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OPTIMAL THREE-BALL INEQUALITIES AND QUANTITATIVE UNIQUENESS FOR THE STOKES SYSTEM

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Dedicated to Louis Nirenberg on the occasion of his 85th birthday

Abstract. We study the local behavior of a solution to the Stokes system with singular coefficients in R^n with $n = 2, 3$. One of our main results a the bound on the vanishing order of a nontrivial solution u satisfying the Stokes system, which is a quantitative version of the strong unique continuation property for u. Different from the previous known results, our strong unique continuation result only involves the velocity field u. Our proof relies on some delicate Carleman-type estimates. We first use these estimates to derive crucial *optimal* three-ball inequalities for u. Taking advantage of the optimality, we then derive an upper bound on the vanishing order of any nontrivial solution u to the Stokes system from those three-ball inequalities. As an application, we derive a minimal decaying rate at infinity of any nontrivial u satisfying the Stokes equation under some a priori assumptions.

1. **Introduction.** Assume that Ω is a connected open set containing 0 in $\mathbb{R}^n$ with $n = 2, 3$. In this paper we are interested in the local behavior of u satisfying the following Stokes system:

$$\begin{cases} \Delta u + A(x)\cdot\nabla u + B(x)u + \nabla p = 0 \quad \text{in} \quad \Omega, \\ \nabla\cdot u = 0 \quad \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

where A and B are measurable satisfying

$$|A(x)| \le \lambda_1 |x|^{-1} |\log|x||^{-3}, |B(x)| \le \lambda_1 |x|^{-2} |\log|x||^{-3} \quad \forall\, x \in \Omega \tag{1.2}$$

and $A\cdot\nabla u = (A\cdot\nabla u_1, \cdots, A\cdot\nabla u_n)$.

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For the Stokes system (1.1) with essentially bounded coefficients $A(x)$, the weak unique continuation property has been shown by Fabre and Lebeau [6]. On the other hand, when $A(x)$ satisfies $|A(x)| = O(|x|^{-1+\epsilon})$ with $\epsilon > 0$, the strong unique continuation property was proved by Regbaoui [20]. The results in [6] and [20] concern only the qualitative unique continuation theorem and both results require the vanishing property for u and p. In this work we aim to derive a quantitative estimate of the strong unique continuation for u satisfying (1.1) with an appropriate p.

For the second order elliptic operator, using Carleman or frequency functions methods, quantitative estimates of the strong unique continuation (in the form of doubling inequality) under different assumptions on coefficients were derived in [4], [7], [8], [15], [17]. For the power of Laplacian, a quantitative estimate was obtained in [18]. We refer to [17] and references therein for development in this direction.

Since there is no equation for p in the Stokes system (1.1), we apply the curl operator $\nabla\times$ on the first equation and obtain

$$\Delta q + \nabla \cdot F = 0, \tag{1.3}$$

where $q = \nabla \times u$ is the vorticity and for $n = 2$, $\nabla \times u = \partial_1 u_2 - \partial_2 u_1$. For $n = 3$, $\nabla \cdot F$ is a vector function defined by $(\nabla \cdot F)_i = \sum_{j=1}^3 \partial_j F_{ij}$, $i = 1, 2, 3$, where $F_{ij} = \sum_{k,\ell=1}^3 \tilde{A}_{ijk\ell}(x)\partial_k u_\ell + \sum_{k=1}^3 \tilde{B}_{ijk}(x)u_k$ with appropriate $\tilde{A}_{ijk\ell}(x)$ and $\tilde{B}_{ijk}(x)$ satisfying

$$|\tilde{A}_{ijk\ell}(x)| \le C_0 |\log|x||^{-3}|x|^{-1}, |\tilde{B}_{ijk}(x)| \le C_0 |\log|x||^{-3}|x|^{-2} \quad \forall\, x \in \Omega. \tag{1.4}$$

When $n = 2$, $\nabla \cdot F$ is a scalar and we simply drop the suffix i in the definition above. Now we define $\nabla^\perp \times G = \nabla \times G$ for any three-dimensional vector function G and $\nabla^\perp \times g = (\partial_2 g, -\partial_1 g)$ for a scalar function g if $n = 2$. It is easy to check that $\Delta u = \nabla(\nabla \cdot u) - \nabla^\perp \times (\nabla \times u)$ and thus we have

$$\Delta u + \nabla^\perp \times q = 0 \tag{1.5}$$

if $\nabla \cdot u = 0$. However, equations (1.3) and (1.5) do not give us a decoupled system. The frequency functions method does not seem to work in this case. So we prove our results using Carleman inequalities. On the other hand, since the coefficient $A(x)$ is more singular than the one considered in [20]. Carleman-type estimates derived in [20] can not be applied to the case here. Hence we need to derive new Carleman-type estimates for our purpose. The key is to use weights which are slightly less singular than the negative powers of $|x|$ (see estimates (2.4) and (2.15)). The estimate (2.15) is to handle (1.3) and the idea is due to Fabre and Lebeau [6].

We can derive certain three-ball inequalities which are *optimal* in the sense explained in [5] using (2.4) and (2.15). We would like to remark that usually the three-ball inequality can be regarded as the quantitative estimate of the weak unique continuation property. However, when the three-ball inequality is optimal, one is able to deduce the strong unique continuation from it. It seems reasonable to expect that one could derive a bound on the vanishing order of a nontrivial solution from the optimal three-ball inequality. A recent result by Bourgain and Kenig [3] (more precisely, Kenig's lecture notes for 2006 CNA Summer School [14]) indicates that this is indeed possible, at least for the Schrödinger operator. In this paper, we show that by the optimal three-ball inequality, we can obtain a bound on the vanishing order of a nontrivial solution to (1.1) containing "nearly" optimal singular coefficients. Finally, we would like to mention that quantitative estimates of the strong unique continuation are useful in studying the nodal sets of solutions for elliptic or parabolic equations [4], [9], [16], or the inverse problem [1].

We now state the main results of this paper. Their proofs will be given in the subsequent sections. Assume that there exists $0 < R_0 \le 1$ such that $B_{R_0} \subset \Omega$. Hereafter B_r denotes an open ball of radius $r > 0$ centered at the origin.

