UNIQUENESS IN AN INVERSE BOUNDARY PROBLEM FOR A MAGNETIC SCHRÖDINGER OPERATOR WITH A BOUNDED MAGNETIC POTENTIAL

KATSIARYNA KRUPCHYK AND GUNTHER UHLMANN

Abstract. We show that the knowledge of the set of the Cauchy data on the boundary of a bounded open set in $\mathbb{R}^n$, $n \geq 3$, for the magnetic Schrödinger operator with L^∞ magnetic and electric potentials determines the magnetic field and electric potential inside the set uniquely. The proof is based on a Carleman estimate for the magnetic Schrödinger operator with a gain of two derivatives.

1. Introduction and statement of result

Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded open set, and let $u \in C_0^\infty(\Omega)$. We consider the magnetic Schrödinger operator,

$$\begin{aligned} L_{A,q}(x,D)u(x) &:= \sum_{j=1}^{n} (D_j + A_j(x))^2 u(x) + q(x)u(x) \\ &= -\Delta u(x) + A(x) \cdot Du(x) + D \cdot (A(x)u(x)) + ((A(x))^2 + q(x))u(x), \end{aligned}$$

where $D = i^{-1}\nabla$, $A \in L^\infty(\Omega, \mathbb{C}^n)$ is the magnetic potential, and $q \in L^\infty(\Omega, \mathbb{C})$ is the electric potential. We have $Au \in L^\infty(\Omega, \mathbb{C}^n) \cap \mathcal{E}'(\Omega, \mathbb{C}^n)$, and therefore,

$$L_{A,q} : C_0^\infty(\Omega) \to H^{-1}(\mathbb{R}^n) \cap \mathcal{E}'(\Omega)$$

is a bounded operator. Here $\mathcal{E}'(\Omega) = \{v \in \mathcal{D}'(\Omega) : \text{supp } (v) \text{ is compact}\}$.

Let us now introduce the Cauchy data for an $H^1(\Omega)$ solution u to the equation

$$L_{A,q}u = 0 \quad \text{in} \quad \Omega, \tag{1.1}$$

in the sense of distributions. First, following [1, 17], we define the trace space of the space $H^1(\Omega)$ as the quotient space $H^1(\Omega)/H_0^1(\Omega)$. The associated trace map $T : H^1(\Omega) \to H^1(\Omega)/H_0^1(\Omega)$, $Tu = [u]$, is the quotient map. Here $H_0^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the $H^1(\Omega)$–topology.

Notice that if Ω has a Lipschitz boundary, then the space $H^1(\Omega)/H_0^1(\Omega)$ can be naturally identified with the Sobolev space $H^{1/2}(\partial\Omega)$. Indeed, in this case the kernel of the continuous surjective map $H^1(\Omega) \to H^{1/2}(\partial\Omega)$, $u \mapsto u|_{\partial\Omega}$ is precisely $H_0^1(\Omega)$, see [12, Theorems 3.37 and 3.40].

For $u\in H^1(\Omega)$ satisfying (1.1), we can define $N_{A,q}u$, formally given by $N_{A,q}u = (\partial_\nu u + i(A\cdot\nu)u)|_{\partial\Omega}$, as an element of the dual space $(H^1(\Omega)/H^1_0(\Omega))'$ as follows. For $[g]\in H^1(\Omega)/H^1_0(\Omega)$, we set

$$(N_{A,q}u,[g])_\Omega := \int_\Omega (\nabla u\cdot\nabla g + iA\cdot(u\nabla g - g\nabla u) + (A^2+q)ug)\,dx. \tag{1.2}$$

As u is a solution to (1.1), $N_{A,q}u$ is a well-defined element of $(H^1(\Omega)/H^1_0(\Omega))'$.

We define the set of the Cauchy data for solutions of the magnetic Schrödinger equation as follows,

$$C_{A,q} := \{(Tu, N_{A,q}u) : u\in H^1(\Omega) \text{ and } L_{A,q}u = 0 \text{ in } \Omega\}.$$

The inverse boundary value problem for the magnetic Schrödinger operator $L_{A,q}$ is to determine A and q in Ω from the set of the Cauchy data $C_{A,q}$.

Similarly to [20], there is an obstruction to uniqueness in this problem given by the following gauge equivalence of the set of the Cauchy data: if $\psi\in W^{1,\infty}$ in a neighborhood of $\overline{\Omega}$ and $\psi|_{\partial\Omega}=0$, then $C_{A,q} = C_{A+\nabla\psi,q}$, see Lemma 3.1 below. Hence, the map $A\mapsto A+\nabla\psi$ transforms the magnetic potential into a gauge equivalent one but preserves the induced magnetic field dA, which is defined by

$$dA = \sum_{1\le j<k\le n} (\partial_{x_j}A_k - \partial_{x_k}A_j)dx_j\wedge dx_k,$$

in the sense of distributions. Here $A=(A_1,\dots,A_n)$. In view of this, one may hope to recover the magnetic field dA and the electric potential q in Ω from the set of the Cauchy data $C_{A,q}$.

As it has been shown by several authors, the knowledge of the set of the Cauchy data $C_{A,q}$ for the magnetic Schrödinger operator $L_{A,q}$ does determine the magnetic field dA and the electric potential q in Ω uniquely, under certain regularity assumptions on A and q. In [20], this result was established for magnetic potentials in $W^{2,\infty}$, satisfying a smallness condition, and L^∞ electric potentials. In [13], the smallness condition was eliminated for smooth magnetic and electric potentials, and for compactly supported C^2 magnetic potentials and L^∞ electric potentials. The uniqueness results were subsequently extended to C^1 magnetic potentials in [22], to some less regular but small potentials in [14], and to Dini continuous magnetic potentials in [17].

The purpose of this paper is to extend the uniqueness result to the case of magnetic Schrödinger operators with magnetic potentials that are of class L^∞. Our main result is as follows.

Theorem 1.1. *Let $\Omega\subset\mathbb{R}^n$, $n\ge 3$, be a bounded open set, and let $A_1,A_2\in L^\infty(\Omega,\mathbb{C}^n)$ and $q_1,q_2\in L^\infty(\Omega,\mathbb{C})$. If $C_{A_1,q_1}=C_{A_2,q_2}$, then $dA_1=dA_2$ and $q_1=q_2$ in Ω.*

Notice in particular that in Theorem 1.1 no regularity assumptions on the boundary of Ω are required.

The key ingredient in the proof of Theorem 1.1 is a construction of complex geometric optics solutions for the magnetic Schrödinger operator $L_{A,q}$ with $A \in L^\infty(\Omega, \mathbb{C}^n)$ and $q \in L^\infty(\Omega, \mathbb{C})$. When constructing such solutions, we shall first derive a Carleman estimate for the magnetic Schrödinger operator $L_{A,q}$, with a gain of two derivatives, which is based on the corresponding Carleman estimate for the Laplacian, obtained in [19]. Another crucial observation, which allows us to handle the case of L^∞ magnetic potentials is that it is in fact sufficient to approximate the magnetic potential by a sequence of smooth vector fields, in the L^2 sense.

We would also like to mention that another important inverse boundary value problem, for which the issues of regularity have been studied extensively, is Calderón's problem for the conductivity equation, see [4]. The unique identifiability of C^2 conductivities from boundary measurements was established in [21]. The regularity assumptions were relaxed to conductivities having $3/2+\varepsilon$ derivatives in [2], and the uniqueness for conductivities having exactly 3/2 derivatives was obtained in [15], see also [3]. In [8], uniqueness for conormal conductivities in $C^{1+\varepsilon}$ was shown. The recent work [9] proves a uniqueness result for Calderón's problem with conductivities of class C^1 and with Lipschitz continuous conductivities, which are close to the identity in a suitable sense.

The paper is organized as follows. Section 2 contains the construction of complex geometric optics solutions for the magnetic Schrödinger operator with L^∞ magnetic and electric potentials. The proof of Theorem 1.1 is then completed in Section 3.

2. Construction of complex geometric optics solutions

Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded open set. Following [5, 11], we shall use the method of Carleman estimates to construct complex geometric optics solutions for the magnetic Schrödinger equation $L_{A,q}u = 0$ in Ω, with $A \in L^\infty(\Omega, \mathbb{C}^n)$ and $q \in L^\infty(\Omega, \mathbb{C})$.

