Quantitative strong unique continuation for the Lamé system with less regular coefficients

C-L Lin* G Nakamura† G Uhlmann‡ J-N Wang§

Abstract

We prove a quantitative form of the strong unique continuation property for the Lamé system when the Lamé coefficients μ is Lipschitz and λ is essentially bounded in dimension $n \geq 2$. This result is an improvement of the earlier result [5] in which both μ and λ were assumed to be Lipschitz.

1 Introduction

Assume that Ω is a connected open set containing 0 in $\mathbb{R}^n$ for $n \geq 2$. Let $\mu(x) \in C^{0,1}(\Omega)$ and $\lambda(x), \rho(x) \in L^\infty(\Omega)$ satisfy

$$\begin{cases} \mu(x) \geq \delta_0, \qquad \lambda(x) + 2\mu(x) \geq \delta_0 \quad \forall\, x \in \Omega, \\ \|\mu\|_{C^{0,1}(\Omega)} + \|\lambda\|_{L^\infty(\Omega)} \leq M_0, \quad \|\rho\|_{L^\infty(\Omega)} \leq M_0 \end{cases} \tag{1.1}$$

with positive constants δ_0, M_0, where we define

$$\|f\|_{C^{0,1}(\Omega)} = \|f\|_{L^\infty(\Omega)} + \|\nabla f\|_{L^\infty(\Omega)}.$$

*Department of Mathematics, NCTS, National Cheng Kung University, Tainan 701, Taiwan. Email:cllin2@mail.ncku.edu.tw

†Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan. Partially supported by Grant-in-Aid for Scientific Research (B)(2)(No. 14340038) of Japan Society for Promotion of Science. (Email: gnaka@math.sci.hokudai.ac.jp)

‡Department of Mathematics, University of Washington, Box 354350, Seattle 98195-4350, USA. Email:gunther@math.washington.edu

§Department of Mathematics, Taida Institute of Mathematical Sciences, NCTS (Taipei), National Taiwan University, Taipei 106, Taiwan. Email:jnwang@math.ntu.edu.tw

The isotropic elasticity system, which represents the displacement equation of equilibrium, is given by

$$\operatorname{div}(\mu(\nabla u+(\nabla u)^t))+\nabla(\lambda\operatorname{div}u)+\rho u=0\quad\text{in }\Omega, \tag{1.2}$$

where $u=(u_1,u_2,\cdots,u_n)^t$ is the displacement vector and $(\nabla u)_{jk}=\partial_k u_j$ for $j,k=1,2,\cdots,n$.

We are interested in the strong unique continuation property (SUCP) of (1.2). More precisely, we would like to show that any nontrivial solution of (1.2) can only vanish of finite order at any point of Ω. We also give an estimate of the vanishing order for u, which can be seen as a quantitative description of the SUCP for (1.2). Here we list some of the known results on the SUCP for (1.2):

- $\lambda,\mu\in C^{1,1}$, $n\geq 2$ (quantitative): Alessandrini and Morassi [1].
- $\lambda,\mu\in C^{0,1}$, $n=2$ (qualitative): Lin and Wang [4].
- $\lambda\in L^\infty,\mu\in C^{0,1}$, $n=2$ (qualitative): Escauriaza [2].
- $\lambda,\mu\in C^{0,1}$, $n\geq 2$ (quantitative): Lin, Nakamura, and Wang [5].

In this paper, we relax the regularity assumption on λ in [5] to $\lambda\in L^\infty(\Omega)$. In view of counterexamples by Plis [7] or Miller [3], this regularity assumption seems to be optimal. This improvement was inspired by our recent work on the Stokes system [6]. We now state the main results of the paper. Assume that there exists $0<R_0\leq 1$ such that $B_{R_0}\subset\Omega$. Hereafter B_r denotes an open ball of radius $r>0$ centered at the origin.

