Local analytic regularity in the linearized Calderón problem

Johannes Sjöstrand*
Institut de Mathématiques de Bourgogne, Université de Bourgogne
9 avenue Alain Savary - BP 47870
21078 Dijon cedex
johannes.sjostrand@u-bourgogne.fr
and UMR 5584 du CNRS

Gunther Uhlmann†
Department of Mathematics, University of Washington
Box 354350
Seattle, WA, 98195
gunther@math.washington.edu

Abstract

We consider the linearization of the Dirichlet-to-Neumann (DN) map as a function of the potential. We show that it is injective at a real analytic potential for measurements made at an open subset of analyticity of the boundary. More generally, we relate the analyticity up to the boundary of the variations of the potential to the analyticity of the symbols of the corresponding variations of the DN-map.

Résumé

Nous considérons la linéarisée de l'application de Dirichlet–Neumann (DN) comme fonction du potentiel en un point donné par un potentiel analytique. Nous montrons qu'elle est injective pour des mésures faites dans un ouvert où le bord est analytique. Plus généralement, nous lions l'analyticité jusqu'au bord des variations infinitésimales du potentiel à celle des symboles des variations correspondantes le l'application DN.

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Contents

1 Introduction

In this paper we consider the *linearized* Calderón problem with local partial data and related problems. We review first briefly Calderón's problem including the case of partial data. For a more complete review see [21].

Calderón's problem is, roughly speaking, the question of whether one can determine the electrical conductivity of a medium by making voltage and current measurements at the boundary of the medium. This inverse method is also called Electrical Impedance Tomography (EIT). We describe the problem more precisely below.

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with smooth boundary. The electrical conductivity of Ω is represented by a bounded and positive function $\gamma(x)$. In the absence of sinks or sources of current the equation for the potential is given by

$$\nabla \cdot (\gamma \nabla u) = 0 \text{ in } \Omega \tag{1.1}$$

since, by Ohm's law, $\gamma \nabla u$ represents the current flux. Given a potential $f \in H^{\frac{1}{2}}(\partial\Omega)$ on the boundary the induced potential $u \in H^1(\Omega)$ solves the Dirichlet problem

$$\begin{aligned} \nabla \cdot (\gamma \nabla u) &= 0 \text{ in } \Omega, \\ u\big|_{\partial\Omega} &= f. \end{aligned} \tag{1.2}$$

The Dirichlet to Neumann (DN) map, or voltage to current map, is given by

$$\Lambda_\gamma(f) = \left(\gamma \frac{\partial u}{\partial \nu}\right)\Big|_{\partial\Omega} \tag{1.3}$$

where ν denotes the unit outer normal to $\partial\Omega$. The inverse problem is to determine γ knowing Λ_γ.

The local Calderón problem, or the Calderón problem with partial data, is the question of whether one can determine the conductivity by measuring the DN map on subsets of the boundary for voltages supported in subsets of the boundary. In this paper we consider the case when the support of the voltages and the induced current fluxes are measured in the same open subset Γ. More conditions on this open set will be stated later. If $\gamma \in C^\infty(\overline{\Omega})$ the DN map is a classical pseudodifferential operator of order 1. It was shown in [19] that its full symbol computed in boundary normal coordinates near a point of Γ determines the Taylor series of γ at the point giving another proof of the result of Kohn and Vogelius [11]. In particular this shows that real-analytic conductivities can be determined by the local DN map. This result was generalized in [14] to the case of anisotropic conductivities using a factorization method related to the methods of this paper. Interior determination was shown in dimension $n \geq 3$ for C^2 conductivities [20]. This was extended to C^1 conductivities in [5]. In two dimensions uniqueness was proven for C^2 conductivities in [16] and for merely L^∞ conductivities in [1]. The case of partial data in dimension $n \geq 3$ was considered in [2], [15], [8], [9], [7]. The two dimensional case was solved in [6]. See [10] for a review. However it is not known at the present whether one can determine uniquely the conductivity if one measures the DN map on an arbitrarily open subset of the boundary applied to functions supported in the same set. We refer to these type of measurements as the local DN map.

The $\gamma \to \Lambda_\gamma$ is not linear. In this paper we consider the linearization of the partial data problem at a real-analytic conductivity for real-analytic Γ. We prove that the linearized map is injective. In fact we prove a more general statement, see Theorem 1.6.)

As in many works on Calderón's problem one can reduce the problem to a similar one for the Schrödinger equation (see for instance [21]). This result uses that one can determine from the DN map the conductivity and the normal derivative of the conductivity. This result is only valid for the local DN map. One can then consider the more general problem of determining a potential from the corresponding DN map. The same is valid for the case of partial data and the linearization. It was shown in [4] that the linearization of the local DN map at the 0 potential is injective. We consider the linearization of the local DN map at any real analytic potential assuming that the local DN map is measured an an open real-analytic set. We now describe more precisely our results in this setting.

Consider the Schrödinger operator $P = \Delta - V$ on the open set $\Omega \Subset \mathbf{R}^n$ where the boundary $\partial\Omega$ is smooth (and later assumed to be analytic in the most interesting region). Assume that 0 is not in the spectrum of the Dirichlet realization of P. Let G and K denote the corresponding Green and Poisson operators. Let $\gamma : C^\infty(\overline{\Omega}) \to C^\infty(\partial\Omega)$ be the restriction operator, ν the exterior normal. If $x_0 \in \partial\Omega$, we can choose local coordinates $y = (y_1, ..., y_n)$, centered at x_0 so that Ω is given by $y_n > 0$, $\partial_\nu = -\partial_{y_n}$. If $\partial\Omega$ is analytic near x_0, we can choose the coordinates to be analytic.

The Dirichlet to Neumann (DN) operator is

$$\Lambda = \gamma\partial_\nu(x, \partial_x)K. \tag{1.4}$$

Consider a smooth deformation of smooth potentials, possibly complex-valued,

$$\begin{aligned} &\operatorname{neigh}(0, \mathbf{R}) \ni t \mapsto P_t = \Delta - V_t, \\ &V_t(x) = V(t, x) \in C^\infty(\operatorname{neigh}(0, \mathbf{R}) \times \overline{\Omega}; \mathbf{R}). \end{aligned} \tag{1.5}$$

Let G_t, K_t be the Green and Poisson kernels for P_t, so that

$$\begin{pmatrix} P_t \\ \gamma \end{pmatrix} : \ C^\infty(\overline{\Omega}) \to C^\infty(\overline{\Omega}) \times C^\infty(\partial\Omega)$$

has the inverse

$$\begin{pmatrix} G_t & K_t \end{pmatrix}.$$

Then, denoting t-derivatives by dots,

$$\begin{pmatrix} \dot{G}_t & \dot{K}_t \end{pmatrix} = -\begin{pmatrix} G_t & K_t \end{pmatrix} \begin{pmatrix} \dot{P}_t \\ 0 \end{pmatrix} \begin{pmatrix} G_t & K_t \end{pmatrix} = -\begin{pmatrix} G_t\dot{P}_tG_t & G_t\dot{P}_tK_t \end{pmatrix}$$

that is,

$$\dot{G} = -G\dot{P}G, \quad \dot{K} = -G\dot{P}K, \tag{1.6}$$

and consequently,

$$\dot{\Lambda} = -\gamma\partial_\nu G\dot{P}K. \tag{1.7}$$

Using the Green formula, we see that

$$\gamma\partial_\nu G = K^{\mathrm{t}}, \tag{1.8}$$

where K^{t} denotes the transposed operator.

In fact, write the Green formula,

$$\int_\Omega ((Pu_1)u_2 - u_1Pu_2)dx = \int_{\partial\Omega} (\partial_\nu u_1 u_2 - u_1\partial_\nu u_2)S(dx)$$

and put $u_1 = Gv$, $u_2 = Kw$ for $v \in C^\infty(\overline{\Omega})$, $w \in C^\infty(\partial\Omega)$

$$\int_\Omega vKwdx = \int_{\partial\Omega} (\gamma\partial_\nu Gv)wS(dx)$$

and (1.8) follows.

(1.7) becomes

$$\dot{\Lambda} = -K^{\rm t}\dot{P}K = K^{\rm t}\dot{V}K. \tag{1.9}$$

The linearized Calderón problem is the following: If $V_t = V + tq$, determine q from $\dot{\Lambda}_{t=0}$. The corresponding partial data problem is to recover q or some information about q from local information about $\dot{\Lambda}_{t=0}$. From now on, we restrict the attention to $t = 0$. In this paper we shall study the following linearized baby problem: Assume that V and $\partial\Omega$ are analytic near some point $x_0 \in \partial\Omega$. We also assume that V is smooth. If $\dot{\Lambda}$ (for $t = 0$) is an analytic pseudodifferential operator near x_0, can we conclude that q is analytic near x_0? Here,

$$\dot{\Lambda} = K^{\rm t}qK \tag{1.10}$$

and we shall view the right hand side as a Fourier integral operator acting on q.

Actually this problem is overdetermined in the sense that the symbol of a pseudodifferential operator on the boundary is a function of $2(n-1)$ variables, while q is a function of n variables and we have $2(n-1) \geq n$ for $n \geq 2$ with equality precisely for $n = 2$. In order to have a non-overdetermined problem we shall only consider the symbol $\sigma_{\dot{\Lambda}}(y', \eta')$ of $\dot{\Lambda}$ along a half-ray in η', i.e. we look at $\sigma_{\dot{\Lambda}}(y', t\eta'_0)$ for some fixed $\eta'_0 \neq 0$ and for some local coordinates as above. Assuming this restricted symbol to be a classical analytic symbol near $y' = 0$ and the potential $V = V_0$ to be analytic near $y = 0$ (i.e. near x_0), we shall show that q is real-analytic up to the boundary near x_0 (corresponding to $y = 0$).

To formulate the result more precisely, we first make some remarks about the analytic singular support of the Schwartz kernels of K and $K^{\rm t}qK$, then we recall the notion of classical analytic pseudodifferential operators. Assume that $W \subset \mathbf{R}^n$ is an open neighborhood of $x_0 \in \partial\mathcal{O}$ and that

$$\partial\Omega \text{ and } V \text{ are analytic in } W. \tag{1.11}$$

For simplicity we shall use the same symbol to denote operators and their Schwartz kernels. Then we have

Lemma 1.1 *The Schwartz kernel $K(x, y')$ is analytic with respect to y', locally uniformly on the set*

$$\{(x, y') \in \overline{\Omega} \times (\partial\Omega \cap W);\ x \neq y'\}.$$

Proof. Using (1.8) we can write $K(x, y') = \gamma\partial_\nu u(y')$, where $u = G(x, .)$ solves the Dirichlet problem

$$(\Delta - V)u = \delta(\cdot - x),\ \gamma u = 0,$$

and from analytic regularity for elliptic boundary value problems, we get the lemma. (When $x \in \partial\Omega$, we view $G(x, y)$ away from $y = x$ as the limit of $G(x_j, y)$ when $\Omega \ni x_j \to x$.) □

Lemma 1.2 *The Schwartz kernel* $(K^{\mathrm{t}}qK)(x', y')$ *is analytic on the set*

$$\{(x', y') \in (\partial\Omega \cap W)^2;\ x' \neq y'\}. \tag{1.12}$$

Proof. Let (x'_0, y'_0) belong to the set (1.12). After decomposing q into a sum of two terms we may assume that $x'_0 \notin \mathrm{supp}\,(q)$ or that $y'_0 \notin \mathrm{supp}\,(q)$. In the first case it follows from Lemma 1.1, that $(K^{\mathrm{t}}qK)(x', y')$ is analytic in x' uniformly for (x', y') in a neighborhood of (x'_0, y'_0) and since the kernel is symmetric we can exchange the roles of x', y' and conclude that $(K^{\mathrm{t}}qK)(x', y')$ is analytic in y' uniformly for (x', y') in a neighborhood of (x'_0, y'_0). In the second case we have the same conclusion about analyticity in x' and in y' separately. It then follows that $(KqK)(x', y')$ is analytic near (x'_0, y'_0) (by using the FBI-definition [17] of the analytic wave-front set and can also (most likely) be deduced from a classical result on logarithmic convexity of Reinhardt domains [13], Theorem 2.4.6). □

Remark 1.3 *By the same proof,* $K^{\mathrm{t}}qK(x', y')$ *is analytic near*

$$\{(x', x') \in (\partial\Omega \cap W)^2;\ x' \notin \mathrm{supp}\, q\}.$$

We next define the notion of symbol up to exponentially small contributions. For that purpose we assume that X is an analytic manifold and consider an operator

$$A:\ C_0^\infty(X) \to C^\infty(X) \tag{1.13}$$

which is also continuous

$$\mathcal{E}'(X) \to \mathcal{D}'(X). \tag{1.14}$$

Assume (as we have verified for $K^{\mathrm{t}}qK$ with n replaced by $n-1$ and with $X = \partial\Omega \cap W$) that the distribution kernel $A(x, y)$ is analytic away from the diagonal. After restriction to a local analytic coordinate chart, we may assume that $X \subset \mathbf{R}^n$ is an open set. The symbol of A is formally given on T^*X by

$$\sigma_A(x, \xi) = e^{-ix\cdot\xi} A(e^{i(\cdot)\cdot\xi}) = \int e^{-i(x-y)\cdot\xi} A(x, y)dy.$$

In the usual case of C^∞-theory, we give a meaning to this symbol up to $\mathcal{O}(\langle\xi\rangle^{-\infty})$ by introducing a cutoff $\chi(x,y) \in C^\infty(X \times X)$ which is properly supported and equal to 1 near the diagonal. In the analytic category we would like to have an exponentially small undeterminacy, and the use of special cut-offs becoming more complicated, we prefer to make a contour deformation.