Theorem 1.1. *There exists a positive number $\tilde{R} < 1$, depending only on n, such that if $0 < R_1 < R_2 < R_3 \le R_0$ and $R_1/R_3 < R_2/R_3 < \tilde{R}$, then*

$$\int_{|x|<R_2} |u|^2 dx \le C \left(\int_{|x|<R_1} |u|^2 dx \right)^{\tau} \left(\int_{|x|<R_3} |u|^2 dx \right)^{1-\tau} \tag{1.6}$$

for $(u,p) \in (H^1(B_{R_0}))^{n+1}$ satisfying (1.1) in B_{R_0}, where the constant C depends on R_2/R_3, n, and $0 < \tau < 1$ depends on R_1/R_3, R_2/R_3, n. Moreover, for fixed R_2 and R_3, the exponent τ behaves like $1/(-\log R_1)$ when R_1 is sufficiently small.

Remark 1.2. *It is important to emphasize that C is independent of R_1 and τ has the asymptotic $(-\log R_1)^{-1}$. These facts are crucial in deriving an vanishing order of a nontrivial (u,p) to (1.1). Due to the behavior of τ, the three-ball inequality is called optimal [5].*

It should be emphasized that three-ball inequalities (1.6) involve only the velocity field u. This is important in the application to inverse problems for the Stokes system, for example, see [2] and [10]. Using (1.6), we can also derive an upper bound of the vanishing order for any nontrivial u satisfying (1.1), which is a quantitative form of the strong unique continuation property for u. Let us now pick any $R_2 < R_3$ such that $R_3 \le R_0$ and $R_2/R_3 < \tilde{R}$.

Theorem 1.3. *Let $(u,p) \in (H^1(B_{R_0}))^{n+1}$ be a nontrivial solution to (1.1), then there exist positive constants K and m, depending on n and u, such that*

$$\int_{|x|<R} |u|^2 dx \ge K R^m \tag{1.7}$$

for all R with $R < R_2$.

Remark 1.4. *Based on Theorem 1.1, the constants K and m in (1.7) are given by*

$$K = \int_{|x|<R_3} |u|^2 dx$$

and

$$m = \tilde{C} \log \Big(\frac{\int_{|x|<R_3} |u|^2 dx}{\int_{|x|<R_2} |u|^2 dx} \Big),$$

where $\tilde{C}$ is a positive constant depending on λ_1, n and R_2/R_3.

From Theorem 1.3, we immediately conclude that if $(u,p) \in (H^1_{loc}(\Omega))^{n+1}$ satisfies (1.1) and for any $N \in \mathbb{N}$, there exists $C_N > 0$ such that

$$\int_{|x|<r} |u|^2 dx \le C_N r^N,$$

then u vanishes identically in Ω. Consequently, p is a constant in Ω. This is a new strong unique continuation result for the Stokes system with singular coefficients.

By three-ball inequalities (1.6), one can also study the minimal decaying rate of any nontrivial velocity u to (1.1) with a suitable assumption on the coefficients A

and B (see [3] for a related result for the Schrödinger equation). Consider (u, p) satisfying (1.1) with $\Omega = \mathbb{R}^n$, $n = 2, 3$. Assume here that

$$\|u\|_{L^\infty(\mathbb{R}^n)} + \|A\|_{L^\infty(\mathbb{R}^n)} + \|B\|_{L^\infty(\mathbb{R}^n)} \leq \lambda_2. \tag{1.8}$$

Denote

$$M_r(t) = \inf_{|x|=t} \int_{|y-x|<r} |u(y)|^2 dy.$$

Then we can prove that

Theorem 1.5. *Let $(u, p) \in (H^1_{loc}(\mathbb{R}^n))^{n+1}$ be a nontrivial solution to (1.1). Assume that (1.8) holds. Then for any $r < 1$, there exists $c > 0$ such that*

$$M_r(t) \geq r^{c\zeta^{(1+\frac{t}{r})}},$$

where c depends on λ_2, n, $\int_{|x|<r} |u|^2 dx$ and $\zeta = 1 + 2\tilde{C}\log(1/r)$ with $\tilde{C}$ given in Remark 1.4.

We can apply Theorem 1.5 to the stationary Navier-Stokes equation.

Corollary 1.6. *Let $(u, p) \in (H^1_{loc}(\mathbb{R}^n))^{n+1}$ be a nontrivial solution of the stationary Navier-Stokes equation:*

$$-\nabla u + u \cdot \nabla u + \rho u + \nabla p = 0, \ \nabla \cdot u = 0, \quad in \quad \mathbb{R}^n$$

with $n = 2, 3$. Assume that

$$\|u\|_{L^\infty(\mathbb{R}^n)} + \|\rho\|_{L^\infty(\mathbb{R}^n)} \leq \lambda_3.$$

Then for any $r < 1$, there exists $\tilde{c} > 0$ such that

$$M_r(t) \geq r^{\tilde{c}\zeta^{(1+\frac{t}{r})}},$$

where $\tilde{c}$ depends on λ_3, n, and $\int_{|x|<r} |u|^2 dx$.

This paper is organized as follows. In Section 2, we derive suitable Carleman-type estimates. A technical interior estimate is proved in Section 3. Section 4 is devoted to the proofs of Theorem 1.1, 1.3. The proof of Theorem 1.5 is given in Section 5.

2. **Carleman estimates.** Similar to the arguments used in [11], we introduce polar coordinates in $\mathbb{R}^n \backslash \{0\}$ by setting $x = r\omega$, with $r = |x|$, $\omega = (\omega_1, \cdots, \omega_n) \in S^{n-1}$. Furthermore, using new coordinate $t = \log r$, we can see that

$$\frac{\partial}{\partial x_j} = e^{-t}(\omega_j \partial_t + \Omega_j), \quad 1 \leq j \leq n,$$

where Ω_j is a vector field in S^{n-1}. We could check that the vector fields Ω_j satisfy

$$\sum_{j=1}^n \omega_j \Omega_j = 0 \quad \text{and} \quad \sum_{j=1}^n \Omega_j \omega_j = n - 1.$$

Since $r \to 0$ iff $t \to -\infty$, we are mainly interested in values of t near $-\infty$.

