constructed so that

$$e_1+e_2\equiv 1 \text{ on } \overline{\Omega}, e_2 \text{ vanishes in some neighborhood of } \mathcal{H}\setminus\partial\Omega \text{ and } e_1 \text{ vanishes in a neighborhood of } \partial\Omega \tag{3.10}$$

and we set

$$w_1=-\frac{1}{4}e^{\tau\Phi}\widetilde{R}_{\Phi,\tau}(e_1(\partial_{\overline{z}}^{-1}(aq_1)-M_1(z)))-\frac{1}{4}e^{\tau\overline{\Phi}}R_{\Phi,-\tau}(e_1(\partial_z^{-1}(\overline{a(z)}q_1)-M_3(\overline{z})))$$

and

$$w_2=-\frac{e^{\tau\Phi}}{\tau}\frac{e_2(\partial_{\overline{z}}^{-1}(aq_1)-M_1(z))}{4\partial_z\Phi}-\frac{e^{\tau\overline{\Phi}}}{\tau}\frac{e_2(\partial_z^{-1}(\overline{a(z)}q_1)-M_3(\overline{z}))}{4\overline{\partial_z\Phi}}.$$

Finally a_0, a_1 are holomorphic functions such that

$$(a_0(z)+\overline{a_1(z)})|_{\Gamma_0}=\frac{(\partial_{\overline{z}}^{-1}(aq_1)-M_1(z))}{4\partial_z\Phi}+\frac{(\partial_z^{-1}(\overline{a(z)}q_1)-M_3(\overline{z}))}{4\overline{\partial_z\Phi}}.$$

Then, noting that $\partial_{\overline{z}}\overline{\Phi}=\overline{\partial_z\Phi}$, (2.8) and (2.9), we have

$$\begin{aligned}
&\Delta w_1=4\partial_z\partial_{\overline{z}}w_1\\
=\ &-\partial_{\overline{z}}(e^{\tau\Phi}\partial_z\widetilde{R}_{\Phi,\tau}(e_1(\partial_{\overline{z}}^{-1}(aq_1)-M_1(z)))+(\tau\partial_z\Phi)e^{\tau\Phi}\widetilde{R}_{\Phi,\tau}(e_1(\partial_{\overline{z}}^{-1}(aq_1)-M_1(z)))\\
-\ &\partial_z(e^{\tau\overline{\Phi}}\partial_{\overline{z}}R_{\Phi,-\tau}(e_1(\partial_z^{-1}(\overline{a}q_1)-M_3(\overline{z})))+(\tau\overline{\partial_z\Phi})e^{\tau\overline{\Phi}}R_{\Phi,-\tau}(e_1(\partial_z^{-1}(\overline{a}q_1)-M_3(\overline{z})))\\
=\ &-\partial_{\overline{z}}(e^{\tau\Phi}e_1(\partial_{\overline{z}}^{-1}(aq_1)-M_1(z)))-\partial_z(e^{\tau\overline{\Phi}}e_1(\partial_z^{-1}(\overline{a}q_1)-M_3(\overline{z}))).
\end{aligned}$$

Moreover

$$\begin{aligned}
&\Delta w_2=4\partial_z\partial_{\overline{z}}w_2\\
=\ &-\partial_{\overline{z}}(e^{\tau\Phi}(e_2(\partial_{\overline{z}}^{-1}(aq_1)-M_1(z)))-\partial_z(e^{\tau\overline{\Phi}}e_2(\partial_z^{-1}(\overline{a}q_1)-M_3(\overline{z})))\\
-\ &e^{\tau\Phi}\Delta\left(\frac{e_2(\partial_{\overline{z}}^{-1}(aq_1)-M_1(z))}{4\tau\partial_z\Phi}\right)-e^{\tau\overline{\Phi}}\Delta\left(\frac{e_2(\partial_z^{-1}(\overline{a(z)}q_1)-M_3(\overline{z}))}{4\tau\overline{\partial_z\Phi}}\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
\Delta(u_{11}e^{\tau\varphi})=\Delta(w_1+w_2)=-aq_1e^{\tau\Phi}-\overline{a}q_1e^{\tau\overline{\Phi}}\\
-e^{\tau\Phi}\Delta\left(\frac{e_2(\partial_{\overline{z}}^{-1}(aq_1)-M_1(z))}{4\tau\partial_z\Phi}\right)-e^{\tau\overline{\Phi}}\Delta\left(\frac{e_2(\partial_z^{-1}(\overline{a(z)}q_1)-M_3(\overline{z}))}{4\tau\overline{\partial_z\Phi}}\right).
\end{aligned}\tag{3.11}$$

By (3.4) and (3.3) observe that

$$(e^{\tau\Phi(z)}a(z)+e^{\tau\overline{\Phi(z)}}\overline{a(z)})|_{\Gamma_0}=0. \tag{3.12}$$

Let u_{12} be solution to the inhomogeneous problem

$$\Delta(u_{12}e^{\tau\varphi})+q_1u_{12}e^{\tau\varphi}=-q_1u_{11}e^{\tau\varphi}+h_1e^{\tau\varphi}\quad\text{in }\Omega, \tag{3.13}$$

$$u_{12}=\frac{1}{4}R_{\Phi,\tau}(e_1(\partial_{\overline{z}}^{-1}(aq_1)-M_1(z)))+\frac{1}{4}R_{\Phi,-\tau}(e_1(\partial_z^{-1}(\overline{a(z)}q_1)-M_3(\overline{z})))\quad\text{on }\Gamma_0, \tag{3.14}$$

has a solution where

$$h_1 = e^{\tau i\psi}\Delta\left(\frac{e_2(\partial_{\overline{z}}^{-1}(a(z)q_1) - M_1(z))}{4\tau\partial_z\Phi}\right) + e^{-\tau i\psi}\Delta\left(\frac{e_2(\partial_z^{-1}(\overline{a(z)}q_1) - M_3(\overline{z}))}{4\tau\overline{\partial_z\Phi}}\right)$$
$$-a_0q_1e^{i\tau\psi}/\tau - \overline{a_1}q_1e^{-i\tau\psi}/\tau. \tag{3.15}$$

By (3.4) and (3.11) - (3.15), we conclude that (3.1) is satisfied.

By Proposition 2.1 there exists a positive τ_0 such that for all $|\tau| > \tau_0$ there exists a solution to (3.13), (3.14) satisfying

$$\|u_{12}\|_{L^2(\Omega)} = o(\frac{1}{\tau}) \quad \text{as } \tau \to +\infty. \tag{3.16}$$

This can be done because

$$\|q_1u_{11} + h_1\|_{L^2(\Omega)} \le C(\delta)/|\tau|^{1-\delta} \quad \forall\delta \in (0,1);\ \|u_{11}\|_{L^2(\partial\Omega)} = o(\frac{1}{\tau})$$

and $(\nabla\varphi, \nu) = 0$ on Γ_0. The latter fact can be seen as follows: On $\partial\Omega$, the Cauchy-Riemann equations imply

$$(\nabla\varphi, \nu) = \nu_1\partial_{x_1}\varphi + \nu_2\partial_{x_2}\varphi = \nu_1\partial_{x_2}\psi - \nu_2\partial_{x_1}\psi = -\frac{\partial\psi}{\partial\vec{\tau}},$$

which is the tangential derivative of $\psi = \operatorname{Im}\Phi$ on $\partial\Omega$. By (3.3) the tangential derivative of ψ vanishes on Γ_0.

Consider now the Schrödinger equation

$$L_2v = \Delta v + q_2v = 0 \quad \text{in } \Omega. \tag{3.17}$$

We will construct solutions to (3.17) of the form

$$v(x) = e^{-\tau\Phi(z)}(a(z) + b_0(z)/\tau) + e^{-\tau\overline{\Phi(z)}}\overline{(a(z) + b_1(z)/\tau)} + e^{-\tau\varphi}v_{11} + e^{-\tau\varphi}v_{12}, \quad v|_{\Gamma_0} = 0. \tag{3.18}$$

The construction of v repeats the corresponding steps of the construction of u_1. The only difference is that instead of q_1 and τ, we use q_2 and $-\tau$ respectively. We provide the details for the sake of completeness. The function v_{11} is given by

$$v_{11} = -\frac{1}{4}e^{-i\tau\psi}\widetilde{R}_{\Phi,-\tau}(e_1(\partial_{\overline{z}}^{-1}(q_2a(z)) - M_2(z))) - \frac{1}{4}e^{i\tau\psi}R_{\Phi,\tau}(e_1(\partial_z^{-1}(q_2\overline{a(z)}) - M_4(\overline{z})))$$
$$+\frac{e^{-i\tau\psi}}{\tau}\frac{e_2(\partial_{\overline{z}}^{-1}(aq_2) - M_2(z))}{4\partial_z\Phi} + \frac{e^{i\tau\psi}}{\tau}\frac{e_2(\partial_z^{-1}(\overline{a(z)}q_2) - M_4(\overline{z}))}{4\overline{\partial_z\Phi}}, \tag{3.19}$$

where

$$\partial_z^j(\partial_{\overline{z}}^{-1}(aq_2) - M_2(z)) = 0, \quad x \in \mathcal{H},\ j = 0,1,2, \tag{3.20}$$

$$\partial_z^j\partial_{\overline{z}}^k(\partial_{\overline{z}}^{-1}(aq_2) - M_2(z)) = 0, \quad x \in \mathcal{H}\cap\partial\Omega,\ j,k \in \{0,1,2\},\ k+j \le 2, \tag{3.21}$$

$$\partial_{\overline{z}}^j(\partial_z^{-1}(\overline{a}q_2)(z) - M_4(\overline{z})) = 0, \quad x \in \mathcal{H},\ j = 0,1,2, \tag{3.22}$$

$$\partial_z^j\partial_{\overline{z}}^k(\partial_z^{-1}(\overline{a}q_2)(z) - M_4(\overline{z})) = 0, \quad x \in \mathcal{H}\cap\partial\Omega,\ j,k \in \{0,1,2\},\ k+j \le 2. \tag{3.23}$$

Finally b_0, b_1 are holomorphic functions such that

$$(b_0+\overline{b}_1)|_{\Gamma_0} = -\frac{(\partial_{\overline{z}}^{-1}(aq_2)-M_2(z))}{4\partial_z\Phi} - \frac{(\partial_z^{-1}(\overline{a(z)}q_2)-M_4(\overline{z}))}{4\overline{\partial_z\Phi}}.$$

Denote

$$h_2 = e^{-\tau i\psi}\Delta\left(\frac{e_2(\partial_{\overline{z}}^{-1}(a(z)q_2)-M_2(z))}{4\tau\partial_z\Phi}\right) + e^{\tau i\psi}\Delta\left(\frac{e_2(\partial_z^{-1}(\overline{a(z)}q_2)-M_4(\overline{z}))}{4\tau\overline{\partial_z\Phi}}\right) - \frac{b_0(z)}{\tau}q_2e^{-i\tau\psi(z)} - \frac{\overline{b_1(z)}}{\tau}q_2e^{i\tau\psi(z)}.$$

The function v_{12} is a solution to the problem:

$$\Delta(v_{12}e^{-\tau\varphi}) + q_2v_{12}e^{-\tau\varphi} = -q_2v_{11}e^{-\tau\varphi} - h_2e^{-\tau\varphi} \quad \text{in } \Omega, \tag{3.24}$$

$$v_{12}|_{\Gamma_0} = \frac{1}{4}\widetilde{R}_{\Phi,-\tau}(e_1(\partial_{\overline{z}}^{-1}(q_2a(z))-M_2(z))) + \frac{1}{4}R_{\Phi,\tau}(e_1(\partial_z^{-1}(q_2\overline{a(z)})-M_4(\overline{z}))) \tag{3.25}$$

such that

$$\|v_{12}\|_{L^2(\Omega)} = o(\frac{1}{\tau}) \quad \text{as } \tau\to+\infty. \tag{3.26}$$

4. **Proof of the theorem.**

We first apply stationary phase with a general phase function Φ and then we construct an appropriate weight function.