Theorem 1.1 (Optimal three-ball inequalities) There exists a positive number $\tilde R<1$, depending only on n,M_0,δ_0, such that if $0<R_1<R_2<R_3\leq R_0$ and $R_1/R_3<R_2/R_3<\tilde R$, then

$$\int_{|x|<R_2}|u|^2dx\leq C\left(\int_{|x|<R_1}|u|^2dx\right)^\tau\left(\int_{|x|<R_3}|u|^2dx\right)^{1-\tau} \tag{1.3}$$

for $u\in H^1_{loc}(B_{R_0})$ satisfying (1.2) in B_{R_0}, where the constant C depends on R_2/R_3, n, M_0,δ_0, and $0<\tau<1$ depends on R_1/R_3, R_2/R_3, n,M_0,δ_0. Moreover, for fixed R_2 and R_3, the exponent τ behaves like $1/(-\log R_1)$ when R_1 is sufficiently small.

Theorem 1.2 *Let* $u \in H^1_{loc}(\Omega)$ *be a nontrivial solution of* (1.2)*, then there exist positive constants* K *and* m*, depending on* n, M_0, δ_0 *and* u*, such that*

$$\int_{|x|<R} |u|^2 dx \geq KR^m \tag{1.4}$$

for all R *sufficiently small.*

Remark 1.3 *Based on Theorem 1.1, the constants* K *and* m *in* (2.2) *are explicitly given by*

$$K = \int_{|x|<R_3} |u|^2 dx$$

and

$$m = \tilde{C} + \log\Big(\frac{\int_{|x|<R_3} |u|^2 dx}{\int_{|x|<R_2} |u|^2 dx}\Big),$$

where $\tilde{C}$ *is a positive constant depending on* n, M_0, δ_0 *and* R_2/R_3*.*

2 Reduced system and estimates

Here we want to find a reduced system from (1.2). This is a crucial step in our approach. Let us write (1.2) into a non-divergence form:

$$\mu\Delta u + \nabla((\lambda+\mu)\ \mathrm{div} u) + (\nabla u + (\nabla u)^t)\nabla\mu - \mathrm{div} u \nabla\mu + \rho u = 0. \tag{2.1}$$

Dividing (2.1) by μ yields

$$\begin{aligned} & \Delta u + \frac{1}{\mu}\nabla((\lambda+\mu)\ \mathrm{div} u) + (\nabla u + (\nabla u)^t)\frac{\nabla\mu}{\mu} - \mathrm{div} u\frac{\nabla\mu}{\mu} + \frac{\rho}{\mu}u \\ = \ & \Delta u + \nabla(\frac{\lambda+\mu}{\mu}\ \mathrm{div} u) + (\nabla u + (\nabla u)^t)\frac{\nabla\mu}{\mu} - \mathrm{div} u(\frac{\nabla\mu}{\mu} + (\lambda+\mu)\nabla(\frac{1}{\mu})) \\ & + \frac{\rho}{\mu}u \\ = \ & \Delta u + \nabla(a(x)v) + G \\ = \ & 0, \end{aligned} \tag{2.2}$$

where

$$a(x) = \frac{\lambda+\mu}{\lambda+2\mu} \in L^\infty(\Omega), \quad v = \frac{\lambda+2\mu}{\mu}\ \mathrm{div} u$$

and

$$G = (\nabla u + (\nabla u)^t)\frac{\nabla\mu}{\mu} - \mathrm{div}u(\frac{\nabla\mu}{\mu} + (\lambda+\mu)\nabla(\frac{1}{\mu})) + \frac{\rho}{\mu}u.$$

Taking the divergence on (2.2) gives

$$\Delta v + \mathrm{div}G = 0. \tag{2.3}$$

Our reduced system now consists of (2.2) and (2.3). It follows easily from (2.3) that if $u \in H^1_{loc}(\Omega)$, then $v \in H^1_{loc}(\Omega)$.