It is easy to see that

$$\frac{\partial^2}{\partial x_j \partial x_\ell} = e^{-2t}(\omega_j \partial_t - \omega_j + \Omega_j)(\omega_\ell \partial_t + \Omega_\ell), \quad 1 \leq j, \ell \leq n.$$

and, therefore, the Laplacian becomes

$$e^{2t}\Delta = \partial_t^2 + (n-2)\partial_t + \Delta_\omega, \tag{2.1}$$

where $\Delta_\omega = \Sigma_{j=1}^n \Omega_j^2$ denotes the Laplace-Beltrami operator on S^{n-1}. We recall that the eigenvalues of $-\Delta_\omega$ are $k(k+n-2), k \in \mathbb{N}$, and the corresponding eigenspaces are E_k, where E_k is the space of spherical harmonics of degree k. It follows that

$$\iint |\Delta_\omega v|^2 dt d\omega = \sum_{k\geq 0} k^2(k+n-2)^2 \iint |v_k|^2 dt d\omega \tag{2.2}$$

and

$$\sum_j \iint |\Omega_j v|^2 dt d\omega = \sum_{k\geq 0} k(k+n-2) \iint |v_k|^2 dt d\omega, \tag{2.3}$$

where v_k is the projection of v onto E_k. Let

$$\Lambda = \sqrt{\frac{(n-2)^2}{4} - \Delta_\omega},$$

then Λ is an elliptic first-order positive pseudodifferential operator in $L^2(S^{n-1})$. The eigenvalues of Λ are $k + \frac{n-2}{2}$ and the corresponding eigenspaces are E_k. Denote

$$L^\pm = \partial_t + \frac{n-2}{2} \pm \Lambda.$$

Then it follows from (2.1) that

$$e^{2t}\Delta = L^+L^- = L^-L^+.$$

Motivated by the ideas in [19], we will derive Carleman-type estimates with weights $\varphi_\beta = \varphi_\beta(x) = \exp(-\beta\tilde{\psi}(x))$, where $\beta > 0$ and $\tilde{\psi}(x) = \log|x| + \log((\log|x|)^2)$. Note that φ_β is less singular than $|x|^{-\beta}$, For simplicity, we denote $\psi(t) = t + \log t^2$, i.e., $\tilde{\psi}(x) = \psi(\log|x|)$. From now on, the notation $X \lesssim Y$ or $X \gtrsim Y$ means that $X \leq CY$ or $X \geq CY$ with some constant C depending only on n.

Lemma 2.1. *There exist a sufficiently small $r_0 > 0$ depending on n and a sufficiently large $\beta_0 > 1$ depending on n such that for all $u \in U_{r_0}$ and $\beta \geq \beta_0$, we have that*

$$\beta \int \varphi_\beta^2 (\log|x|)^{-2}|x|^{-n}(|x|^2|\nabla u|^2 + |u|^2)dx \lesssim \int \varphi_\beta^2 |x|^{-n}|x|^4|\Delta u|^2 dx, \tag{2.4}$$

where $U_{r_0} = \{u \in C_0^\infty(\mathbb{R}^n \setminus \{0\}) : \mathrm{supp}(u) \subset B_{r_0}\}$.

Proof. Using polar coordinates as described above, we have

$$\begin{aligned} & \int \varphi_\beta^2 |x|^{4-n}|\Delta u|^2 dx \\ = & \iint e^{-2\beta\psi(t)} e^{4t}|\Delta u|^2 dt d\omega \\ = & \iint |e^{-\beta\psi(t)} e^{2t}\Delta u|^2 dt d\omega. \end{aligned} \tag{2.5}$$

If we set $u = e^{\beta\psi(t)}v$ and use (2.1), then

$$e^{-\beta\psi(t)}e^{2t}\Delta u = \partial_t^2 v + b\partial_t v + av + \Delta_\omega v =: P_\beta v, \tag{2.6}$$

where $a = (1+2t^{-1})^2\beta^2 + (n-2)\beta + 2(n-2)t^{-1}\beta - 2t^{-2}\beta$ and $b = n-2+2\beta+4t^{-1}\beta$. By (2.5) and (2.6), (2.4) holds if for t near $-\infty$ we have

$$\sum_{j+|\alpha|\leq 1} \beta^{3-2(j+|\alpha|)} \iint |t|^{-2}|\partial_t^j \Omega^\alpha v|^2 dt d\omega \leq \tilde{C}_1 \iint |P_\beta v|^2 dt d\omega, \tag{2.7}$$

where $\tilde{C}_1$ is a positive constant depending on n.

From (2.6), using the integration by parts, for $t < t_0$ and $\beta > \beta_0$, where $t_0 < -1$ and $\beta_0 > 0$ depend on n, we have that

$$\begin{aligned}
&\iint |P_\beta v|^2 dt d\omega \\
= \;& \iint |\partial_t^2 v|^2 dt d\omega + \iint |b\partial_t v|^2 dt d\omega + \iint |av|^2 dt d\omega + \iint |\Delta_\omega v|^2 dt d\omega \\
&- \iint \partial_t b |\partial_t v|^2 dt d\omega - 2\iint a|\partial_t v|^2 dt d\omega + \iint \partial_t^2 a|v|^2 dt d\omega \\
&- \iint \partial_t(ab)|v|^2 dt d\omega + 2\sum_j \iint |\partial_t \Omega_j v|^2 dt d\omega \\
&+ \sum_j \iint \partial_t b|\Omega_j v|^2 dt d\omega - 2\sum_j \iint a|\Omega_j v|^2 dt d\omega \\
\geq \;& \iint |\Delta_\omega v|^2 dt d\omega + \iint \{b^2 - \partial_t b - 2a\}|\partial_t v|^2 dt d\omega \\
&+ \sum_j \iint \{\partial_t b - 2a\}|\Omega_j v|^2 dt d\omega + \iint \{a^2 + \partial_t^2 a - \partial_t(ab)\}|v|^2 dt d\omega \\
\geq \;& \iint |\Delta_\omega v|^2 dt d\omega + \sum_j \iint \{-4t^{-2}\beta - 2a\}|\Omega_j v|^2 dt d\omega \\
&+ \iint \{a^2 + 11t^{-2}\beta^3\}|v|^2 dt d\omega + \iint \beta^2 |\partial_t v|^2 dt d\omega. \qquad (2.8)
\end{aligned}$$