Let us start by recalling the Carleman estimate for the semiclassical Laplace operator $-h^2\Delta$ with a gain of two derivatives, established in [19], see also [11]. Here $h > 0$ is a small semiclassical parameter. Let $\widetilde{\Omega}$ be an open set in $\mathbb{R}^n$ such that $\Omega \subset\subset \widetilde{\Omega}$ and let $\varphi \in C^\infty(\widetilde{\Omega}, \mathbb{R})$. Consider the conjugated operator

$$P_\varphi = e^{\frac{\varphi}{h}}(-h^2\Delta)e^{-\frac{\varphi}{h}},$$

with the semiclassical principal symbol

$$p_\varphi(x,\xi) = \xi^2 + 2i\nabla\varphi\cdot\xi - |\nabla\varphi|^2, \quad x \in \widetilde{\Omega}, \quad \xi \in \mathbb{R}^n.$$

We have for $(x,\xi)\in\overline{\Omega}\times\mathbb{R}^n$, $|\xi|\ge C\gg 1$, that $|p_\varphi(x,\xi)|\sim|\xi|^2$ so that P_φ is elliptic at infinity, in the semiclassical sense. Following [11], we say that φ is a limiting Carleman weight for $-h^2\Delta$ in $\widetilde{\Omega}$, if $\nabla\varphi\ne 0$ in $\widetilde{\Omega}$ and the Poisson bracket of $\operatorname{Re}p_\varphi$ and $\operatorname{Im}p_\varphi$ satisfies,

$$\{\operatorname{Re}p_\varphi,\operatorname{Im}p_\varphi\}(x,\xi)=0\quad\text{when}\quad p_\varphi(x,\xi)=0,\quad (x,\xi)\in\widetilde{\Omega}\times\mathbb{R}^n.$$

Examples of limiting Carleman weights are linear weights $\varphi(x)=\alpha\cdot x$, $\alpha\in\mathbb{R}^n$, $|\alpha|=1$, and logarithmic weights $\varphi(x)=\log|x-x_0|$, with $x_0\notin\widetilde{\Omega}$. In this paper we shall only use the linear weights.

Our starting point is the following result due to [19].

Proposition 2.1. *Let φ be a limiting Carleman weight for the semiclassical Laplacian on $\widetilde{\Omega}$, and let $\varphi_\varepsilon=\varphi+\frac{h}{2\varepsilon}\varphi^2$. Then for $0<h\ll\varepsilon\ll 1$ and $s\in\mathbb{R}$, we have*

$$\frac{h}{\sqrt{\varepsilon}}\|u\|_{H^{s+2}_{\mathrm{scl}}(\mathbb{R}^n)}\le C\|e^{\varphi_\varepsilon/h}(-h^2\Delta)e^{-\varphi_\varepsilon/h}u\|_{H^s_{\mathrm{scl}}(\mathbb{R}^n)},\quad C>0,\tag{2.1}$$

for all $u\in C_0^\infty(\Omega)$.

Here

$$\|u\|_{H^s_{\mathrm{scl}}(\mathbb{R}^n)}=\|\langle hD\rangle^s u\|_{L^2(\mathbb{R}^n)},\quad \langle\xi\rangle=(1+|\xi|^2)^{1/2},$$

is the natural semiclassical norm in the Sobolev space $H^s(\mathbb{R}^n)$, $s\in\mathbb{R}$.

Next we shall derive a Carleman estimate for the magnetic Schrödinger operator $L_{A,q}$ with $A\in L^\infty(\Omega,\mathbb{C}^n)$ and $q\in L^\infty(\Omega,\mathbb{C})$. To that end we shall use the estimate (2.1) with $s=-1$, and with $\varepsilon>0$ being sufficiently small but fixed, i.e. independent of h. We have the following result.

Proposition 2.2. *Let $\varphi\in C^\infty(\widetilde{\Omega},\mathbb{R})$ be a limiting Carleman weight for the semiclassical Laplacian on $\widetilde{\Omega}$, and assume that $A\in L^\infty(\Omega,\mathbb{C}^n)$ and $q\in L^\infty(\Omega,\mathbb{C})$. Then for $0<h\ll 1$, we have*

$$h\|u\|_{H^1_{\mathrm{scl}}(\mathbb{R}^n)}\le C\|e^{\varphi/h}(h^2L_{A,q})e^{-\varphi/h}u\|_{H^{-1}_{\mathrm{scl}}(\mathbb{R}^n)},\tag{2.2}$$

for all $u\in C_0^\infty(\Omega)$.

Proof. In order to prove the estimate (2.2) it will be convenient to use the following characterization of the semiclassical norm in the Sobolev space $H^{-1}(\mathbb{R}^n)$,

$$\|v\|_{H^{-1}_{\mathrm{scl}}(\mathbb{R}^n)}=\sup_{0\ne\psi\in C_0^\infty(\mathbb{R}^n)}\frac{|\langle v,\psi\rangle_{\mathbb{R}^n}|}{\|\psi\|_{H^1_{\mathrm{scl}}(\mathbb{R}^n)}},\tag{2.3}$$

where $\langle\cdot,\cdot\rangle_{\mathbb{R}^n}$ is the distribution duality on $\mathbb{R}^n$.

Let $\varphi_\varepsilon=\varphi+\frac{h}{2\varepsilon}\varphi^2$ be the convexified weight with $\varepsilon>0$ such that $0<h\ll\varepsilon\ll 1$, and let $u\in C_0^\infty(\Omega)$. Then for all $0\neq\psi\in C_0^\infty(\mathbb{R}^n)$, we have

$$\begin{aligned}|\langle e^{\varphi_\varepsilon/h}h^2A\cdot D(e^{-\varphi_\varepsilon/h}u),\psi\rangle_{\mathbb{R}^n}|&\le\int_{\mathbb{R}^n}\Big|hA\cdot\Big(-u\Big(1+\frac{h}{\varepsilon}\varphi\Big)D\varphi+hDu\Big)\psi\Big|dx\\&\le\mathcal{O}(h)\|u\|_{H^1_{\mathrm{scl}}(\mathbb{R}^n)}\|\psi\|_{H^1_{\mathrm{scl}}(\mathbb{R}^n)}.\end{aligned}$$

We also obtain that

$$\begin{aligned}|\langle e^{\varphi_\varepsilon/h}h^2D\cdot(Ae^{-\varphi_\varepsilon/h}u),\psi\rangle_{\mathbb{R}^n}|&\le\int_{\mathbb{R}^n}|h^2Ae^{-\varphi_\varepsilon/h}u\cdot D(e^{\varphi_\varepsilon/h}\psi)|dx\\&\le\mathcal{O}(h)\|u\|_{H^1_{\mathrm{scl}}(\mathbb{R}^n)}\|\psi\|_{H^1_{\mathrm{scl}}(\mathbb{R}^n)}.\end{aligned}$$

Hence, using (2.3), we get

$$\|e^{\varphi_\varepsilon/h}h^2A\cdot D(e^{-\varphi_\varepsilon/h}u)+e^{\varphi_\varepsilon/h}h^2D\cdot(Ae^{-\varphi_\varepsilon/h}u)\|_{H^{-1}_{\mathrm{scl}}(\mathbb{R}^n)}\le\mathcal{O}(h)\|u\|_{H^1_{\mathrm{scl}}(\mathbb{R}^n)}. \tag{2.4}$$

Notice that the implicit constant in (2.4) only depends on $\|A\|_{L^\infty(\Omega)}$, $\|\varphi\|_{L^\infty(\Omega)}$ and $\|D\varphi\|_{L^\infty(\Omega)}$. Now choosing $\varepsilon>0$ sufficiently small but fixed, i.e. independent of h, we conclude from the estimate (2.1) with $s=-1$ and the estimate (2.4) that for all $h>0$ small enough,

$$\begin{aligned}\|e^{\varphi_\varepsilon/h}(-h^2\Delta)e^{-\varphi_\varepsilon/h}u+e^{\varphi_\varepsilon/h}h^2A\cdot D(e^{-\varphi_\varepsilon/h}u)+e^{\varphi_\varepsilon/h}h^2D\cdot(Ae^{-\varphi_\varepsilon/h}u)\|_{H^{-1}_{\mathrm{scl}}(\mathbb{R}^n)}\\\ge\frac{h}{C}\|u\|_{H^1_{\mathrm{scl}}(\mathbb{R}^n)},\quad C>0.\end{aligned} \tag{2.5}$$

Furthermore, the estimate

$$\|h^2(A^2+q)u\|_{H^{-1}_{\mathrm{scl}}(\mathbb{R}^n)}\le\mathcal{O}(h^2)\|u\|_{H^1_{\mathrm{scl}}(\mathbb{R}^n)}$$

and the estimate (2.5) imply that for all $h>0$ small enough,

$$\|e^{\varphi_\varepsilon/h}(h^2L_{A,q})e^{-\varphi_\varepsilon/h}u\|_{H^{-1}_{\mathrm{scl}}(\mathbb{R}^n)}\ge\frac{h}{C}\|u\|_{H^1_{\mathrm{scl}}(\mathbb{R}^n)},\quad C>0.$$

Using that

$$e^{-\varphi_\varepsilon/h}u=e^{-\varphi/h}e^{-\varphi^2/(2\varepsilon)}u,$$

we obtain (2.2). The proof is complete.