For x in a compact subset of X, let $r > 0$ be small enough and define for $\xi \neq 0$,

$$\sigma_A(x,\xi) = \int_{x+\Gamma_{r,\xi}} e^{i(y-x)\cdot\xi} A(x,y)dy, \tag{1.15}$$

where

$$\Gamma_{r,\xi} : B(0,r) \ni t \mapsto t + i\chi\left(\frac{t}{r}\right) r\frac{\xi}{|\xi|} \in \mathbf{C}^n$$

and $\chi \in C^\infty(B(0,1);[0,1])$ is a radial function which vanishes on $B(0,1/2)$ and is equal to 1 near $\partial B(0,1)$. Thus the contour $x + \Gamma_{r,\xi}$ coincides with $\mathbf{R}^n$ near $y = x$ and becomes complex for t close to the boundary of $B(0,r)$. Along this contour,

$$|e^{i(y-x)\cdot\xi}| = e^{-\chi(t/r)r|\xi|}$$

is bounded by 1 and for t close to $\partial B(0,r)$ it is exponentially decaying in $|\xi|$. Thus from Stokes' formula it is clear that $\sigma_A(x,\xi)$ will change only by an exponentially small term if we modify r. More generally, for (x,ξ) in a conic neighborhood of a fixed point $(x_0,\xi_0) \in X \times S^{n-1}$ we change $\sigma_A(x,\xi)$ only by an exponentially small term if we replace the contour in (1.15) by $x_0 + \Gamma_{r,\xi_0}$ and we then get a function which has a holomorphic extension to a conic neighborhood of (x_0,ξ_0) in $\mathbf{C}^n \times (\mathbf{C}^n \setminus \{0\})$.

Remark 1.4 *Instead of using contour deformation to define σ_A, we can use an almost analytic cut-off in the following way. Choose $C > 0$ so that*

$$1 = \int Ch^{-\frac{n}{2}} e^{-\frac{(y-t)^2}{2h}} dt,$$

and put

$$e_t(y) = \widetilde{\chi}(y-t)Ch^{-\frac{n}{2}} e^{-\frac{(y-t)^2}{2h}},$$

where $\widetilde{\chi} \in C_0^\infty(\mathbf{R}^n)$ is equal to 1 near 0 and has its support in a small neighborhood of that point. Then if χ is another cut-off of the same type, we see by contour deformation that

$$\sigma_A(x,\xi) = e^{-ix\cdot\xi} A(\int \chi(t-x) e_t e^{i(\cdot)\cdot\xi} dt)$$

up to an exponentially decreasing term.

In the following definition we recall the notion of analytic symbols in the sense of L. Boutet de Monvel and P. Krée [3]. We will avoid to use the corresponding notion of analytic pseudodifferential operator, since it involves some special facts about the distribution kernel of the operator that will not be needed.

Definition 1.5 *We say that σ_A is a classical analytic symbol (cl.a.s.) of order m on $X \times \mathbf{R}^n$ if the following holds:*

There exist holomorphic functions $p_{m-j}(x,\xi)$ on a fixed complex conic neighborhood V of $X \times \dot{\mathbf{R}}^n$ such that

$$p_k(x,\xi) \text{ is positively homogeneous of degree } k \text{ in } \xi, \tag{1.16}$$

$$\begin{aligned} &\forall K \Subset V \cap \{(x,\xi);\, |\xi| = 1\},\ \exists C = C_K \text{ such that} \\ &|p_{m-j}(x,\xi)| \le C^{j+1} j^j, \text{ on } K. \end{aligned} \tag{1.17}$$

$$\begin{aligned} &\forall K \Subset X, \text{ and every } C_1 > 0, \text{ large enough, } \exists C_2 > 0, \\ &\text{such that } |\sigma_A(x,\xi) - \sum_{0 \le j \le |\xi|/C_1} p_{m-j}(x,\xi)| \le C_2 e^{-|\xi|/C_2}, \\ &(x,\xi) \in K \times \mathbf{R}^n,\ |\xi| \ge 1. \end{aligned} \tag{1.18}$$

The formal sum $\sum_0^\infty p_{m-j}(x,\xi)$ is called a formal cl.a.s. when (1.16), (1.17) hold. We define cl.a.s. and formal cl.a.s. on open conic subsets of $X \times \dot{\mathbf{R}}^n$ and on other similar sets by the obvious modifications of the above definitions. If $p(x,\xi)$ is a cl.a.s. on $X \times \mathbf{R}^n$ and if $\xi_0 \in \dot{\mathbf{R}}^n$, then

$$q(x,\tau) := p(x,\tau\xi_0)$$

is a cl.a.s. on $X \times \mathbf{R}_+$.

The main result of this work is

Theorem 1.6 *Let $x_0 \in \partial\Omega$ and assume that $\partial\Omega$ and V are analytic near that point. Let $q \in L^\infty(\Omega)$. Choose local analytic coordinates $y' = (y_1, ..., y_{n-1})$ on* neigh $(x_0, \partial\Omega)$, *centered at x_0, so that the symbol $\sigma_{\dot\Lambda}(y',\eta')$ becomes well-defined up to an exponentially small term on* neigh $(0) \times \dot{\mathbf{R}}^{n-1}$. *Let $\eta_0' \in \dot{\mathbf{R}}^{n-1}$.*

If $\sigma_{\dot\Lambda}(y',\tau\eta_0')$ is a cl.a.s. on neigh $(0, \mathbf{R}^{n-1}) \times \mathbf{R}_+$, *then q is analytic up to the boundary in a neighborhood of x_0.*

We have a simpler direct result.

Proposition 1.7 *Let x_0, $\partial\Omega$, V be as in Theorem 1.6 and choose analytic coordinates as there. If $q \in L^\infty(\Omega)$ is analytic up to the boundary near x_0, then $\sigma_{\dot\Lambda}$ is a cl.a.s. near $y' = 0$.*

We get the following immediate consequence

Corollary 1.8 *Under the conditions of the previous theorem the map*

$$q \to \dot{\Lambda}$$

is injective.

This follows from the previous result since q must be analytic on W and the Taylor series of q vanishes on W then $q = 0$ on the set where q is analytic.

Most of the paper will be devoted to the proof of Theorem 1.6, and in Section 7 we will prove Proposition 1.7.

2 Heuristics and some remarks about the Laplace transform

Let us first explain heuristically why some kind of Laplace transform will appear. Assume that $x_0 \in \partial\Omega$ and that V and $\partial\Omega$ are analytic near that point. Choose local analytic coordinates

$$y = (y_1, ..., y_{n-1}, y_n) = (y', y_n)$$

centered at x_0 such that the set Ω coincides near x_0 (i.e. $y = 0$) with the half-space $\mathbf{R}^n_+ = \{y \in \mathbf{R}^n;\, y_n > 0\}$. Assume also (for this heuristic discussion) that we know that $q(y) = q(y', y_n)$ is analytic in y' and that the original Laplace operator remains the standard Laplace operator also in the y-coordinates. Then up to a smoothing operator, the Poisson operator is of the form

$$Ku(y) = \frac{1}{(2\pi)^{n-1}} \int e^{i(y'-w')\cdot\eta' - y_n|\eta'|} a(y, \eta') u(w') dw' d\eta',$$

where the symbol a is equal to 1 to leading order. We can view K, q, K^{t} as pseudodifferential operators in y' with operator valued symbols. K has the operator valued symbol

$$K(y', \eta') : \mathbf{C} \ni z \mapsto z e^{-y_n|\eta'|} a(y, \eta') \in L^2([0, +\infty[_{y_n}). \tag{2.1}$$

The symbol of multiplication with q is independent of η' and equals multiplication with $q(y', \cdot)$. The symbol of K^{t} is to leading order

$$K^{\mathrm{t}}(y', \eta') :\ L^2([0, +\infty[_{y_n}) \ni f(y_n) \mapsto \int_0^\infty e^{-y_n|\eta'|} a(y, -\eta') f(y_n) dy_n \in \mathbf{C}. \tag{2.2}$$

For simplicity we set $a = 1$ in the following discussion. To leading order the symbol of $\dot{\Lambda}$ is

$$\sigma_{\dot{\Lambda}}(y', \eta') = \int_0^\infty e^{-2y_n|\eta'|} q(y', y_n) dy_n = (\mathcal{L}q(y', \cdot))(2|\eta'|), \tag{2.3}$$

where

$$\mathcal{L}f(\tau) = \int_0^\infty e^{-t\tau} f(t) dt$$

is the Laplace transform.

Now we fix $\eta_0' \in \dot{\mathbf{R}}^{n-1}$ and assume that $\sigma_{\dot{\Lambda}}(y', \tau\eta_0')$ is a cl.a.s. on neigh $(0, \mathbf{R}^{n-1}) \times \mathbf{R}_+$,

$$\sigma_{\dot{\Lambda}}(y', \tau\eta_0') \sim \sum_1^\infty n_k(y', \tau), \tag{2.4}$$

where n_k is analytic in y' in a fixed complex neighborhood of 0, (positively) homogeneous of degree $-k$ in τ and satisfying

$$|n_k(y', \tau)| \leq C^{k+1} k^k |\tau|^{-k}. \tag{2.5}$$

More precisely for $C > 0$ large enough, there exists $\widetilde{C} > 0$ such that

$$|\sigma_{\dot{\Lambda}}(y', \tau\eta_0') - \sum_1^{[|\eta'|/C]} n_k(y', \tau)| \leq \widetilde{C} \exp(-\tau/\widetilde{C}) \tag{2.6}$$

on the real domain.

From (2.3) we also have

$$|(\mathcal{L}q(y', \cdot))(2|\eta_0'|\tau) - \sum_1^{[|\eta'|/C]} n_k(y', \tau)| \leq \exp(-\tau/\widetilde{C}), \tag{2.7}$$

for $y' \in$ neigh $(0, \mathbf{R}^{n-1})$, $\tau \geq 1$. In this heuristic discussion we assume that (2.7) extends to $y' \in$ neigh $(0, \mathbf{C}^{n-1})$. It then follows that $q(y', y_n)$ is analytic for y_n in a neighborhood of 0, from the following certainly classic result about Borel transforms.

Proposition 2.1 *Let $q \in L^\infty([0, 1])$ and assume that for some $C, \widetilde{C} > 0$,*

$$|\mathcal{L}q(\tau) - \sum_0^{[\tau/C]} q_k \tau^{-(k+1)}| \leq e^{-\tau/\widetilde{C}}, \ \tau > 0, \tag{2.8}$$

$$|q_k| \leq \widetilde{C}^{k+1} k^k. \tag{2.9}$$

Then q is analytic in a neighborhood of $t = 0$. The converse also holds.

Proof. We shall first show the converse statement, namely that if q is analytic near $t = 0$, then (2.8), (2.9) hold. We start by computing the Laplace transform of powers of t.

For $\tau > 0$, $a > 0$, $k \in \mathbf{N}$, we have

$$\int_0^\infty e^{-t\tau} t^k dt = \frac{k!}{\tau^{k+1}}. \tag{2.10}$$

In fact, the integral to the left is equal to

$$(-\partial_\tau)^k \left(\int_0^\infty e^{-t\tau} dt \right) = (-\partial_\tau)^k \left(\frac{1}{\tau} \right).$$

Next, for $a > 0$, we look at

$$\begin{aligned} \frac{1}{k!} \int_0^a e^{-t\tau} t^k dt &= \frac{1}{\tau^{k+1}} \left(1 - \frac{\tau^{k+1}}{k!} \int_a^\infty e^{-t\tau} t^k dt \right) \\ &= \frac{1}{\tau^{k+1}} \left(1 - \int_{a\tau}^\infty e^{-s} \frac{s^k}{k!} ds \right). \end{aligned} \tag{2.11}$$

Let first $\tau \in]0, \infty[$ be large. For $0 < \theta < 1$, we write for $s \geq 0$,

$$\frac{s^k}{k!} e^{-s} = \theta^{-k} \underbrace{\frac{(\theta s)^k}{k!} e^{-\theta s}}_{\leq 1} e^{-(1-\theta)s} \leq \theta^{-k} e^{-(1-\theta)s}.$$

Thus,

$$\int_{a\tau}^\infty e^{-s} \frac{s^k}{k!} ds \leq \theta^{-k} \int_{a\tau}^\infty e^{-(1-\theta)s} ds = \frac{\theta^{-k} e^{-(1-\theta)a\tau}}{1 - \theta}. \tag{2.12}$$

We will estimate this for $k \leq a\tau/\mathcal{O}(1)$. Under the apriori assumption that $\theta \leq 1 - \frac{1}{\mathcal{O}(1)}$, we look for θ that minimizes the enumerator

$$\theta^{-k} e^{-(1-\theta)a\tau} = e^{-[(1-\theta)a\tau + k \ln \theta]}.$$

Setting the derivative of the exponent equal to zero, we are led to the choice $\theta = \frac{k}{a\tau}$. Assume that

$$\frac{k}{a\tau} \leq \theta_0 < 1. \tag{2.13}$$

Then,

$$(1-\theta)a\tau + k \ln \theta = a\tau \left(1 - \frac{k}{a\tau} + \frac{k}{a\tau} \ln \frac{k}{a\tau} \right) = a\tau (1 - f(\frac{k}{a\tau})),$$

where

$$f(x) = x + x\ln\frac{1}{x},\ \ 0 \le x \le 1.$$

Clearly $f(0) = 0$, $f(1) = 1$ and for $0 < x < 1$ we have $f'(x) = \ln\frac{1}{x} > 0$, so f is strictly increasing on $[0,1]$. In view of (2.13) we have

$$(1-\theta)a\tau + k\ln\theta \ge a\tau(1 - f(\theta_0)),$$

and (2.12) gives

$$\int_{a\tau}^{\infty} e^{-s}\frac{s^k}{k!}ds \le \frac{e^{-a\tau(1-f(\theta_0))}}{1-\theta_0}. \tag{2.14}$$

Using this in (2.11), we get

$$\begin{aligned} &\frac{1}{k!}\int_0^a e^{-t\tau}t^k dt = \frac{1}{\tau^{k+1}}(1 + \mathcal{O}(1)e^{-a\tau/C(\theta_0)}),\\ &\text{for } \frac{k}{a\tau} \le \theta_0 < 1,\ \text{where } C(\theta_0) > 0. \end{aligned} \tag{2.15}$$

Now, assume that $q \in C([0,1])$ is analytic near $t = 0$. Then for $t \in [0, 2a]$, $0 < a \ll 1$, we have

$$q(t) = \sum_0^\infty \frac{q^{(k)}(0)}{k!}t^k,$$

where

$$\frac{|q^{(k)}(0)|}{k!} \le \widetilde{C}\frac{1}{(2a)^k}, \tag{2.16}$$

so

$$|q(t) - \sum_0^{[\tau/C]}\frac{q^{(k)}(0)}{k!}t^k| \le \widetilde{C}e^{-\tau/\widetilde{C}},\ \ 0 \le t \le a.$$

Hence,

$$\mathcal{L}q = \sum_0^{[\tau/C]}\frac{q^{(k)}(0)}{\tau^{k+1}} + \mathcal{O}(e^{-\tau/\widetilde{C}}) + \underbrace{\mathcal{L}(1_{[a,1]}q)(\tau)}_{=\mathcal{O}(e^{-\tau/\widetilde{C}})}$$

and we obtain (2.8) with $q_k = q^{(k)}(0)$, while (2.9) follows from (2.16).