To prove the main results, we rely on suitable Carleman estimates. Denote $\varphi_\beta = \varphi_\beta(x) = \exp(-\beta\tilde{\psi}(x))$, where $\beta > 0$ and $\tilde{\psi}(x) = \log|x| + \log((\log|x|)^2)$. Note that φ_β is less singular than $|x|^{-\beta}$. We use the notation $X \lesssim Y$ or $X \gtrsim Y$ to mean that $X \le CY$ or $X \ge CY$ with some constant C depending only on n.

Lemma 2.1 [5, Lemma 2.4] *There exist a sufficiently small number $r_1 > 0$ depending on n and a sufficiently large number $\beta_1 > 3$ depending on n such that for all $w \in U_{r_1}$ and $f = (f_1, \cdots, f_n) \in (U_{r_1})^n$, $\beta \ge \beta_1$, we have that*

$$\begin{aligned}&\int \varphi_\beta^2 (\log|x|)^2(\beta|x|^{4-n}|\nabla w|^2 + \beta^3|x|^{2-n}|w|^2)dx \\ \lesssim\ &\int \varphi_\beta^2(\log|x|)^4|x|^{2-n}[(|x|^2\Delta w + |x|\mathrm{div}f)^2 + \beta^2\|f\|^2]dx,\end{aligned} \tag{2.4}$$

where $U_{r_1} = \{w \in C_0^\infty(\mathbb{R}^n \setminus \{0\}) : \mathrm{supp}(w) \subset B_{r_0}\}$.

Next, replacing β by $\beta + 1$ in (2.4), we get another Carleman estimate.

Lemma 2.2 *There exist a sufficiently small number $r_1 > 0$ depending on n and a sufficiently large number $\beta_1 > 2$ depending on n such that for all $w \in U_{r_1}$ and $f = (f_1, \cdots, f_n) \in (U_{r_1})^n$, $\beta \ge \beta_1$, we have that*

$$\begin{aligned}&\int \varphi_\beta^2 (\log|x|)^{-2}(\beta|x|^{2-n}|\nabla w|^2 + \beta^3|x|^{-n}|w|^2)dx \\ \lesssim\ &\int \varphi_\beta^2|x|^{-n}[(|x|^2\Delta w + |x|\mathrm{div}f)^2 + \beta^2\|f\|^2]dx.\end{aligned} \tag{2.5}$$

In addition to Carleman estimates, we also need the following Caccioppoli's type inequality.

Lemma 2.3 *Let $u \in (H^1_{loc}(\Omega))^n$ be a solution of* (1.1)*. Then for any $0 < a_3 < a_1 < a_2 < a_4$ such that $B_{a_4 r} \subset \Omega$ and $|a_4 r| < 1$, we have*

$$\int_{a_1 r<|x|<a_2 r} |x|^4|\nabla v|^2 + |x|^2|v|^2 + |x|^2|\nabla u|^2 dx \le C_0 \int_{a_3 r<|x|<a_4 r} |u|^2 dx \quad (2.6)$$

where the constant C_0 is independent of r and u. Here v is defined in (2.2)*.*

The proof of Lemma 2.3 will be given in the next section. Here we would like to outline how to proceed the proofs of main theorems. The detailed arguments can be found in [5] or [6]. Firstly, applying (2.5) to $w = u$, $f = |x|a(x)v$ and using (2.2), we have that

$$\begin{aligned} & \int \varphi_\beta^2 (\log|x|)^{-2}(\beta|x|^{2-n}|\nabla u|^2 + \beta^3|x|^{-n}|u|^2)dx \\ \lesssim & \int \varphi_\beta^2|x|^{-n}[\big(|x|^2\Delta u + |x|\mathrm{div}(|x|a(x)v)\big)^2 + \beta^2\||x|a(x)v\|^2]dx. \end{aligned} \quad (2.7)$$

Next, applying (2.4) to $w = v$, $f = |x|G$ and using (2.3), we get that

$$\begin{aligned} & \int \varphi_\beta^2 (\log|x|)^{2}(\beta|x|^{4-n}|\nabla v|^2 + \beta^3|x|^{2-n}|v|^2)dx \\ \lesssim & \int \varphi_\beta^2(\log|x|)^4|x|^{2-n}[\big(|x|^2\Delta v + |x|\mathrm{div}(|x|G)\big)^2 + \beta^2\||x|G\|^2]dx. \end{aligned} \quad (2.8)$$

Finally, adding $\beta\times$(2.7) and (2.8) together and using (2.6), we can prove Theorem 1.1 and 1.2 as in [5] and [6].