□

Let $\varphi\in C^\infty(\widetilde{\Omega},\mathbb{R})$ be a limiting Carleman weight for $-h^2\Delta$ and set $L_\varphi=e^{\varphi/h}(h^2L_{A,q})e^{-\varphi/h}$. Then we have

$$\langle L_\varphi u,\overline{v}\rangle_\Omega=\langle u,\overline{L_\varphi^*v}\rangle_\Omega,\quad u,v\in C_0^\infty(\Omega),$$

where $L_\varphi^*=e^{-\varphi/h}(h^2L_{\overline{A},\overline{q}})e^{\varphi/h}$ is the formal adjoint of L_φ and $\langle\cdot,\cdot\rangle_\Omega$ is the distribution duality on Ω. We have

$$L_\varphi^*:C_0^\infty(\Omega)\to H^{-1}(\mathbb{R}^n)\cap\mathcal{E}'(\Omega)$$

is bounded, and the estimate (2.2) holds for L_φ^*, since $-\varphi$ is a limiting Carleman weight as well.

To construct complex geometric optics solutions for the magnetic Schrödinger operator we need to convert the Carleman estimate (2.2) for L_φ^* into the following solvability result. The proof is essentially well-known, and is included here for the convenience of the reader. We shall write

$$\|u\|^2_{H^1_{\mathrm{scl}}(\Omega)} = \|u\|^2_{L^2(\Omega)} + \|hDu\|^2_{L^2(\Omega)},$$
$$\|v\|_{H^{-1}_{\mathrm{scl}}(\Omega)} = \sup_{0\neq\psi\in C_0^\infty(\Omega)} \frac{|\langle v,\psi\rangle_\Omega|}{\|\psi\|_{H^1_{\mathrm{scl}}(\Omega)}}.$$

Proposition 2.3. *Let $A\in L^\infty(\Omega,\mathbb{C}^n)$, $q\in L^\infty(\Omega,\mathbb{C})$, and let φ be a limiting Carleman weight for the semiclassical Laplacian on $\widetilde{\Omega}$. If $h>0$ is small enough, then for any $v\in H^{-1}(\Omega)$, there is a solution $u\in H^1(\Omega)$ of the equation*

$$e^{\varphi/h}(h^2L_{A,q})e^{-\varphi/h}u = v \quad in \quad \Omega,$$

which satisfies

$$\|u\|_{H^1_{\mathrm{scl}}(\Omega)} \le \frac{C}{h}\|v\|_{H^{-1}_{\mathrm{scl}}(\Omega)}.$$

Proof. Let $v\in H^{-1}(\Omega)$ and let us consider the following complex linear functional,

$$L : L_\varphi^* C_0^\infty(\Omega)\to\mathbb{C}, \quad L_\varphi^* w\mapsto \langle w,\overline{v}\rangle_\Omega.$$

By the Carleman estimate (2.2) for L_φ^*, the map L is well-defined. Let $w\in C_0^\infty(\Omega)$. Then we have

$$\begin{aligned}|L(L_\varphi^* w)| = |\langle w,\overline{v}\rangle_\Omega| &\le \|w\|_{H^1_{\mathrm{scl}}(\mathbb{R}^n)}\|v\|_{H^{-1}_{\mathrm{scl}}(\Omega)}\\ &\le \frac{C}{h}\|v\|_{H^{-1}_{\mathrm{scl}}(\Omega)}\|L_\varphi^* w\|_{H^{-1}_{\mathrm{scl}}(\mathbb{R}^n)}.\end{aligned}$$

By the Hahn-Banach theorem, we may extend L to a linear continuous functional $\widetilde{L}$ on $H^{-1}(\mathbb{R}^n)$, without increasing its norm. By the Riesz representation theorem, there exists $u\in H^1(\mathbb{R}^n)$ such that for all $\psi\in H^{-1}(\mathbb{R}^n)$,

$$\widetilde{L}(\psi) = \langle\psi,\overline{u}\rangle_{\mathbb{R}^n}, \quad \text{and} \quad \|u\|_{H^1_{\mathrm{scl}}(\mathbb{R}^n)} \le \frac{C}{h}\|v\|_{H^{-1}_{\mathrm{scl}}(\Omega)}.$$

Let us now show that $L_\varphi u = v$ in Ω. To that end, let $w\in C_0^\infty(\Omega)$. Then

$$\langle L_\varphi u,\overline{w}\rangle_\Omega = \langle u,\overline{L_\varphi^* w}\rangle_{\mathbb{R}^n} = \overline{\widetilde{L}(L_\varphi^* w)} = \overline{\langle w,\overline{v}\rangle_\Omega} = \langle v,\overline{w}\rangle_\Omega.$$

The proof is complete. □

Let $A\in L^\infty(\Omega,\mathbb{C}^n)$. We shall extend A to $\mathbb{R}^n$ by defining it to be zero in $\mathbb{R}^n\setminus\Omega$, and denote this extension by the same letter. Then $A\in(L^\infty\cap\mathcal{E}')(\mathbb{R}^n,\mathbb{C}^n)\subset L^p(\mathbb{R}^n,\mathbb{C}^n)$, $1\le p\le\infty$.

Let $\Psi_\tau(x) = \tau^{-n}\Psi(x/\tau)$, $\tau > 0$, be the usual mollifier with $\Psi \in C_0^\infty(\mathbb{R}^n)$, $0 \le \Psi \le 1$, and $\int \Psi dx = 1$. Then $A^\sharp = A * \Psi_\tau \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^n)$ and

$$\|A - A^\sharp\|_{L^2(\mathbb{R}^n)} = o(1), \quad \tau \to 0. \tag{2.6}$$

A direct computation shows that

$$\|\partial^\alpha A^\sharp\|_{L^\infty(\mathbb{R}^n)} = \mathcal{O}(\tau^{-|\alpha|}), \quad \tau \to 0, \quad \text{for all} \quad \alpha, \quad |\alpha| \ge 0. \tag{2.7}$$

We shall now construct complex geometric optics solutions for the magnetic Schrödinger equation

$$L_{A,q} u = 0 \quad \text{in} \quad \Omega, \tag{2.8}$$

with $A \in L^\infty(\Omega, \mathbb{C}^n)$ and $q \in L^\infty(\Omega, \mathbb{C})$, using the solvability result of Proposition 2.3 and the approximation (2.6). Complex geometric optics solutions are solutions of the form,

$$u(x, \zeta; h) = e^{x\cdot\zeta/h}(a(x, \zeta; h) + r(x, \zeta; h)), \tag{2.9}$$

where $\zeta \in \mathbb{C}^n$, $\zeta \cdot \zeta = 0$, $|\zeta| \sim 1$, a is a smooth amplitude, r is a correction term, and $h > 0$ is a small parameter.

It will be convenient to introduce the following bounded operator,

$$m_A : H^1(\Omega) \to H^{-1}(\Omega), \quad m_A(u) = D \cdot (Au),$$

where the distribution $m_A(u)$ is given by

$$\langle m_A(u), v\rangle_\Omega = -\int_\Omega Au \cdot Dv dx, \quad v \in C_0^\infty(\Omega).$$

Let us conjugate $h^2 L_{A,q}$ by $e^{x\cdot\zeta/h}$. First, let us compute $e^{-x\cdot\zeta/h} \circ h^2 m_A \circ e^{x\cdot\zeta/h}$. When $u \in H^1(\Omega)$ and $v \in C_0^\infty(\Omega)$, we get

$$\begin{aligned}\langle e^{-x\cdot\zeta/h} h^2 m_A(e^{x\cdot\zeta/h} u), v\rangle_\Omega &= -\int_\Omega h^2 A e^{x\cdot\zeta/h} u \cdot D(e^{-x\cdot\zeta/h} v) dx \\ &= -\int_\Omega (hi\zeta \cdot Auv + h^2 Au \cdot Dv) dx,\end{aligned}$$

and therefore,

$$e^{-x\cdot\zeta/h} \circ h^2 m_A \circ e^{x\cdot\zeta/h} = -hi\zeta \cdot A + h^2 m_A.$$

Furthermore, we obtain that

$$\begin{aligned} e^{-x\cdot\zeta/h} \circ (-h^2\Delta) \circ e^{x\cdot\zeta/h} &= -h^2\Delta - 2ih\zeta \cdot D, \\ e^{-x\cdot\zeta/h} \circ h^2(A \cdot D) \circ e^{x\cdot\zeta/h} &= h^2 A \cdot D - hi\zeta \cdot A.\end{aligned}$$

Hence, we have

$$e^{-x\cdot\zeta/h} \circ h^2 L_{A,q} \circ e^{x\cdot\zeta/h} = -h^2\Delta - 2ih\zeta \cdot D + h^2 A \cdot D - 2hi\zeta \cdot A + h^2 m_A + h^2(A^2 + q). \tag{2.10}$$