Proposition 4.1. *Suppose that Φ satisfies (2.1),(2.2) and (3.3). Let $\{\widetilde{x}_1,\dots,\widetilde{x}_\ell\}$ be the set of critical points of the function $Im\Phi$. Then for any potentials $q_1,q_2\in C^{2+\alpha}(\overline{\Omega})$, $\alpha>0$ with the same Cauchy data on $\widetilde{\Gamma}$. For any holomorphic function a satisfying (3.4) and $M_1(z), M_2(z), M_3(\overline{z}), M_4(\overline{z})$ as in Section 3, we have*

$$\begin{aligned} &2\sum_{k=1}^{\ell}\frac{\pi(q|a|^2)(\widetilde{x}_k)Re\, e^{2i\tau Im\Phi(\widetilde{x}_k)}}{|(det\, Im\Phi'')(\widetilde{x}_k)|^{\frac12}} + \int_\Omega q(a(a_0+b_0)+\overline{a}(\overline{a}_1+\overline{b}_1))dx \\ &+ \frac14\int_\Omega\left(qa\frac{\partial_{\overline{z}}^{-1}(aq_2)-M_2(z)}{\partial_z\Phi} + q\overline{a}\frac{\partial_z^{-1}(q_2\overline{a})-M_4(\overline{z})}{\overline{\partial_z\Phi}}\right)dx \\ &- \frac14\int_\Omega\left(qa\frac{(\partial_{\overline{z}}^{-1}(aq_1)-M_1(z))}{\partial_z\Phi} + q\overline{a}\frac{(\partial_z^{-1}(\overline{a}q_1)-M_3(\overline{z}))}{\overline{\partial_z\Phi}}\right)dx = 0, \quad \tau>0, \end{aligned} \tag{4.1}$$

where

$$q = q_1 - q_2.$$

Proof. Let u_1 be a solution to (3.1) and satisfy (3.2), and u_2 be a solution to the following equation

$$\Delta u_2 + q_2u_2 = 0 \quad \text{in } \Omega, \quad u_2|_{\partial\Omega} = u_1|_{\partial\Omega}.$$

Since the Dirichlet-to-Neumann maps are equal, we have

$$\nabla u_2 = \nabla u_1 \quad \text{on } \widetilde{\Gamma}.$$

Denoting $u = u_1 - u_2$, we obtain

$$\Delta u + q_2 u = -q u_1 \quad \text{in } \Omega, \quad u|_{\partial\Omega} = \frac{\partial u}{\partial \nu}|_{\widetilde{\Gamma}} = 0. \tag{4.2}$$

Let v satisfy (3.17) and (3.18). We multiply (4.2) by v, integrate over Ω and we use $v|_{\Gamma_0} = 0$ and $\frac{\partial u}{\partial \nu} = 0$ on $\widetilde{\Gamma}$ to obtain $\int_\Omega q u_1 v dx = 0$. By (3.2), (3.16), (3.18) and (3.26), we have

$$\begin{aligned} 0 = \int_\Omega q u_1 v dx = \int_\Omega & q(a^2 + \overline{a}^2 + |a|^2 e^{\tau(\Phi - \overline{\Phi})} + |a|^2 e^{\tau(\overline{\Phi} - \Phi)} \\ & + \frac{1}{\tau}(a(a_0 + b_0) + \overline{a}(\overline{a}_1 + \overline{b}_1)) + u_{11} e^{\tau\varphi}(a e^{-\tau\Phi} + \overline{a} e^{-\tau\overline{\Phi}}) \\ & + (a e^{\tau\Phi} + \overline{a} e^{\tau\overline{\Phi}}) v_{11} e^{-\tau\varphi}) dx + o\left(\frac{1}{\tau}\right), \quad \tau > 0. \end{aligned} \tag{4.3}$$

The first and second terms in the asymptotic expansion of (4.3) are independent of τ, so that

$$\int_\Omega q(a^2 + \overline{a}^2) dx = 0. \tag{4.4}$$

Using stationary phase argument (see p.215 in [13]. cf. [16]) and functions e_1, e_2 defined in (3.10) we obtain

$$\begin{aligned} \int_\Omega q(|a|^2 e^{\tau(\Phi - \overline{\Phi})} + |a|^2 e^{\tau(\overline{\Phi} - \Phi)}) dx &= \int_\Omega e_1 q(|a|^2 e^{\tau(\Phi - \overline{\Phi})} + |a|^2 e^{\tau(\overline{\Phi} - \Phi)}) dx \\ + \quad \int_\Omega e_2 q(|a|^2 e^{\tau(\Phi - \overline{\Phi})} + |a|^2 e^{\tau(\overline{\Phi} - \Phi)}) dx. \end{aligned}$$

By the Cauchy-Riemann equations, we see that $\operatorname{sgn}(\operatorname{Im} \Phi''(\widetilde{x}_k)) = 0$, where $\operatorname{sgn} A$ denotes the signature of the matrix A, that is, the number of positive eigenvalues of A minus the number of negative eigenvalues (e.g., [13], p.210). Moreover we note that

$$\det \operatorname{Im} \Phi''(z) = -(\partial_{x_1}\partial_{x_2}\varphi)^2 - (\partial_{x_1}^2 \varphi)^2 \neq 0.$$

To see this, suppose that $\det \operatorname{Im} \Phi''(z) = 0$. Then $\partial_{x_1}\partial_{x_2}\varphi(\operatorname{Re} z, \operatorname{Im} z) = \partial_{x_1}^2 \varphi(\operatorname{Re} z, \operatorname{Im} z) = 0$ and the Cauchy-Riemann equations imply that all second order partial derivatives of functions φ, ψ at the point z are zero. This fact contradicts the assumption that critical points of the function Φ are nondegenerate.

Using stationary phase (see p.215 in [13]. cf. [16]), we obtain

$$\int_\Omega e_1 q(|a|^2 e^{\tau(\Phi - \overline{\Phi})} + |a|^2 e^{\tau(\overline{\Phi} - \Phi)}) dx = 2 \sum_{k=1}^{\ell} \frac{\pi q |a|^2(\widetilde{x}_k) \operatorname{Re} e^{2\tau i \operatorname{Im} \Phi(\widetilde{x}_k)}}{\tau |(\det \operatorname{Im} \Phi'')(\widetilde{x}_k)|^{\frac{1}{2}}} + o\left(\frac{1}{\tau}\right). \tag{4.5}$$

Let $\tilde{x}_1, \dots \tilde{x}_{k'}$ be the set of critical points of the function Φ on Γ_0. Integrating by parts we have

$$\begin{aligned}
&\int_\Omega e_2 q(|a|^2 e^{\tau(\Phi-\overline{\Phi})} + |a|^2 e^{\tau(\overline{\Phi}-\Phi)})dx \\
= &\int_\Omega e_2 q|a|^2 \left(\frac{(\nabla\psi, \nabla e^{\tau(\Phi-\overline{\Phi})})}{2i\tau|\nabla\psi|^2} - \frac{(\nabla\psi, \nabla e^{\tau(\overline{\Phi}-\Phi)})}{2i\tau|\nabla\psi|^2} \right) dx \\
= &\lim_{\delta\to+0} \int_{\Omega\setminus\cup_{k=1}^{k'} B(\tilde{x}_k,\delta)} e_2 q|a|^2 \left(\frac{(\nabla\psi, \nabla e^{\tau(\Phi-\overline{\Phi})})}{2i\tau|\nabla\psi|^2} - \frac{(\nabla\psi, \nabla e^{\tau(\overline{\Phi}-\Phi)})}{2i\tau|\nabla\psi|^2} \right) dx \\
= &\lim_{\delta\to+0} \left\{ - \int_{\Omega\setminus\cup_{k=1}^{k'} B(\tilde{x}_k,\delta)} \mathrm{div} \left(\frac{e_2 q|a|^2 \nabla\psi}{2i\tau|\nabla\psi|^2} \right) (e^{\tau(\Phi-\overline{\Phi})} - e^{\tau(\overline{\Phi}-\Phi)})dx \right. \\
&\left. + \int_{\Omega\cap\cup_{k=1}^{k'} S(\tilde{x}_k,\delta)} e_2 q|a|^2 \left(\frac{(\nabla\psi, \nu)}{2i\tau|\nabla\psi|^2} - \frac{(\nabla\psi, \nu)}{2i\tau|\nabla\psi|^2} \right) e^{\tau(\overline{\Phi}-\Phi)} \right\} \\
= &- \int_\Omega \mathrm{div} \left(\frac{e_2 q|a|^2 \nabla\psi}{2i\tau|\nabla\psi|^2} \right) (e^{\tau(\Phi-\overline{\Phi})} - e^{\tau(\overline{\Phi}-\Phi)})dx \\
+ &\int_{\partial\Omega} \frac{q|a|^2}{2i\tau|\nabla\psi|^2} \frac{\partial\psi}{\partial\nu} (e^{\tau(\Phi-\overline{\Phi})} - e^{\tau(\overline{\Phi}-\Phi)})d\sigma \\
= &- \int_{\mathrm{supp}\, e_2} \mathrm{div} \left(\frac{e_2 q|a|^2 \nabla\psi}{2i\tau|\nabla\psi|^2} \right) (e^{\tau(\Phi-\overline{\Phi})} - e^{\tau(\overline{\Phi}-\Phi)})dx.
\end{aligned}$$

In the last equality, we used that $e^{\tau(\Phi-\overline{\Phi})} - e^{\tau(\overline{\Phi}-\Phi)} = 0$ on Γ_0 which follows since $\mathrm{Im}\,\Phi = 0$ on Γ_0, and $q = 0$ on $\widetilde{\Gamma}$ and (3.4) in order to show that $\mathrm{div}\left(\frac{e_2 q|a|^2\nabla\psi}{2i\tau|\nabla\psi|^2}\right)$ and $\frac{q|a|^2}{2i\tau|\nabla\psi|^2}$ are bounded functions. The latter fact follows from the unique boundary determination of potentials from the Dirichlet-to- Neumann map (see for instance [12], [29]). Applying Proposition 2.4 we obtain

$$\int_\Omega e_2 q(|a|^2 e^{\tau(\Phi-\overline{\Phi})} + |a|^2 e^{\tau(\overline{\Phi}-\Phi)})dx = o(\frac{1}{\tau}) \quad \text{as } |\tau| \to +\infty.$$

Therefore

$$\int_\Omega q(|a|^2 e^{\tau(\Phi-\overline{\Phi})} + |a|^2 e^{\tau(\overline{\Phi}-\Phi)})dx = o\left(\frac{1}{\tau}\right). \tag{4.6}$$

We calculate the two remaining terms in (4.3). We have:

$$
\begin{aligned}
(4.7)\qquad & \int_\Omega q u_{11} e^{\tau\varphi}(a e^{-\tau\Phi} + \overline{a} e^{-\tau\overline{\Phi}})dx \\
= & -\frac{1}{4}\int_\Omega q\left\{ e^{\tau\Phi}\widetilde{R}_{\Phi,\tau}(e_1(\partial_{\overline{z}}^{-1}(aq_1) - M_1(z)))\right. \\
+ & \left. e^{\tau\overline{\Phi}} R_{\Phi,-\tau}(e_1(\partial_z^{-1}(\overline{a}q_1) - M_3(\overline{z})))\right\}(a e^{-\tau\Phi} + \overline{a} e^{-\tau\overline{\Phi}})dx \\
- & \int_\Omega \left(\frac{e^{\tau\Phi}}{\tau}\frac{e_2(\partial_{\overline{z}}^{-1}(aq_1) - M_1(z))}{4\partial_z\Phi} + \frac{e^{\tau\overline{\Phi}}}{\tau}\frac{e_2(\partial_z^{-1}(\overline{a(z)}q_1) - M_3(\overline{z}))}{4\overline{\partial_z\Phi}}\right) q(a e^{-\tau\Phi} + \overline{a} e^{-\tau\overline{\Phi}})dx \\
= & -\frac{1}{4}\int_\Omega (qa\widetilde{R}_{\Phi,\tau}(e_1(\partial_{\overline{z}}^{-1}(aq_1) - M_1(z))) + q\overline{a}R_{\Phi,-\tau}(e_1(\partial_z^{-1}(\overline{a}q_1) - M_3(\overline{z}))))dx \\
- & \frac{1}{4}\int_\Omega (q\overline{a}\widetilde{R}_{\Phi,\tau}(e_1(\partial_{\overline{z}}^{-1}(aq_1) - M_1(z)))e^{\tau(\Phi-\overline{\Phi})} + qaR_{\Phi,-\tau}(e_1(\partial_z^{-1}(\overline{a}q_1) - M_3(\overline{z})))e^{-\tau(\Phi-\overline{\Phi})})dx \\
- & \int_\Omega q\left(\frac{e^{\tau(\Phi-\overline{\Phi})}}{\tau}\frac{\overline{a}e_2(\partial_{\overline{z}}^{-1}(aq_1) - M_1(z))}{4\partial_z\Phi} + \frac{e^{\tau(\overline{\Phi}-\Phi)}}{\tau}\frac{ae_2(\partial_z^{-1}(\overline{a(z)}q_1) - M_3(\overline{z}))}{4\overline{\partial_z\Phi}}\right)dx \\
- & \int_\Omega q\left(\frac{a}{\tau}\frac{e_2(\partial_{\overline{z}}^{-1}(aq_1) - M_1(z))}{4\partial_z\Phi} + \frac{\overline{a}}{\tau}\frac{e_2(\partial_z^{-1}(\overline{a(z)}q_1) - M_3(\overline{z}))}{4\overline{\partial_z\Phi}}\right)dx \\
\equiv & \; I_1 + I_2 + I_3 + I_4.
\end{aligned}
$$

We estimate I_1 and I_2 separately. Using Proposition 2.7, (3.6) and Proposition 2.4 we get

$$
\begin{aligned}
(4.8)\qquad & I_2 = -\frac{1}{4}\int_\Omega (q\overline{a}\widetilde{R}_{\Phi,\tau}(e_1(\partial_{\overline{z}}^{-1}(aq_1) - M_1(z)))e^{\tau(\Phi-\overline{\Phi})} \\
+ & \; qaR_{\Phi,-\tau}(e_1(\partial_z^{-1}(\overline{a}q_1) - M_3(\overline{z})))e^{-\tau(\Phi-\overline{\Phi})})dx \\
= & -\frac{1}{4}\int_\Omega \left(\frac{e_1 q\overline{a}}{\tau\partial_z\Phi}(\partial_{\overline{z}}^{-1}(aq_1) - M_1(z))e^{2i\tau \mathrm{Im}\Phi} + \frac{e_1 qa}{\tau\overline{\partial_z\Phi}}(\partial_z^{-1}(\overline{a}q_1) - M_3(\overline{z}))e^{-2i\tau\mathrm{Im}\Phi}\right)dx \\
+ & \; o\left(\frac{1}{\tau}\right) = o\left(\frac{1}{\tau}\right) \quad \text{as } |\tau| \to +\infty.
\end{aligned}
$$