In view of (2.8), using (2.2),(2.3), we see that

$$\begin{aligned}
&\iint |\Delta_\omega v|^2 dt d\omega - 2\sum_j \iint a|\Omega_j v|^2 dt d\omega + \iint a^2|v|^2 dt d\omega \\
= \;& \sum_{k\geq 0} \iint [a - k(k+n-2)]^2 |v_k|^2 dt d\omega. \qquad (2.9)
\end{aligned}$$

Substituting (2.9) into (2.8) yields

$$\begin{aligned}
&\iint |P_\beta v|^2 dt d\omega \\
\geq \;& \sum_{k\geq 0} \iint \{11t^{-2}\beta^3 - 4t^{-2}\beta k(k+n-2) + [a - k(k+n-2)]^2\}|v_k|^2 dt d\omega \\
&+ \iint \beta^2 |\partial_t v|^2 dt d\omega \\
= \;& \Big(\sum_{k, k(k+n-2)\geq 2\beta^2} + \sum_{k, k(k+n-2) < 2\beta^2} \Big) \iint \{11t^{-2}\beta^3 - 4t^{-2}\beta k(k+n-2) \\
&\quad + [a - k(k+n-2)]^2\}|v_k|^2 dt d\omega + \iint \beta^2 |\partial_t v|^2 dt d\omega. \qquad (2.10)
\end{aligned}$$

For k such that $k(k+n-2) < 2\beta^2$, we have

$$11t^{-2}\beta^3 - 4t^{-2}\beta k(k+n-2) \geq t^{-2}\beta^3 + t^{-2}\beta k(k+n-2). \qquad (2.11)$$

On the other hand, if $2\beta^2 < k(k+n-2)$, then, by taking t even smaller, if necessary, we get that

$$-4t^{-2}\beta k(k+n-2) + [a - k(k+n-2)]^2 \gtrsim t^{-2}\beta k(k+n-2). \tag{2.12}$$

Finally, using formula (2.3) and estimates (2.11), (2.12) in (2.10), we immediately obtain (2.7) and the proof of the lemma is complete. □

To handle the auxiliary equation corresponding to the vorticity q, we need another Carleman estimate. The derivation here follows the line in [20].

Lemma 2.2. *There exists a sufficiently small number $t_0 < 0$ depending on n such that for all $u \in V_{t_0}$, $\beta > 1$, we have that*

$$\sum_{j+|\alpha|\le 1} \beta^{1-2(j+|\alpha|)} \iint t^{-2}\varphi_\beta^2 |\partial_t^j \Omega^\alpha u|^2 dtd\omega \lesssim \iint \varphi_\beta^2 |L^- u|^2 dtd\omega, \tag{2.13}$$

where $V_{t_0} = \{u(t,\omega) \in C_0^\infty((-\infty, t_0) \times S^{n-1})\}$.

Proof. If we set $u = e^{\beta\psi(t)}v$, then simple integration by parts implies

$$\begin{aligned}
&\iint \varphi_\beta^2 |L^- u|^2 dtd\omega \\
= &\iint |\partial_t v - \Lambda v + \beta v + 2\beta t^{-1} v + (n-2)v/2|^2 dtd\omega \\
= &\iint |\partial_t v|^2 dtd\omega + \iint |-\Lambda v + \beta v + 2\beta t^{-1} v + (n-2)v/2|^2 dtd\omega \\
&+\beta \iint t^{-2}|v|^2 dtd\omega.
\end{aligned}$$

By the definition of Λ, we have

$$\begin{aligned}
&\iint |-\Lambda v + \beta v + 2\beta t^{-1} v + (n-2)v/2|^2 dtd\omega \\
= &\sum_{k\ge 0} \iint |-kv_k + \beta v_k + 2\beta t^{-1} v_k|^2 dtd\omega \\
= &\sum_{k\ge 0} \iint (-k + \beta + 2\beta t^{-1})^2 |v_k|^2 dtd\omega,
\end{aligned}$$

where, as before, v_k is the projection of v on E_k. Note that

$$(-k + \beta + 2\beta t^{-1})^2 + \beta t^{-2} \ge \frac{1}{8\beta}(2\beta t^{-1})^2 + \frac{1}{16\beta}(\beta - k)^2.$$

Considering $\beta > (1/2)k$ and $\beta \le (1/2)k$, we can get that

$$\begin{aligned}
&\iint \varphi_\beta^2 |L^- u|^2 dtd\omega \\
= &\iint |\partial_t v|^2 dtd\omega + \Sigma_{k\ge 0} \iint [(-k + \beta + 2\beta t^{-1})^2 + \beta t^{-2}] |v_k|^2 dtd\omega \\
\gtrsim &\iint |\partial_t v|^2 dtd\omega + \Sigma_{k\ge 0} \iint (\beta^{-1} t^{-2} k(k+n-2) + \beta t^{-2}) |v_k|^2 dtd\omega.
\end{aligned} \tag{2.14}$$

The estimate (2.13) then follows from (2.3). □

Next we need a technical lemma. We then use this lemma to derive another Carleman estimate.

Lemma 2.3. *There exists a sufficiently small number $t_1 < -2$ depending on n such that for all $u \in V_{t_1}$, $g = (g_0, g_1, \cdots, g_n) \in (V_{t_1})^{n+1}$ and $\beta > 0$, we have that*

$$\iint \varphi_\beta^2 |u|^2 dt d\omega \lesssim \iint \varphi_\beta^2 (|L^+ u + \partial_t g_0 + \sum_{j=1}^n \Omega_j g_j|^2 + \|g\|^2) dt d\omega.$$

Proof. This lemma can be proved by exactly the same arguments used in Lemma 2.2 of [20]. So we omit the proof here. □

Lemma 2.4. *There exist a sufficiently small number $r_1 > 0$ depending on n and a sufficiently large number $\beta_1 > 2$ depending on n such that for all $w \in U_{r_1}$ and $f = (f_1, \cdots, f_n) \in (U_{r_1})^n$, $\beta \geq \beta_1$, we have that*

$$\begin{aligned} &\int \varphi_\beta^2 (\log|x|)^2 (|x|^{4-n} |\nabla w|^2 + |x|^{2-n} |w|^2) dx \\ \lesssim \quad &\beta \int \varphi_\beta^2 (\log|x|)^4 |x|^{2-n} [(|x|^2 \Delta w + |x| \mathrm{div} f)^2 + \|f\|^2] dx, \end{aligned} \tag{2.15}$$

where U_{r_1} is defined as in Lemma 2.1.