We shall consider ζ depending slightly on h, i.e. $\zeta = \zeta_0 + \zeta_1$ with ζ_0 being independent of h and $\zeta_1 = \mathcal{O}(h)$ as $h \to 0$. We also assume that $|\mathrm{Re}\,\zeta_0| = |\mathrm{Im}\,\zeta_0| = 1$. Then we write (2.10) as follows,

$$\begin{aligned} e^{-x\cdot\zeta/h} \circ h^2 L_{A,q} \circ e^{x\cdot\zeta/h} = & - h^2\Delta - 2ih\zeta_0\cdot D - 2ih\zeta_1\cdot D + h^2 A\cdot D - 2hi\zeta_0\cdot A^\sharp \\ & - 2hi\zeta_0\cdot(A - A^\sharp) - 2hi\zeta_1\cdot A + h^2 m_A + h^2(A^2+q). \end{aligned}$$

In order that (2.9) be a solution of (2.8), we require that

$$\zeta_0\cdot Da + \zeta_0\cdot A^\sharp a = 0 \quad \text{in} \quad \mathbb{R}^n, \tag{2.11}$$

and

$$\begin{aligned} e^{-x\cdot\zeta/h} h^2 L_{A,q} e^{x\cdot\zeta/h} r = & -(-h^2\Delta a + h^2 A\cdot Da + h^2 m_A(a) + h^2(A^2+q)a) \\ & + 2ih\zeta_1\cdot Da + 2hi\zeta_0\cdot(A-A^\sharp)a + 2hi\zeta_1\cdot Aa =: g \quad \text{in} \quad \Omega. \end{aligned} \tag{2.12}$$

The equation (2.11) is the first transport equation and one looks for its solution in the form $a = e^{\Phi^\sharp}$, where $\Phi^\sharp$ solves the equation

$$\zeta_0\cdot\nabla\Phi^\sharp + i\zeta_0\cdot A^\sharp = 0 \quad \text{in} \quad \mathbb{R}^n. \tag{2.13}$$

As $\zeta_0\cdot\zeta_0 = 0$ and $|\mathrm{Re}\,\zeta_0| = |\mathrm{Im}\,\zeta_0| = 1$, the operator $N_{\zeta_0} := \zeta_0\cdot\nabla$ is the $\bar\partial$–operator in suitable linear coordinates. Let us introduce an inverse operator defined by

$$(N_{\zeta_0}^{-1} f)(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{f(x - y_1\mathrm{Re}\,\zeta_0 - y_2\mathrm{Im}\,\zeta_0)}{y_1 + iy_2} dy_1 dy_2, \quad f\in C_0(\mathbb{R}^n).$$

We have the following result, see [17, Lemma 4.6].

Lemma 2.4. *Let $f\in W^{k,\infty}(\mathbb{R}^n)$, $k\ge 0$, with* supp $(f)\subset B(0,R)$. *Then $\Phi = N_{\zeta_0}^{-1} f \in W^{k,\infty}(\mathbb{R}^n)$ satisfies $N_{\zeta_0}\Phi = f$ in $\mathbb{R}^n$, and we have*

$$\|\Phi\|_{W^{k,\infty}(\mathbb{R}^n)} \le C\|f\|_{W^{k,\infty}(\mathbb{R}^n)}, \tag{2.14}$$

where $C = C(R)$. If $f\in C_0(\mathbb{R}^n)$, then $\Phi\in C(\mathbb{R}^n)$.

Thanks to Lemma 2.4, the function $\Phi^\sharp(x,\zeta_0;\tau) := N_{\zeta_0}^{-1}(-i\zeta_0\cdot A^\sharp) \in C^\infty(\mathbb{R}^n)$ satisfies the equation (2.13). Furthermore, the estimates (2.7) and (2.14) imply that

$$\|\partial^\alpha\Phi^\sharp\|_{L^\infty(\mathbb{R}^n)} \le C_\alpha \tau^{-|\alpha|}, \quad \text{for all} \quad \alpha, \quad |\alpha|\ge 0. \tag{2.15}$$

Owing to [21, Lemma 3.1], we have the following result, where we use the norms

$$\|f\|^2_{L^2_\delta(\mathbb{R}^n)} = \int_{\mathbb{R}^n} (1+|x|^2)^\delta |f(x)|^2 dx.$$

Lemma 2.5. *Let $-1<\delta<0$ and let $f\in L^2_{\delta+1}(\mathbb{R}^n)$. Then there exists a constant $C>0$, independent of ζ_0, such that*

$$\|N_{\zeta_0}^{-1} f\|_{L^2_\delta(\mathbb{R}^n)} \le C\|f\|_{L^2_{\delta+1}(\mathbb{R}^n)}.$$

Setting $\Phi(\cdot,\zeta_0) := N_{\zeta_0}^{-1}(-i\zeta_0\cdot A)\in L^\infty(\mathbb{R}^n)$, it follows from Lemma 2.5 and the estimate (2.6) that $\Phi^\sharp(\cdot,\zeta_0;\tau)$ converges to $\Phi(\cdot,\zeta_0)$ in $L^2_{\mathrm{loc}}(\mathbb{R}^n)$ as $\tau\to 0$.

Let us turn now to the equation (2.12). First notice that the right hand side g of (2.12) belongs to $H^{-1}(\Omega)$ and we would like to estimate $\|g\|_{H^{-1}_{\mathrm{scl}}(\Omega)}$. To that end, let $0\ne\psi\in C_0^\infty(\Omega)$. Then using (2.15) and the fact that $\zeta_1=\mathcal{O}(h)$, we get by the Cauchy–Schwarz inequality,

$$
\begin{aligned}
&|\langle h^2\Delta a,\psi\rangle_\Omega|\le \mathcal{O}(h^2/\tau^2)\|\psi\|_{L^2(\Omega)}\le \mathcal{O}(h^2/\tau^2)\|\psi\|_{H^1_{\mathrm{scl}}(\Omega)},\\
&|\langle h^2A\cdot Da,\psi\rangle_\Omega|\le \mathcal{O}(h^2/\tau)\|\psi\|_{H^1_{\mathrm{scl}}(\Omega)},\\
&|\langle 2ih\zeta_1\cdot Da,\psi\rangle_\Omega|\le \mathcal{O}(h^2/\tau)\|\psi\|_{H^1_{\mathrm{scl}}(\Omega)},\\
&|\langle 2hi\zeta_1\cdot Aa,\psi\rangle_\Omega|\le \mathcal{O}(h^2)\|\psi\|_{H^1_{\mathrm{scl}}(\Omega)}.
\end{aligned}
$$

Using (2.6) and (2.15), we have

$$
\begin{aligned}
|\langle 2hi\zeta_0\cdot(A-A^\sharp)a,\psi\rangle_\Omega| &\le \mathcal{O}(h)\|a\|_{L^\infty(\mathbb{R}^n)}\|A-A^\sharp\|_{L^2(\Omega)}\|\psi\|_{L^2(\Omega)}\\
&\le \mathcal{O}(h)o_{\tau\to 0}(1)\|\psi\|_{H^1_{\mathrm{scl}}(\Omega)}.
\end{aligned}
$$

With the help of (2.6), (2.7), and (2.15), we obtain that

$$
\begin{aligned}
|\langle h^2m_A(a),\psi\rangle_\Omega| &\le \left|\int_\Omega h^2A^\sharp a\cdot D\psi dx\right| + \left|\int_\Omega h^2(A-A^\sharp)a\cdot D\psi dx\right|\\
&\le \left|\int_\Omega h^2(D\cdot(A^\sharp a))\psi dx\right| + \mathcal{O}(h)\|A-A^\sharp\|_{L^2(\Omega)}\|hD\psi\|_{L^2(\Omega)}\\
&\le (\mathcal{O}(h^2/\tau)+\mathcal{O}(h)o_{\tau\to 0}(1))\|\psi\|_{H^1_{\mathrm{scl}}(\Omega)}.
\end{aligned}
$$

We also have $\|h^2(A^2+q)a\|_{L^2(\Omega)}\le\mathcal{O}(h^2)$. Thus, from the above estimates, we conclude that

$$
\|g\|_{H^{-1}_{\mathrm{scl}}(\Omega)}\le \mathcal{O}(h^2/\tau^2)+\mathcal{O}(h)o_{\tau\to 0}(1).
$$

Choosing now $\tau=h^\sigma$ with some σ, $0<\sigma<1/2$, we get

$$
\|g\|_{H^{-1}_{\mathrm{scl}}(\Omega)} = o(h)\quad \text{as}\quad h\to 0. \tag{2.16}
$$

Thanks to Proposition 2.3 and (2.16), for $h>0$ small enough, there exists a solution $r\in H^1(\Omega)$ of (2.12) such that $\|r\|_{H^1_{\mathrm{scl}}(\Omega)}=o(1)$ as $h\to 0$.

The discussion led in this section can be summarized in the following proposition.