By Proposition 2.7, we obtain

(4.9)
$$
I_1 = -\frac{1}{4\tau}\int_\Omega e_1\left(qa\frac{(\partial_{\overline{z}}^{-1}(aq_1) - M_1(z))}{\partial_z\Phi} + q\overline{a}\frac{(\partial_z^{-1}(\overline{a}q_1) - M_3(\overline{z}))}{\overline{\partial_z\Phi}}\right)dx + o\left(\frac{1}{\tau}\right) \quad \text{as } |\tau| \to +\infty.
$$

By Proposition 2.4 (4.6), we conclude that

$$
(4.10)\qquad I_3 = o\left(\frac{1}{\tau}\right) \quad \text{as } |\tau| \to +\infty.
$$

Similarly

$$\begin{aligned}
\int_\Omega qv_{11}e^{-\tau\varphi}(ae^{\tau\Phi}+\overline{a}e^{\tau\overline{\Phi}})dx &= -\tfrac14\int_\Omega q\Big\{e^{-\tau\Phi}\widetilde{R}_{\Phi,-\tau}(e_1(\partial_{\overline{z}}^{-1}(aq_2)-M_2(z)))\\
&\quad + e^{-\tau\overline{\Phi}}R_{\Phi,\tau}(e_1(\partial_z^{-1}(\overline{a}q_2)-M_4(\overline{z})))\Big\}(ae^{\tau\Phi}+\overline{a}e^{\tau\overline{\Phi}})dx\\
&\quad + \int_\Omega q\left(\tfrac{e^{-\tau\Phi}}{\tau}\tfrac{e_2(\partial_{\overline{z}}^{-1}(aq_2)-M_2(z)))}{4\partial_z\Phi}+\tfrac{e^{-\tau\overline{\Phi}}}{\tau}\tfrac{e_2(\partial_z^{-1}(\overline{a(z)}q_2)-M_4(\overline{z}))}{4\overline{\partial_z\Phi}}\right)(ae^{\tau\Phi}+\overline{a}e^{\tau\overline{\Phi}})dx\\
&= -\tfrac14\int_\Omega(qa\widetilde{R}_{\Phi,-\tau}(e_1(\partial_{\overline{z}}^{-1}(aq_2)-M_2(z)))+q\overline{a}R_{\Phi,\tau}(e_1(\partial_z^{-1}(\overline{a}q_2)-M_4(\overline{z}))))dx\\
&\quad -\tfrac14\int_\Omega[q\overline{a}e^{\tau(\overline{\Phi}-\Phi)}(\widetilde{R}_{\Phi,-\tau}(e_1(\partial_{\overline{z}}^{-1}(aq_2)-M_2(z)))+qae^{\tau(\overline{\Phi}-\Phi)}R_{\Phi,\tau}(e_1(\partial_z^{-1}(\overline{a}q_2)-M_4(\overline{z})))]dx\\
&\quad + \int_\Omega q\left(\tfrac{e^{-\tau(\Phi-\overline{\Phi})}}{\tau}\tfrac{\overline{a}e_2(\partial_{\overline{z}}^{-1}(aq_2)-M_2(z)))}{4\partial_z\Phi}+\tfrac{e^{\tau(\Phi-\overline{\Phi})}}{\tau}\tfrac{ae_2(\partial_z^{-1}(\overline{a(z)}q_2)-M_4(\overline{z}))}{4\overline{\partial_z\Phi}}\right)dx\\
&\quad + \int_\Omega q\left(\tfrac{a}{\tau}\tfrac{e_2(\partial_{\overline{z}}^{-1}(aq_2)-M_2(z)))}{4\partial_z\Phi}+\tfrac{\overline{a}}{\tau}\tfrac{e_2(\partial_z^{-1}(\overline{a(z)}q_2)-M_4(\overline{z}))}{4\overline{\partial_z\Phi}}\right)dx\\
&= J_1+J_2+J_3+J_4.
\end{aligned}$$

By (3.20) and Proposition 2.7, we have

(4.11)
$$J_1=\frac{1}{4\tau}\int_\Omega e_1\left(qa\frac{\partial_{\overline{z}}^{-1}(aq_2)-M_2(z)}{\partial_z\Phi}+q\overline{a}\frac{\partial_z^{-1}(\overline{a}q_2)-M_4(\overline{z})}{\overline{\partial_z\Phi}}\right)dx+o\left(\frac1\tau\right)\quad\text{as } |\tau|\to+\infty.$$

Proposition 2.4 , (3.20), and Proposition 2.7 yield

(4.12)
$$J_2=-\frac14\int_\Omega[q\overline{a}e^{\tau(\overline{\Phi}-\Phi)}\widetilde{R}_{\Phi,-\tau}(e_1(\partial_{\overline{z}}^{-1}(aq_2)-M_2(z)))+qae^{\tau(\overline{\Phi}-\Phi)}R_{\Phi,\tau}(e_1(\partial_z^{-1}(\overline{a}q_2)-M_4(\overline{z})))]dx=o\left(\frac1\tau\right).$$

By Proposition 2.4 we see that

$$J_3=o\left(\frac1\tau\right)\quad\text{as } |\tau|\to+\infty. \tag{4.13}$$

Therefore, applying (4.6), (4.8), (4.11), (4.12), (4.10) and (4.13) in (4.3), we conclude that

$$\begin{aligned}
&2\sum_{k=1}^{\ell}\frac{\pi(q|a|^2)(\widetilde{x}_k)\mathrm{Re}\,e^{2i\tau\mathrm{Im}\Phi(\widetilde{x}_k)}}{|(\det\mathrm{Im}\Phi'')(\widetilde{x}_k)|^{\frac12}}+\int_\Omega q(a(a_0+b_0)+\overline{a}(\overline{a}_1+\overline{b}_1))dx\\
&+\ \frac14\int_\Omega\left(qa\frac{\partial_{\overline{z}}^{-1}(a(z)q_2)-M_2(z)}{\partial_z\Phi}+q\overline{a}\frac{\partial_z^{-1}(q_2\overline{a(z)})-M_4(\overline{z})}{\overline{\partial_z\Phi}}\right)dx\\
&-\frac14\int_\Omega\left(qa\frac{\partial_{\overline{z}}^{-1}(q_1a)-M_1(z)}{\partial_z\Phi}+q\overline{a}\frac{\partial_z^{-1}(q_1\overline{a})-M_3(\overline{z})}{\overline{\partial_z\Phi}}\right)dx=o(1)
\end{aligned}\tag{4.14}$$

as $\tau\to+\infty$. Passing to the limit in this equality and applying Bohr's theorem (e.g., [4], p.393), we finish the proof of the proposition. □

We need the following proposition in the construction of the phase function Φ.

Let $\widetilde{y}_0,\widetilde{y}_1,\dots,\widetilde{y}_m\in\Omega$ and $\widetilde{y}_{m+1},\dots,\widetilde{y}_{m+\hat m}\in\Gamma_0$.

Denote by $\mathcal{R}=(\mathcal{R}(\widetilde{y}_1),\dots,\mathcal{R}(\widetilde{y}_m),\mathcal{R}_1(\widetilde{y}_{m+1}),\dots,\mathcal{R}_1(\widetilde{y}_{m+\hat m}))$ the following operator:

$$\mathcal{R}(\tilde{y}_k)g=(u(\widetilde{y}_k),\partial_zu(\widetilde{y}_k),\partial_z^2u(\widetilde{y}_k)),\quad \mathcal{R}_1(\hat{y}_k)g=(\mathrm{Re}\,u(\hat{y}_k),\partial_zu(\hat{y}_k)/(\nu_2+i\nu_1)),$$

where

$$\partial_{\bar z}u = 0 \quad \text{in } \Omega, \quad \operatorname{Re} u(\widetilde{y}_0) = 0, \quad \operatorname{Im} u|_{\Gamma_0} = 0, \quad \operatorname{Im} u|_{\widetilde{\Gamma}} = g. \tag{4.15}$$

For any $g \in C_0^\infty(\widetilde{\Gamma})$ problem (4.15) has at most one solution. We have

Proposition 4.2. *The operator $\mathcal{R} : D(\mathcal{R}) \subset C_0^\infty(\widetilde{\Gamma}) \to \mathbb{C}^{3m} \times \mathbb{R}^{2\hat m}$ satisfies $Im\mathcal{R} = \mathbb{C}^{3m} \times \mathbb{R}^{2\hat m}$.*

Proof. We note that $\operatorname{Im}\mathcal{R} = \mathbb{C}^{3m}\times\mathbb{R}^{2\hat m}$ if and only if the closure of $\operatorname{Im}\mathcal{R}$ is equal to $\mathbb{C}^{3m}\times\mathbb{R}^{2\hat m}$. This follows immediately from Corollary 5.1. Let $\vec H$ be an arbitrary element of the space $\mathbb{C}^{3m} \times \mathbb{R}^{2\hat m}$. Consider the problem (5.17) where

$$\hat x_1 = \tilde y_j \quad j \in \{1, \dots, m\}, \quad \hat x_{m+1} = \tilde y_0,$$

$$c_{0,1} = h_1, c_{1,1} = h_2, c_{2,1} = h_3, \dots c_{0,m} = h_{3m-2}, c_{1,m} = h_{3m-1}, c_{2,m} = h_{3m}, c_{0,m+1} = 0.$$

Taking into account that $\partial_z u|_{\Gamma_0} = (\nu_2 + i\nu_1)\partial_{\vec\tau}\operatorname{Re} u$, we take a function b such that

$$b(\tilde y_{m+1}) = h_{m+1}, \partial_{\vec\tau} b(\tilde y_{m+1}) = h_{m+2}, \dots, b(\tilde y_{m+\hat m}) = h_{m+2\hat m-1}, \partial_{\vec\tau} b(\tilde y_{m+\hat m}) = h_{m+2\hat m}.$$

According to Proposition 5.1 (5.17) with such initial data can be solved approximately. If necessary we can add to these solutions a real constants such that $u_\epsilon(\tilde y_0) = 0$. The proof of the proposition is complete.

□

End of proof of Theorem 1.1

Proof. We will construct a complex geometrical optics solution of the form (3.2) where Φ and a satisfy (2.1), (2.2), (3.3) and (3.4).

Let $\tilde\Omega$ be a bounded domain in $\mathbb{R}^2$ such that $\overline{\Omega} \subset \overline{\tilde\Omega}$, $\Gamma_0 \subset \partial\tilde\Omega$, $\partial\tilde\Omega \cap \tilde\Gamma = \emptyset$. Let $\widehat{x}$ be an arbitrary point in Ω. By Proposition 4.2 and Proposition 5.1 there exists a holomorphic function $u \in C^2(\overline{\Omega})$ such that

$$\operatorname{Im} u|_{\Gamma_0} = 0, \ \ \operatorname{Im} u(\widehat{x}) \neq 0, \ \ \partial_z u(\widehat{x}) = 0, \ \text{ and } \partial_z^2 u(\widehat{x}) \neq 0. \tag{4.16}$$

$$\frac{\partial \operatorname{Im} u}{\partial \nu}|_{\overline{\Gamma_0\cap\gamma_j}} < \alpha' < 0, \quad \text{if } Int((\partial\Omega \setminus \Gamma_0) \cap \gamma_j) \neq \emptyset. \tag{4.17}$$

$$\begin{aligned} &\text{In the case } \operatorname{Int}((\partial\Omega \setminus \Gamma_0) \cap \gamma_j) = \emptyset \text{ then } \{x \in \gamma_j | \partial_{\vec\tau}\operatorname{Re} u = 0\} = \{y_{1,j}, y_{2,j}\},\\ &\qquad\qquad \text{and } \partial^2_{\vec\tau}\operatorname{Re} u(y_{1,j}) \neq 0, \quad \partial^2_{\vec\tau}\operatorname{Re} u(y_{2,j}) \neq 0. \end{aligned} \tag{4.18}$$

Here $y_{1,j}, y_{2,j}$ are of maximum and minimum points of the function $\operatorname{Re} u$ on the boundary contour γ_j. In fact, the existence of such u is proved as follows. By Proposition 5.1 and the Cauchy-Riemann equations, there exists a sequence of holomorphic functions u_ε in Ω such that

$$u_\varepsilon \in C^2(\overline{\Omega}), \quad \operatorname{Im} u_\varepsilon|_{\Gamma_0} = 0,$$

$$\frac{\partial \operatorname{Im} u_\varepsilon}{\partial \nu}|_{\overline{\Gamma_0\cap\gamma_j}} < \alpha' < 0, \quad \text{if } \operatorname{Int}((\partial\Omega \setminus \Gamma_0) \cap \gamma_j) \neq \emptyset.$$

$$\text{In the case} \operatorname{Int}((\partial\Omega \setminus \Gamma_0) \cap \gamma_j) = \emptyset \ \text{ then} \operatorname{Re} u_\varepsilon \to \tilde b_j \quad \text{in } C^2(\gamma_j), \tag{4.19}$$

where $\tilde b_j \in C^2(\gamma_j)$ is a function such that

$$\{x\in\gamma_j \vert \partial_{\vec\tau}\tilde b_j=0\}=\{y_{1,j},y_{2,j}\}, \text{and } \partial^2_{\vec\tau}\tilde b_j(y_{1,j})\ne 0,\quad \partial^2_{\vec\tau}\tilde b_j(y_{2,j})\ne 0.$$

$$\operatorname{Im} u_\varepsilon(\widehat x)\to 1,\quad \partial_z u_\varepsilon(\widehat x):=c_\varepsilon\to 0,\quad \partial_z^2 u_\varepsilon(\widehat x)\to 1\quad \text{as } \epsilon\to 0.$$