We now prove the direct statement in the proposition, so we take $q \in L^\infty([0,1])$ satisfying (2.8), (2.9). Put for $a > 0$ small,

$$\widetilde{q}(t) = q(t) - 1_{[0,a]}(t)\sum_0^\infty \frac{q_k}{k!}t^k.$$

The proof of the converse part shows that

$$|\mathcal{L}\widetilde{q}(\tau)| \le e^{-\tau/\widetilde{C}}, \tag{2.17}$$

where $\widetilde{C}$ is a new positive constant, and it suffices to show that

$$\widetilde{q} \text{ vanishes in a neighborhood of } 0. \tag{2.18}$$

We notice that $\mathcal{L}\widetilde{q}$ is a bounded holomorphic function in the right half-plane. We can therefore apply the Phragmén-Lindelöf theorem in each sector $\arg\tau \in [0, \frac{\pi}{2}]$ and $\arg\tau \in [-\frac{\pi}{2}, 0]$ to the holomorphic function

$$e^{\tau/\widetilde{C}}\mathcal{L}\widetilde{q}(\tau)$$

and conclude that this function is bounded in the right half-plane:

$$|\mathcal{L}\widetilde{q}(\tau)| \le \mathcal{O}(1)e^{-\Re\tau/\widetilde{C}},\ \Re\tau \ge 0. \tag{2.19}$$

Now, $\mathcal{L}\widetilde{q}(i\sigma) = \mathcal{F}\widetilde{q}(\sigma)$, where $\mathcal{F}$ denotes the Fourier transform, and Paley-Wiener's theorem allows us to conclude that $\operatorname{supp}\widetilde{q} \subset [\frac{1}{\widetilde{C}}, 1]$. $\square$

3 The Fourier integral operator $q \mapsto \sigma_{\dot\Lambda}$

Assume that $\partial\Omega$ and V are analytic near the boundary point x_0. Let $y' = (y_1, ..., y_{n-1})$ be local analytic coordinates on $\partial\Omega$, centered at x_0. Then we can extend y' to analytic coordinates $y = (y_1, ..., y_{n-1}, y_n) = (y', y_n)$ in a full neighborhood of x_0, where y' are extensions of the given coordinates on the boundary and such that Ω is given (near x_0) by $y_n > 0$ and

$$-P = D_{y_n}^2 + R(y, D_{y'}), \tag{3.1}$$

where R is a second order elliptic differential operator in y' with positive principal symbol $r(y, \eta')$. (Here we neglect a contribution $f(y)\partial_{y_n}$ which can be eliminated by conjugation.) Then there is a neighborhood $W \subset \mathbf{R}^n$ of $y = 0$ and a cl.a.s. $a(y, \xi')$ on $W \times \mathbf{R}^{n-1}$ of order 0 such that

$$Ku(y) = \frac{1}{(2\pi)^{n-1}} \iint e^{i(\phi(y,\xi') - \widetilde{y}'\cdot\xi')} a(y, \xi')u(\widetilde{y}')d\widetilde{y}'d\xi' + K_a u(y), \tag{3.2}$$

for $y \in W$, $u \in C_0^\infty(W \cap \partial\Omega)$. The distribution kernel of K_a is analytic on $W \times (W \cap \partial\Omega)$ and we choose a realization of a which is analytic in y. ϕ is the solution of the Hamilton-Jacobi problem

$$\begin{aligned} &(\partial_{y_n}\phi)^2 + r(y, \phi'_{y'}) = 0,\ \Im\partial_{y_n}\phi > 0, \\ &\phi(y', 0, \xi') = y' \cdot \xi'. \end{aligned} \tag{3.3}$$

This means that we choose ϕ to be the solution of

$$\partial_{y_n}\phi - ir(y,\phi'_{y'})^{1/2} = 0, \tag{3.4}$$

with the natural branch of $r^{1/2}$ with a cut along the real negative axis.

To see this, recall (by the analytic WKB-method, cf. [17], Ch. 9) that we can construct the first term $K_{\mathrm{fop}}u$ in the right hand side of (3.2) such that PK_{fop} has analytic distribution kernel and $\gamma K_{\mathrm{fop}} = 1$. It then follows from local analytic regularity in elliptic boundary value problems, that the remainder operator K_a has analytic distribution kernel.

We notice that

$$K(e^{ix'\cdot\xi'}) = e^{i\phi(y,\xi')}a(y,\xi') + \mathcal{O}(e^{-|\xi'|/C}), \tag{3.5}$$

since the first term to the right solves the problem

$$Pu = 0, \quad u_{|_{y_n=0}} = e^{iy'\cdot\xi'}$$

with an exponentially small error in the first equation. When V is real, then K is real, so $K(e^{ix'\cdot(-\xi')}) = \overline{K(e^{ix'\cdot\xi'})}$. It follows that

$$\phi(y,-\xi') = -\overline{\phi(y,\xi')},\ a(y,-\xi') = \overline{a(y,\xi')} \tag{3.6}$$

without any error in the last equation when viewing a as a formal cl.a.s. Now ϕ and the leading homogeneous term a_0 in a are independent of V, so if we drop the reality assumption on V, the first part of (3.6) remains valid and the second part is valid to leading order.

We shall now view $\dot{\Lambda} = K^{\mathrm{t}}qK = K^*qK$ as a pseudodifferential operator in the classical quantization. In this section we proceed formally in order to study the associated geometry. A more efficient analytic description will be given later for the left composition with an FBI-transform in x'. The symbol becomes

$$\begin{aligned}&\sigma_{\dot{\Lambda}}(x',\xi') = e^{-ix'\cdot\xi'}\dot{\mathcal{N}}(e^{i(\cdot)\cdot\xi'})\\ &= (2\pi)^{1-n}\iint e^{i(x'\cdot(\eta'-\xi')-\phi^*(y,\eta')+\phi(y,\xi'))}a(y,-\eta')a(y,\xi')q(y)dyd\eta',\end{aligned}$$

where in general we write $f^*(z) = \overline{f(\overline{z})}$ for the holomorphic extension of the complex conjugate of a function f. Here we use (cf. (3.6)) that

$$K^{\mathrm{t}}v(x') = (2\pi)^{1-n}\iint e^{i(x'\cdot\eta'-\phi^*(y,\eta'))}a(y,-\eta')v(y)dyd\eta'.$$

Actually, rather than letting ξ' tend to ∞ we replace ξ' with ξ'/h where the new ξ' is of length $\asymp 1$ and $h \to 0$. This amounts to viewing $\dot{\Lambda}$ as a semi-classical pseudodifferential operator with semi-classical symbol $\sigma_{\dot{\Lambda}}(x', \xi'; h) = \sigma_{\dot{\Lambda}}(x', \xi'/h)$. Thus,

$$\begin{aligned}
&\sigma_{\dot{\Lambda}}(x', \xi'; h) = e^{-ix'\cdot\xi'/h}\dot{\Lambda}(e^{i(\cdot)\cdot\xi'/h}) \\
&= (2\pi h)^{1-n} \iint e^{\frac{i}{h}(x'\cdot(\eta'-\xi')-\phi^*(y,\eta')+\phi(y,\xi'))} a(y, -\eta'; h) a(y, \xi'; h) q(y) dy d\eta',
\end{aligned}$$

where $a(y, \xi'; h) = a(y, \xi'/h)$.

We have

$$\phi(y, \xi') = y' \cdot \xi' + \psi(y, \xi'),\ \ \phi^*(y, \eta') = y' \cdot \eta' + \psi^*(y, \eta'), \tag{3.7}$$

where

$$\Im\psi \asymp y_n,\ \ \Re\psi = \mathcal{O}(y_n^2), \tag{3.8}$$

uniformly on every compact set which does not intersect the zero section. (3.6) tells us that $\Re\psi$ is odd and $\Im\psi$ is even with respect to the fiber variables ξ' (and also positively homogeneous of degree 1 of course). Using (3.7) in the formula for the symbol of $\dot{\Lambda}$, we get

$$\begin{aligned}
&\sigma_{\dot{\Lambda}}(x', \xi'; h) \\
&= (2\pi h)^{1-n} \iint e^{\frac{i}{h}\Phi_M(x',\xi',y,\eta')} a(y, -\eta'; h) a(y, \xi'; h) q(y) dy d\eta' \\
&=: Mq(x', \xi'; h),
\end{aligned} \tag{3.9}$$

where

$$\Phi_M(x', \xi', y, \eta') = (x' - y') \cdot (\eta' - \xi') + \psi(y, \xi') - \psi^*(y, \eta'), \tag{3.10}$$

and η' are the fiber variables. We shall see that this is a nondegenerate phase function in the sense of L. Hörmander [12] except for the fact that Φ_M is not homogeneous in η' alone, so $q \mapsto M_h q(x', \xi') := Mq(x', \xi'; h)$ is a semi-classical Fourier integral operator, at least formally.

We fix a vector $\xi'_0 \in \dot{\mathbf{R}}^{n-1}$ and consider Φ_M in a neighborhood of $(x', y, \xi', \eta') = (0, 0, \xi'_0, \xi'_0) \in \mathbf{C}^{4(n-1)+1} = \mathbf{C}^{4n-3}$. The critical set C_{Φ_M} of the phase Φ_M is given by $\partial_{\eta'}\Phi_M = 0$, that is $x' - y' - \partial_{\eta'}\psi^*(y, \eta') = 0$ or equivalently,

$$x' = y' + \partial_{\eta'}\psi^*(y, \eta'). \tag{3.11}$$

This is a smooth submanifold of codimension $n-1$ in $\mathbf{C}^{4n-3}$, parametrized by $(y, \eta', \xi') \in \operatorname{neigh}((0, \xi'_0, \xi'_0), \mathbf{C}^{3n-2})$. We also see that Φ_M is a nondegenerate phase function in the sense that $d\partial_{\eta'_1}\Phi_M, ..., d\partial_{\eta'_{n-1}}\Phi_M$ are linearly independent on C_{Φ_M}. Using the above parametrization, we express the graph of the corresponding canonical relation $\kappa : \mathbf{C}^{2n}_{y,y^*} \to \mathbf{C}^{4(n-1)}_{x',\xi',x'^*,\xi'^*}$ (where we notice that $4(n-1) \geq 2n$ with equality for $n = 2$ and strict inequality for $n \geq 3$):

$$\begin{aligned}&\operatorname{graph}(\kappa) = \{(x', \xi', \partial_{x'}\Phi_M, \partial_{\xi'}\Phi_M; y, -\partial_y\Phi_M);\ (x', \xi', y, \eta') \in C_{\Phi_M}\}\\ &= \{(y' + \partial_{\eta'}\psi^*(y, \eta'), \xi', \eta' - \xi', \partial_{\xi'}\psi(y, \xi') - \partial_{\eta'}\psi^*(y, \eta');\\ &y, -\partial_{y'}\psi(y, \xi') + \partial_{y'}\psi^*(y, \eta') + \eta' - \xi', -\partial_{y_n}\psi(y, \xi') + \partial_{y_n}\psi^*(y, \eta'))\}\end{aligned} \tag{3.12}$$

The restriction to $y_n = 0$ of this graph is the set of points

$$(y', \xi', \eta' - \xi', 0; y', 0, \eta' - \xi', -\partial_{y_n}\psi(y', 0, \xi') + \partial_{y_n}\psi^*(y', 0, \eta')). \tag{3.13}$$

It contains the point

$$(0, \xi'_0, 0, 0; 0, 0, -2\partial_{y_n}\psi(0, \xi'_0)) = (0, \xi'_0, 0, 0; 0, 0, -2ir(0, \xi'_0)^{\frac{1}{2}}) \tag{3.14}$$

The tangent space at a point where $y_n = 0$ is given by

$$\begin{aligned}&\{(\delta_{y'} + {\psi^*}''_{\eta',y_n}\delta_{y_n}, \delta_{\xi'}, \delta_{\eta'} - \delta_{\xi'}, (\psi''_{\xi',y_n}(y, \xi') - {\psi^*}''_{\eta',y_n}(y, \eta'))\delta_{y_n};\\ &\quad \delta_y, (-\psi''_{y',y_n}(y, \xi') + {\psi^*}''_{y',y_n}(y, \eta'))\delta_{y_n} + \delta_{\eta'} - \delta_{\xi'},\\ &\quad (-\psi''_{y_n,y}(y, \xi') + {\psi^*}''_{y_n,y}(y, \eta'))\delta_y + (-\psi''_{y_n,\xi'}\delta_{\xi'} + {\psi^*}''_{y_n,\eta'}\delta_{\eta'}))\}\end{aligned} \tag{3.15}$$

From (3.15) we see that at every point of $\operatorname{graph}\kappa$ with $y_n = 0$ and with $\eta' \approx \xi'$,

1) The projection $\operatorname{graph}(\kappa) \to \mathbf{C}^{2n}_{y,y^*}$ has surjective differential,

2) The projection $\operatorname{graph}(\kappa) \to \mathbf{C}^{4(n-1)}_{x',\xi',x'^*,\xi'^*}$ has injective differential.