3 Proof of Lemma 2.3

Define $b_1 = (a_1 + a_3)/2$ and $b_2 = (a_2 + a_4)/2$. Let $X = B_{a_4 r}\backslash \bar{B}_{a_3 r}$, $Y = B_{b_2 r}\backslash \bar{B}_{b_1 r}$ and $Z = B_{a_2 r}\backslash \bar{B}_{a_1 r}$. Let $\xi(x) \in C_0^\infty(\mathbb{R}^n)$ satisfy $0 \le \xi(x) \le 1$ and

$$\xi(x) = \begin{cases} 0, & |x| \le a_3 r, \\ 1, & b_1 r < |x| < b_2 r, \\ 0, & |x| \ge a_4 r. \end{cases} \quad (3.1)$$

From (1.2), we have that

$$
\begin{aligned}
0 &= -\int [\mathrm{div}(\mu(\nabla u + (\nabla u)^t)) + \nabla(\lambda \mathrm{div} u) + \rho u] \cdot (\xi^2 \bar{u}) dx \\
&= \int \sum_{ijkl=1}^{n} [\lambda \delta_{ij}\delta_{kl} + \mu(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl})] \partial_{x_l} u_k \partial_{x_j} (\xi^2 \bar{u}_i) dx - \int \rho \xi^2 |u|^2 dx \\
&= \int \xi^2 \sum_{ijkl=1}^{n} [\lambda \delta_{ij}\delta_{kl} + \mu(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl})] \partial_{x_l} u_k \partial_{x_j} \bar{u}_i dx \\
&\quad + \int \sum_{ijkl=1}^{n} \partial_{x_j}(\xi^2) [\lambda \delta_{ij}\delta_{kl} + \mu(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl})] \partial_{x_l} u_k \bar{u}_i dx - \int \rho \xi^2 |u|^2 dx \\
&= I_1 + I_2, \qquad (3.2)
\end{aligned}
$$

where

$$
I_1 = \int \xi^2 [\sum_{ij=1}^{n} \lambda \partial_{x_j} u_j \partial_{x_i} \bar{u}_i + \sum_{ij=1}^{n} \mu (\partial_{x_i} u_j \partial_{x_j} \bar{u}_i + \partial_{x_j} u_i \partial_{x_j} \bar{u}_i)] dx
$$

and

$$
I_2 = \int \sum_{ijkl=1}^{n} \partial_{x_j}(\xi^2) [\lambda \delta_{ij}\delta_{kl} + \mu(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl})] \partial_{x_l} u_k \bar{u}_i dx - \int \rho \xi^2 |u|^2 dx.
$$

Observe that

$$
\begin{aligned}
&\int \xi^2 (2\mu - \frac{\delta_0}{2}) \partial_{x_i} u_j \partial_{x_j} \bar{u}_i dx \\
&= -\int \partial_{x_j} [\xi^2 (2\mu - \frac{\delta_0}{2})] \partial_{x_i} u_j \bar{u}_i dx - \int \xi^2 (2\mu - \frac{\delta_0}{2}) \partial^2_{x_i x_j} u_j \bar{u}_i dx \\
&= -\int \partial_{x_j} [\xi^2 (2\mu - \frac{\delta_0}{2})] \partial_{x_i} u_j \bar{u}_i dx + \int \partial_{x_i} [\xi^2 (2\mu - \frac{\delta_0}{2})] \partial_{x_j} u_j \bar{u}_i dx \\
&\quad + \int \xi^2 (2\mu - \frac{\delta_0}{2}) \partial_{x_j} u_j \partial_{x_i} \bar{u}_i dx. \qquad (3.3)
\end{aligned}
$$