Let $\mathcal R$ be the operator similar to one introduced in Proposition 4.2:

$$\mathcal R g=(u(\hat x),\partial_z u(\hat x),\partial_z^2 u(\hat x)),$$

where

$$\partial_{\bar z} u=0\quad \text{in } \tilde\Omega,\quad \operatorname{Re} u(x_0)=0,\quad \operatorname{Im} u\vert_{\Gamma_0}=0,\quad \operatorname{Im} u\vert_{\partial\tilde\Omega\setminus\Gamma_0}=g$$

and $x_0\in\Omega$, $x_0\ne\hat x$. Obviously we can consider it as operator from the space $D(\mathcal R)\subset C_0^3(\widetilde\Gamma)\to\mathbb C_0^3$. We have (e.g., p.79 in [1]) that there exists a mapping $M:\mathbb C^3\to C_0^3(\widetilde\Gamma)$ such that $\mathcal R M=I$ and

$$\|My\|_{C_0^3(\widetilde\Gamma)}\le C|y|,\quad y\in\mathbb C^3$$

with some constant $C>0$. We consider the sequence $y_\varepsilon=(0,-c_\varepsilon,0)\in\mathbb C^3$. Let $g_\epsilon=M(y_\varepsilon)\to 0\quad$ in $\quad C_0^3(\widetilde\Gamma)$. Denote by w_ε the function which satisfies

$$\partial_{\bar z} w_\varepsilon=0\quad \text{in } \tilde\Omega,\quad \operatorname{Re} w_\varepsilon(x_0)=0,\quad \operatorname{Im} w_\varepsilon\vert_{\Gamma_0}=0,\quad \operatorname{Im} w_\varepsilon\vert_{\partial\tilde\Omega\setminus\Gamma_0}=g_\varepsilon,$$

$$w_\varepsilon(\widehat x)=0,\quad \partial_z w_\varepsilon(\widehat x)=-c_\varepsilon,\quad \partial_z^2 w_\varepsilon(\widehat x)=0.$$

Hence $\operatorname{Im}(u_\varepsilon+w_\varepsilon)(\widehat x)\to 1$, $\partial_z(u_\varepsilon+w_\varepsilon)(\widehat x)=0$ and $\partial_z^2(u_\varepsilon+w_\varepsilon)(\widehat x)\to 1$ and

$$w_\varepsilon\to 0\quad \text{in } C^2(\overline\Omega).$$

Hence $u_\varepsilon+w_\varepsilon$ is the function which we are looking for provided that ε is sufficiently small.

In general, the function u may have critical points on the part of the boundary $\partial\Omega\setminus\Gamma_0$.

Next we construct a holomorphic function $p\in C^2(\overline\Omega)$ such that $u+\epsilon p$ does not have critical points on $\overline{\partial\Omega\setminus\Gamma_0}$ for all sufficiently small positive ϵ and $\operatorname{Im} p\vert_{\Gamma_0}=0$.

If u does not have critical points on $\partial\Omega\setminus\Gamma_0$ we set $p\equiv 0$. Otherwise, since u is holomorphic in $\tilde\Omega$ the number of such critical points on $\partial\Omega\setminus\Gamma_0$ is finite and the function $|\nabla u|^2$ has zero of finite order at these points. By using a conformal transformation, if necessary, we may assume that $\partial\Omega\setminus\Gamma_0$ is a segment on the line $\{x_2=0\}$. Let $\{(y_k,0)\}_{k=1}^{\tilde N}$ be the set of critical points of the function u on the boundary $\partial\Omega\setminus\Gamma_0$.

We divide the set $\{y_k\}_{k=1}^{\tilde N}$ into two sets $\mathcal O_1$ and $\mathcal O_2$ in the following way: Let us fix some point y_k. By Taylor's formula $\frac{\partial \operatorname{Re} u}{\partial x_1}(x_1,0)=c_1(x_1-y_k)^{\kappa_1+1}+o((x_1-y_k)^{\kappa_1+1})$ and $\frac{\partial \operatorname{Im} u}{\partial x_1}(x_1,0)=c_2(x_1-y_k)^{\kappa_2+1}+o((x_1-y_k)^{\kappa_2+1})$ with some $(c_1,c_2)\ne 0$. If $c_2\ne 0$ and $\kappa_2\le\kappa_1$, then we say that $y_k\in\mathcal O_1$. If $c_1\ne 0$ and $\kappa_2>\kappa_1$, then we say that $y_k\in\mathcal O_2$.

Now we construct a set of $\mathcal S$ open in $C^2(\overline{\Gamma_0})\times C^2(\overline{\Gamma_0})$ such that if $(b_1,b_2)\in\mathcal S$ and the holomorphic function p which satisfies $\operatorname{Im} p\vert_{\Gamma_0}=b_1, \operatorname{Im} p=b_2$ (if such function p exists) then the function $u+\epsilon p$ does not have critical points on $\overline{\tilde\Gamma}$ for all small positive ϵ.

Let us consider the two cases. Assume $y_k\in\mathcal O_1$. If κ_2 is odd, then we take Cauchy data such that the holomorphic function p satisfies the following: b_1 is small and $\frac{\partial b_2}{\partial\vec\tau}$ is positive near y_k if c_2 is positive, $\frac{\partial b_2}{\partial\vec\tau}$ is negative near y_k if c_2 is negative and small on $\partial\Omega\setminus\Gamma_0$. If κ_2 is even and $\kappa_1\ne\kappa_2$, then we take Cauchy data such that $\frac{\partial b_2}{\partial\vec\tau}(y_k)-1$, $\frac{\partial b_1}{\partial\vec\tau}(y_k)-1$ are small and otherwise $\frac{1}{c_2}\frac{\partial b_2}{\partial\vec\tau}(y_k)\ne\frac{1}{c_1}\frac{\partial b_1}{\partial\vec\tau}(y_k)$.

Assume $y_k \in \mathcal{O}_2$. If κ_1 is odd, then we take the Cauchy data for the holomorphic function p such that $\frac{\partial b_1}{\partial \vec\tau}$ is positive near y_k if c_1 is positive, $\frac{\partial b_1}{\partial \vec\tau}$ is negative near y_k if c_1 is negative. If κ_1 is even, then we take $\frac{\partial b_1}{\partial \vec\tau}(y_k) - 1$, $\frac{\partial b_2}{\partial \vec\tau}(y_k) - 1$ to be small. Now we have finished the construction of Cauchy data on Γ_0 and in a neighborhood $\mathcal{U}$ of the set $\{(y_k, 0)\}_{k=1}^{\tilde N}$. On the part of the boundary $\partial\Omega \setminus (\Gamma_0 \cup \mathcal{U})$ we continue functions b_1, b_2 as smooth functions. By Proposition 5.1 and general results on solvability of the boundary value problem for $\partial_{\bar z}$ (see e.g. [32]) there exists a holomorphic function p which satisfies the above choice of the Cauchy data. For all small positive ϵ the function $u + \epsilon p$ does not have critical points on $\overline{\partial\Omega \setminus \Gamma_0}$.

Denote by $\mathcal{H}_\epsilon$ the set of critical points of the function $u+\epsilon p$ in Ω. By the implicit function theorem, there exists a neighborhood of $\widehat{x}$ such that for all small ϵ in this neighborhood the function $u + \epsilon p$ has only one critical point $\widehat{x}(\epsilon)$, this critical point is nondegenerate and

$$\widehat{x}(\epsilon) \to \widehat{x} \text{ as } \quad \epsilon \to 0. \tag{4.20}$$

Let us fix a sufficiently small ϵ. Let $\mathcal{H}_\epsilon = \{x_{k,\epsilon}\}_{1\le k\le N(\epsilon)}$. By Proposition 4.2, there exists a function w holomorphic in Ω, such that

$$\operatorname{Im} w|_{\Gamma_0} = 0, \ w|_{\mathcal{H}_\epsilon} = \partial_z w|_{\mathcal{H}_\epsilon} = 0, \quad \partial_z^2 w|_{\mathcal{H}_\epsilon} \ne 0. \tag{4.21}$$

Denote $\Phi_\delta = u + \epsilon p + \delta w$. For all sufficiently small positive constants δ, we have

$$\mathcal{H}_\epsilon \subset \mathcal{G}_\delta \equiv \{x \in \Omega | \partial_z \Phi_\delta(x) = 0\}.$$

We show now that for all small positive δ, the critical points of the function Φ_δ are nondegenerate. Let $\widetilde{x}$ be a critical point of the function $u + \epsilon p$. If $\widetilde{x}$ is a nondegenerate critical point, by the implicit function theorem, there exists a ball $B(\widetilde{x}, \delta_1)$ such that the function Φ_δ in this ball has only one nondegenerate critical point for all sufficiently small δ. Let $\widetilde{x}$ be a degenerate critical point of $u + \epsilon p$. Without loss of generality we may assume that $\widetilde{x} = 0$. In some neighborhood of 0, we have $\partial_z \Phi_\delta = \sum_{k=1}^\infty c_k z^{k+\hat k} - \delta \sum_{k=1}^\infty b_k z^k$ for some natural positive number $\hat k$ and some $c_1 \ne 0$. Moreover (4.21) implies $b_1 \ne 0$. Let $(x_{1,\delta}, x_{2,\delta}) \in \mathcal{G}_\delta$ and $z_\delta = x_{1,\delta} + i x_{2,\delta} \to 0$. Then either

$$z_\delta = 0 \ \text{ or } \ z_\delta^{\hat k} = \delta b_1 / c_1 + o(\delta). \tag{4.22}$$

Therefore $\partial_z^2 \Phi(z_\delta) \ne 0$ for all sufficiently small δ.

Observe that by (4.16) $\operatorname{Im} \Phi_\delta(\widehat{x}(\epsilon)) \ne 0$. Moreover, without loss of generality we may assume that

$$\operatorname{Im} \Phi_\delta(\widehat{x}(\epsilon)) \ne \operatorname{Im} \Phi_\delta(x) \ \ \forall x \in \mathcal{G}_\delta \ \text{ such that } \widehat{x}(\epsilon) \ne x. \tag{4.23}$$

To see this we argue as follows. If (4.23) is not valid, then we add to the function Φ_δ a function $\delta_1 \tilde w$ such that δ_1 is a small parameter and $\tilde w$ holomorphic in Ω, such that

$$\operatorname{Im} \tilde w|_{\Gamma_0} = 0, \ \operatorname{Im} w(\widehat{x}(\epsilon)) = 1, \ w|_{\mathcal{G}_\delta \setminus \{\widehat{x}(\epsilon)\}} = \partial_z \tilde w|_{\mathcal{G}_\delta} = 0, \quad \partial_z^2 \tilde w|_{\mathcal{G}_\delta} \ne 0.$$

Since the function Φ_δ was constructed as the approximation of the function u, by (4.17), (4.18) we have

$$\frac{\partial \operatorname{Im} \Phi_\delta}{\partial \nu}\Big|_{\overline{\Gamma_0 \cap \gamma_j}} < \alpha'' < 0, \quad \text{if } Int((\partial\Omega \setminus \Gamma_0) \cap \gamma_j) \ne \emptyset. \tag{4.24}$$

In the case $\mathrm{Int}\,((\partial\Omega\setminus\Gamma_0)\cap\gamma_j)=\emptyset$ then $\{x\in\gamma_j|\partial_{\vec\tau}\mathrm{Re}\,\Phi_\delta=0\}=\{y_{1,j}(\delta),y_{2,j}(\delta)\}$,

(4.25) and $\partial^2_{\vec\tau}\mathrm{Re}\,\Phi_\delta(y_{1,j}(\delta))\neq 0, \partial^2_{\vec\tau}\mathrm{Re}\,\Phi_\delta(y_{2,j}(\delta))\neq 0.$

Thanks to (4.25) we can claim that all critical points of Φ_δ are nondegenerate.

By (4.24), (4.25) we can apply Proposition 4.2. Hence there exists a function $a_\delta\in C^2(\overline{\Omega})$ such that

$$\partial_{\bar z}a_\delta=0 \quad \text{in } \Omega, \quad \mathrm{Re}\,a_\delta|_{\Gamma_0}=0,$$

and

$$a_\delta(x)|_{\mathcal{G}_\delta\cap\partial\Omega}=\partial_z a_\delta(x)|_{\mathcal{G}_\delta\cap\partial\Omega}=0,\ a_\delta(\hat x_\epsilon)\neq 0.$$

Hence we can apply Proposition 4.1 to conclude

$$\sum_{x\in\mathcal{G}_\delta} q(x)c(x)e^{2i\tau\mathrm{Im}\,\Phi_\delta(x)}=C(q).$$

By (4.1) $c(\widehat{x}(\epsilon))$ is not equal to zero.