In fact, since κ is a canonical relation, 1) and 2) are pointwise equivalent, so it suffices to verify 2). In other words, we have to show that if

$$\begin{aligned}0 &= \delta_{y'} + {\psi^*}''_{\eta',y_n}\delta_{y_n},\\ 0 &= \delta_{\xi'},\\ 0 &= \delta_{\eta'} - \delta_{\xi'},\\ 0 &= (\psi''_{\xi',y_n}(y, \xi') - {\psi^*}''_{\eta',y_n}(y, \eta'))\delta_{y_n},\end{aligned} \tag{3.16}$$

then $\delta_{y'} = 0$, $\delta_{y_n} = 0$, $\delta_{\xi'} = 0$, $\delta_{\eta'} = 0$.

When $y_n = 0$ we have $\partial_{y_n}\psi^* = -\partial_{y_n}\psi$ and when in addition $\eta' \approx \xi'$ we see that the $(n-1) \times 1$ matrix in the 4:th equation is non-vanishing, so this equation implies that $\delta_{y_n} = 0$. Then the first equation gives $\delta_{y'} = 0$ and from the 2:nd and the 3:d equations we get $\delta_{\xi'} = 0$ and $\delta_{\eta'} = 0$ and we have verified 2).

As an exercise, let us determine the image under κ of the complexified conormal bundle of the boundary, given by $y_n = 0$, $y^{*\prime} = 0$. From (3.13) we see that it is equal to the set of all points

$$(x', \xi', 0, 0). \tag{3.17}$$

The subset of real points in (3.17) is the image of the set of points $(y', 0, 0, y_n^*)$ such that y' is real and $y_n^* \in -i\mathbf{R}_+$.

Now restrict (x', ξ') to the set of $(x', t\eta_0')$, $x' \in \mathbf{C}^{n-1}$, $t \in \mathbf{C}$, where $0 \neq \eta_0' \in \mathbf{R}^{n-1}$. This means that we restrict the symbol of $\dot{\Lambda}$ to the radial direction $\xi' \in \mathbf{C}\eta_0'$ and consider

$$\begin{aligned} &\sigma_{\dot{\Lambda}}(x', t\eta_0'; h) = Mq(x', t\eta_0'; h) =: M_{\text{new}} q(x', t; h) \\ &= (2\pi h)^{1-n} \iint e^{i\Phi_{M_{\text{new}}}(x', t, y, \eta')/h} a(y, -\eta'; h) a(y, \xi'; h) q(y) dy d\eta', \end{aligned} \tag{3.18}$$

where

$$\begin{aligned} &\Phi_{M_{\text{new}}}(x', t, y; \eta') = \Phi_M(x', t\eta_0', y; \eta') \\ &= \psi(y, t\eta_0') - \psi^*(y, \eta') + (x' - y') \cdot (\eta' - t\eta_0') \end{aligned} \tag{3.19}$$

We will soon drop the subscripts "new" when no confusion is possible. This is again a nondegenerate phase function. The new canonical relation $\kappa_{\text{new}} : \mathbf{C}^{2n}_{y,y^*} \to \mathbf{C}^{2n}_{x',t,x'^*,t^*}$ has the graph

$$\begin{aligned} &\text{graph}\,(\kappa_{\text{new}}) \\ &= \{(y' + \partial_{\eta'}\psi^*(y, \eta'), t, \eta' - t\eta_0', \eta_0' \cdot \partial_{\xi'}\psi(y, t\eta_0') - \eta_0' \cdot \partial_{\xi'}\psi^*(y, \eta'); \\ &y, -\partial_{y'}\psi(y, t\eta_0') + \partial_{y'}\psi^*(y, \eta') + \eta' - t\eta_0', -\partial_{y_n}\psi(y, t\eta_0') + \partial_{y_n}\psi^*(y, \eta'))\}. \end{aligned} \tag{3.20}$$

This graph is conic with respect to the dilations

$$\mathbf{R}_+ \ni \lambda \mapsto (x', \lambda t, \lambda x'^*, t^*; y, \lambda y^*)$$

The restriction of the graph to $y_n = 0$ is

$$\{(y', t, \eta' - t\eta_0', 0; y', 0, \eta' - t\eta_0', -\partial_{y_n}\psi(y', 0, t\eta_0') + \partial_{y_n}\psi^*(y', 0, \eta'))\},$$

where

$$\partial_{y_n}\psi(y', 0, \xi') = ir(y', 0, \xi')^{1/2},\ \partial_{y_n}\psi^*(y', 0, \xi') = -ir(y', 0, \xi')^{1/2}$$

so the restriction is

$$\{(y',t,\eta'-t\eta_0',0;y',0,\eta'-t\eta_0',-i(r^{\frac{1}{2}}(y',0,t\eta_0')+r^{\frac{1}{2}}(y',0,\eta')))\}. \tag{3.21}$$

If we take $\eta = t\eta_0'$ and use that $r^{\frac{1}{2}}$ is homogeneous of degree 1 in the fiber variables, we get

$$\{(y',t,0,0;y',0,0,-2itr^{\frac{1}{2}}(y',0,\eta_0'))\}. \tag{3.22}$$

We assume, to fix the ideas, that $r(0,\eta_0') = 1/4$. This is the graph of a diffeomorphism

$$\mathrm{neigh}\,(0,\partial\Omega)\times(-i\mathbf{R}^+_{y_n^*})\to \mathrm{neigh}\,(0;\partial\Omega)\times\mathbf{R}^+_t.$$

The tangent space of (3.20) at a point where $y_n = 0$ is given by

$$\begin{aligned}\{(&\delta_{y'}+(\psi^*)''_{\eta',y_n}\delta_{y_n},\delta_t,\delta_{\eta'}-\delta_t\eta_0',\eta_0'\cdot(\psi''_{\xi',y_n}-(\psi^*)''_{\eta',y_n})\delta_{y_n};\\ &\delta_y,(-\psi''_{y',y_n}+(\psi^*)''_{y',y_n})\delta_{y_n}+\delta_{\eta'}-\delta_t\eta_0',\\ &\quad(-\psi''_{y_n,y}+(\psi^*)''_{y_n,y})\delta_y-\psi''_{y_n,\xi'}\delta_t\eta_0'+(\psi^*)''_{y_n,\eta'}\delta_{\eta'})\}.\end{aligned} \tag{3.23}$$

The projection onto the first component is injective as can be seen exactly as in the proof of the property 2) stated after (3.15). Now $\kappa_{\rm new}$ is a canonical relation between spaces of the same dimension so we conclude that $\kappa_{\rm new}$ is a canonical transformation locally near each point of its graph. Combining this with the observation right after (3.22), we get

Proposition 3.1 *(3.20) is the graph of a bijective canonical transformation*

$$\kappa_{\rm new}:\ \mathrm{neigh}\,((0;0,-i),\mathbf{C}^n_y\times\mathbf{C}^n_{y^*})\to\mathrm{neigh}\,((0,1;0),\mathbf{C}^n_{x',t}\times\mathbf{C}^n_{x'^*,t^*}).$$

The neighborhoods can be taken conic with respect to the actions $\mathbf{R}_+\ni\lambda\mapsto(y,\lambda y^*)$ and $\mathbf{R}_+\ni\lambda\mapsto(x,\lambda t,\lambda x'^*,t^*)$ and $\kappa_{\rm new}$ intertwines the two actions (so $\kappa_{\rm new}$ is positively homogeneous of degree 1, with y^* as the fiber variables on the departure side and with t, x'^* as the fiber variables on the arrival side).

Basically, the same exercise as the one leading to (3.17) shows that the image under $\kappa_{\rm new}$ of the complexified conormal bundle, given by $y_n = 0$, $(y^*)' = 0$, is the zero section

$$\{(x',t;(x'^*,t^*)=0)\}. \tag{3.24}$$

Consider the image of $T^*\partial\Omega\times i\mathbf{R}^-_{y_n^*} = \{(y,y^*);\ y',(y^*)'\in\mathbf{R}^{n-1},\ y_n = 0,\ y_n^*\in i\mathbf{R}^-\}$ under $\kappa_{\rm new}$ and recall (3.21). If we restrict the attention to $t\in\mathbf{R}_+$, so that $\eta' = (y^*)'+t\eta_0'\in\mathbf{R}^{n-1}$, we see that

$$y_n^* = -i(r^{\frac{1}{2}}(y',0,t\eta_0')+r^{\frac{1}{2}}(y',0,\eta'))\in i\mathbf{R}^-.$$

Thus the image contains locally

$$\{(x', t, (x^*)', 0);\ x', (x^*)' \in \mathbf{R}^{n-1},\ t \in \mathbf{R}^+\},$$

which has the right dimension $2(n-1)+1$

Similarly, the image of $T^*\partial\Omega \times \mathrm{neigh}\,(i\mathbf{R}^-_{y_n^*}, \mathbf{C}_{y_n^*})$ is obtained by dropping the reality condition on t but keeping that on $\eta' - t\eta_0'$, and we get

$$\begin{aligned} &\kappa_{\mathrm{new}}(T^*\partial\Omega \times \mathrm{neigh}\,(i\mathbf{R}^-_{y_n^*}, \mathbf{C}_{y_n^*})) \\ &= \{(x', t, x'^*, 0);\ x', (x^*)' \in \mathbf{R}^{n-1},\ t \in \mathrm{neigh}\,(\mathbf{R}^+, \mathbf{C})\}. \end{aligned} \tag{3.25}$$

4 Some function spaces and their FBI-transforms

We continue to work locally near a point x_0 where the boundary is analytic and we use analytic coordinates y centered at x_0 as specified in the beginning of Section 3.

We start by defining some piecewise smooth I-Lagrange manifolds.

- The cotangent space $T^*\Omega$ that we identify with $(\mathrm{neigh}\,(0) \cap \mathbf{R}^n_+) \times \mathbf{R}^n$.
- The real conormal bundle $N^*\partial\Omega \subset T^*\mathbf{R}^n$. In the local coordinates y,

$$N^*\partial\Omega = \{(y, \eta) \in \mathbf{R}^{2n};\ y_n = 0,\ \eta' = 0\}.$$

  It will sometimes be convenient to write $N^*\partial\Omega = \partial\Omega \times \mathbf{R}$ where of course the second expression appeals to the use of special coordinates as above. More invariantly, $N^*\partial\Omega$ is the inverse image of the zero-section in $T^*\partial\Omega$ for the natural projection map $\pi_{T^*\partial\Omega} : T^*_{\partial\Omega}\mathbf{R}^n \to T^*\partial\Omega$.

We will also need some complex sets.

- The complexified zero section in the complexification $\widetilde{T^*\mathbf{R}^n} = \mathbf{C}^n_y \times \mathbf{C}^n_\eta$ is defined to be

$$\mathrm{neigh}\,(0, \mathbf{C}^n) \times \{\eta = 0\} \subset \mathbf{C}^n_y \times \mathbf{C}^n_\eta.$$

  We denote it by $\mathbf{C}^n_y \times 0_\eta$ for short.

- The complexification $\widetilde{N^*\partial\Omega}$ of $N^*\partial\Omega$ is defined to be

$$\{(y, \eta) \in \mathbf{C}^n_y \times \mathbf{C}^n_\eta;\ y \in \mathrm{neigh}\,(0, \mathbf{C}^n),\ y_n = 0,\ \eta' = 0\}.$$

- The space $\pi^{-1}(T^*\partial\Omega)$, where $\pi : T^*_{\partial\Omega}\mathbf{R}^n \otimes \mathbf{C} \to T^*\partial\Omega \otimes \mathbf{C}$ is the natural projection and $\otimes\mathbf{C}$ indicates fiberwise complexification. In special coordinates it is $\{(y,\eta);\ (y',\eta') \in \mathbf{R}^{2(n-1)},\ y_n = 0,\ \eta_n \in \mathbf{C}\}$. We will denote it by $T^*\partial\Omega \times \mathbf{C}$ or $T^*\partial\Omega \times \mathbf{C}_{\eta_n}$ for simplicity. It contains the subset $T^*\partial\Omega \times \mathbf{C}^-_{\eta_n}$ (easy to define invariantly), where $\mathbf{C}^-$ is the open lower half-plane. Notice that

$$T^*\partial\Omega \times \partial\mathbf{C}_- = T^*\partial\Omega \times \mathbf{R} = T^*_{\partial\Omega}\mathbf{R}^n.$$

- The piecewise smooth (Lipschitz) manifold

$$F = \overline{T^*\Omega} \cup (T^*\partial\Omega \times \mathbf{C}^-_{\eta_n}).$$

  Notice that the two components to the right have $T^*_{\partial\Omega}\mathbf{R}^n$ as their common boundary.