It follows from (3.3) that

$$
\begin{aligned}
I_1 &= \int \xi^2[\sum_{ij=1}^n \lambda \partial_{x_j} u_j \partial_{x_i} \bar{u}_i + \sum_{ij=1}^n (2\mu - \frac{\delta_0}{2})(\partial_{x_i} u_j \partial_{x_j} \bar{u}_i)]dx \\
&\quad + \int \sum_{ij=1}^n \xi^2 (\mu - \frac{\delta_0}{2})(\partial_{x_j} u_i \partial_{x_j} \bar{u}_i - \partial_{x_i} u_j \partial_{x_j} \bar{u}_i)dx \\
&\quad + \frac{\delta_0}{2} \int \sum_{ij=1}^n \xi^2 \partial_{x_j} u_i \partial_{x_j} \bar{u}_i dx \\
&= \int (2\mu + \lambda - \frac{\delta_0}{2}) \xi^2 \sum_{ij=1}^n (\partial_{x_j} u_j \partial_{x_i} \bar{u}_i) dx \\
&\quad + \int \sum_{ij=1}^n \xi^2 (\mu - \frac{\delta_0}{2})(\partial_{x_j} u_i \partial_{x_j} \bar{u}_i - \partial_{x_i} u_j \partial_{x_j} \bar{u}_i)dx \\
&\quad + \frac{\delta_0}{2} \int \sum_{ij=1}^n \xi^2 \partial_{x_j} u_i \partial_{x_j} \bar{u}_i dx + I_3, \qquad (3.4)
\end{aligned}
$$

where

$$
I_3 = \sum_{ij=1}^n \int \partial_{x_i}[\xi^2(2\mu - \frac{\delta_0}{2})]\partial_{x_j} u_j \bar{u}_i - \partial_{x_j}[\xi^2(2\mu - \frac{\delta_0}{2})]\partial_{x_i} u_j \bar{u}_i dx.
$$

Since

$$
\begin{aligned}
&\int \sum_{ij=1}^n \xi^2 (\mu - \frac{\delta_0}{2})(\partial_{x_j} u_i \partial_{x_j} \bar{u}_i - \partial_{x_i} u_j \partial_{x_j} \bar{u}_i)dx \\
= \; &\frac{1}{2} \int \sum_{ij=1}^n \xi^2 (\mu - \frac{\delta_0}{2}) |\partial_{x_j} u_i - \partial_{x_i} u_j|^2 dx,
\end{aligned}
$$

we obtain that

$$
I_1 \geq \frac{\delta_0}{2} \int |\xi \nabla u|^2 dx + I_3. \qquad (3.5)
$$

Combining (3.2) and (3.5), we have that

$$
\int_Y |\nabla u|^2 dx \leq \int_X |\xi \nabla u|^2 dx \leq C_1 \int_X |x|^{-2} |u|^2 dx,
$$

which implies

$$\int_Y |x|^2|\nabla u|^2 dx \le C_2 \int_X |u|^2 dx. \tag{3.6}$$

Here and below all constants $C_1, C_2, \cdots$ depend on δ_0, M_0.