Proposition 2.6. *Let $\Omega\subset\mathbb{R}^n$, $n\ge 3$, be a bounded open set. Let $A\in L^\infty(\Omega,\mathbb{C}^n)$, $q\in L^\infty(\Omega,\mathbb{C})$, and let $\zeta\in\mathbb{C}^n$ be such that $\zeta\cdot\zeta=0$, $\zeta=\zeta_0+\zeta_1$ with ζ_0 being independent of $h>0$, $|\mathrm{Re}\,\zeta_0|=|\mathrm{Im}\,\zeta_0|=1$, and $\zeta_1=\mathcal{O}(h)$ as $h\to 0$. Then for all $h>0$ small enough, there exists a solution $u(x,\zeta;h)\in H^1(\Omega)$ to the magnetic Schrödinger equation $L_{A,q}u=0$ in Ω, of the form*

$$
u(x,\zeta;h)=e^{x\cdot\zeta/h}(e^{\Phi^\sharp(x,\zeta_0;h)}+r(x,\zeta;h)).
$$

The function $\Phi^\sharp(\cdot,\zeta_0;h)\in C^\infty(\mathbb{R}^n)$ *satisfies* $\|\partial^\alpha\Phi^\sharp\|_{L^\infty(\mathbb{R}^n)}\le C_\alpha h^{-\sigma|\alpha|}$, $0<\sigma<1/2$, *for all* α, $|\alpha|\ge 0$, *and* $\Phi^\sharp(\cdot,\zeta_0;h)$ *converges to* $\Phi(\cdot,\zeta_0):=N_{\zeta_0}^{-1}(-i\zeta_0\cdot A)\in L^\infty(\mathbb{R}^n)$ *in* $L^2_{\text{loc}}(\mathbb{R}^n)$ *as* $h\to 0$. *Here we have extended* A *by zero to* $\mathbb{R}^n\setminus\Omega$. *The remainder* r *is such that* $\|r\|_{H^1_{\text{scl}}(\Omega)}=o(1)$ *as* $h\to 0$.

3. Proof of Theorem 1.1

Let us begin by recalling the following auxiliary, essentially well-known, result which shows that the set of the Cauchy data for the magnetic Schrödinger operator remains unchanged if the gradient of a function, vanishing along the boundary, is added to the magnetic potential, see [17, Lemma 4.1], [20].

Lemma 3.1. *Let* $\Omega\subset\mathbb{R}^n$ *be a bounded open set, let* $A\in L^\infty(\Omega,\mathbb{C}^n)$, $q\in L^\infty(\Omega,\mathbb{C})$, *and let* $\psi\in W^{1,\infty}$ *in a neighborhood of* $\overline{\Omega}$. *Then we have*

$$e^{-i\psi}\circ L_{A,q}\circ e^{i\psi}=L_{A+\nabla\psi,q}. \tag{3.1}$$

If furthermore, $\psi|_{\partial\Omega}=0$ *then*

$$C_{A,q}=C_{A+\nabla\psi,q}. \tag{3.2}$$

Proof. Let us notice first that the assumption that $\psi\in W^{1,\infty}$ in a neighborhood of $\overline{\Omega}$ implies that ψ is Lipschitz continuous on $\overline{\Omega}$, so that $\psi|_{\partial\Omega}$ is well-defined pointwise.

Since (3.1) follows by a direct computation, only (3.2) has to be established. To that end, let $u\in H^1(\Omega)$ be a solution to $L_{A,q}u=0$ in Ω. Then $e^{-i\psi}u\in H^1(\Omega)$ satisfies $L_{A+\nabla\psi,q}(e^{-i\psi}u)=0$ in Ω. Let us show that $T(e^{-i\psi}u)=Tu$. In other words, we have to check that

$$u(e^{-i\psi}-1)\in H^1_0(\Omega). \tag{3.3}$$

Since the function $e^{-i\psi}-1$ is Lipschitz continuous on $\overline{\Omega}$ and vanishes along $\partial\Omega$, we have $|e^{-i\psi(x)}-1|\le Cd(x)$ for any $x\in\Omega$ and some constant $C>0$. Here $d(x)$ is the distance from x to the boundary of Ω. Then (3.3) follows from the following fact: if $v\in H^1(\Omega)$ and $v/d\in L^2(\Omega)$, then $v\in H^1_0(\Omega)$, see [6, Theorem 3.4, p. 223].

Let us now show that $N_{A+\nabla\psi,q}(e^{-i\psi}u)=N_{A,q}u$. To that end, first as above, one observes that for $g\in H^1(\Omega)$, we have $[g]=[e^{i\psi}g]$. Thus,

$$(N_{A+\nabla\psi,q}(e^{-i\psi}u),[g])_\Omega=(N_{A+\nabla\psi,q}(e^{-i\psi}u),[e^{i\psi}g])_\Omega=(N_{A,q}(u),[g])_\Omega,$$

for any $[g]\in H^1(\Omega)/H^1_0(\Omega)$, and therefore, $C_{A,q}\subset C_{A+\nabla\psi,q}$. The proof is complete. □

The first step in the proof of Theorem 1.1 is the derivation of the following integral identity based on the fact that $C_{A_1,q_1}=C_{A_2,q_2}$, see also [17, Lemma 4.3].

Proposition 3.2. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded open set. Assume that $A_1, A_2 \in L^\infty(\Omega, \mathbb{C}^n)$ and $q_1, q_2 \in L^\infty(\Omega, \mathbb{C})$. If $C_{A_1,q_1} = C_{A_2,q_2}$, then the following integral identity*

$$\int_\Omega i(A_1 - A_2)\cdot(u_1\nabla\overline{u_2} - \overline{u_2}\nabla u_1)dx + \int_\Omega (A_1^2 - A_2^2 + q_1 - q_2)u_1\overline{u_2}dx = 0 \quad (3.4)$$

holds for any $u_1, u_2 \in H^1(\Omega)$ satisfying $L_{A_1,q_1}u_1 = 0$ in Ω and $L_{\overline{A_2},\overline{q_2}}u_2 = 0$ in Ω, respectively.

Proof. Let $u_1, u_2 \in H^1(\Omega)$ be solutions to $L_{A_1,q_1}u_1 = 0$ in Ω and $L_{\overline{A_2},\overline{q_2}}u_2 = 0$ in Ω, respectively. Then the fact that $C_{A_1,q_1} = C_{A_2,q_2}$ implies that there is $v_2 \in H^1(\Omega)$ satisfying $L_{A_2,q_2}v_2 = 0$ in Ω such that

$$Tu_1 = Tv_2 \quad \text{and} \quad N_{A_1,q_1}u_1 = N_{A_2,q_2}v_2.$$

This together with (1.2) shows that

$$(N_{A_1,q_1}u_1, [\overline{u_2}])_\Omega = (N_{A_2,q_2}v_2, [\overline{u_2}])_\Omega = \overline{(N_{\overline{A_2},\overline{q_2}}u_2, [\overline{v_2}])_\Omega} = \overline{(N_{\overline{A_2},\overline{q_2}}u_2, [\overline{u_1}])_\Omega}.$$

Then the integral identity (3.4) follows from the definition (1.2) of $N_{A_1,q_1}u_1$ and $N_{\overline{A_2},\overline{q_2}}u_2$. The proof is complete. ☐

We shall use the integral identity (3.4) with u_1 and u_2 being complex geometric optics solutions for the magnetic Schrödinger equations in Ω. To construct such solutions, let $\xi, \mu_1, \mu_2 \in \mathbb{R}^n$ be such that $|\mu_1| = |\mu_2| = 1$ and $\mu_1\cdot\mu_2 = \mu_1\cdot\xi = \mu_2\cdot\xi = 0$. Similarly to [20], we set

$$\zeta_1 = \frac{ih\xi}{2} + \mu_1 + i\sqrt{1 - h^2\frac{|\xi|^2}{4}}\mu_2, \quad \zeta_2 = -\frac{ih\xi}{2} - \mu_1 + i\sqrt{1 - h^2\frac{|\xi|^2}{4}}\mu_2, \quad (3.5)$$

so that $\zeta_j\cdot\zeta_j = 0$, $j = 1, 2$, and $(\zeta_1 + \overline{\zeta_2})/h = i\xi$. Here $h > 0$ is a small enough semiclassical parameter. Moreover, $\zeta_1 = \mu_1 + i\mu_2 + \mathcal{O}(h)$ and $\zeta_2 = -\mu_1 + i\mu_2 + \mathcal{O}(h)$ as $h \to 0$.