- The piecewise smooth (Lipschitz) manifold $(\mathbf{C}^n_y \times 0_\eta) \cup \widetilde{N^*\partial\Omega}$ where the two constituents contain $\widetilde{\partial\Omega} \times 0_\eta$. Here $\widetilde{\partial\Omega}$ denotes a complexification of the boundary (near x_0).

Let

$$Tu(z;h) = Ch^{-\frac{3n}{4}} \int_{\mathbf{R}^n} e^{\frac{i}{h}\phi(z,y)} u(y)dy,\ z \in \mathbf{C}^n, \tag{4.1}$$

be a standard FBI transform ([17]), sending distributions with compact support on $\mathbf{R}^n$ to holomorphic functions on (in general some subdomains of) $\mathbf{C}^n$. For simplicity we let ϕ be a holomorphic quadratic form so that T can also be viewed as a generalized Bargmann transform and a metaplectic Fourier integral operator (see for instance [18]). We work under the standard assumptions

$$\Im\phi''_{y,y} > 0, \quad \det\phi''_{z,y} \neq 0. \tag{4.2}$$

We let $C > 0$ be the unique positive constant for which $T : L^2(\mathbf{R}^2) \to H_{\Phi_0}(\mathbf{C}^n)$ is unitary, where

$$\Phi_0(z) = \sup_{y\in\mathbf{R}^n} -\Im\phi(z,y) = -\Im\phi(z,y(z)) \tag{4.3}$$

is a strictly pluri-subharmonic (real) quadratic form on $\mathbf{C}^n$ and H_{Φ_0} is the complex Hilbert space $\mathrm{Hol}\,(\mathbf{C}^n) \cap L^2(e^{-2\Phi_0/h}L(dz))$, $L(dz)$ denoting the Lebesgue measure on $\mathbf{C}^n \simeq \mathbf{R}^{2n}$. Let

$$\kappa_T : \mathbf{C}^{2n} \ni (y, -\phi'_y(z,y)) \mapsto (z, \phi'_z(z,y)) \in \mathbf{C}^{2n} \tag{4.4}$$

be the complex (linear) canonical transformation associated to T and let $\Lambda_{\Phi_0} = \{(z, \frac{2}{i}\frac{\partial \Phi_0}{\partial z}(z));\ z \in \mathbf{C}^n\}$ be the R-symplectic[1] and I-Lagrangian[2] manifold of $\mathbf{C}^{2n}$, actually a real-linear subspace since ϕ is quadratic. Then we know that

$$\Lambda_{\Phi_0} = \kappa_T(\mathbf{R}^{2n}). \tag{4.5}$$

More explicitly,

$$\kappa_T^{-1}(z, \frac{2}{i}\frac{\partial \Phi_0}{\partial z}) = (y(z), \eta(z)) \in \mathbf{R}^{2n}, \tag{4.6}$$

where $y(z)$ appeared in (4.3).

In this paper, we shall deal with FBI-transforms and H_Φ locally and we recall some definitions and facts from [17]. The local H_Φ-spaces are defined in Chapter 1 of that work:

Let $\Omega \subset \mathbf{C}^n$ be an open set, $\Phi : \Omega \to \mathbf{R}$ a continous function. A function $u(z;h)$ on $\Omega\times]0, h_0]$, $0 < h_0 \leq 1$, is said to belong to $H_\Phi^{\rm loc}(\Omega)$ if

$$u \text{ is holomorphic in } z \text{ for each } h \in]0, h_0], \tag{4.7}$$

$$\begin{aligned} &\text{For every compact set } K \subset \Omega \text{ and every } \epsilon > 0,\\ &\text{there exists a constant } C = C_{\epsilon,K} > 0, \text{ such that}\\ &|u(z;h)| \leq Ce^{\frac{1}{h}(\Phi(z)+\epsilon)}|,\ (z,h) \in K\times]0,h_0]. \end{aligned} \tag{4.8}$$

In general, we shall not distingguish between two elements $u, v \in H_\Phi^{\rm loc}(\Omega)$, if their difference is exponentially small relative to $e^{\Phi(z)/h}$. More precisely, if $u, v \in H_\Phi^{\rm loc}(\Omega)$ we say that they are equivalent ($u \sim v$) if there exists a continous function $\widetilde{\Phi} < \Phi$ on Ω, such that

$$u - v \in H_{\widetilde{\Phi}}^{\rm loc}(\Omega). \tag{4.9}$$

This is clearly an equivalence relation and sometimes we do not distinguish between $H_\Phi^{\rm loc}(\Omega)$ and the corresponding set of equivalence classes.

It will also be convenient to work with germs of H_Φ-functions. If $z_0 \in \mathbf{C}^n$ and $\Phi \in C(\text{neigh}\,(z_0, \mathbf{C}^n); \mathbf{R})$, then by definition, an element $u \in H_{\Phi,z_0}$ is an element $u \in H_\Phi^{\rm loc}(\Omega)$, where Ω is a neighborhood of z_0. We say that $u, v \in H_{\Phi,z_0}$ are equivalent, $u \sim v$, if they are equivalent in $H_\Phi^{\rm loc}(W)$ for some neighborhood W of z_0.

The corresponding microlocal version of FBI-transforms is then the following: Let $\phi \in \text{Hol}\,(\text{neigh}\,((z_0, y_0), \mathbf{C}^{2n})$ satisfy (4.2) at the point (z_0, y_0). Also assume that

$$\eta_0' := -\phi_y'(z_0, y_0) \in \mathbf{R}^n. \tag{4.10}$$

[1] i.e. symplectic with respect to $\Re\sigma$, where $\sigma = d\zeta \wedge dz$ is the complex symplectic form
[2] i.e. Lagrangian with respect to $\Im\sigma$

Then (cf.(4.3)) we can defined $\Phi_0 \in C^\infty(\text{neigh}\,(z_0, \mathbf{C}^n); \mathbf{R})$ by

$$\Phi_0(z) = \sup_{y \in \text{neigh}\,(y_0, \mathbf{R}^n)} (-\Im\phi(z, y)), \tag{4.11}$$

and Φ_0 becomes strictly plurisubharmonic. As after (4.3), we can define the canonical transformation

$$\kappa_T : \text{neigh}\,((y_0, \eta_0), \mathbf{C}^{2n}) \to \text{neigh}\,((z_0, \zeta_0), \mathbf{C}^{2n}),$$

where $\zeta_0 = \phi'(z_0, y_0) = \frac{2}{i}\partial_z \Phi_0(z_0)$ and we have natural local versions of (4.5). If $u \in \mathcal{D}'(\text{neigh}\,(y_0, \mathbf{R}^n))$ is independent of h or more generally h-dependent but of temperate growth in $\mathcal{D}'$ as a function of h, then by throwing in a cutoff $\chi \in C_0^\infty(\text{neigh}\,(y_0, \mathbf{R}^n))$ with $y_0 \notin \text{supp}\,(1-\chi)$ into the formula (4.1), we can define $Tu \in H_{\Phi_0, z_0}$ up to $\sim$.

We now return to the FBI-transform (4.1) with quadratic phase. Let

$$\Phi_1^{\text{ext}}(z) = \sup_{y \in \partial \mathbf{R}^n_+} -\Im\phi(z, y) = -\Im\phi(z, \widetilde{y}(z)), \tag{4.12}$$

where $\widetilde{y}(z) = (\widetilde{y}'(z), 0)$ and $\widetilde{y}'(z)$ is the unique point of maximum in $\mathbf{R}^{n-1}$ of $y' \mapsto -\Im\phi(z, y', 0)$. If $\text{supp}\, u \subset \{y \in \mathbf{R}^n;\ y_n \geq 0\}$, then $Tu \in H^{\text{loc}}_{\Phi_1}$, where

$$\Phi_1(z) = \sup_{y \in \mathbf{R}^n_+} -\Im\phi(z, y) = \begin{cases} \Phi_0(z), & \text{if } y_n(z) \geq 0, \\ \Phi_1^{\text{ext}}(z), & \text{if } y_n(z) \leq 0. \end{cases} \tag{4.13}$$

Notice that

- $-\Im\partial_{y_n}\phi(z, \widetilde{y}(z)) \geq 0$ in the first case,
- $-\Im\partial_{y_n}\phi(z, \widetilde{y}(z)) \leq 0$ in the second case.

Moreover,

$$\frac{2}{i}\frac{\partial \Phi_1}{\partial z}(z) = \frac{2}{i}(\frac{\partial}{\partial z}(-\Im\phi))(z, \widetilde{y}(z)) = \phi'_z(z, \widetilde{y}(z))$$

and $\widetilde{\eta}(z) = -\phi'_y(z, \widetilde{y}(z))$ satisfies $\widetilde{\eta}'(z) \in \mathbf{R}^{n-1}$. When $\Phi_1(z) = \Phi_1^{\text{ext}}(z)$ we have

$$\widetilde{\eta}'(z) \in \mathbf{R}^{n-1},\ \Im\widetilde{\eta}_n(z) \leq 0. \tag{4.14}$$

This means that

$$\Lambda_{\Phi_1^{\text{ext}}} = \kappa_T(T^*\partial\Omega \times \mathbf{C}_{\eta_n}),$$

and that

$$\Lambda_{\Phi_1} = \kappa_T(F), \tag{4.15}$$

near $(z, \frac{2}{i}\partial_z \Phi_1(z))$ in case of strict inequality in (4.14). Here the Lipschitz manifold F was defined above,

$$F = \overline{T^*(\Omega)} \cup \{(y', 0; \eta', \eta_n);\ (y', \eta') \in T^*\partial\Omega,\ \Im\eta_n \leq 0\}. \tag{4.16}$$

The second component is a union of complex half-lines, consequently in the region where $\Phi_1 < \Phi_0$, Λ_{Φ_1} is a union of complex half-lines. If we project these lines to the complex z-space we get a foliation of $\mathbf{C}^n_z$ into complex half-lines and the restriction of Φ_1 to each of these is harmonic. We have the corresponding local statements.

If $y_0 = (y_0', 0) \in \partial\mathbf{R}^n_+$, $(y_0, \eta_0) \in F$ and $z_0 = \pi_z \kappa_T(y_0, \eta_0)$, then for $u \in \mathcal{D}'(\mathrm{neigh}\,(y_0, \mathbf{R}^n))$ with $\mathrm{supp}\, u \subset \mathbf{R}^n_+$, we have that Tu is well-defined up to equivalence in H_{Φ_1, z_0}.

We introduce the real hyperplane

$$H = \pi_z \kappa_T(T^*_{\partial\Omega}\mathbf{R}^n),$$

which is the common boundary of the two half-spaces

$$H_+ = \pi_z \kappa_T(T^*\Omega),$$
$$H_- = \pi_z \kappa_T(\{(y', 0; \eta);\ (y', \eta') \in T^*\partial\Omega,\ \Im\eta_n < 0\}).$$

Here, $\pi_z : \mathbf{C}^n_z \times \mathbf{C}^n_\zeta \to \mathbf{C}^n$ is the natural projection. We have

$$\Phi_0 - \Phi_1 \begin{cases} = 0 \text{ in } H_+, \\ \asymp \mathrm{dist}\,(z, H)^2 \text{ in } H_-. \end{cases} \tag{4.17}$$

Similarly, recall the definition of the complexified normal bundle $\widetilde{N^*\partial\Omega}$ at the beginning of this section. It is a $\mathbf{C}$-Lagrangian manifold.[3] We have $\kappa_T(\widetilde{N^*\partial\Omega}) = \Lambda_{\Phi_3}$, where Φ_3 is pluriharmonic and given by

$$\Phi_3(z) = \mathrm{v.c.}_{y' \in \mathbf{C}^{n-1}}(-\Im\phi(z, y', 0)),$$

v.c. = “critical value of”.

Similarly $\kappa_T(\mathbf{C}^n_y \times 0_\eta)$ (with the notation from the beginning of this section) is of the form Λ_{Φ_4}, where

$$\Phi_4(z) = \mathrm{v.c.}_{y \in \mathbf{C}^n}(-\Im\phi(z, y)).$$

The complex zero-section $\mathbf{C}_y \times 0_\eta$ and $T^*\mathbf{R}^n$ intersect transversally along the real zero-section $\mathbf{R}^n_y \times 0_\eta$. Correspondingly, we check that

$$\Phi_0(z) - \Phi_4(z) \asymp \mathrm{dist}\,(z, \pi_z \circ \kappa_T(\mathbf{R}^n \times 0_\eta))^2. \tag{4.18}$$

[3] i.e. a holomorphic manifold which is Lagrangian for the complex symplectic form σ.

Similarly,

$$\Phi_1^{\rm ext}(z)-\Phi_3(z)\asymp {\rm dist}\,(z,\pi_z\circ\kappa_T((\partial\Omega\times 0)\times \mathbf{C}^*_{\eta_n}))^2, \tag{4.19}$$

where $\partial\Omega\times 0$ denotes the zero section in $T^*\partial\Omega$, so that

$$(\partial\Omega\times 0)\times\mathbf{C}^*_{\eta_n}=N^*\partial\Omega\otimes\mathbf{C}$$

is the fiber-wise complexification of $N^*\partial\Omega$. (Here we work locally near $y=0$.)