Proof. Replacing β by $\beta + 1$ in (2.15), we see that it suffices to prove

$$\begin{aligned} &\int \varphi_\beta^2 (\log|x|)^{-2} (|x|^2 |\nabla w|^2 + |w|^2) |x|^{-n} dx \\ \lesssim \quad &\beta \int \varphi_\beta^2 [(|x|^2 \Delta w + |x| \mathrm{div} f)^2 + \|f\|^2] |x|^{-n} dx. \end{aligned} \tag{2.16}$$

Working in polar coordinates and using the relation $e^{2t}\Delta = L^+ L^-$, (2.16) is equivalent to

$$\begin{aligned} &\sum_{j+|\alpha| \leq 1} \iint \beta^{2-2(j+|\alpha|)} t^{-2} \varphi_\beta^2 |\partial_t^j \Omega^\alpha u|^2 dt d\omega \\ \lesssim \quad &\beta \iint \varphi_\beta^2 (|L^+ L^- w + \partial_t (\sum_{j=1}^n \omega_j f_j) + \sum_{j=1}^n \Omega_j f_j|^2 + \|f\|^2) dt d\omega. \end{aligned} \tag{2.17}$$

Applying Lemma 2.3 to $u = L^- w$ and $g = (\sum_{j=1}^n \omega_j f_j, f_1, \cdots, f_n)$ yields

$$\begin{aligned} &\beta \iint \varphi_\beta^2 |L^- w|^2 dt d\omega \\ \lesssim \quad &\beta \iint \varphi_\beta^2 (|L^+ L^- w + \partial_t (\sum_{j=1}^n \omega_j f_j) + \sum_{j=1}^n \Omega_j f_j|^2 + \|f\|^2) dt d\omega. \end{aligned} \tag{2.18}$$

Now (2.17) is an easy consequence of (2.13) and (2.18). □

3. **Interior estimates.** To establish the three-ball inequality for (1.1), the following interior estimate is useful.

Lemma 3.1. *Let $(u, p) \in (H^1_{loc}(\Omega))^{n+1}$ be a solution of (1.1). Then for any $0 < a_3 < a_1 < a_2 < a_4$ such that $B_{a_4 r} \subset \Omega$ and $|a_4 r| < 1$, we have*

$$\int_{a_1 r < |x| < a_2 r} |x|^4 |\nabla q|^2 + |x|^2 |q|^2 + |x|^2 |\nabla u|^2 dx \leq C' \int_{a_3 r < |x| < a_4 r} |u|^2 dx \tag{3.1}$$

where the constant C' is independent of r and u. Here $q = \nabla \times u$.

Proof. The proof of this lemma is motivated by ideas used in [12]. Let $X = B_{a_4 r}\backslash\bar{B}_{a_3 r}$ and $d(x)$ be the distant from $x\in X$ to $\mathbb{R}^n\backslash X$. By the elliptic regularity, we obtain from (1.1) that $u\in H^2_{loc}(\Omega\backslash\{0\})$. It is trivial that

$$\|v\|_{H^1(\mathbb{R}^n)}\lesssim\|\Delta v\|_{L^2(\mathbb{R}^n)}+\|v\|_{L^2(\mathbb{R}^n)} \tag{3.2}$$

for all $v\in H^2(\mathbb{R}^n)$. By changing variables $x\to E^{-1}x$ in (3.2), we will have

$$\textstyle\sum_{|\alpha|\le 1}E^{2-|\alpha|}\|D^\alpha v\|_{L^2(\mathbb{R}^n)}\lesssim(\|\Delta v\|_{L^2(\mathbb{R}^n)}+E^2\|v\|_{L^2(\mathbb{R}^n)}) \tag{3.3}$$

for all $v\in H^2(\mathbb{R}^n)$. To apply (3.3) on u, we need to cut-off u. So let $\xi(x)\in C_0^\infty(\mathbb{R}^n)$ satisfy $0\le\xi(x)\le 1$ and

$$\xi(x)=\begin{cases}1, & |x|<1/4,\\ 0, & |x|\ge 1/2.\end{cases}$$

Let us denote $\xi_y(x)=\xi((x-y)/d(y))$. For $y\in X$, we apply (3.3) to $\xi_y(x)u(x)$ and use equation (1.5) to get that

$$\begin{aligned}
&E^2\int_{|x-y|\le d(y)/4}|\nabla u|^2dx\\
\le\;& C_1'\int_{|x-y|\le d(y)/2}|\nabla q|^2dx+C_1'\int_{|x-y|\le d(y)/2}d(y)^{-2}|\nabla u|^2dx\\
&+C_1'(E^4+d(y)^{-4})\int_{|x-y|\le d(y)/2}|u|^2dx.
\end{aligned} \tag{3.4}$$

Now taking $E=Md(y)^{-1}$ for some positive constant M and multiplying $d(y)^4$ on both sides of (3.4), we have

$$\begin{aligned}
&M^2d(y)^2\int_{|x-y|\le d(y)/4}|\nabla u|^2dx\\
\le\;& C_1'\int_{|x-y|\le d(y)/2}d(y)^4|\nabla q|^2dx+C_1'\int_{|x-y|\le d(y)/2}d(y)^2|\nabla u|^2dx\\
&+C_1'(M^4+1)\int_{|x-y|\le d(y)/2}|u|^2dx.
\end{aligned} \tag{3.5}$$

Integrating $d(y)^{-n}dy$ over X on both sides of (3.5) and using Fubini's Theorem, we get that

$$\begin{aligned}
&M^2\int_X\int_{|x-y|\le d(y)/4}d(y)^{2-n}|\nabla u|^2dydx\\
\le\;& C_1'\int_X\int_{|x-y|\le d(y)/2}d(y)^4|\nabla q(x)|^2d(y)^{-n}dydx\\
&+C_1'\int_X\int_{|x-y|\le d(y)/2}d(y)^{2-n}|\nabla u|^2dydx\\
&+2C_1'M^4\int_X\int_{|x-y|\le d(y)/2}|u|^2d(y)^{-n}dydx.
\end{aligned} \tag{3.6}$$