By Proposition 2.6, for all $h > 0$ small enough, there exists a solution $u_1(x, \zeta_1; h) \in H^1(\Omega)$ to the magnetic Schrödinger equation $L_{A_1,q_1}u_1 = 0$ in Ω, of the form

$$u_1(x, \zeta_1; h) = e^{x\cdot\zeta_1/h}(e^{\Phi_1^\sharp(x,\mu_1+i\mu_2;h)} + r_1(x, \zeta_1; h)), \quad (3.6)$$

where $\Phi_1^\sharp(\cdot, \mu_1 + i\mu_2; h) \in C^\infty(\mathbb{R}^n)$ satisfies the estimate

$$\|\partial^\alpha\Phi_1^\sharp\|_{L^\infty(\mathbb{R}^n)} \leq C_\alpha h^{-\sigma|\alpha|}, \quad 0 < \sigma < 1/2, \quad (3.7)$$

for all α, $|\alpha| \geq 0$, $\Phi_1^\sharp(\cdot, \mu_1 + i\mu_2; h)$ converges to

$$\Phi_1(\cdot, \mu_1 + i\mu_2) := N_{\mu_1+i\mu_2}^{-1}(-i(\mu_1 + i\mu_2)\cdot A_1) \in L^\infty(\mathbb{R}^n) \quad (3.8)$$

in $L^2_{\mathrm{loc}}(\mathbb{R}^n)$ as $h \to 0$, and

$$\|r_1\|_{H^1_{\mathrm{scl}}(\Omega)} = o(1) \quad \text{as} \quad h \to 0. \quad (3.9)$$

Similarly, for all $h>0$ small enough, there exists a solution $u_2(x,\zeta_2;h)\in H^1(\Omega)$ to the magnetic Schrödinger equation $L_{\overline{A_2},\overline{q_2}}u_2=0$ in Ω, of the form

$$u_2(x,\zeta_2;h)=e^{x\cdot\zeta_2/h}(e^{\Phi_2^\sharp(x,-\mu_1+i\mu_2;h)}+r_2(x,\zeta_2;h)), \quad (3.10)$$

where $\Phi_2^\sharp(\cdot,-\mu_1+i\mu_2;h)\in C^\infty(\mathbb{R}^n)$ satisfies the estimate

$$\|\partial^\alpha\Phi_2^\sharp\|_{L^\infty(\mathbb{R}^n)}\le C_\alpha h^{-\sigma|\alpha|},\quad 0<\sigma<1/2, \quad (3.11)$$

for all α, $|\alpha|\ge 0$. Furthermore, $\Phi_2^\sharp(\cdot,-\mu_1+i\mu_2;h)$ converges to

$$\Phi_2(\cdot,-\mu_1+i\mu_2):=N^{-1}_{-\mu_1+i\mu_2}(-i(-\mu_1+i\mu_2)\cdot\overline{A_2})\in L^\infty(\mathbb{R}^n) \quad (3.12)$$

in $L^2_{\text{loc}}(\mathbb{R}^n)$ as $h\to 0$, and

$$\|r_2\|_{H^1_{\text{scl}}(\Omega)}=o(1)\quad\text{as}\quad h\to 0. \quad (3.13)$$

We shall next substitute u_1 and u_2, given by (3.6) and (3.10), into the integral identity (3.4), multiply it by h, and let $h\to 0$. We first compute

$$\begin{aligned}hu_1\nabla\overline{u_2}=&\overline{\zeta_2}e^{ix\cdot\xi}(e^{\Phi_1^\sharp+\overline{\Phi_2^\sharp}}+e^{\Phi_1^\sharp}\overline{r_2}+r_1e^{\overline{\Phi_2^\sharp}}+r_1\overline{r_2})\\&+he^{ix\cdot\xi}(e^{\Phi_1^\sharp}\nabla e^{\overline{\Phi_2^\sharp}}+e^{\Phi_1^\sharp}\nabla\overline{r_2}+r_1\nabla e^{\overline{\Phi_2^\sharp}}+r_1\nabla\overline{r_2}).\end{aligned}$$

Recall that $\overline{\zeta_2}=-\mu_1-i\mu_2+\mathcal{O}(h)$. We shall show that

$$(\mu_1+i\mu_2)\cdot\int_\Omega(A_1-A_2)e^{ix\cdot\xi}e^{\Phi_1^\sharp+\overline{\Phi_2^\sharp}}dx\to(\mu_1+i\mu_2)\cdot\int_\Omega(A_1-A_2)e^{ix\cdot\xi}e^{\Phi_1+\overline{\Phi_2}}dx,$$

as $h\to 0$, where Φ_1 and Φ_2 are defined by (3.8) and (3.12), respectively. To that end, we have

$$\begin{aligned}\Big|(\mu_1+i\mu_2)\cdot\int_\Omega(A_1-A_2)e^{ix\cdot\xi}\big(e^{\Phi_1^\sharp+\overline{\Phi_2^\sharp}}-e^{\Phi_1+\overline{\Phi_2}}\big)dx\Big|&\le C\big\|e^{\Phi_1^\sharp+\overline{\Phi_2^\sharp}}-e^{\Phi_1+\overline{\Phi_2}}\big\|_{L^2(\Omega)}\\&\le C\|\Phi_1^\sharp+\overline{\Phi_2^\sharp}-\Phi_1-\overline{\Phi_2}\|_{L^2(\Omega)}\to 0,\end{aligned}$$

as $h\to 0$. Here we have used the inequality

$$|e^z-e^w|\le|z-w|e^{\max(\operatorname{Re}z,\operatorname{Re}w)},\quad z,w\in\mathbb{C}, \quad (3.14)$$

obtained by integration of e^z from z to w, and the fact that $\Phi_j,\Phi_j^\sharp\in L^\infty(\mathbb{R}^n)$, $j=1,2$, and $\|\Phi_j^\sharp\|_{L^\infty(\mathbb{R}^n)}\le C$ uniformly in h.

Now using the estimates (3.7), (3.9), (3.11) and (3.13), we get

$$\begin{aligned}&\Big|\int_\Omega i(A_1-A_2)\cdot\overline{\zeta_2}e^{ix\cdot\xi}(e^{\Phi_1^\sharp}\overline{r_2}+r_1e^{\overline{\Phi_2^\sharp}}+r_1\overline{r_2})dx\Big|\\&\quad\le C\|A_1-A_2\|_{L^\infty}(\|e^{\Phi_1^\sharp}\|_{L^2}\|\overline{r_2}\|_{L^2}+\|r_1\|_{L^2}\|e^{\overline{\Phi_2^\sharp}}\|_{L^2}+\|r_1\|_{L^2}\|\overline{r_2}\|_{L^2})=o(1),\end{aligned}$$

as $h\to 0$. We also obtain that

$$\Big|\int_\Omega hi(A_1-A_2)\cdot e^{ix\cdot\xi}(e^{\Phi_1^\sharp}\nabla e^{\overline{\Phi_2^\sharp}}+e^{\Phi_1^\sharp}\nabla\overline{r_2}+r_1\nabla e^{\overline{\Phi_2^\sharp}}+r_1\nabla\overline{r_2})dx\Big|$$
$$\le \mathcal{O}(h)(h^{-\sigma}+h^{-1}o(1)+o(1)h^{-\sigma}+o(1)h^{-1})=o(1),$$

as $h\to 0$. Here $0<\sigma<1/2$. Furthermore,

$$\Big|h\int_\Omega(A_1^2-A_2^2+q_1-q_2)e^{ix\cdot\xi}(e^{\Phi_1^\sharp+\overline{\Phi_2^\sharp}}+e^{\Phi_1^\sharp}\overline{r_2}+r_1e^{\overline{\Phi_2^\sharp}}+r_1\overline{r_2})dx\Big|=\mathcal{O}(h),$$

as $h\to 0$. Hence, substituting u_1 and u_2, given by (3.6) and (3.10), into the integral identity (3.4), multiplying it by h, and letting $h\to 0$, we get

$$(\mu_1+i\mu_2)\cdot\int_{\mathbb{R}^n}(A_1-A_2)e^{ix\cdot\xi}e^{\Phi_1(x,\mu_1+i\mu_2)+\overline{\Phi_2(x,-\mu_1+i\mu_2)}}dx=0, \tag{3.15}$$

where

$$\Phi_1=N^{-1}_{\mu_1+i\mu_2}(-i(\mu_1+i\mu_2)\cdot A_1)\in L^\infty(\mathbb{R}^n),$$
$$\Phi_2=N^{-1}_{-\mu_1+i\mu_2}(-i(-\mu_1+i\mu_2)\cdot\overline{A_2})\in L^\infty(\mathbb{R}^n).$$

Notice that the integration in (3.15) is extended to all of $\mathbb{R}^n$, since $A_1=A_2=0$ on $\mathbb{R}^n\setminus\Omega$.

The next step is to remove the function $e^{\Phi_1+\overline{\Phi_2}}$ in the integral (3.15). First using the following properties of the Cauchy transform,

$$\overline{N_\zeta^{-1}f}=N_{\overline{\zeta}}^{-1}\overline{f},\quad N_{-\zeta}^{-1}f=-N_\zeta^{-1}f,$$

we see that

$$\Phi_1+\overline{\Phi_2}=N^{-1}_{\mu_1+i\mu_2}(-i(\mu_1+i\mu_2)\cdot(A_1-A_2)). \tag{3.16}$$

We have the following result.