Let u be real-analytic in a neighborhood of $y_0\in\partial\mathbf{R}^n_+$ and consider

$$v(z)=T(1_\Omega u)(z), \tag{4.20}$$

where we restrict the attention to $z\in\mathbf{C}^n$ such that the critical point $y_{\Phi_4}(z)$ in the definition of $\Phi_4(z)$ belongs to a small complex neighborhood of $y_0\in\partial\mathbf{R}^n_+$ or equivalently to $z\in\mathbf{C}^n$ in a small neighborhood of $\kappa_T(\{y_0\}\times 0_\eta)$. By the "méthode du col" we see that $v\in H^{\rm loc}_{\Phi_5}$, where first of all $\Phi_5\le\Phi_1$ and further,

$$\Phi_5(z)=\Phi_4(z),\ \text{ when }\begin{cases}\Re y_{\Phi_4}(z)\in\Omega \text{ and}\\ |\Im y_{\Phi_4}(z)|\ll {\rm dist}\,(\Re y_{\Phi_4}(z),\partial\Omega),\end{cases} \tag{4.21}$$

$$\Phi_5(z)=\Phi_3(z),\ \text{ when }\begin{cases}\Re y_{\Phi_4}(z)\not\in\Omega \text{ and}\\ |\Im y_{\Phi_4}(z)|\ll {\rm dist}\,(\Re y_{\Phi_4}(z),\partial\Omega).\end{cases} \tag{4.22}$$

Actually, in the last case we can relax the condition that $y_{\Phi_4}(z)$ belongs to a small (u-dependent) neighborhood of $\overline{\Omega}$. The appropriate restriction is then that the critical point $y_{\Phi_3}(z)\in\widetilde{\partial\Omega}$ in the definition of Φ_3 belongs to a small (u-dependent) neighborhood of $\partial\Omega$.

5 Expressing M with the help of FBI-transforms

From now on we work with $M_{\rm new}$, $\Phi_{M_{\rm new}}$, $\kappa_{\rm new}$ and we drop the corresponding subscript "new". Then (3.18) reads

$$Mq(x',t)=\frac{1}{(2\pi h)^{n-1}}\iint e^{\frac{i}{h}\Phi_M(x',t,y,\eta')}a(y,-\eta';h)a(y,t\eta_0';h)q(y)dyd\eta'. \tag{5.1}$$

with Φ_M given in (3.19).

We want to express Mq with the help of Tq, where T is as in (4.1) and we start by recalling some general facts about metaplectic Fourier integral operators of this form, following [17] for the local theory, and [18] for the simplified global theory in the metaplectic frame work (i.e. all phases are quadratic and all amplitudes are constant). To start with, we weaken the assumptions on the quadratic phase in T and assume only that $\phi(x,y)$ is a holomorphic quadratic form on $\mathbf{C}^n \times \mathbf{C}^n$, satisfying the second part of (4.2):

$$\det \phi''_{x,y}(x,y) \neq 0. \tag{5.2}$$

To T we can still associate a linear canonical transformation κ_T as in (4.4). Let Φ_1, Φ_2 be plurisubharmonic quadratic forms on $\mathbf{C}^n$ related by

$$\Lambda_{\Phi_2} = \kappa_T(\Lambda_{\Phi_1}) \tag{5.3}$$

Then we can define $T : H_{\Phi_1} \to H_{\Phi_2}$ as a bounded operator as in (4.1) with the modification that $\mathbf{R}^n$ should be replaced by a so called good contour, which is an affine subspace of $\mathbf{C}^n$ of real dimension n, passing through the nondegenerate critical point $y_c(x)$ the function

$$y \mapsto -\Im\phi(x,y) + \Phi_1(y) \tag{5.4}$$

and along which this function is $\Phi_2(x) - (\asymp |y - y_c(x)|^2)$. (Actually in this situation it would have been better to replace the power $h^{-3n/4}$ by $h^{-n/2}$ since we would then get a uniform bound on the norm.)

Remark 5.1 Recall also that if only Φ_1 is given as above, the existence of a quadratic form Φ_2 as in (5.3) is equivalent to the fact that (5.4) has a non-degenerate critical point and the plurisubharmonicity of Φ_2 is equivalent to the fact that the signature of the critical point is $(n, -n)$ (which represents the maximal number of negative eigenvalues of the Hessian of a plurisubharmonic quadratic form). This in turn is equivalent to the existence of an affine good contour as above.

In this situation $T : H_{\Phi_1} \to H_{\Phi_2}$ is bijective with the inverse

$$Sv(y) = T^{-1}v(y) = \widetilde{C}h^{-\frac{n}{4}} \int e^{-\frac{i}{h}\phi(z,y)}v(z)dz, \tag{5.5}$$

which can be realized the same way with a good contour and here the constant $\widetilde{C}$ does not depend on the choice of Φ_j, $j = 1, 2$.

Remark 5.2 Let us introduce the formal adjoints of T and S,

$$T^{\mathrm{t}}v(y) = Ch^{-\frac{3n}{4}} \int_{\mathbf{R}^n} e^{\frac{i}{h}\phi(z,y)}v(z)dz,\ y \in \mathbf{C}^n,$$

$$S^{\mathrm{t}}u(z) = \widetilde{C}h^{-\frac{n}{4}} \int e^{-\frac{i}{h}\phi(z,y)}u(y)dy.$$

Let Ψ_1, Ψ_2 be pluri-subharmonic quadratic forms such that $\kappa_{S^{\mathrm{t}}}(\Lambda_{\Psi_1}) = \Lambda_{\Psi_2}$. Then as above, $T^{\mathrm{t}} : H_{\Psi_2} \to H_{\Psi_1}$, $S^{\mathrm{t}} : H_{\Psi_1} \to H_{\Psi_2}$ are bijective and $S^{\mathrm{t}} = \mathrm{const.}(T^{\mathrm{t}})^{-1}$. We claim that S^{t} is the inverse of T^{t}. In fact, this statement is independent of the choice of Φ_j, Ψ_j as above and we can choose them to be pluri-harmonic in such a way that Λ_{Φ_j} intersects $\Lambda_{-\Psi_j}$ transversally for one value of j and then automatically for the other value. Then for $j = 1, 2$ we can define

$$\langle u|v\rangle = \int_{\gamma_j} u(x)v(x)dx,$$

for $u \in H_{\Phi_j}$, $v \in H_{\Psi_j}$ (or rather for functions that are $\mathcal{O}(e^{\Phi_j/h})$ and $e^{\Psi_j/h}$ respectively, the space of such functions is of dimension 1 which suffices for our purposes) if we let γ_j be a good contour for $\Phi_j + \Psi_j$. For $u = \mathcal{O}(e^{\Phi_2/h})$, $v = \mathcal{O}(e^{\Psi_2/h})$ non zero, we have

$$0 \neq \langle u|v\rangle = \langle TSu|v\rangle = \langle Su|T^{\mathrm{t}}v\rangle = \langle u|S^{\mathrm{t}}T^{\mathrm{t}}v\rangle$$

and knowing already that $S^{\mathrm{t}}T^{\mathrm{t}}$ is a multiple of the identity, we see that it has to be equal to the identity.

Now return to the discussion of an FBI-transform T whose phase satisfies (4.2). When letting T act on suitable H_Φ-spaces it has the inverse S in (5.5). However, if we let T act on $L^2(\mathbf{R}^n)$ so that $Tu \in H_{\Phi_0}$ (with $\Lambda_{\Phi_0} = \kappa_T(\mathbf{R}^{2n})$), the best possible contour in (5.5) is

$$\Gamma(y) = \{z \in \mathbf{C}^n;\, y(z) = y\}.$$

This follows from the property

$$\Phi_0(z) + \Im\phi(z, y) \asymp \mathrm{dist}\,(z, \Gamma(y))^2 \asymp |y(z) - y|^2, \tag{5.6}$$

so $\Phi_0(z) + \Im\phi(z, y) = 0$ on $\Gamma(y)$ and $e^{-\frac{i}{h}\phi(z,y)+\frac{1}{h}\Phi_0(z)}$ is bounded there. This is not sufficient for a straight forward definition of $Sv(y)$, $v \in H_{\Phi_0}$ since we would need some extra exponential decay along the contour near infinity, but it does suffice to give a precise meaning up to exponentially small errors of the formula

$$\widetilde{T}u = (\widetilde{T}S)Tu, \tag{5.7}$$

in a local situation, where $\widetilde{T} : L^2 \to H_{\widetilde{\Phi}_0}$ is a second FBI-transform and where $\widetilde{T}S : H_{\Phi_0} \to H_{\widetilde{\Phi}_0}$ is defined by means of a good contour.

Proposition 5.3 *Let $(y_0, \eta_0) \in \mathbf{R}^{2n}$, $(z_0, \zeta_0) = \kappa_T(y_0, \eta_0)$, $(w_0, \omega_0) = \kappa_{\widetilde{T}}(y_0, \eta_0)$. We realize Tu, $\widetilde{T}u$, $\widetilde{T}Su$ (modulo exponentially small terms) in H_{Φ_0, z_0}, $H_{\widetilde{\Phi}_0, w_0}$, $H_{\widetilde{\Phi}_0, w_0}$ respectively, by choosing good contours restricted to neighborhoods of*

y_0, y_0, z_0 respectively. Then (5.7) holds (modulo an exponentially small error) in $H_{\widetilde{\Phi}_0,w_0}$. Here $u \in \mathcal{D}'(\mathbf{R}^n)$ is either independent of h or of temperate growth in $\mathcal{D}'(\mathbf{R}^n)$ as a function of h.

Proof. The left hand side of (5.7) is

$$\mathrm{Const.}\, h^{-\frac{3n}{4}-n} \iiint e^{\frac{i}{h}(\widetilde{\phi}(w,x)-\phi(z,x)+\phi(z,y))} u(y)dydzdx$$

where the composed contour is good, and all good contours being homotopic, we can write it as

$$\widetilde{C}h^{-\frac{3n}{4}} \int \left(\mathrm{Const.}\, h^{-n} \iint e^{\frac{i}{h}(-\phi(z,x)+\phi(z,y))} e^{\frac{i}{h}\widetilde{\phi}(w,x)} dxdz\right) u(y)dy.$$

The expression in the big parenthesis is nothing but $T^{\mathrm{t}}S^{\mathrm{t}}(e^{\frac{i}{h}\widetilde{\phi}(w,\cdot)})(y)$, which by Remark 5.2 is equal to $e^{\frac{i}{h}\widetilde{\phi}(w,y)}$ and (5.7) follows. (In the proof we have chosen not to spell out the various exponentially small errors due to the fact that the integration contours are confined to various small neighborhoods of certain points.) $\square$

We now return to the operator M in (5.1). Choose adapted analytic coordinates centered at x_0 as in the beginning of Section 3. In that section (cf (3.25)) we have seen that there is a well defined canonical transformation κ_M from a neighborhood of $(0,0,-i) \in \mathbf{C}^{2n}_{y,\eta}$ to a neighborhood of $(0,1,0,0)$ in $\mathbf{C}^{n-1}_{x'} \times \mathbf{C}_t \times \mathbf{C}^{n-1}_{x'^*} \times \mathbf{C}_{t^*}$ mapping $T^*\partial\Omega \times i\mathbf{R}_-$ to $\mathbf{R}^{n-1}_{x'} \times \mathbf{R}_t \times \mathbf{R}^{n-1}_{x'^*} \times \{t^* = 0\}$. This means that we have a microlocal description of Mq near $(0,1,0,0)$ and not a local one near $x' = 0$, $t = 0$. We shall therefore microlocalize in (x', x'^*) by means of an FBI-transform in the x'-variables.

Let

$$\widehat{T}u(w') = \widehat{C}h^{\frac{1-n}{2}} \int_{\mathbf{R}^{n-1}} e^{\frac{i}{h}\widehat{\phi}(w',x')} u(x')dx', \; w' \in \mathbf{C}^{n-1} \tag{5.8}$$

be a second FBI-transform as in (4.1) though acting in $n-1$ variables and with a different normalization. Assume (to fix the ideas) that

$$\kappa_{\widehat{T}}(\mathbf{C}^{n-1}_{x'} \times \{0\}) = \mathbf{C}^{n-1}_{w'} \times \{0\}. \tag{5.9}$$

Then

$$\kappa_{\widehat{T}}(T^*\mathbf{R}^{n-1}) = \Lambda_{\widehat{\Phi}_0}, \tag{5.10}$$

where $\widehat{\Phi}_0$ is a strictly plurisubharmonic quadratic form. In view of (5.9) and the fact that the zero-section $\mathbf{C}^{n-1} \times \{0\}$ is strictly positive with respect to the real phase space, we also know that

$$\widehat{\Phi}_0(w') \asymp |w'|^2, \tag{5.11}$$

or equivalently that the quadratic form $\widehat{\Phi}_0$ is strictly convex.