Note that $|d(x)-d(y)|\le|x-y|$. If $|x-y|\le d(x)/3$, then

$$2d(x)/3\le d(y)\le 4d(x)/3. \tag{3.7}$$

On the other hand, if $|x-y|\le d(y)/2$, then

$$d(x)/2\le d(y)\le 3d(x)/2. \tag{3.8}$$

By (3.7) and (3.8), we have

$$\begin{cases} \int_{|x-y|\le d(y)/4} d(y)^{-n}dy \ge (3/4)^n \int_{|x-y|\le d(x)/6} d(x)^{-n}dy \ge 8^{-n}\int_{|y|\le 1} dy, \\ \int_{|x-y|\le d(y)/2} d(y)^{-n}dy \le 2^n \int_{|x-y|\le 3d(x)/4} d(x)^{-n}dy \le (3/2)^n \int_{|y|\le 1} dy \end{cases} \tag{3.9}$$

Combining (3.6)–(3.9), we obtain

$$\begin{aligned} & M^2 \int_X d(x)^2|\nabla u|^2dx \\ \le\ & C_2' \int_X d(x)^2|\nabla u(x)|^2dx + C_2'\int_X d(x)^4|\nabla q|^2dx + C_2'M^4\int_X |u|^2dx. \end{aligned} \tag{3.10}$$

On the other hand, we have from (1.3) that

$$\begin{aligned} & \sum_{i=1}^n \int |\xi_y(x)\nabla q_i|^2dx = \sum_{i=1}^n \int \nabla q_i\cdot\nabla(\xi_y^2(x)\bar{q}_i)dx - \sum_{i=1}^n 2\int \xi_y\nabla q_i\cdot\bar{q}_i\nabla\xi_y dx \\ \le\ & C_3'\sum_{i=1}^n \left|\int (\mathrm{div}F)_i\xi_y^2 q_i dx\right| + \sum_{i=1}^n 2\int |\xi_y\nabla q_i\cdot\bar{q}_i\nabla\xi_y|dx \\ \le\ & C_3'\sum_{i=1}^n \left|\int\sum_{j=1}^n F_{ij}\cdot\partial_j(\xi_y^2 q_i)dx\right| + \frac{1}{4}\sum_{i=1}^n\int|\xi_y\nabla q_i|^2dx + 4\int_{|x-y|\le d(y)/2} d(y)^{-2}|q|^2dx \\ \le\ & C_4'\int_{|x-y|\le d(y)/2}|F|^2dx + \frac{1}{4}\sum_{i=1}^n\int|\xi_y\nabla q_i|^2dx + C_4'\int_{|x-y|\le d(y)/2} d(y)^{-2}|q|^2dx \\ & +\frac{1}{4}\sum_{i=1}^n\int|\xi_y\nabla q_i|^2dx + C_4'\int_{|x-y|\le d(y)/2} d(y)^{-2}|q|^2dx. \end{aligned} \tag{3.11}$$

Therefore, we get that

$$\begin{aligned} & \int_{|x-y|\le d(y)/4} |\nabla q|^2dx \\ \le\ & \int |\xi_y(x)\nabla q|^2dx \\ \le\ & C_5'\int_{|x-y|\le d(y)/2}|F|^2dx + C_5'\int_{|x-y|\le d(y)/2} d(y)^{-2}|q|^2dx. \end{aligned} \tag{3.12}$$

Multiply $d(y)^4$ on both sides of (3.12), we obtain that

$$\begin{aligned} & \int_{|x-y|\le d(y)/4} d(y)^4|\nabla q|^2dx \\ \le\ & C_6'\int_{|x-y|\le d(y)/2} d(y)^4|\tilde{A}|^2|\nabla u|^2dx + C_6'\int_{|x-y|\le d(y)/2} d(y)^4|\tilde{B}|^2|u|^2dx \\ & +C_6'\int_{|x-y|\le d(y)/2} d(y)^2|q|^2dx. \end{aligned} \tag{3.13}$$

Repeating (3.6)$\sim$(3.10), we have that

$$\begin{aligned}
&\int_X d(x)^4|\nabla q|^2dx\\
\le\ & C_7'\int_X d(x)^4|\tilde{A}|^2|\nabla u|^2dx+C_7'\int_X d(x)^4|\tilde{B}|^2|u|^2dx\\
&+C_7'\int_X d(x)^2|q|^2dx. \qquad (3.14)
\end{aligned}$$

Combining $K\times$(3.14), (3.10) and $\int_X d(x)^2|q|^2dx$, we obtain that

$$\begin{aligned}
&M^2\int_X d(x)^2|\nabla u|^2dx+K\int_X d(x)^4|\nabla q|^2dx+\int_X d(x)^2|q|^2dx\\
\le\ & \int_X (C_2'd(x)^2+C_7'Kd(x)^4|\tilde{A}|^2)|\nabla u(x)|^2dx+C_7'K\int_X d(x)^4|\tilde{B}|^2|u|^2dx\\
&+C_2'M^4\int_X |u|^2dx+C_2'\int_X d(x)^4|\nabla q|^2dx+(C_7'K+1)\int_X d(x)^2|q|^2dx.
\end{aligned}$$

(3.15)

Taking $K=2C_2'$, one can eliminate $\int_X d(x)^4|\nabla q|^2dx$ on the right hand side of (3.15). Observe that

$$\int_X d(x)^2|q|^2dx\le C_8'\int_X d(x)^2|\nabla u(x)|^2dx.$$

So, by choosing M large enough, we can ignore $\int_X d(x)^2|\nabla u(x)|^2dx$ on the right hand side of (3.15). Finally, we get that

$$\begin{aligned}
&M^2\int_X d(x)^2|\nabla u|^2dx+K\int_X d(x)^4|\nabla q|^2dx+\int_X d(x)^2|q|^2dx\\
\le\ & C_9'\int_X |u|^2dx. \qquad (3.16)
\end{aligned}$$

We recall that $X=B_{a_4r}\backslash\bar{B}_{a_3r}$ and note that $d(x)\ge\tilde{C}r$ if $x\in B_{a_2r}\backslash\bar{B}_{a_1r}$, where $\tilde{C}$ is independent of r. Hence, (3.1) is an easy consequence of (3.16). $\square$