Since the exponents are linearly independent functions of τ, thanks to (4.23), we have $q(\widehat{x}(\epsilon))=0$. Thus (4.20) implies $q(\widehat{x})=0$. Thus the proof is completed. □

5. **Appendix.**

Consider the Cauchy problem for the Cauchy-Riemann equations

$$L(\phi,\psi)=(\frac{\partial\phi}{\partial x_1}-\frac{\partial\psi}{\partial x_2},\frac{\partial\phi}{\partial x_2}+\frac{\partial\psi}{\partial x_1})=0 \quad \text{in } \Omega, \quad (\phi,\psi)\,|_{\Gamma_0}=(b_1(x),b_2(x)), \tag{5.1}$$
$$(\phi+i\psi)(\hat x_j)=c_{0,j}, \quad \partial_z(\phi+i\psi)(\hat x_j)=c_{1,j}, \quad \partial_z^2(\phi+i\psi)(\hat x_j)=c_{2,j} \quad \forall j\in\{1,\dots N\}.$$

Here $\hat x_1,\dots\hat x_N$ be an arbitrary fixed points in Ω. We consider the pair b_1,b_2 and complex numbers $\vec C=(c_{0,1},c_{1,1},c_{2,1},\dots c_{0,N},c_{1,N},c_{2,N})$ as initial data for (5.1). The following proposition establishes the solvability of (5.1) for a dense set of Cauchy data.

Proposition 5.1. *There exists a set $\mathcal{O}\subset C^2(\overline{\Gamma_0})^2\times\mathbb{C}^{3N}$ such that for each $(b_1,b_2,\vec C)\in\mathcal{O}$, (5.1) has at least one solution $(\phi,\psi)\in(C^2(\overline{\Omega}))^2$ and $\overline{\mathcal{O}}=C^2(\overline{\Gamma})^2\times\mathbb{C}^{3N}$.*

Proof. Denote $B=(b_1,b_2)$ an arbitrary element of the space $C^3(\overline{\Gamma_0})\times C^3(\overline{\Gamma_0})$. Consider the following extremal problem

$$J_\epsilon(\phi,\psi)=\|(\phi,\psi)-B\|^4_{B_4^{\frac{11}{4}}(\Gamma_0)}+\epsilon\|(\phi,\psi)\|^4_{B_4^{\frac{11}{4}}(\partial\Omega)}+\frac{1}{\epsilon}\,\|\Delta L(\phi,\psi)\|^4_{L^4(\Omega)}$$
$$+\sum_{j=1}^N\sum_{k=0}^2|\partial_z^k(\phi+i\psi)(\hat x_j)-c_{k,j}|^2\to\inf, \tag{5.2}$$

$$(\phi,\psi)\in\mathcal{X}. \tag{5.3}$$

Here $\mathcal{X}=\Big\{\tilde\delta=(\tilde\delta_1,\tilde\delta_2)|\tilde\delta\in W_4^3(\Omega),\Delta L\tilde\delta\in L^4(\Omega),L\tilde\delta|_{\partial\Omega}=0,\tilde\delta|_{\partial\Omega}\in B_4^{\frac{11}{4}}(\partial\Omega)\Big\}$, and B^l_k denotes the Besov space of corresponding order.

For each $\epsilon > 0$ there exists a unique solution to (5.2), (5.3) which we denote as $(\widehat{\phi}_\epsilon, \widehat{\psi}_\epsilon)$. This fact can be proved using standard arguments. We fix $\epsilon > 0$. Denote by $\mathcal{U}_{ad}$ the set of admissible elements of the problem (5.2), (5.3), namely

$$\mathcal{U}_{ad} = \{(\phi, \psi) \in \mathcal{X} | J_\epsilon(\phi, \psi) < \infty\}.$$

Denote $\hat{J}_\epsilon = \inf_{(\phi,\psi)\in\mathcal{X}} J_\epsilon(\phi, \psi)$. Clearly the pair $(0, 0) \in \mathcal{U}_{ad}$. Therefore there exists a minimizing sequence $\{(\phi_k, \psi_k)\}_{k=1}^N \subset \mathcal{X}$ such that

$$\hat{J}_\epsilon = \lim_{k\to+\infty} J_\epsilon(\phi_k, \psi_k).$$

Observe that the minimizing sequence is bounded in $W_4^3(\Omega)$. Indeed, since $L(\phi_k, \psi_k)$ is bounded in $L^4(\Omega)$ and thanks to the zero Dirichlet boundary conditions for the function $L(\phi_k, \psi_k)$, the standard elliptic estimate implies that the sequence $\{L(\phi_k, \psi_k)\}$ is bounded in the space $W_4^2(\Omega)$. Taking into account that the the sequence traces of the functions (ϕ_k, ψ_k) is bounded in the Besov space $B_4^{\frac{11}{4}}(\partial\Omega)$ and applying the estimates for elliptic operators one more time we obtain that $\{(\phi_k, \psi_k)\}$ bounded in $W_4^3(\Omega)$. By the Sobolev imbedding theorem the sequence $\{(\phi_k, \psi_k)\}$ is bounded in $C^1(\overline{\Omega})$. Then taking if necessary a subsequence, (which we denote again as $\{(\phi_k, \psi_k)\}$) we obtain

$$(\phi_k, \psi_k) \to (\widehat{\phi}_\epsilon, \widehat{\psi}_\epsilon)\text{weakly in}\, W_4^3(\Omega), \quad (\phi_k, \psi_k) \to (\widehat{\phi}_\epsilon, \widehat{\psi}_\epsilon) \text{ weakly in } B_4^{\frac{11}{4}}(\partial\Omega),$$

$$\partial_z^k(\phi + i\psi)(\hat{x}_j) - c_{k,j} \to C_{k,j,\epsilon},$$

$$\Delta L(\phi_k, \psi_k) \to r_\epsilon \text{ weakly in } L^4(\Omega), \quad L(\phi_k, \psi_k) \to \tilde{r}_\epsilon \text{ weakly in}\, W_4^2(\Omega).$$

Obviously, $r_\epsilon = \Delta L(\widehat{\phi}_\epsilon, \widehat{\psi}_\epsilon), \tilde{r}_\epsilon = L(\widehat{\phi}_\epsilon, \widehat{\psi}_\epsilon)$. Then, since the norms in the spaces $L^4(\Omega)$ and $B_4^{\frac{11}{4}}(\partial\Omega), B_4^{\frac{11}{4}}(\Gamma_0)$ are lower semicontinuous with respect to weak convergence we obtain that

$$J_\epsilon(\widehat{\phi}_\epsilon, \widehat{\psi}_\epsilon) \le \lim_{k\to+\infty} J_\epsilon(\phi_k, \psi_k) = \hat{J}_\epsilon.$$

Thus the pair $(\widehat{\phi}_\epsilon, \widehat{\psi}_\epsilon)$ is solution the extremal problem (5.2), (5.3). Since the set of an admissible elements is convex and the functional J_ϵ is strictly convex this solution is unique.

By Fermat's theorem (see e.g. [1] p. 155) we have

$$J'_\epsilon(\widehat{\phi}_\epsilon, \widehat{\psi}_\epsilon)[\tilde{\delta}] = 0, \quad \forall\tilde{\delta} \in \mathcal{X}.$$

This equality can be written in the form

$$I'_{\Gamma_0}((\widehat{\phi}_\epsilon, \widehat{\psi}_\epsilon) - B)[\tilde{\delta}] + \epsilon I'_{\partial\Omega}((\widehat{\phi}_\epsilon, \widehat{\psi}_\epsilon))[\tilde{\delta}] + (p_\epsilon, \Delta L\tilde{\delta})_{L^2(\Omega)} \tag{5.4}$$

$$+\frac{1}{2}\sum_{j=1}^N\sum_{k=0}^2(\partial_z^k(\widehat{\phi}_\epsilon + i\widehat{\psi}_\epsilon)(\hat{x}_j) - c_{k,j})\overline{\partial_z^k(\tilde{\delta}_1 + i\tilde{\delta}_2)(\hat{x}_j)} + \overline{(\partial_z^k(\widehat{\phi}_\epsilon + i\widehat{\psi}_\epsilon)(\hat{x}_j) - c_{k,j})}\partial_z^k(\tilde{\delta}_1 + i\tilde{\delta}_2)(\hat{x}_j) = 0,$$

where $p_\epsilon = \frac{2}{\epsilon}((\Delta(\frac{\partial\hat{\phi}_\epsilon}{\partial x_1} - \frac{\partial\hat{\psi}_\epsilon}{\partial x_2}))^3, (\Delta(\frac{\partial\hat{\phi}_\epsilon}{\partial x_2} + \frac{\partial\hat{\psi}_\epsilon}{\partial x_1}))^3)$. $I'_{\Gamma^*}(\hat{w})$ denotes the derivative of the functional $w \to \|w\|^4_{B_4^{\frac{11}{4}}(\Gamma^*)}$ at space element $\hat{w}$.

Observe that the pair $J_\epsilon(\widehat{\phi}_\epsilon,\widehat{\psi}_\epsilon)\le J_\epsilon(0,0)=\|B\|^2_{B_4^{\frac{11}{4}}(\Gamma_0)}+\sum_{j=1}^N\sum_{k=0}^2|c_{k,j}|^2$. This implies that the sequence $\{(\widehat{\phi}_\epsilon,\widehat{\psi}_\epsilon)\}$ is bounded in $B_4^{\frac{11}{4}}(\Gamma_0)$, the sequences $\{\partial_z^k(\widehat{\phi}_\epsilon+i\widehat{\psi}_\epsilon)(\hat x_j)-c_{k,j}\}$ are bounded in $\mathbb{C}$, the sequence $\{\epsilon(\widehat{\phi}_\epsilon,\widehat{\psi}_\epsilon)\}$ converges to zero in $B_4^{\frac{11}{4}}(\partial\Omega)$. Then (5.4) implies that the sequence $\{p_\epsilon\}$ is bounded in $L^{\frac43}(\Omega)$.

Therefore, there exist $\mathcal{B}\in B_4^{\frac{11}{4}}(\Gamma_0)$, $C_{0,j},C_{1,j},C_{2,j}\in\mathbb{C}$ and $p=(p_1,p_2)\in L^{\frac43}(\Omega)$ such that

$$(\widehat{\phi}_\epsilon,\widehat{\psi}_\epsilon)-B\rightharpoonup\mathcal{B}\quad\text{weakly in } B_4^{\frac{11}{4}}(\Gamma_0),\quad p_\epsilon\rightharpoonup p\quad\text{weakly in } L^{\frac43}(\Omega),\tag{5.5}$$

$$\partial_z^k(\widehat{\phi}_\epsilon+i\widehat{\psi}_\epsilon)(\hat x_j)-c_{k,j}\rightharpoonup C_{k,j}\quad k\in\{0,1,2\},\ j\in\{1,\dots,N\}.\tag{5.6}$$

Passing to the limit in (5.4) we get

$$I'_{\Gamma_0}(\mathcal{B})[\tilde\delta]+(p,\Delta L\tilde\delta)_{L^2(\Omega)}+\mathrm{Re}\sum_{j=1}^N\sum_{k=0}^2 C_{k,j}\overline{\partial_z^k(\tilde\delta_1+i\tilde\delta_2)}(\hat x_j)=0\quad\forall\tilde\delta\in\mathcal{X}.\tag{5.7}$$

Next we claim that

$$\Delta p=0\quad\text{in }\Omega\setminus\cup_{j=1}^N\{\hat x_j\}\tag{5.8}$$

in the sense of distributions. Suppose that (5.8) is already proved. This implies

$$(p,\Delta L\tilde\delta)_{L^2(\Omega)}+\mathrm{Re}\sum_{j=1}^N\sum_{k=0}^2 C_{k,j}\overline{\partial_z^k(\tilde\delta_1+i\tilde\delta_2)}(\hat x_j)=0\quad\forall\tilde\delta_1,\tilde\delta_2\in C_0^\infty(\Omega).$$

If $p=(p_1,p_2)$, denoting $P=p_1-ip_2$, we have

$$2\mathrm{Re}\,(\Delta P,\partial_{\bar z}(\tilde\delta_1+i\tilde\delta_2))_{L^2(\Omega)}+\mathrm{Re}\sum_{j=1}^N\sum_{k=0}^2\overline{C_{k,j}}\partial_z^k(\tilde\delta_1+i\tilde\delta_2)(\hat x_j)=0\quad\forall\tilde\delta_1,\tilde\delta_2\in C_0^\infty(\Omega).$$

Since by (5.8) supp $\Delta P\subset\cup_{j=1}^N\{\hat x_j\}$ there exist some constants $m_{\beta,j}$ and $\hat\ell_j$ such that $\Delta P=\sum_{j=1}^N\sum_{|\beta|=1}^{\hat\ell}m_{\beta,j}D^\beta\delta(x-\hat x_j)$. The above equality can be written in the form

$$-2\sum_{|\beta|=1}^{\hat\ell_j}m_{\beta,j}\partial_{\bar z}D^\beta\delta(x-\hat x_j)=\sum_{k=0}^2(-1)^k\overline{C_{k,j}}\partial_z^k\delta(x-\hat x_j).$$

From this we obtain

$$C_{0,j}=C_{1,j}=C_{2,j}=0\quad j\in\{1,\dots,N\}.\tag{5.9}$$

Therefore

$$\Delta p=0\quad\text{in }\Omega.\tag{5.10}$$

This implies

$$(p,\Delta L\tilde\delta)_{L^2(\Omega)}=0\quad\forall\tilde\delta\in W_4^3(\Omega),\quad L\tilde\delta|_{\partial\Omega}=\frac{\partial L\tilde\delta}{\partial\nu}|_{\partial\Omega}=0.$$