Proposition 3.3. *Let $\xi,\mu_1,\mu_2\in\mathbb{R}^n$, $n\ge 3$, be such that $|\mu_1|=|\mu_2|=1$ and $\mu_1\cdot\mu_2=\mu_1\cdot\xi=\mu_2\cdot\xi=0$. Let $W\in(L^\infty\cap\mathcal{E}')(\mathbb{R}^n,\mathbb{C}^n)$ and $\phi=N^{-1}_{\mu_1+i\mu_2}(-i(\mu_1+i\mu_2)\cdot W)$. Then*

$$(\mu_1+i\mu_2)\cdot\int_{\mathbb{R}^n}W(x)e^{ix\cdot\xi}e^{\phi(x)}dx=(\mu_1+i\mu_2)\cdot\int_{\mathbb{R}^n}W(x)e^{ix\cdot\xi}dx. \tag{3.17}$$

Proof. The statement of the proposition for $W\in C_0(\mathbb{R}^n,\mathbb{C}^n)$ is due to [7], with similar ideas appearing in [20]. See also [18, Lemma 6.2]. For the completeness and convenience of the reader, we shall give a complete proof of the proposition here.

Assume first that $W\in C_0^\infty(\mathbb{R}^n,\mathbb{C}^n)$. Then by Lemma 2.4 we have

$$\phi=N^{-1}_{\mu_1+i\mu_2}(-i(\mu_1+i\mu_2)\cdot W)\in C^\infty(\mathbb{R}^n). \tag{3.18}$$

We can always assume that $\mu_1=(1,0,\dots,0)$ and $\mu_2=(0,1,0,\dots,0)$, so that $\xi=(0,0,\xi'')$, $\xi''\in\mathbb{R}^{n-2}$, and therefore,

$$(\partial_{x_1}+i\partial_{x_2})\phi=-i(\mu_1+i\mu_2)\cdot W \quad \text{in} \quad \mathbb{R}^n.$$

Hence, writing $x=(x',x'')$, $x'=(x_1,x_2)$, $x''\in\mathbb{R}^{n-2}$, we get

$$\begin{aligned}(\mu_1+i\mu_2)\cdot\int_{\mathbb{R}^n}W(x)e^{ix\cdot\xi}e^{\phi(x)}dx&=i\int_{\mathbb{R}^n}e^{ix''\cdot\xi''}e^{\phi(x)}(\partial_{x_1}+i\partial_{x_2})\phi(x)dx\\&=i\int_{\mathbb{R}^{n-2}}e^{ix''\cdot\xi''}h(x'')dx'',\end{aligned}$$

where

$$\begin{aligned}h(x'')=\int_{\mathbb{R}^2}(\partial_{x_1}+i\partial_{x_2})e^{\phi(x)}dx'&=\lim_{R\to\infty}\int_{|x'|\le R}(\partial_{x_1}+i\partial_{x_2})e^{\phi(x)}dx'\\&=\lim_{R\to\infty}\int_{|x'|=R}e^{\phi(x)}(\nu_1+i\nu_2)dS_R(x').\end{aligned}$$

Here $\nu=(\nu_1,\nu_2)$ is the unit outer normal to the circle $|x'|=R$, and we have used the Gauss theorem.

It follows from (3.18) that $|\phi(x',x'')|=\mathcal{O}(1/|x'|)$ as $|x'|\to\infty$. Hence, we have

$$e^{\phi}=1+\phi+\mathcal{O}(|\phi|^2)=1+\phi+\mathcal{O}(|x'|^{-2}) \quad \text{as} \quad |x'|\to\infty.$$

Since

$$\int_{|x'|=R}(\nu_1+i\nu_2)dS_R(x')=\int_{|x'|\le R}(\partial_{x_1}+i\partial_{x_2})(1)dx'=0,$$

$$\left|\int_{|x'|=R}\mathcal{O}(|x'|^{-2})(\nu_1+i\nu_2)dS_R(x')\right|\le\mathcal{O}(R^{-1}) \quad \text{as} \quad R\to\infty,$$

we obtain that

$$\begin{aligned}h(x'')=\lim_{R\to\infty}\int_{|x'|=R}\phi(x)(\nu_1+i\nu_2)dS_R(x')&=\lim_{R\to\infty}\int_{|x'|\le R}(\partial_{x_1}+i\partial_{x_2})\phi(x)dx'\\&=-\int_{\mathbb{R}^2}i(\mu_1+i\mu_2)\cdot W(x)dx',\end{aligned}$$

which shows (3.17) for $W\in C_0^\infty(\mathbb{R}^n,\mathbb{C}^n)$.

To prove (3.17) for $W\in(L^\infty\cap\mathcal{E}')(\mathbb{R}^n,\mathbb{C}^n)$, consider the regularizations $W_j=\chi_j*W\in C_0^\infty(\mathbb{R}^n)$. Here $\chi_j(x)=j^n\chi(jx)$ is the usual mollifier with $0\le\chi\in C_0^\infty(\mathbb{R}^n)$ such that $\int\chi dx=1$. Then $W_j\to W$ in $L^2(\mathbb{R}^n)$ as $j\to\infty$ and

$$\|W_j\|_{L^\infty(\mathbb{R}^n)}\le\|W\|_{L^\infty(\mathbb{R}^n)}\|\chi_j\|_{L^1(\mathbb{R}^n)}=\|W\|_{L^\infty(\mathbb{R}^n)}, \quad j=1,2,\dots. \tag{3.19}$$

Furthermore, there is a compact set $K\subset\subset\mathbb{R}^n$ such that $\text{supp }(W_j),\text{supp }(W)\subset K$, $j=1,2,\dots$.

We set $\phi_j = N^{-1}_{\mu_1+i\mu_2}(-i(\mu_1+i\mu_2)\cdot W_j) \in C^\infty(\mathbb{R}^n)$. Then by Lemma 2.5, we know that $\phi_j \to \phi$ in $L^2_{\text{loc}}(\mathbb{R}^n)$ as $j \to \infty$. Lemma 2.4 together with the estimate (3.19) implies that

$$\|\phi_j\|_{L^\infty(\mathbb{R}^n)} \le C\|W_j\|_{L^\infty(\mathbb{R}^n)} \le C\|W\|_{L^\infty(\mathbb{R}^n)}, \quad j = 1, 2, \dots. \tag{3.20}$$

For $j = 1, 2, \dots$, we have

$$(\mu_1 + i\mu_2) \cdot \int_K W_j(x)e^{ix\cdot\xi}e^{\phi_j(x)}dx = (\mu_1 + i\mu_2) \cdot \int_K W_j(x)e^{ix\cdot\xi}dx. \tag{3.21}$$

The fact that the integral in right hand side of (3.21) converges to the integral in the right hand side of (3.17) as $j \to \infty$ follows from the estimate

$$\left|(\mu_1 + i\mu_2) \cdot \int_K (W_j(x) - W(x))e^{ix\cdot\xi}dx\right| \le C\|W_j - W\|_{L^2(K)} \to 0, \quad j \to \infty.$$

In order to show that the integral in the left hand side of (3.21) converges to the integral in the left hand side of (3.17) as $j \to \infty$, we establish that $I_1 + I_2 \to 0$ as $j \to \infty$, where

$$I_1 := (\mu_1 + i\mu_2) \cdot \int_K (W_j(x) - W(x))e^{ix\cdot\xi}e^{\phi_j(x)}dx,$$
$$I_2 := (\mu_1 + i\mu_2) \cdot \int_K W(x)e^{ix\cdot\xi}(e^{\phi_j(x)} - e^{\phi(x)})dx.$$

Using (3.20), we have

$$|I_1| \le Ce^{\|\phi_j\|_{L^\infty(\mathbb{R}^n)}} \int_K |W_j(x) - W(x)|dx \le C\|W_j - W\|_{L^2(K)} \to 0, \quad j \to \infty.$$

Using (3.14) and (3.20), we get

$$|I_2| \le C\|W\|_{L^\infty(\mathbb{R}^n)}\|e^{\phi_j(x)} - e^{\phi(x)}\|_{L^2(K)} \le C\|\phi_j - \phi\|_{L^2(K)} \to 0, \quad j \to \infty.$$

Here we have also used (3.20) and the fact that $\phi_j \to \phi$ in $L^2_{\text{loc}}(\mathbb{R}^n)$ as $j \to \infty$. Hence, passing to the limit as $j \to \infty$ in (3.21), we obtain the identity (3.17). The proof is complete. □

By Proposition 3.3 we conclude from (3.15) and (3.16) that

$$(\mu_1 + i\mu_2) \cdot \int_{\mathbb{R}^n} (A_1(x) - A_2(x))e^{ix\cdot\xi}dx = 0. \tag{3.22}$$

It follows from (3.22) that $\mu\cdot(\widehat{A}_1(\xi) - \widehat{A}_2(\xi)) = 0$ whenever $\mu, \xi \in \mathbb{R}^n$ are such that $\mu\cdot\xi = 0$. Here $\widehat{A}_j$ is the Fourier transform of A_j, $j = 1, 2$. Let $\mu_{jk}(\xi) = \xi_j e_k - \xi_k e_j$ for $j \ne k$, where $e_1, \dots, e_n$ is the standard basis of $\mathbb{R}^n$. Then $\mu_{jk}(\xi) \cdot \xi = 0$, and therefore,

$$\xi_j(\widehat{A}_{1,k}(\xi) - \widehat{A}_{2,k}(\xi)) - \xi_k(\widehat{A}_{1,j}(\xi) - \widehat{A}_{2,j}(\xi)) = 0.$$

Hence, $\partial_{x_j}(A_{1,k}-A_{2,k})-\partial_{x_k}(A_{1,j}-A_{2,j})=0$ in $\mathbb{R}^n$ in the sense of distributions, for $j\neq k$, and thus, $d(A_1-A_2)=0$ in $\mathbb{R}^n$.