4. **Proof of Theorem 1.1 and Theorem 1.3.** This section is devoted to the proofs of Theorem 1.1 and Theorem 1.3. To begin, we first consider the case where $0<R_1<R_2<R<1$ and $B_R\subset\Omega$. The small constant R will be determined later. Since $(u,p)\in(H^1(B_{R_0}))^{n+1}$, the elliptic regularity theorem implies $u\in H^2_{loc}(B_{R_0}\setminus\{0\})$. Therefore, to use estimate (2.4), we simply cut-off u. So let $\chi(x)\in C_0^\infty(\mathbb{R}^n)$ satisfy $0\le\chi(x)\le1$ and

$$\chi(x)=\begin{cases}0, & |x|\le R_1/e,\\ 1, & R_1/2<|x|<eR_2,\\ 0, & |x|\ge 3R_2,\end{cases}$$

where $e=\exp(1)$. We remark that we first choose a small R such that $R\le\min\{r_0,r_1\}/3=\tilde{R}_0$, where r_0 and r_1 are constants appeared in (2.4) and (2.15). Hence $\tilde{R}_0$ depends on n. It is easy to see that for any multiindex α

$$\begin{cases}|D^\alpha\chi|=O(R_1^{-|\alpha|})\text{ for all }R_1/e\le|x|\le R_1/2\\ |D^\alpha\chi|=O(R_2^{-|\alpha|})\text{ for all }eR_2\le|x|\le 3R_2.\end{cases}\qquad (4.1)$$

Applying (2.4) to χu gives

$$C_1\beta\int(\log|x|)^{-2}\varphi_\beta^2|x|^{-n}(|x|^2|\nabla(\chi u)|^2+|\chi u|^2)dx\le\int\varphi_\beta^2|x|^{-n}|x|^4|\Delta(\chi u)|^2dx. \tag{4.2}$$

From now on, $C_1, C_2, \cdots$ denote general constants whose dependence will be specified whenever necessary. Next applying (2.15) to $w=\chi q$ and $f=|x|\chi F$, we get that

$$\begin{aligned}
&C_2\int\varphi_\beta^2(\log|x|)^2(|x|^{4-n}|\nabla(\chi q)|^2+|x|^{2-n}|\chi q|^2)dx\\
\le\ &\beta\int\varphi_\beta^2(\log|x|)^4|x|^{2-n}[|x|^2\Delta(\chi q)+|x|\mathrm{div}(|x|\chi F)]^2dx\\
&+\beta\int\varphi_\beta^2(\log|x|)^4|x|^{2-n}\||x|\chi F\|^2dx.
\end{aligned}\tag{4.3}$$

Multiplying by M_1 on (4.2) and combining (4.3), we obtain that

$$\begin{aligned}
&M_1\beta\int_{R_1/2<|x|<eR_2}(\log|x|)^{-2}\varphi_\beta^2|x|^{-n}(|x|^2|\nabla u|^2+|u|^2)dx\\
&+\int_{R_1/2<|x|<eR_2}(\log|x|)^2\varphi_\beta^2|x|^{-n}(|x|^4|\nabla q|^2+|x|^2|q|^2)dx\\
\le\ &M_1\beta\int\varphi_\beta^2(\log|x|)^{-2}|x|^{-n}(|x|^2\nabla(\chi u)|^2+|\chi u|^2)dx\\
&+\int(\log|x|)^2\varphi_\beta^2|x|^{-n}(|x|^4|\nabla(\chi q)|^2+|x|^2|\chi q|^2)dx\\
\le\ &M_1C_3\int\varphi_\beta^2|x|^{-n}|x|^4|\Delta(\chi u)|^2dx\\
&+\beta C_3\int(\log|x|)^4\varphi_\beta^2|x|^{-n}[|x|^3\Delta(\chi q)+|x|^2\mathrm{div}(|x|\chi F)]^2dx\\
&+\beta C_3\int(\log|x|)^4\varphi_\beta^2|x|^{-n}\||x|^2\chi F\|^2dx.
\end{aligned}\tag{4.4}$$

By (1.2), (1.3), (1.4), and estimates (4.1), we deduce from (4.4) that

$$\begin{aligned}
&M_1\beta\int_{R_1/2<|x|<eR_2}(\log|x|)^{-2}\varphi_\beta^2|x|^{-n}(|x|^2|\nabla u|^2+|u|^2)dx\\
&+\int_{R_1/2<|x|<eR_2}(\log|x|)^2\varphi_\beta^2|x|^{-n}(|x|^4|\nabla q|^2+|x|^2|q|^2)dx\\
\le\ &C_4M_1\int_{R_1/2<|x|<eR_2}\varphi_\beta^2|x|^{-n}|x|^4|\nabla q|^2dx\\
&+C_4\beta\int_{R_1/2<|x|<eR_2}(\log|x|)^{-2}\varphi_\beta^2|x|^{-n}(|x|^2|\nabla u|^2+|u|^2)dx\\
&+C_4M_1\int_{\{R_1/e\le|x|\le R_1/2\}\cup\{eR_2\le|x|\le 3R_2\}}\varphi_\beta^2|x|^{-n}|\tilde U|^2dx\\
&+C_4\beta\int_{\{R_1/e\le|x|\le R_1/2\}\cup\{eR_2\le|x|\le 3R_2\}}(\log|x|)^4\varphi_\beta^2|x|^{-n}|\tilde U|^2dx,
\end{aligned}\tag{4.5}$$

where $|\tilde U(x)|^2=|x|^4|\nabla q|^2+|x|^2|q|^2+|x|^2|\nabla u|^2+|u|^2$ and the positive constant C_4 only depends on n.