This equality and (5.7) yield

$$I'_{\Gamma_0}(\mathcal{B})[\tilde\delta]=0\quad\forall\tilde\delta\in W_4^3(\Omega),\quad L\tilde\delta|_{\partial\Omega}=\frac{\partial L\tilde\delta}{\partial\nu}|_{\partial\Omega}=0.\tag{5.11}$$

Then using the trace theorem we conclude that $\mathcal{B}=0$. Using this and (5.5) we obtain

$$(\widehat{\phi}_{\epsilon_k},\widehat{\psi}_{\epsilon_k})-B\rightharpoonup 0\quad\text{weakly in } B_4^{\frac{11}{4}}(\Gamma_0). \tag{5.12}$$

From (5.6), (5.9) we get

$$\partial_z^k(\widehat{\phi}_\epsilon+i\widehat{\psi}_\epsilon)(\hat{x})\rightharpoonup c_{k,j}\quad k\in\{0,1,2\},\ j\in\{1,\dots,N\}.$$

By the Sobolev embedding theorem $B_4^{\frac{11}{4}}(\Gamma_0)\subset\subset C^2(\overline{\Gamma_0})$. Therefore (5.12) implies

$$(\widehat{\phi}_{\epsilon_k},\widehat{\psi}_{\epsilon_k})-B\to 0\quad\text{in } C^2(\overline{\Gamma_0}). \tag{5.13}$$

Let the pair $(\tilde{\phi}_{\epsilon_k},\tilde{\psi}_{\epsilon_k})$ be a solution to the boundary value problem

$$L(\tilde{\phi}_{\epsilon_k},\tilde{\psi}_{\epsilon_k})=L(\hat{\phi}_{\epsilon_k},\hat{\psi}_{\epsilon_k})\quad\text{in }\Omega,\quad \tilde{\psi}_{\epsilon_k}|_{\partial\Omega}=\psi^*_{\epsilon_k}. \tag{5.14}$$

Here $\psi^*_{\epsilon_k}$ is a smooth function such that $\psi^*_{\epsilon_k}|_{\Gamma_0}=0$ and the pair $(L(\hat{\phi}_{\epsilon_k},\hat{\psi}_{\epsilon_k}),\psi^*_{\epsilon_k})$ is orthogonal to all solutions of the adjoint problem (see [32]). Moreover since $L(\hat{\phi}_{\epsilon_k},\tilde{\psi}_{\epsilon_k})\to 0$ in $W_4^2(\Omega)$ we may assume $\psi^*_{\epsilon_k}\to 0$ in $C^4(\partial\Omega)$. Among all possible solutions to problem (5.14) (clearly there is no unique solution to this problem) we choose one such that $\int_\Omega\tilde{\phi}_{\epsilon_k}dx=0$. Thus we obtain

$$(\tilde{\phi}_{\epsilon_k},\tilde{\psi}_{\epsilon_k})\to 0\quad\text{in } W_4^3(\Omega). \tag{5.15}$$

Therefore the sequence $\{(\widehat{\phi}_{\epsilon_k}-\tilde{\phi}_{\epsilon_k},\widehat{\psi}_{\epsilon_k}-\tilde{\psi}_{\epsilon_k})\}$ represents the desired approximation for the solution of the Cauchy problem (5.1).

Now we prove (5.8). Let $\widetilde{x}$ be an arbitrary point in $\Omega\setminus\cup_{j=1}^N\{\hat{x}_j\}$ and let $\widetilde{\chi}$ be a smooth function such that it is zero in some neighborhood of $\Gamma_0\cup\cup_{j=1}^N\{\hat{x}_j\}$ and the set $\mathcal{A}=\{x\in\Omega|\widetilde{\chi}(x)=1\}$ contains an open connected subset $\mathcal{F}$ such that $\widetilde{x}\in\mathcal{F}$ and $\tilde{\Gamma}\cap\overline{\mathcal{F}}$ is an open set in $\partial\Omega$. By (5.7) we have

$$0=(p,\Delta L(\widetilde{\chi}\tilde{\delta}))_{L^2(\Omega)}=(\widetilde{\chi}p,\Delta L\tilde{\delta})_{L^2(\Omega)}+(p,[\Delta L,\widetilde{\chi}]\tilde{\delta})_{L^2(\Omega)}.$$

That is,

$$(\widetilde{\chi}p,\Delta L\tilde{\delta})_{L^2(\Omega)}+([\Delta L,\widetilde{\chi}]^*p,\tilde{\delta})_{L^2(\Omega)}=0\quad\forall\tilde{\delta}\in\mathcal{X}. \tag{5.16}$$

from this we conclude that $\widetilde{\chi}p\in W^1_{\frac{4}{3}}(\Omega)$.

Next we take another smooth cut off function $\widetilde{\chi}_1$ such that supp $\widetilde{\chi}_1\subset\mathcal{A}$. A neighborhood of $\widetilde{x}$ belongs to $\mathcal{A}_1=\{x|\widetilde{\chi}_1=1\}$, the interior of $\mathcal{A}_1$ is connected, and $\overline{\text{Int }\mathcal{A}_1}\cap\tilde{\Gamma}$ contains an open subset $\mathcal{O}$ in $\partial\Omega$. Similarly to (5.16) we have

$$(\widetilde{\chi}_1p,\Delta L\tilde{\delta})_{L^2(\Omega)}+([\Delta L,\widetilde{\chi}_1]^*p,\tilde{\delta})_{L^2(\Omega)}=0.$$

This equality implies that $\widetilde{\chi}_1p\in W^2_{\frac{4}{3}}(\Omega)$. Let ω be a domain such that $\omega\cap\Omega=\emptyset$, $\partial\omega\cap\partial\Omega\subset\mathcal{O}$ contains an open set in $\partial\Omega$.

We extend p on ω by zero. Then

$$(\Delta(\widetilde{\chi}_1p),L\tilde{\delta})_{L^2(\Omega\cup\omega)}+([\Delta L,\widetilde{\chi}_1]^*p,\tilde{\delta})_{L^2(\Omega\cup\omega)}=0.$$

Hence

$$L^*\Delta(\widetilde{\chi}_1p)=0\quad\text{in Int }\mathcal{A}_1\cup\omega,\quad p|_\omega=0.$$

By Holmgren's theorem $\Delta(\widetilde{\chi}_1 p)|_{\mathrm{Int}\ \mathcal{A}_1} = 0$, that is, $(\Delta p)(\widetilde{x}) = 0$. $\square$

Consider now the Cauchy problem for the Cauchy-Riemann equations

$$L(\phi,\psi) = (\frac{\partial\phi}{\partial x_1} - \frac{\partial\psi}{\partial x_2}, \frac{\partial\phi}{\partial x_2} + \frac{\partial\psi}{\partial x_1}) = 0 \quad \text{in } \Omega, \quad (\phi,\psi)\,|_{\Gamma_0} = (b(x),0), \tag{5.17}$$

$$(\phi + i\psi)(\hat{x}_j) = c_{0,j}, \quad \partial_z(\phi+i\psi)(\hat{x}_j) = c_{1,j}, \quad \partial_z^2(\phi+i\psi)(\hat{x}_j) = c_{2,j} \quad \forall j \in \{1,\dots N\}.$$

Here $\hat{x}_1, \dots \hat{x}_N$ are arbitrary fixed points in Ω. We consider the function b and complex numbers $\vec{C} = (c_{0,1}, c_{1,1}, c_{2,1}, \dots c_{0,N}, c_{1,N}, c_{2,N})$ as an initial data for (5.17). We get as a corollary of Proposition 5.1 the solvability of (5.17) for a dense set of Cauchy data.

Corollary 5.1. *There exists a set $\mathcal{O}_0 \subset C^2(\overline{\Gamma_0}) \times \mathbb{C}^{3N}$ such that for each $(b,\vec{C}) \in \mathcal{O}_0$, (5.17) has at least one solution $(\phi,\psi) \in (C^2(\overline{\Omega}))^2$ and $\overline{\mathcal{O}_0} = C^2(\overline{\Gamma_0}) \times \mathbb{C}^{3N}$.*

We have

Proposition 5.2. *Let Φ satisfy (2.1) and (2.2). Let $\widetilde{f} \in L^2(\Omega)$ and $\widetilde{v}$ be a solution to*

$$2\partial_z \widetilde{v} - \tau(\partial_z \Phi)\widetilde{v} = \widetilde{f} \quad \text{in } \Omega \tag{5.18}$$

or $\widetilde{v}$ be a solution to

$$2\partial_{\overline{z}} v - \tau(\partial_{\overline{z}}\overline{\Phi})\widetilde{v} = \widetilde{f} \quad \text{in } \Omega. \tag{5.19}$$

In the case that $\tilde{v}$ solves (5.18) we have

$$\begin{aligned} &\|\partial_{x_1}(e^{-i\tau\psi}\widetilde{v})\|^2_{L^2(\Omega)} - \tau\int_{\partial\Omega}(\nabla\varphi,\nu)|\widetilde{v}|^2 d\sigma \\ &+Re\int_{\partial\Omega} i\left(\left(\nu_2\frac{\partial}{\partial x_1} - \nu_1\frac{\partial}{\partial x_2}\right)\widetilde{v}\right)\overline{\widetilde{v}}d\sigma + \|\partial_{x_2}(e^{-i\tau\psi}\widetilde{v})\|^2_{L^2(\Omega)} = \|\widetilde{f}\|^2_{L^2(\Omega)}. \end{aligned} \tag{5.20}$$

In the case that $\widetilde{v}$ solves (5.19) we have

$$\begin{aligned} \|\partial_{x_1}(e^{i\tau\psi}\widetilde{v})\|_{L^2(\Omega)} - \tau\int_{\partial\Omega}(\nabla\varphi,\nu)|\widetilde{v}|^2 d\sigma + Re\int_{\partial\Omega} i\left(\left(-\nu_2\frac{\partial}{\partial x_1} + \nu_1\frac{\partial}{\partial x_2}\right)\widetilde{v}\right)\overline{\widetilde{v}}d\sigma \\ +\|\partial_{x_2}(e^{i\tau\psi}\widetilde{v})\|^2_{L^2(\Omega)} = \|\widetilde{f}\|^2_{L^2(\Omega)}. \end{aligned} \tag{5.21}$$

Proof. We prove the statement of the proposition first for the equation $2\frac{\partial\widetilde{v}}{\partial z} - \tau\frac{\partial\Phi}{\partial z}\widetilde{v} = \widetilde{f}$. Since $2\frac{\partial}{\partial z} - \tau\frac{\partial\Phi}{\partial z} = (\frac{\partial}{\partial x_1} - i\frac{\partial\psi}{\partial x_1}\tau) + (\frac{\partial}{i\partial x_2} - \frac{\partial\psi}{\partial x_2}\tau)$, taking the L^2- norms of the right and the left hand sides of (5.18) we get

$$\begin{aligned} \left\|\left(\frac{\partial}{\partial x_1} - i\frac{\partial\psi}{\partial x_1}\tau\right)\widetilde{v}\right\|^2_{L^2(\Omega)} + 2\mathrm{Re}\left(\left(\frac{\partial}{\partial x_1} - i\frac{\partial\psi}{\partial x_1}\tau\right)\widetilde{v}, \left(-i\frac{\partial}{\partial x_2} - \frac{\partial\psi}{\partial x_2}\tau\right)\widetilde{v}\right)_{L^2(\Omega)} \\ + \left\|\left(-i\frac{\partial}{\partial x_2} - \frac{\partial\psi}{\partial x_2}\tau\right)\widetilde{v}\right\|^2_{L^2(\Omega)} = \|\widetilde{f}\|^2_{L^2(\Omega)}. \end{aligned}$$

Since the commutator vanishes$[(\frac{\partial}{\partial x_1}-i\frac{\partial\psi}{\partial x_1}\tau),(\frac{\partial}{i\partial x_2}-\frac{\partial\psi}{\partial x_2}\tau)]\equiv 0$, we obtain

$$\left\|\left(\frac{\partial}{\partial x_1}-i\frac{\partial\psi}{\partial x_1}\tau\right)\widetilde{v}\right\|^2_{L^2(\Omega)}+\left(\left(\frac{\partial}{\partial x_1}-i\frac{\partial\psi}{\partial x_1}\tau\right)\widetilde{v},\overline{(-i\nu_2\widetilde{v})}\right)_{L^2(\partial\Omega)}+\left(\overline{\nu_1\widetilde{v}},\left(-i\frac{\partial}{\partial x_2}-\frac{\partial\psi}{\partial x_2}\tau\right)\widetilde{v}\right)_{L^2(\partial\Omega)}$$
$$+\left\|\left(i\frac{\partial}{\partial x_2}+\frac{\partial\psi}{\partial x_2}\tau\right)\widetilde{v}\right\|^2_{L^2(\Omega)}=\|\widetilde{f}\|^2_{L^2(\Omega)}.$$

This equality implies

$$\left\|\left(\frac{\partial}{\partial x_1}-i\frac{\partial\psi}{\partial x_1}\tau\right)\widetilde{v}\right\|^2_{L^2(\Omega)}-\tau\int_{\partial\Omega}(\frac{\partial\psi}{\partial x_2}\nu_1-\frac{\partial\psi}{\partial x_1}\nu_2)|\widetilde{v}|^2d\sigma+\int_{\partial\Omega}i\left(\left(\nu_2\frac{\partial}{\partial x_1}-\nu_1\frac{\partial}{\partial x_2}\right)\widetilde{v}\right)\overline{\widetilde{v}}d\sigma$$
$$+\left\|\left(i\frac{\partial}{\partial x_2}+\frac{\partial\psi}{\partial x_2}\tau\right)\widetilde{v}\right\|^2_{L^2(\Omega)}=\|\widetilde{f}\|^2_{L^2(\Omega)}.$$

Finally by (2.1) we observe that

$$\frac{\partial\psi}{\partial x_2}=\frac{\partial\varphi}{\partial x_1}\quad\text{and}\quad\frac{\partial\psi}{\partial x_1}=-\frac{\partial\varphi}{\partial x_2}. \tag{5.22}$$

Thus (5.20) follows immediately.