Our next goal is to show that $q_1=q_2$ in Ω. First, viewing A_1-A_2 as a 1–current and using the Poincaré lemma for currents, we conclude that there is $\psi\in\mathcal{D}'(\mathbb{R}^n)$ such that $d\psi=A_1-A_2\in(L^\infty\cap\mathcal{E}')(\mathbb{R}^n)$ in $\mathbb{R}^n$, see [16]. It follows from [10, Theorem 4.5.11] that ψ is continuous on $\mathbb{R}^n$, and since ψ is constant near infinity, we have $\psi\in L^\infty(\mathbb{R}^n)$. Therefore, $\psi\in W^{1,\infty}(\mathbb{R}^n)$, and without loss of generality, we may assume that there is an open ball B such that $\Omega\subset\subset B$ and supp $(\psi)\subset B$.

We want to add $\nabla\psi$ to the potential A_2 without changing the set of the Cauchy data for L_{A_2,q_2} on the ball B. To that end, we shall need the following result, which is due to [17, Lemma 4.2].

Proposition 3.4. *Let $\Omega,\Omega'\subset\mathbb{R}^n$ be bounded open sets such that $\Omega\subset\subset\Omega'$. Let $A_1,A_2\in L^\infty(\Omega',\mathbb{C}^n)$, and $q_1,q_2\in L^\infty(\Omega',\mathbb{C})$. Assume that*

$$A_1=A_2 \quad \textit{and} \quad q_1=q_2 \quad \textit{in} \quad \Omega'\setminus\Omega. \tag{3.23}$$

If $C_{A_1,q_1}=C_{A_2,q_2}$ then $C'_{A_1,q_1}=C'_{A_2,q_2}$, where C'_{A_j,q_j} is the set of the Cauchy data for L_{A_j,q_j} in Ω', $j=1,2$.

Proof. Let $u_1'\in H^1(\Omega')$ be a solution to $L_{A_1,q_1}u_1'=0$ in Ω' and let $u_1=u_1'|_\Omega\in H^1(\Omega)$. As $C_{A_1,q_1}=C_{A_2,q_2}$, there exists $u_2\in H^1(\Omega)$ satisfying $L_{A_2,q_2}u_2=0$ in Ω such that

$$Tu_2=Tu_1 \quad \text{and} \quad N_{A_2,q_2}u_2=N_{A_1,q_1}u_1 \quad \text{in} \quad \Omega.$$

In particular, $\varphi:=u_2-u_1\in H^1_0(\Omega)\subset H^1_0(\Omega')$. We define

$$u_2'=u_1'+\varphi\in H^1(\Omega'),$$

so that $u_2'=u_2$ on Ω. It follows that $Tu_2'=Tu_1'$ in Ω'.

Let us show now that $L_{A_2,q_2}u_2'=0$ in Ω'. To that end, let $\psi\in C_0^\infty(\Omega')$, and write

$$\begin{aligned}\langle L_{A_2,q_2}u_2',\psi\rangle_{\Omega'}=&\int_{\Omega'}\Big((\nabla u_1'+\nabla\varphi)\cdot\nabla\psi+A_2\cdot(Du_1'+D\varphi)\psi\Big)dx\\&+\int_{\Omega'}\Big(-A_2(u_1'+\varphi)\cdot D\psi+(A_2^2+q_2)(u_1'+\varphi)\psi\Big)dx.\end{aligned}$$

Using (3.23), we have

$$\begin{aligned}\langle L_{A_2,q_2}u_2',\psi\rangle_{\Omega'}=&\int_{\Omega}(\nabla u_2\cdot\nabla\psi+A_2\cdot(Du_2)\psi-A_2u_2\cdot D\psi+(A_2^2+q_2)u_2\psi)dx\\&+\int_{\Omega'\setminus\Omega}(\nabla u_1'\cdot\nabla\psi+A_1\cdot(Du_1')\psi-A_1u_1'\cdot D\psi+(A_1^2+q_1)u_1'\psi)dx\\&+\int_{\Omega'\setminus\Omega}(\nabla\varphi\cdot\nabla\psi+A_1\cdot(D\varphi)\psi-A_1\varphi\cdot D\psi+(A_1^2+q_1)\varphi\psi)dx.\end{aligned}$$

As $\varphi \in H_0^1(\Omega)$, we get

$$\int_{\Omega'\setminus\Omega} (\nabla\varphi\cdot\nabla\psi + A_1\cdot(D\varphi)\psi - A_1\varphi\cdot D\psi + (A_1^2+q_1)\varphi\psi)dx = 0.$$

This together with the fact $N_{A_2,q_2}u_2 = N_{A_1,q_1}u_1$ in Ω implies that

$$\begin{aligned}\langle L_{A_2,q_2}u_2', \psi\rangle_{\Omega'} &= (N_{A_2,q_2}u_2, [\psi|_\Omega])_\Omega \\ &\quad + \int_{\Omega'\setminus\Omega}(\nabla u_1'\cdot\nabla\psi + A_1\cdot(Du_1')\psi - A_1u_1'\cdot D\psi + (A_1^2+q_1)u_1'\psi)dx \\ &= \langle L_{A_1,q_1}u_1', \psi\rangle_{\Omega'} = 0,\end{aligned}$$

which shows that $L_{A_2,q_2}u_2' = 0$ in Ω'.

Arguing similarly, we see that $N_{A_2,q_2}u_2' = N_{A_1,q_1}u_1'$ in Ω', which allows us to conclude that $C'_{A_1,q_1} \subset C'_{A_2,q_2}$. The same argument in the other direction gives the claim. □

Let us extend q_j, $j = 1,2$, to the open ball B by defining $q_j = 0$ in $B\setminus\Omega$. Then using Proposition 3.4, Lemma 3.1 and the fact that $\psi|_{\partial B} = 0$, we obtain that

$$C'_{A_1,q_1} = C'_{A_2,q_2} = C'_{A_2+\nabla\psi,q_2} = C'_{A_1,q_2}.$$

This implies the following integral identity,

$$\int_B (q_1-q_2)u_1\overline{u_2}dx = 0, \tag{3.24}$$

valid for any $u_1, u_2 \in H^1(B)$ satisfying $L_{A_1,q_1}u_1 = 0$ in B and $L_{\overline{A_1},\overline{q_2}}u_2 = 0$ in B, respectively.

Let us choose u_1 and u_2 to be the complex geometric optics solutions in B, given by (3.6) and (3.10), respectively. In this case, it follows from (3.16) that $\Phi_1^\sharp(\cdot,\mu_1+i\mu_2;h) + \overline{\Phi_2^\sharp(\cdot,-\mu_1+i\mu_2;h)}$ converges to zero in $L^2_{\text{loc}}(\mathbb{R}^n)$ as $h\to 0$.

Plugging u_1 and u_2 into (3.24) gives

$$\int_B (q_1-q_2)e^{ix\cdot\xi}e^{\Phi_1^\sharp+\overline{\Phi_2^\sharp}}dx = -\int_B (q_1-q_2)e^{ix\cdot\xi}(e^{\Phi_1^\sharp}\overline{r_2} + r_1e^{\overline{\Phi_2^\sharp}} + r_1\overline{r_2})dx.$$

Letting $h\to 0$, and using (3.7), (3.9), (3.11), and (3.13), we get

$$\int_B (q_1-q_2)e^{ix\cdot\xi}dx = 0,$$

and therefore, $q_1 = q_2$ in Ω. The proof of Theorem 1.1 is complete.

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K. Krupchyk, Department of Mathematics and Statistics, University of Helsinki, P.O. Box 68, FI-00014 Helsinki, Finland

E-mail address: katya.krupchyk@helsinki.fi

G. Uhlmann, Department of Mathematics, University of Washington, Seattle, WA 98195-4350, USA

E-mail address: gunther@math.washington.edu