By slight abuse of notation we also let $\widehat{T}$ act on functions of n variables by

$$\widehat{T}(u)(w',t) = (\widehat{T}u(\cdot,t))(w').$$

The presence of $\widehat{T}$ leads to a formula for $\widehat{T}M$ that is simpler than the one for M in (5.1).

$$\begin{aligned}\widehat{T}Mq(w',t) = \widehat{T}\left(e^{-\frac{i}{h}(\cdot)\cdot t\eta_0'}K^{\mathrm{t}}qK\left(e^{\frac{i}{h}(\cdot)\cdot t\eta_0'}\right)\right)(w') = \\ \widehat{C}h^{\frac{1-n}{2}}\iiint e^{\frac{i}{h}(\widehat{\phi}(w',\widetilde{x}')-\widetilde{x}'\cdot t\eta_0')}K(y,\widetilde{x}')q(y)K(y,x')e^{\frac{i}{h}x'\cdot t\eta_0'}dx'dyd\widetilde{x}' \\ = \int K\left(e^{\frac{i}{h}(\widehat{\phi}(w',\cdot)-(\cdot)\cdot t\eta_0')}\right)(y)q(y)K\left(e^{\frac{i}{h}(\cdot)\cdot t\eta_0'}\right)(y)dy.\end{aligned}$$

Up to exponentially small errors we have (cf. (3.5))

$$K\left(e^{\frac{i}{h}(\cdot)\cdot t\eta_0'}\right)(y) = e^{\frac{i}{h}\phi(y,t\eta_0')}a(y,t\eta_0';h)$$

and

$$K\left(e^{\frac{i}{h}\widehat{\phi}(w',\cdot)}\right)(y) = e^{\frac{i}{h}\widetilde{\psi}(w',t\eta_0',y)}b(w',y,t\eta_0';h),$$

where b is an elliptic analytic symbol of order 0 and ψ is the solution of the following eikonal equation in y,

$$\partial_{y_n}\widetilde{\psi} = ir(y,\partial_{y'}\psi)^{\frac{1}{2}}, \quad \widetilde{\psi}_{|y_n=0} = \widehat{\phi}(w',y') - y'\cdot t\eta_0'.$$

Thus, up to exponentially small errors, we get for $q \in L^\infty(\Omega)$,

$$\begin{aligned}&\widehat{T}Mq(w',t) = \int e^{\frac{i}{h}\psi(w',t,y)}c(w',t,y;h)q(y)dy, \\ &(w',t) \in \operatorname{neigh}((0,1),\mathbf{C}^{n-1}\times\mathbf{C}),\end{aligned} \tag{5.12}$$

where c is an elliptic analytic symbol of order 0 and

$$\psi(w',t,y) = \widetilde{\psi}(w',t,y) + \phi(y,t\eta_0')$$

satisfies

$$\psi_{|_{y_n=0}} = \widehat{\phi}(w',y'), \tag{5.13}$$

$$\partial_{y_n}\psi_{|_{y_n=0}} = i\left(r\left(y',0,\partial_{y'}\widehat{\phi}(w',y') - t\eta_0'\right)^{\frac{1}{2}} + r(y',0,t\eta_0')^{\frac{1}{2}}\right). \tag{5.14}$$

Assume for simplicity that $r(0,0,\eta_0') = 1/4$. Then, at the point $w' = 0$, $t = 1$, $y = 0$, we have

$$(\partial_{w'}\psi, \partial_t\psi, -\partial_{y'}\psi, -\partial_{y_n}\psi) = (0,0,0,-i),$$

so $\kappa_{\widehat{T}M}(0,0,-i) = (0,1,0,0)^4$. Also, $\kappa_{\widehat{T}M} = \kappa_{\widehat{T}} \circ \kappa_M$ and

$$\begin{aligned}&\kappa_M(0,0,-i) = (0,1,0,0),\\ &\kappa_{\widehat{T}}(0,1,0,0) = (0,1,0,0).\end{aligned}$$

Recall from (3.25) that

$$\begin{aligned}&\kappa_M : \operatorname{neigh}((0;0,-i), T^*\partial\Omega \times \mathbf{C}^-_{y_n^*}) \to \\ &\operatorname{neigh}((0,1;0,0), \mathbf{R}^{n-1}_{x'} \times \mathbf{C}_t \times \mathbf{R}^{n-1}_{x'^*} \times \{t^* = 0\}),\end{aligned}$$

so

$$\kappa_{\widehat{T}M} : \operatorname{neigh}((0,0,-i), T^*\partial\Omega \times \mathbf{C}^-_{y_n^*}) \to \operatorname{neigh}((0,1,0,0), \Lambda_{\widehat{\Phi}_0 \oplus 0}).$$

On the other hand, we have seen in Section 4 that $\kappa_T(F) = \Lambda_{\Phi_1}$ and that the part $T^*\partial\Omega \times \mathbf{C}^-_{y_n^*}$ of F is mapped to $\Lambda_{\Phi_1^{\mathrm{ext}}}$. More locally,

$$\begin{aligned}&\kappa_T : \ \operatorname{neigh}((0,0,-i), T^*\partial\Omega \times \mathbf{C}^-_{y_n^*}) \to \operatorname{neigh}(\kappa_T(0,0,-i), \Lambda_{\Phi_1^{\mathrm{ext}}})\\ &\kappa_S : \ \operatorname{neigh}(\kappa_T(0,0,-i), \Lambda_{\Phi_1^{\mathrm{ext}}}) \to \operatorname{neigh}((0,0,-i), T^*\partial\Omega \times \mathbf{C}^-_{y_n^*}).\end{aligned}$$

Using also (3.25), we get

$$\kappa_{\widehat{T}MS} : \ \operatorname{neigh}(\pi_z\kappa_T(0,0,-i), \Lambda_{\Phi_1^{\mathrm{ext}}}) \to \operatorname{neigh}((0,1,0,0), \Lambda_{\widehat{\Phi}_0 \oplus 0}). \tag{5.15}$$

We then also know that

$$\widehat{\Phi}_0(w') = \mathrm{vc}_{y,z}(-\Im\psi(w',t,y) + \Im\phi_T(z,y)).$$

This means ([17]) that the formal composition

$$\widehat{T}MSv(w',t) = \widetilde{C}h^{-\frac{n}{4}} \iint e^{\frac{i}{h}(\psi(w',t,y) - \phi_T(z,y))} c(w',t,y;h) v(z) dz dy \tag{5.16}$$

gives a well-defined operator

$$\widehat{T}MS : \ H_{\Phi_1^{\mathrm{ext}}, \pi_z\kappa_T(0,0,-i)} \to H_{\widehat{\Phi}_0 \oplus 0, (0,1)}, \tag{5.17}$$

that can be realized with the help of a good contour.

We shall next show that

$$\widehat{T}Mu = (\widehat{T}MS)Tu \text{ in } H_{\widehat{\Phi}_0 \oplus 0, (0,1)} \tag{5.18}$$

[4] We can verify directly that $\det \partial_{w',t}\partial_y\psi \neq 0$.

when u is supported in $\{y_n \geq 0\}$. The proof is the same as the one for (5.7). The right hand side in (5.18) is equal to

$$\mathrm{Const}.h^{-n} \iiint e^{\frac{i}{h}(\psi(w',t,x)-\phi_T(z,x)+\phi_T(z,y))} c(w',t,x;h)u(y)dydzdx,$$

where the y-integration is over $\mathbf{R}^n_+$, and we may assume without loss of generality, that u has its support in a small neighborhood of $y=0$. The $dzdx$ integration is, to start with, over the good contour in (5.16). This last integration can be viewed as $T^{\mathrm{t}}S^{\mathrm{t}}$ acting on $e^{\frac{i}{h}\psi(w',t,\cdot)}c(w',t,\cdot;h)$ and here $T^{\mathrm{t}}S^{\mathrm{t}}$ is the identity operator, that can be realized with a good contour, so we get

$$(\widehat{T}MS)Tu(w',t) = \int e^{\frac{i}{h}\psi(w',t,x)} c(w',t,x;h)u(x)dx = \widehat{T}Mu(w',t),$$

and we have verified (5.18).

Above, we have established (5.17) as the quantum version of (5.15). It follows by an easy adaptation of the exercise leading to (3.17) that

$$\begin{aligned} &\kappa_M(\mathrm{neigh}\,((0,0,-i), \mathbf{C}^{n-1}\times\{0\}\times\mathbf{C}^-_{\eta_n})) \\ &\ = \mathrm{neigh}\,((0,0,1,0), \mathbf{C}^{n-1}_{x'}\times\{x'^*=0\}\times\mathbf{C}_t\times\{t^*=0\}), \end{aligned} \tag{5.19}$$

and hence

$$\kappa_{\widehat{T}MS}(\mathrm{neigh}\,(\kappa_T(0,0,-i),\Lambda_{\Phi_3})) = \mathrm{neigh}\,((0,0,1,0),\Lambda_{0\oplus 0}). \tag{5.20}$$

The quantum version of (5.20) is

$$\widehat{T}MS:\ H^{\mathrm{loc}}_{\Phi_3,\pi_z\kappa_T(0,0,-i)} \to H^{\mathrm{loc}}_{0\oplus 0,(0,1)}. \tag{5.21}$$

We also know that $\widehat{T}MS$ is an elliptic Fourier integral operator. Consequently, (5.17), (5.21) have continuous inverses. We also have the following result:

Proposition 5.4 *Assume that* $u \in H^{\mathrm{loc}}_{\Phi_1,\pi_z\kappa_T(0,0,-i)}$ *and that* $\widehat{T}MSu \in H^{\mathrm{loc}}_{0\oplus 0,(0,1)}$. *Then* $u \in H^{\mathrm{loc}}_{\Phi_3,\pi_z\kappa_T(0,0,-i)}$.

6 End of the proof of the main result

We will work with FBI and Laplace transforms of functions that are independent of h or that have some special h-dependence. Consider a formal Fourier integral operator $u \mapsto Tu$, given by

$$Tu(x;h) = Ch^{\alpha}\int e^{\frac{i}{h}\phi(x,y)}u(y)dy, \tag{6.1}$$

where $\phi = \phi_T$ is a quadratic form on $\mathbf{C}^{2n}_{x,y}$ satisfying

$$\det \phi''_{xy} \neq 0, \tag{6.2}$$

and hence generating a canonical transformation, that will be used below.

Proposition 6.1 *If u is independent of h, we have*

$$(hD_h + \frac{1}{h} P_\alpha(x, hD; h))Tu = 0, \tag{6.3}$$

where

$$P_\alpha = p(x, hD) + ih\left(\alpha + \frac{1}{2}\mathrm{tr}\,(\phi''_{xx}{\phi''_{yx}}^{-1}\phi''_{yy}{\phi''_{xy}}^{-1})\right), \tag{6.4}$$

$$\begin{aligned} p(x,\xi) =& \frac{1}{2}\phi''_{xx}x\cdot x + x\cdot(\xi - \phi''_{xx}x) \\ &+ \frac{1}{2}{\phi''_{yx}}^{-1}\phi''_{yy}{\phi''_{xy}}^{-1}(\xi - \phi''_{xx}x)\cdot(\xi - \phi''_{xx}x) \\ =& -\frac{1}{2}\phi''_{xx}x\cdot x + \frac{1}{2}\phi''_{xx}{\phi''_{yx}}^{-1}\phi''_{yy}{\phi''_{xy}}^{-1}\phi''_{xx}x\cdot x \\ &+ x\cdot\xi - {\phi''_{yx}}^{-1}\phi''_{yy}{\phi''_{xy}}^{-1}\phi''_{xx}x\cdot\xi + \frac{1}{2}{\phi''_{yx}}^{-1}\phi''_{yy}{\phi''_{xy}}^{-1}\xi\cdot\xi. \end{aligned} \tag{6.5}$$

Proof. We have

$$\begin{aligned} hD_h\left(e^{\frac{i}{h}\phi(x,y)}\right) &= -\frac{1}{h}e^{\frac{i}{h}\phi(x,y)}, \\ hD_h\,(h^\alpha) &= \frac{\alpha}{i}h^\alpha, \\ hD_hTu(x;h) &= -\frac{1}{h}h^\alpha\int e^{\frac{i}{h}\phi(x,y)}(ih\alpha + \phi(x,y))u(y)dy, \end{aligned}$$

Try to write $\phi(x,y) = p(x, \phi'_x(x,y))$ for a suitable quadratic form $p(x,\xi)$ (that will turn out to be the one given in (6.5)). We have

$$\phi(x,y) = \frac{1}{2}\phi''_{xx}x\cdot x + \phi''_{xy}y\cdot x + \frac{1}{2}\phi''_{yy}y\cdot y, \tag{6.6}$$

$$\phi'_x = \phi''_{xx}x + \phi''_{xy}y, \text{ i.e. } y = {\phi''_{xy}}^{-1}(\phi'_x - \phi''_{xx}x) \tag{6.7}$$

and using the last relation from (6.7) in (6.6), we get

$$\begin{aligned} \phi(x,y) =& \frac{1}{2}\phi''_{xx}x\cdot x + \phi''_{yx}x\cdot{\phi''_{xy}}^{-1}(\phi'_x - \phi''_{xx}x) \\ &+ \frac{1}{2}\phi''_{yy}{\phi''_{xy}}^{-1}(\phi'_x - \phi''_{xx}x)\cdot{\phi''_{xy}}^{-1}(\phi'_x - \phi''_{xx}x), \end{aligned} \tag{6.8}$$

where the ϕ''_{yx} and ${\phi''_{xy}}^{-1}$ in the second term cancel, and we get $p(x, \phi'_x)$ with p as in (6.5).

To verify (6.4), it suffices to notice that

$$
\begin{aligned}
e^{-\frac{i}{h}\phi(x,y)}p(x,hD_x)\left(e^{\frac{i}{h}\phi(x,y)}\right) - p(x,\phi'_x) \\
&= \frac{1}{2}{\phi''_{yx}}^{-1}\phi''_{yy}{\phi''_{xy}}^{-1}hD_x\cdot(\phi'_x) \\
&= \frac{1}{2}{\phi''_{yx}}^{-1}\phi''_{yy}{\phi''_{xy}}^{-1}hD_x\cdot(\phi''_{xx}x) \\
&= \frac{h}{2i}\phi''_{xx}{\phi''_{yx}}^{-1}\phi''_{yy}{\phi''_{xy}}^{-1}\partial_x\cdot x \\
&= \frac{h}{2i}\mathrm{tr}\left(\phi''_{xx}{\phi''_{yx}}^{-1}\phi''_{yy}{\phi''_{xy}}^{-1}\right).
\end{aligned}
$$

□

Remark 6.2 Let $\kappa_T: (y, -\phi'_y(x,y)) \mapsto (x, \phi'_x(x,y))$ be the canonical transformation associated to T which can also be written

$$\kappa_T: (y, -(\phi''_{yx}x + \phi''_{yy}y) \mapsto (x, \phi''_{xx}x + \phi''_{xy}y),$$

or still $\kappa_T: (y,\eta) \mapsto (x,\xi)$ where

$$
\begin{aligned}
x &= -{\phi''_{yx}}^{-1}(\eta + \phi''_{yy}y) \\
\xi &= (\phi''_{xy} - \phi''_{xx}{\phi''_{yx}}^{-1}\phi''_{yy})y - \phi''_{xx}{\phi''_{yx}}^{-1}\eta.
\end{aligned}
$$

We see that the following three statements are equivalent:

- κ_T maps the Lagrangian space $\eta = 0$ to $\xi = 0$.
- $\phi''_{xy} - \phi''_{xx}{\phi''_{yx}}^{-1}\phi''_{yy} = 0$.
- $p(x,0) = 0$, $p'_\xi(x,0) = 0$, $\forall x$.