Now we prove the statement of the proposition for (5.19). Since $2\frac{\partial}{\partial\bar z}-\tau\frac{\partial\overline{\Phi}}{\partial\bar z}=(\frac{\partial}{\partial x_1}+i\frac{\partial\psi}{\partial x_1}\tau)+(-\frac{\partial}{i\partial x_2}-\frac{\partial\psi}{\partial x_2}\tau)$, taking the L^2- norms of the right and left hand sides of (5.19) we get

$$\left\|\left(\frac{\partial}{\partial x_1}+i\frac{\partial\psi}{\partial x_1}\tau\right)\widetilde{v}\right\|^2_{L^2(\Omega)}+2\mathrm{Re}\left(\left(\frac{\partial}{\partial x_1}+i\frac{\partial\psi}{\partial x_1}\tau\right)\widetilde{v},\left(i\frac{\partial}{\partial x_2}-\frac{\partial\psi}{\partial x_2}\tau\right)\widetilde{v}\right)_{L^2(\Omega)}$$
$$+\left\|\left(i\frac{\partial}{\partial x_2}-\frac{\partial\psi}{\partial x_2}\tau\right)\widetilde{v}\right\|^2_{L^2(\Omega)}=\|\widetilde{f}\|^2_{L^2(\Omega)}.$$

Since $[(\frac{\partial}{\partial x_1}+i\frac{\partial\psi}{\partial x_1}\tau),(\frac{\partial}{i\partial x_2}+\frac{\partial\psi}{\partial x_2}\tau)]\equiv 0$, we obtain

$$\left\|\left(\frac{\partial}{\partial x_1}+i\frac{\partial\psi}{\partial x_1}\tau\right)\widetilde{v}\right\|^2_{L^2(\Omega)}+\left(\left(\frac{\partial}{\partial x_1}+i\frac{\partial\psi}{\partial x_1}\tau\right)\widetilde{v},\overline{(i\nu_2\widetilde{v})}\right)_{L^2(\partial\Omega)}+\left(\overline{\nu_1\widetilde{v}},\left(i\frac{\partial}{\partial x_2}-\frac{\partial\psi}{\partial x_2}\tau\right)\widetilde{v}\right)_{L^2(\partial\Omega)}$$
$$+\left\|\left(i\frac{\partial}{\partial x_2}-\frac{\partial\psi}{\partial x_2}\tau\right)\widetilde{v}\right\|^2_{L^2(\Omega)}=\|\widetilde{f}\|^2_{L^2(\Omega)}.$$

This equality implies

$$\left\|\left(\frac{\partial}{\partial x_1}+i\frac{\partial\psi}{\partial x_1}\tau\right)\widetilde{v}\right\|^2_{L^2(\Omega)}-\tau\int_{\partial\Omega}(\frac{\partial\psi}{\partial x_2}\nu_1-\frac{\partial\psi}{\partial x_1}\nu_2)|\widetilde{v}|^2d\sigma+\int_{\partial\Omega}i\left(\left(-\nu_2\frac{\partial}{\partial x_1}+\nu_1\frac{\partial}{\partial x_2}\right)\widetilde{v}\right)\overline{\widetilde{v}}d\sigma$$
$$+\left\|\left(i\frac{\partial}{\partial x_2}-\frac{\partial\psi}{\partial x_2}\tau\right)\widetilde{v}\right\|^2_{L^2(\Omega)}=\|\widetilde{f}\|^2_{L^2(\Omega)}.$$

Thus estimate (5.21) follows immediately from the above equality and (5.22), finishing the proof of the proposition. ∎

Now we prove a Carleman estimate for the Laplace operator.

Proposition 5.3. *Suppose that Φ satisfies (2.1), (2.2), (2.5). Let $u \in H^1_0(\Omega) \cap H^2(\Omega)$ be a real valued function. Then there exists τ_0 such that for all $|\tau| \ge \tau_0$ we have:*

$$
\begin{aligned}
&|\tau| \|ue^{\tau\varphi}\|^2_{L^2(\Omega)} + \|ue^{\tau\varphi}\|^2_{H^1(\Omega)} + \|\frac{\partial u}{\partial \nu} e^{\tau\varphi}\|^2_{L^2(\Gamma_0)} + \tau^2 \| |\frac{\partial \Phi}{\partial z}| ue^{\tau\varphi}\|^2_{L^2(\Omega)} \\
&\le C(\|(\Delta u)e^{\tau\varphi}\|^2_{L^2(\Omega)} + |\tau| \int_{\tilde\Gamma} |\frac{\partial u}{\partial \nu}|^2 e^{2\tau\varphi} d\sigma).
\end{aligned}
\tag{5.23}
$$

Proof. Without loss of generality, we may assume that $\tau > 0$. Denote $\tilde v = ue^{\tau\varphi}, \Delta u = f$. Observe that $\Delta = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar z}$ and $\varphi(x_1, x_2) = \frac12(\Phi(z) + \overline{\Phi(z)})$. Therefore

$$
e^{\tau\varphi}\Delta e^{-\tau\varphi}\tilde v = (2\frac{\partial}{\partial z} - \tau\frac{\partial \Phi}{\partial z})(2\frac{\partial}{\partial \bar z} - \tau\frac{\partial \bar\Phi}{\partial \bar z})\tilde v = (2\frac{\partial}{\partial \bar z} - \tau\frac{\partial \bar\Phi}{\partial \bar z})(2\frac{\partial}{\partial z} - \tau\frac{\partial \Phi}{\partial z})\tilde v = fe^{\tau\varphi}.
$$

Denote $\tilde w_1 = \overline{Q(z)}(2\frac{\partial}{\partial \bar z} - \tau\frac{\partial\bar\Phi}{\partial \bar z})\tilde v, \tilde w_2 = Q(z)(2\frac{\partial}{\partial z} - \tau\frac{\partial\Phi}{\partial z})\tilde v$, where $Q(z) \in C^2(\overline\Omega)$ is an holomorphic function in $\overline\Omega$. Thanks to the zero Dirichlet boundary condition for u we have

$$
\tilde w_1|_{\partial\Omega} = 2\overline{Q(z)}\partial_{\bar z}\tilde v|_{\partial\Omega} = (\nu_1 + i\nu_2)\overline{Q(z)}\frac{\partial \tilde v}{\partial \nu}|_{\partial\Omega},\ \tilde w_2|_{\partial\Omega} = 2Q(z)\partial_z \tilde v|_{\partial\Omega} = (\nu_1 - i\nu_2)Q(z)\frac{\partial\tilde v}{\partial\nu}|_{\partial\Omega}.
$$

By Proposition 5.2 we obtain

$$
\begin{aligned}
&\|(\frac{\partial}{\partial x_1} - i\tau\frac{\partial\psi}{\partial x_1})\tilde w_1\|^2_{L^2(\Omega)} - \tau\int_{\partial\Omega}(\nabla\varphi, \nu)|Q|^2|\frac{\partial\tilde v}{\partial \nu}|^2 d\sigma + \mathrm{Re}\int_{\partial\Omega} i((\nu_2\frac{\partial}{\partial x_1} - \nu_1\frac{\partial}{\partial x_2})\tilde w_1)\overline{\tilde w_1}d\sigma + \\
&+\|(\frac{\partial}{\partial x_2} - i\tau\frac{\partial \psi}{\partial x_2})\tilde w_1\|^2_{L^2(\Omega)} = \|Qfe^{\tau\varphi}\|^2_{L^2(\Omega)}
\end{aligned}
$$

and

$$
\begin{aligned}
&\|(\frac{\partial}{\partial x_1} + i\frac{\partial\psi}{\partial x_1}\tau)\tilde w_2\|^2_{L^2(\Omega)} - \tau\int_{\partial\Omega}(\nabla\varphi, \nu)|Q|^2|\frac{\partial\tilde v}{\partial \nu}|^2 d\sigma + \mathrm{Re}\int_{\partial\Omega} i((-\nu_2\frac{\partial}{\partial x_1} + \nu_1\frac{\partial}{\partial x_2})\tilde w_2)\overline{\tilde w_2}d\sigma + \\
&+\|(\frac{\partial}{\partial x_2} + i\tau\frac{\partial \psi}{\partial x_2})\tilde w_2\|^2_{L^2(\Omega)} = \|Qfe^{\tau\varphi}\|^2_{L^2(\Omega)}.
\end{aligned}
$$

We simplify the integral $\mathrm{Re} i\int_{\partial\Omega}((\nu_2\frac{\partial}{\partial x_1} - \nu_1\frac{\partial}{\partial x_2})\tilde w_1)\overline{\tilde w_1}d\sigma$. We recall that $\tilde v = ue^{\tau\varphi}$ and $\tilde w_1|_{\partial\Omega} = \overline{Q(z)}(\nu_1 + i\nu_2)\frac{\partial\tilde v}{\partial\nu} = \overline{Q(z)}(\nu_1 + i\nu_2)\frac{\partial u}{\partial \nu}e^{\tau\varphi}$. Denote $A + iB = \overline{Q(z)}(\nu_1 + i\nu_2)$. We get

$$
\begin{aligned}
&\mathrm{Re}\int_{\partial\Omega} i((\nu_2\frac{\partial}{\partial x_1} - \nu_1\frac{\partial}{\partial x_2})\tilde w_1)\overline{\tilde w_1}d\sigma = \\
&\mathrm{Re}\int_{\partial\Omega} i(\nu_2\frac{\partial}{\partial x_1} - \nu_1\frac{\partial}{\partial x_2})[(A + iB)\frac{\partial u}{\partial\nu}e^{\tau\varphi}](A - iB)\frac{\partial u}{\partial \nu}e^{\tau\varphi}d\sigma = \\
&\mathrm{Re}\int_{\partial\Omega} i[(\nu_2\frac{\partial}{\partial x_1} - \nu_1\frac{\partial}{\partial x_2})(A + iB)]|\frac{\partial\tilde v}{\partial\nu}|^2(A - iB)d\sigma + \\
&\mathrm{Re}\int_{\partial\Omega}\frac{i}{2}(A^2 + B^2)((\nu_2\frac{\partial}{\partial x_1} - \nu_1\frac{\partial}{\partial x_2})|\frac{\partial\tilde v}{\partial\nu}|^2 d\sigma = \\
&\int_{\partial\Omega}(\partial_{\vec\tau}AB - \partial_{\vec\tau}BA)|\frac{\partial\tilde v}{\partial\nu}|^2 d\sigma.
\end{aligned}
$$

Now we simplify the integral $\mathrm{Re}\int_{\partial\Omega} i((-\nu_2\frac{\partial}{\partial x_1}+\nu_1\frac{\partial}{\partial x_2})\tilde w_2)\overline{\tilde w_2}d\sigma$. We recall that $\tilde v = ue^{\tau\varphi}$ and $\tilde w_2|_{\partial\Omega} = (\nu_1 - i\nu_2)Q(z)\frac{\partial \tilde v}{\partial\nu} = (\nu_1 - i\nu_2)Q(z)\frac{\partial u}{\partial \nu}e^{\tau\varphi}$. A straightforward computation gives

$$
\begin{aligned}
&\mathrm{Re}\int_{\partial\Omega} i((-\nu_2\frac{\partial}{\partial x_1}+\nu_1\frac{\partial}{\partial x_2})\tilde w_2)\overline{\tilde w_2}d\sigma = \\
&\mathrm{Re}\int_{\partial\Omega} i(-\nu_2\frac{\partial}{\partial x_1}+\nu_1\frac{\partial}{\partial x_2})[(A-iB)\frac{\partial u}{\partial\nu}e^{\tau\varphi}](A+iB)\frac{\partial u}{\partial\nu}e^{\tau\varphi}d\sigma = \\
&\mathrm{Re}\int_{\partial\Omega} i[(-\nu_2\frac{\partial}{\partial x_1}+\nu_1\frac{\partial}{\partial x_2})(A-iB)]|\frac{\partial \tilde v}{\partial\nu}|^2(A+iB)d\sigma - \\
&\mathrm{Re}\int_{\partial\Omega} \frac{i}{2}(A^2+B^2)((\nu_2\frac{\partial}{\partial x_1}-\nu_1\frac{\partial}{\partial x_2})|\frac{\partial \tilde v}{\partial\nu}|^2d\sigma = \\
&\int_{\partial\Omega}(\partial_{\vec\tau}AB-\partial_{\vec\tau}BA)|\frac{\partial\tilde v}{\partial\nu}|^2d\sigma.
\end{aligned} \tag{5.24}
$$