To estimate ∇v, we define $\chi(x) \in C_0^\infty(\mathbb{R}^n)$ satisfy $0 \le \chi(x) \le 1$ and

$$\chi(x) = \begin{cases} 0, & |x| \le b_1 r, \\ 1, & a_1 r < |x| < a_2 r, \\ 0, & |x| \ge b_2 r. \end{cases}$$

By (2.3), we derive that

$$\begin{aligned} & \int |\chi(x)\nabla v|^2 dx \\ = & \int \nabla v \cdot \nabla(\chi^2 \bar{v}) dx - 2\int \chi \nabla v \cdot \bar{v} \nabla \chi dx \\ \le & \ |\int (\mathrm{div} G)\chi^2 \bar{v} dx| + 2\int |\chi \nabla v \cdot \bar{v}\nabla\chi| dx \\ \le & \ |\int (\mathrm{div} G)\chi^2 \bar{v} dx| + \frac{1}{4}\int |\chi\nabla v|^2 dx + C_3 \int_Y |x|^{-2}|v|^2 dx \\ \le & \ C_4 \int_Y |\nabla u|^2 dx + C_4 \int_Y |u|^2 dx + \frac{1}{2}\int |\chi\nabla v|^2 dx + C_4 \int_Y |x|^{-2}|v|^2 dx \\ \le & \ C_5 \int_Y |x|^{-2}|\nabla u|^2 dx + C_4 \int_Y |u|^2 dx + \frac{1}{2}\int |\chi \nabla v|^2 dx. \end{aligned} \tag{3.7}$$

Therefore, we get from (3.7) that

$$\int_Z |\nabla v|^2 dx \le 2C_5 \int_Y |x|^{-2}|\nabla u|^2 dx + 2C_4 \int_Y |u|^2 dx$$

and hence

$$\int_Z |x|^4|\nabla v|^2 dx \le C_6 \int_Y |x|^2|\nabla u|^2 dx + C_6 \int_Y |x|^4|u|^2 dx. \tag{3.8}$$

Putting together $K\times$(3.6) and (3.8), we have that

$$\begin{aligned} & K\int_Y |x|^2|\nabla u|^2 dx + \int_Z |x|^4|\nabla v|^2 dx \\ \le & \ KC_2 \int_X |u|^2 dx + C_6 \int_Y |x|^2|\nabla u|^2 dx + C_6 \int_Y |x|^4|u|^2 dx. \end{aligned} \tag{3.9}$$

Choosing $K = 2C_6$ in (3.9) yields

$$\begin{aligned} & \int_Z |x|^2|v|^2dx + \int_Z |x|^2|\nabla u|^2dx + \int_Z |x|^4|\nabla v|^2dx \\ \leq \;& C_7\int_Y |x|^2|\nabla u|^2dx + C_7\int_Z |x|^4|\nabla v|^2dx \\ \leq \;& C_8\int_X |u|^2dx, \end{aligned}$$

The proof is now complete. □

Acknowledgements

The first and fourth authors were supported in part by the National Science Council of Taiwan. The third author was partially supported by NSF and a Walker Family Endowed Professorship.

References

[1] G. Alessandrini and A. Morassi, *Strong unique continuation for the Lamé system of elasticity*, Comm. in PDE., **26**, 1787-1810, 2001.

[2] L. Escauriaza, *Unique continuation for the system of elasticity in the plane*, Proc. Amer. Math. Soc., **134** (2006), 2015-2018.

[3] K. Miller, *Nonunique continuation for uniformly parabolic and elliptic equations in selfadjoint divergence form with Hölder continuous coefficients*, Bull. AMS, **79** (1973), 350-354.

[4] C.L. Lin and J.N. Wang, *Strong unique continuation for the Lamé system with Lipschitz coefficients*, Math. Ann., **331** (2005), 611-629.

[5] C.L. Lin, G. Nakamura and J.N. Wang *Optimal three-ball inequalities and quantitative uniqueness for the Lamé system with Lipschitz coefficients*, to appear in Duke Math Journal.

[6] C.L. Lin, G. Uhlmann, and J.N. Wang, *Optimal three-ball inequalities and quantitative uniqueness for the Stokes system*, to appear in DCDS-A (Special issue dedicated to Professor Louis Nirenberg on the occasion of his 85th birthday.)

[7] A. Plis, *On non-uniqueness in Cauchy problem for an elliptic second order differential equation*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys., **11** (1963), 95-100.