Example 6.3 Consider

$$\widehat{T}\mathcal{L}u(x;h) = Ch^{\frac{1-n}{2}}\int e^{\frac{i}{h}(\widehat{\phi}(x',y')+ix_ny_n)}u(y)dy,\ \phi = \phi_{\widehat{T}}.$$

If $P'(x', hD_{x'}; h)$ is the operator associated to $\widehat{T}$ in $n-1$ variables, we get when u is independent of h,

$$(hD_h + \frac{1}{h}(P'(x', hD_{x'}; h) + x_n hD_{x_n}))\widehat{T}\mathcal{L}u = 0. \tag{6.9}$$

Similarly (though not a direct consequence of Theorem 6.1 but rather of its method of proof) we have for $\mathcal{L}$ alone that

$$(hD_h + \frac{1}{h} x_n hD_{x_n})\mathcal{L}u = 0. \tag{6.10}$$

Example 6.4 Let T be as above and assume that we are in the situation of Remark 6.2 so that $p(x,0) = 0$, $p'_\xi(x,0) = 0$. Then

$$p(x, hD) = bhD \cdot hD$$

where b is a constant symmetric matrix. Then

$$P_\alpha = p(x, hD) + ih(\alpha + f_0),\ \ f_0 = \frac{n}{2}.$$

and (6.3) reads

$$(hD_h + (hbD \cdot D + i(\alpha + f_0)))Tu = 0. \tag{6.11}$$

If $Tu = \sum_m^\infty h^k v_k \in H_0$ and u is independent of h, we can plug this expression into (6.11) and get the sequence of equations,

$$\begin{aligned}
&\left(\frac{m}{i} + i(\alpha + f_0)\right) v_m = 0,\\
&\left(\frac{m+1}{i} + i(\alpha + f_0)\right) v_{m+1} + bD \cdot Dv_m = 0,\\
&\left(\frac{m+2}{i} + i(\alpha + f_0)\right) v_{m+2} + bD \cdot Dv_{m+1} = 0,\\
&\ldots
\end{aligned}$$

so unless $v_m \equiv 0$, we get $m = \alpha + f_0$. $v_m \in H_0$ can be chosen arbitrary and v_{m+1}, $v_{m+2}, \ldots$ are then uniquely determined.

Now, consider the situation in Theorem 1.6 and let $q \in L^\infty(\Omega)$ be independent of h and such that $\sigma_{\dot\Lambda}(y', t\eta'_0)$ is a cl.a.s. on neigh $(\{0\} \times \mathbf{R}_+, \mathbf{R}^{n-1} \times \mathbf{R}_+)$ of order -1 (cf. (2.4),

$$\sigma_{\dot\Lambda}(y', t\eta'_0) \sim \sum_1^\infty n_k(y', t), \tag{6.12}$$

where $n_k(y', t)$ is homogeneous of degree $-k$ in t.

$$|n_k(y', t)| \le C^{k+1} k^k |t|^{-k},\ \ y' \in \text{neigh}\,(0, \mathbf{C}^{n-1}). \tag{6.13}$$

For the moment we shall only work with formal cl.a.s. and neglect remainders in the asymptotic expansions. For the semi-classical symbol of $\dot{\Lambda}$ we have

$$\begin{aligned}&\sigma_{\dot{\Lambda}}(y', t\eta_0'/h) \sim \sum_1^\infty n_k(y', t/h) = \sum_1^\infty h^k n_k(y', t),\\ &(y', t) \in \operatorname{neigh}((0,1), \mathbf{R}^{n-1} \times \mathbf{R}_+).\end{aligned} \tag{6.14}$$

Recall that $\sigma_{\dot{\Lambda}}(y', t\eta_0'/h) = Mq(y', t; h)$. From (6.14) we infer that $\widehat{T}Mq$ is a cl.a.s. near $w' = 0$, $t = 1$:

$$\widehat{T}Mq \sim \sum_1^\infty h^k m_k(w', t). \tag{6.15}$$

Formally,

$$\widehat{T}M = (\widehat{T}M\mathcal{L}^{-1})\mathcal{L}. \tag{6.16}$$

The canonical transformation $\kappa_{\mathcal{L}}$ is given by

$$(y, \eta) \mapsto (y', i\eta_n, \eta', iy_n).$$

It maps the complex manifold $\eta' = 0$, $y_n = 0$ to the manifold $\{(z, 0)\}$ and the point $(0; 0, -i)$ to $(0, 1; 0)$, so $\kappa_{*^{-1}} = \kappa_{\mathcal{L}}^{-1}$ maps $\zeta = 0$ to $\eta' = 0$, $y_n = 0$ and we noticed in (3.24) (cf. (3.22)), that κ_M takes the complexified conormal bundle to the zero section, and it maps the point $(0; 0, -i)$ to $(0, 1; 0)$. Thus $\kappa_{M\mathcal{L}^{-1}}$ maps the zero section $\zeta = 0$ to the zero section and in particular $(0, 1; 0)$ to $(0, 1; 0)$. (We may notice that this is global in the sense that we can extend z_n to an annulus, and we then get t in an annulus.) Since $\kappa_{\widehat{T}}$ maps the zero section to the zero section, we have the same facts for $\kappa_{\widehat{T}M}$.

From the above, it is clear that $\widehat{T}M\mathcal{L}^{-1}$ maps formal cl.a.s. to formal cl.a.s.

Recalling (6.14) for $\sigma_{\dot{\Lambda}}(y', t\eta_0'/h) = Mq(y', t; h)$ and using that $\widehat{T}M\mathcal{L}^{-1}$ is an elliptic Fourier integral operator whose canonical transformation maps the zero-section to the zero-section, we see that there exists a unique formal cl.a.s.

$$v \sim \sum_1^\infty v_k(z', z_n)h^k,\ \ z \in \operatorname{neigh}((0,1), \mathbf{C}^n), \tag{6.17}$$

such that in the sense of formal stationary phase,

$$\widehat{T}Mq = \widehat{T}M\mathcal{L}^{-1}v. \tag{6.18}$$

Now q is independent of h, so Mq satisfies a compatibility equation of the form

$$\left(hD_h + \frac{1}{h}P_{\widehat{T}M}\right)Mq = 0. \tag{6.19}$$

This gives rise to a similar compatibility condition for v,

$$\left(hD_h + \frac{1}{h}P_{\mathcal{L}M^{-1}\widehat{T}^{-1}\widehat{T}M}\right)v = 0,$$

or simply

$$\left(hD_h + \frac{1}{h}P_{\mathcal{L}}\right)v = 0,$$

which is the same as (6.10):

$$(h\partial_h + z_n\partial_{z_n})v = 0. \tag{6.20}$$

Application of this to (6.17) gives

$$(k + z_n\partial_{z_n})v_k = 0, \tag{6.21}$$

i.e.

$$v_k(z) = q_k(z')z_n^{-k},\ |q_k(z')| \le C^{k+1}k^k. \tag{6.22}$$

Thus,

$$v \sim \sum_1^\infty q_k(z')\left(\frac{h}{z_n}\right)^k = \sum_0^\infty q_{k+1}(z')\left(\frac{h}{z_n}\right)^{k+1},$$

and we see as in Section 2 (with the difference that we now deal with the semi-classical Laplace transform) that

$$v \sim \mathcal{L}\widetilde{q}(z;h),\ \widetilde{q}(y) = 1_{[0,a]}(y_n)\sum_0^\infty \frac{q_{k+1}(y')}{k!}y_n^k, \tag{6.23}$$

with $a > 0$ small enough to ensure the convergence of the power series.

More precisely, (and now we end the limitation to formal symbols), as in (5.18), (5.7), we check that

$$\widehat{T}M\widetilde{q} \equiv (\widehat{T}M\mathcal{L}^{-1})\mathcal{L}\widetilde{q} \text{ in } H_{0,(0,1)} \tag{6.24}$$

(up to an exponentially small error). By the construction of $\widetilde{q}$, the right hand side is $\equiv \widehat{T}Mq$ in the same space.

Put $r = q - \widetilde{q}$. Then

$$\widehat{T}Mr \equiv 0 \text{ in } H_{0,(0,1)}. \tag{6.25}$$

Now, we replace $\mathcal{L}$ with T and consider in the light of (5.18):

$$(\widehat{T}MS)Tr \equiv 0 \text{ in } H_{0,(0,1)}, \tag{6.26}$$

which implies that $Tr \in H_{\Phi_1}$ satisfies

$$Tr \equiv 0 \text{ in } H_{\Phi_1^{\mathrm{ext}}, \pi_z \kappa_T(0;0,-i)}. \tag{6.27}$$

As we saw in Section 4, Λ_{Φ_1} contains the closure $\overline{\Gamma}$ of the complex curve

$$\Gamma = \kappa_T(\{(0;0,\eta_n);\ \Im\eta_n < 0\}),$$

and $\kappa_T((0;0,-i)) \in \Gamma$. Consequently, $\Phi_{1|_{\pi_z\Gamma}}$ is harmonic and (6.27) and the maximum principle imply that

$$Tr \equiv 0 \text{ in } H_{\Phi_1,z},\ z \in \overline{\Gamma}. \tag{6.28}$$

In particular,

$$Tr \equiv 0 \text{ in } H_{\Phi_1,0} \tag{6.29}$$

and a fortiori,

$$Tr \equiv 0 \text{ in } H_{\Phi_0,0}. \tag{6.30}$$

This implies that $r = 0$ near $y = 0$. Hence $q = \widetilde{q}$ near $y = 0$, which gives the theorem.

7 Proof of Proposition 1.7

We choose local coordinates $y = (y', y_n)$ as in the beginning of Section 2. As in Proposition 1.7, we assume that q is analytic in a neighborhood of 0. We adopt the alternative definition of symbols in Remark 1.4. It will also be convenient to consider the semi-classical symbol of $\dot{\Lambda}$, $\sigma_{\dot{\Lambda}}(y', \eta'; h) = \sigma_{\dot{\Lambda}}(y', \eta'/h)$. For $y' \in \mathrm{neigh}\,(0, \mathbf{R}^{n-1})$,

$$\begin{aligned} &\sigma_{\dot{\Lambda}}(y', \eta'; h) = \\ &-\partial_{y_n} GqK\left(\int \chi(t') e_{t'}(\cdot; h) e^{i(\cdot)\cdot\eta'/h} dt'\right)(y', 0) e^{-iy'\cdot\eta'/h}, \end{aligned} \tag{7.1}$$

where χ and e_t were defined in Remark 1.4 with n there replaced by $n-1$. By analytic WKB (as we already used), we have up to an exponentially small error,

$$K(e_{t'}(\cdot; h) e^{i(\cdot)\cdot\eta'/h}) = Ch^{\frac{1-n}{2}} a(y, \eta'; h) e^{i\phi(y,t,\eta')/h}, \tag{7.2}$$

where ϕ is the solution of the eiconal problem

$$\partial_{y_n}\phi = ir(y, \partial_{y'}\phi)^{\frac{1}{2}},\ \phi_{|_{y_n=0}} = y'\cdot\eta' + \frac{i}{2}(y'-t)^2, \tag{7.3}$$

and a is an cl.a.s. of order 0, obtained from solving a sequence of transport equations with the "initial" condition $a(y', 0, \eta'; h) = 1$.

Using again the analytic WKB-method we can find a cl.a.s. b of order 0 in h which solves the following inhomogeneous problem up to exponentially small errors:

$$\begin{cases} (h^2\Delta - h^2 V)(h^{\frac{1-n}{2}+3} b(y,t,\eta';h) e^{\frac{i}{h}\phi(y,t,\eta')} = C h^{\frac{1-n}{2}+2} a e^{\frac{i}{h}\phi} q, \\ b(y', 0, t, \eta'; h) = 0. \end{cases}$$

Then up to exponentially small errors,

$$GqK(e_t(\cdot;h)e^{i(\cdot)\cdot\eta'/h}) \equiv h^{\frac{1-n}{2}+1} b(y,t,\eta';h) e^{\frac{i}{h}\phi(y,t,\eta')}$$

and similarly for the gradients, so

$$-\left(\partial_{y_n}\right)_{y_n=0} GqK(e_t(\cdot;h)e^{i(\cdot)\cdot\eta'/h}) \equiv -h^{\frac{3-n}{2}} (\partial_{y_n} b)(y', 0, t, \eta'; h) e^{\frac{i}{h}(y'\cdot\eta' + \frac{i}{2}(y'-t')^2)}.$$

Multiplying with $\chi(t')$ and integrating in t', we see that $\sigma_{\dot{\Lambda}}(y', \eta'; h)$ is a cl.a.s. in the semi-classical sense and this implies that $\sigma_{\dot{\Lambda}}(y', \eta)$ is a cl.a.s.

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