Using the above formula we obtain

$$
\begin{aligned}
&\|(\frac{\partial}{\partial x_1}+i\frac{\partial\psi}{\partial x_1}\tau)\tilde w_2\|^2_{L^2(\Omega)} + \|(i\frac{\partial}{\partial x_2}+\frac{\partial\psi}{\partial x_2}\tau)\tilde w_2\|^2_{L^2(\Omega)} - 2\tau\int_{\partial\Omega}(\nu,\nabla\varphi)|Q|^2|\frac{\partial\tilde v}{\partial\nu}|^2d\sigma \\
&+\|(\frac{\partial}{\partial x_1}-i\frac{\partial\psi}{\partial x_1}\tau)\tilde w_1\|^2_{L^2(\Omega)} + \|(\frac{\partial}{\partial x_2}-\frac{\partial\psi}{\partial x_2}\tau)\tilde w_1\|^2_{L^2(\Omega)} \\
&+2\int_{\partial\Omega}(\partial_{\vec\tau}AB-\partial_{\vec\tau}BA)|\frac{\partial\tilde v}{\partial\nu}|^2d\sigma = 2\|Qfe^{\tau\varphi}\|^2_{L^2(\Omega)}.
\end{aligned} \tag{5.25}
$$

We can rewrite (5.25) in the form

$$
\begin{aligned}
&\|\frac{\partial}{\partial x_1}(e^{-i\psi\tau}\tilde w_2)\|^2_{L^2(\Omega)} + \|\frac{\partial}{\partial x_2}(e^{-i\psi\tau}\tilde w_2)\|^2_{L^2(\Omega)} - 2\tau\int_{\partial\Omega}(\nu,\nabla\varphi)|Q|^2|\frac{\partial\tilde v}{\partial\nu}|^2d\sigma \\
&+\|\frac{\partial}{\partial x_1}(e^{i\psi\tau}\tilde w_1)\|^2_{L^2(\Omega)} + \|\frac{\partial}{\partial x_2}(e^{i\psi\tau}\tilde w_1)\|^2_{L^2(\Omega)} \\
&+2\int_{\partial\Omega}(\partial_{\vec\tau}AB-\partial_{\vec\tau}BA)|\frac{\partial\tilde v}{\partial\nu}|^2d\sigma = 2\|Qfe^{\tau\varphi}\|^2_{L^2(\Omega)}.
\end{aligned} \tag{5.26}
$$

At this point, in order to estimate the integral $\int_{\partial\Omega}(\partial_{\vec\tau}AB-\partial_{\vec\tau}BA)|\frac{\partial\tilde v}{\partial\nu}|^2d\sigma$, we have to make a choice of the holomorphic function Q. If Ω is simply connected, after an appropriate conformal transformation to the ball, we can take $Q\equiv 1$. Then the function $(\partial_{\vec\tau}AB-\partial_{\vec\tau}BA)$ will be positive.

In the general situation, using Proposition 5.1 we choose the holomorphic function $Q(z)$ such that $(\partial_{\vec\tau}AB-\partial_{\vec\tau}BA)$ is positive on $\bar\Gamma_0$. Such a function can be constructed in the following way. Let γ_j be a contour from $\partial\Omega$. We parameterize it by the smooth curve $x(s) : [0,\ell_j]\to\gamma_j$, satisfying $|x'(s)| = 1$ and $\partial_{\vec\tau}A = \frac{d}{ds}A\circ x(s)$. We take now $A\circ x(s) = \ell_j sin(s/\ell_j), B\circ x(s) = \ell_j cos(s/\ell_j)$. Then

$$
(\partial_{\vec\tau}AB-\partial_{\vec\tau}BA) = \ell_j \quad \text{on } \gamma_j.
$$

Taking into account that $A+iB = \overline{Q(z)}(\nu_1+i\nu_2)$ we set

$$
b_1 = \mathrm{Re}\left\{\frac{\ell_j sin(s/\ell_j) - i\ell_j cos(s/\ell_j)}{(\nu_1-i\nu_2)\circ x(s)}\right\}, \quad b_2 = \mathrm{Im}\left\{\frac{\ell_j sin(s/\ell_j) - i\ell_j cos(s/\ell_j)}{(\nu_1-i\nu_2)\circ x(s)}\right\}. \tag{5.27}
$$

We take Q as a solution to problem (5.1) with the initial data close to one given by (5.27). Then we have the estimate

$$\int_{\partial\Omega}|\frac{\partial\tilde v}{\partial\nu}|^2d\sigma \le (\|Qfe^{\tau\varphi}\|^2_{L^2(\Omega)}+|\tau|\int_{\tilde\Gamma}|\frac{\partial\tilde v}{\partial\nu}|^2d\sigma). \tag{5.28}$$

The function $Q(z)$ which allowed us to establish the estimate (5.28) might be equal to zero at some points of Ω. Thus, from now on , we take $Q(z)\equiv 1$. Of course (5.26) is valid.

Since φ is a harmonic function we have $\int_{\partial\Omega}\frac{\partial\varphi}{\partial\nu}d\sigma=0$. By (2.1), (2.2) the function φ is not constant, so the set $\partial\Omega_-=\{x\in\partial\Omega|(\nu,\nabla\varphi)>0\}$ is not empty.

We now establish a Poincaré type inequality with boundary terms. Let Γ_* be some open subset of $\partial\Omega$. Observe that the functional $\|\nabla W\|_{L^2(\Omega)}+\|W\|_{L^2(\Gamma_*)}$ is the norm on the Sobolev space $H^1(\Omega)$. In order to prove this it suffices to establish the existence of constant C such that

$$\|W\|_{L^2(\Omega)}\le C(\|\nabla W\|_{L^2(\Omega)}+\|W\|_{L^2(\Gamma_*)})\quad \forall W\in H^1(\Omega). \tag{5.29}$$

Suppose that (5.29) is false. Then there exists a sequence $\{W_k\}\subset H^1(\Omega)$ such that $\|W_k\|_{L^2(\Omega)}=1$ and

$$\|\nabla W_k\|_{L^2(\Omega)}+\|W_k\|_{L^2(\Gamma_*)}\to 0. \tag{5.30}$$

On the other hand the sequence W_k is clearly bounded in $H^1(\Omega)$. So taking a subsequence and using the compactness of the embedding of $H^1(\Omega)$ into $L^2(\Omega)$ we see that there exists $\tilde W\in H^1(\Omega)$ such that

$$W_k\to\tilde W\quad\text{in } L^2(\Omega). \tag{5.31}$$

By (5.30) $\tilde W\equiv const$. On the other hand, by (5.30) $\tilde W|_{\Gamma_*}=0$. Therefore $\tilde W\equiv 0$ and we have the contradiction with (5.31) and the fact that $\|W_k\|_{L^2(\Omega)}=1$.

Thus, by (5.29) there exists a positive constant C, independent of τ, such that

$$\begin{aligned}\frac{1}{C}(\|\widetilde w_1\|^2_{L^2(\Omega)}+\|\widetilde w_2\|^2_{L^2(\Omega)})\le \frac12\|\frac{\partial}{\partial x_1}(e^{-i\psi\tau}\widetilde w_2)\|^2_{L^2(\Omega)}+\frac12\|\frac{\partial}{\partial x_2}(e^{-i\psi\tau}\widetilde w_2)\|^2_{L^2(\Omega)}\\ -\tau\int_{\partial\Omega_-}(\nu,\nabla\varphi)|\frac{\partial\widetilde v}{\partial\nu}|^2d\sigma\\ +\frac12\|\frac{\partial}{\partial x_1}(e^{i\psi\tau}\widetilde w_1)\|^2_{L^2(\Omega)}+\frac12\|\frac{\partial}{\partial x_2}(e^{i\psi\tau}\widetilde w_1)\|^2_{L^2(\Omega)}.\end{aligned} \tag{5.32}$$

Since $\tilde v$ is a real-valued function we have

$$\|2\frac{\partial\tilde v}{\partial x_1}+\tau\frac{\partial\psi}{\partial x_2}\tilde v\|^2_{L^2(\Omega)}+\|2\frac{\partial\tilde v}{\partial x_2}-\tau\frac{\partial\psi}{\partial x_1}\tilde v\|^2_{L^2(\Omega)}\le C_0(\|\tilde w_1\|^2_{L^2(\Omega)}+\|\tilde w_2\|^2_{L^2(\Omega)}).$$

Therefore

$$\begin{aligned}4\|\frac{\partial\tilde v}{\partial x_1}\|^2_{L^2(\Omega)}-2\tau\int_\Omega(\frac{\partial}{\partial x_1}\frac{\partial\psi}{\partial x_2}-\frac{\partial}{\partial x_2}\frac{\partial\psi}{\partial x_1})\tilde v^2dx+\|\tau\frac{\partial\psi}{\partial x_2}\tilde v\|^2_{L^2(\Omega)}\\ +4\|\frac{\partial\tilde v}{\partial x_2}\|^2_{L^2(\Omega)}+\|\tau\frac{\partial\psi}{\partial x_1}\tilde v\|^2_{L^2(\Omega)}\le C_1(\|\tilde w_1\|^2_{L^2(\Omega)}+\|\tilde w_2\|^2_{L^2(\Omega)}).\end{aligned} \tag{5.33}$$

Now we claim that there exists a constant C_2 independent of τ such that

$$|\tau|\|\tilde v\|^2_{L^2(\Omega)} \le C_2(\|\tilde v\|^2_{H^1(\Omega)} + \tau^2\||\frac{\partial\Phi}{\partial z}|\tilde v\|^2_{L^2(\Omega)}). \tag{5.34}$$

It suffices to prove inequality (5.34) locally assuming that $\operatorname{supp} v \in B(y,\delta)$ where $y \in \mathcal{H}$ and the radius δ can be taken sufficiently small. If $y \in \mathcal{H}\cap\partial\Omega$ by (2.2) one can take δ such that $v|_{\partial\Omega\cap B(y,\delta)} = 0$. Moreover, if $y \in \mathcal{H}$ is an arbitrary point we may assume, without loss of generality, that $y = 0$. Since all critical points of the function Φ are assumed to be nondegenerate there exists a holomorphic function $\Psi(z)$ such that $\partial_z\Phi(z) = z\Psi(z)$ and $\Psi(0) \ne 0$. Thus for some positive δ there exists a positive constant C_3 such that

$$|\partial_z\Phi| \le C_3|z| \quad \forall(\operatorname{Re} z, \operatorname{Im} z) \in B(0,\delta).$$

Then there exists a positive constant C_4 such that

$$\int_\Omega |v|^2dx = \int_\Omega (\partial_z z)|v|^2dx = -\int_\Omega z(v\partial_z\bar v + \bar v\partial_z v)dx \le C_4\int_\Omega (|\nabla v|^2 + |z|^2|v|^2)dx.$$

By (5.33), (5.34) there exists a positive constant C_5 such that

$$|\tau|\|\tilde v\|^2_{L^2(\Omega)} + \|\tilde v\|^2_{H^1(\Omega)} + \tau^2\||\frac{\partial\Phi}{\partial z}|\tilde v\|^2_{L^2(\Omega)} \le C_5(\|\tilde w_1\|^2_{L^2(\Omega)} + \|\tilde w_2\|^2_{L^2(\Omega)}). \tag{5.35}$$

By (5.35) we obtain from (5.26), (5.32) that there exists a positive constant C_6 such that

$$\frac{1}{C_6}(|\tau|\|\tilde v\|^2_{L^2(\Omega)} + \|\tilde v\|^2_{H^1(\Omega)} + \tau^2\||\frac{\partial\Phi}{\partial z}|\tilde v\|^2_{L^2(\Omega)}) - \tau\int_{\partial\Omega}(\nu,\nabla\varphi)|\frac{\partial\tilde v}{\partial\nu}|^2d\sigma$$
$$+\int_{\partial\Omega} 2(\partial_{\vec\tau}AB - \partial_{\vec\tau}BA)|\frac{\partial\tilde v}{\partial\nu}|^2d\sigma \le \|fe^{\tau\varphi}\|^2_{L^2(\Omega)} + |\tau|\int_{\tilde\Gamma}|(\nu,\nabla\varphi)||\frac{\partial\tilde v}{\partial\nu}|^2d\sigma.$$

This estimate and (5.28) concludes the proof of the proposition. $\square$

Now we give the proof of Proposition 2.1.

Proof. Let us introduce the space

$$H = \left\{v \in H^1_0(\Omega)|\,\Delta v + q_0v \in L^2(\Omega), \frac{\partial v}{\partial\nu}|_{\tilde\Gamma} = 0\right\}$$

with the scalar product

$$(v_1,v_2)_H = \int_\Omega e^{2\tau\varphi}(\Delta v_1 + q_0v_1)(\Delta v_2 + q_0v_2)dx.$$

By Proposition 5.3 H is a Hilbert space. Consider the linear functional on $H : v \to \int_\Omega vf dx + \int_{\Gamma_0} g\frac{\partial v}{\partial\nu}d\sigma$. By (5.23) this is the continuous linear functional with the norm estimated by a constant $C(\|fe^{\tau\varphi}\|_{L^2(\Omega)}/\tau^{\frac12} + \|ge^{\tau\varphi}\|_{L^2(\Gamma_0)})$. Therefore by the Riesz representation theorem there exists an element $\widehat v \in H$ so that

$$\int_\Omega vf dx + \int_{\tilde\Gamma} g\frac{\partial v}{\partial\nu}d\sigma = \int_\Omega e^{2\tau\varphi}(\Delta\widehat v + q_0\widehat v)(\Delta v + q_0v)dx.$$

Then, as a solution to (2.4), we take the function $u = e^{2\tau\varphi}(\Delta\widehat v + q_0\widehat v)$. $\square$

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