Now letting $M_1 = 2 + 2C_4$, $\beta \geq 2 + 2C_4$, and R small enough such that $(\log(eR))^2 \geq 2C_4 M_1$, then the first three terms on the right hand side of (4.5) can be absorbed by the left hand side of (4.5). Also, it is easy to check that there exists $\tilde{R}_1 > 0$, depending on n, such that for all $\beta > 0$, both $(\log|x|)^{-2}|x|^{-n}\varphi_\beta^2(|x|)$ and $(\log|x|)^4|x|^{-n}\varphi_\beta^2(|x|)$ are decreasing functions in $0 < |x| < \tilde{R}_1$. So we choose a small $R < \tilde{R}_2$, where $\tilde{R}_2 = \min\{\exp(-2\sqrt{2C_4M_1} - 1), \tilde{R}_1/3, \tilde{R}_0\}$. It is clear that $\tilde{R}_2$ depends on n. With the choices described above, we obtain from (4.5) that

$$\begin{aligned}
& R_2^{-n}(\log R_2)^{-2}\varphi_\beta^2(R_2)\int_{R_1/2<|x|<R_2}|u|^2dx \\
\leq\ & \int_{R_1/2<|x|<eR_2}(\log|x|)^{-2}\varphi_\beta^2|x|^{-n}|u|^2dx \\
\leq\ & C_5\beta\int_{\{R_1/e\leq|x|\leq R_1/2\}\cup\{eR_2\leq|x|\leq 3R_2\}}(\log|x|)^4\varphi_\beta^2|x|^{-n}|\tilde{U}|^2dx \\
\leq\ & C_5\beta(\log(R_1/e))^4(R_1/e)^{-n}\varphi_\beta^2(R_1/e)\int_{\{R_1/e\leq|x|\leq R_1/2\}}|\tilde{U}|^2dx \\
& +C_5\beta(\log(eR_2))^4(eR_2)^{-n}\varphi_\beta^2(eR_2)\int_{\{eR_2\leq|x|\leq 3R_2\}}|\tilde{U}|^2dx. \qquad (4.6)
\end{aligned}$$

Using (3.1), we can control $|\tilde{U}|^2$ terms on the right hand side of (4.6). In other words, it follows from (3.1) that

$$\begin{aligned}
& R_2^{-2\beta-n}(\log R_2)^{-4\beta-2}\int_{R_1/2<|x|<R_2}|u|^2dx \\
\leq\ & C_6 2^{2\beta+n}(\log(R_1/e))^4(R_1/e)^{-n}\varphi_\beta^2(R_1/e)\int_{\{R_1/4\leq|x|\leq R_1\}}|u|^2dx \\
& +C_6 2^{2\beta+n}(\log(eR_2))^4(eR_2)^{-n}\varphi_\beta^2(eR_2)\int_{\{2R_2\leq|x|\leq 4R_2\}}|u|^2dx \\
=\ & C_6 2^{2\beta+n}(\log(R_1/e))^{-4\beta+4}(R_1/e)^{-2\beta-n}\int_{\{R_1/4\leq|x|\leq R_1\}}|u|^2dx \\
& +C_6 2^{2\beta+n}(\log(eR_2))^{-4\beta+4}(eR_2)^{-2\beta-n}\int_{\{2R_2\leq|x|\leq 4R_2\}}|u|^2dx. \qquad (4.7)
\end{aligned}$$

Replacing $2\beta + n$ by β, (4.7) becomes

$$\begin{aligned}
& R_2^{-\beta}(\log R_2)^{-2\beta+2n-2}\int_{R_1/2<|x|<R_2}|u|^2dx \\
\leq\ & C_7 2^{\beta}(\log(R_1/e))^{-2\beta+2n+4}(R_1/e)^{-\beta}\int_{\{R_1/4\leq|x|\leq R_1\}}|u|^2dx \\
& +C_7 2^{\beta}(\log(eR_2))^{-2\beta+2n+4}(eR_2)^{-\beta}\int_{\{2R_2\leq|x|\leq 4R_2\}}|u|^2dx. \qquad (4.8)
\end{aligned}$$

Dividing $R_2^{-\beta}(\log R_2)^{-2\beta+2n-2}$ on the both sides of (4.8) and providing $\beta \geq n+2$, we have that

$$\begin{aligned}
&\int_{R_1/2<|x|<R_2} |u|^2 dx \\
\leq\ & C_8(\log R_2)^6 (2eR_2/R_1)^\beta \int_{\{R_1/4\leq|x|\leq R_1\}} |u|^2 dx \\
&+C_8(\log R_2)^6(2/e)^\beta [(\log R_2/\log(eR_2))^2]^{\beta-n-2} \int_{\{2R_2\leq|x|\leq 4R_2\}} |u|^2 dx \\
\leq\ & C_8(\log R_2)^6 (2eR_2/R_1)^\beta \int_{\{R_1/4\leq|x|\leq R_1\}} |u|^2 dx \\
&+C_8(\log R_2)^6(4/5)^\beta \int_{\{2R_2\leq|x|\leq 4R_2\}} |u|^2 dx. \qquad (4.9)
\end{aligned}$$

In deriving the second inequality above, we use the fact that

$$\frac{\log R_2}{\log(eR_2)} \to 1 \quad \text{as} \quad R_2 \to 0,$$

and thus

$$\frac{2}{e} \cdot \frac{\log R_2}{\log(eR_2)} < \frac{4}{5}$$

for all $R_2 < \tilde{R}_3$, where $\tilde{R}_3$ is sufficiently small. We now take $\tilde{R} = \min\{\tilde{R}_2, \tilde{R}_3\}$, which depends on n.

Adding $\int_{|x|<R_1/2} |u|^2 dx$ to both sides of (4.9) leads to

$$\begin{aligned}
\int_{|x|<R_2} |u|^2 dx \quad \leq\quad & C_9(\log R_2)^6 (2eR_2/R_1)^\beta \int_{|x|\leq R_1} |u|^2 dx \\
&+C_9(\log R_2)^6 (4/5)^\beta \int_{|x|\leq 1} |u|^2 dx. \qquad (4.10)
\end{aligned}$$

It should be noted that (4.10) holds for all $\beta \geq \tilde{\beta}$ with $\tilde{\beta}$ depending only on n. For simplicity, by denoting

$$E(R_1, R_2) = \log(2eR_2/R_1), \quad B = \log(5/4),$$

(4.10) becomes

$$\begin{aligned}
&\int_{|x|<R_2} |u|^2 dx \\
\leq\ & C_9(\log R_2)^6 \Big\{ \exp(E\beta) \int_{|x|<R_1} |u|^2 dx + \exp(-B\beta) \int_{|x|<1} |u|^2 dx \Big\}. \qquad (4.11)
\end{aligned}$$

To further simplify the terms on the right hand side of (4.11), we consider two cases. If $\int_{|x|<R_1} |u|^2 dx \neq 0$ and

$$\exp(E\tilde{\beta}) \int_{|x|<R_1} |u|^2 dx < \exp(-B\tilde{\beta}) \int_{|x|<1} |u|^2 dx,$$

then we can pick a $\beta > \tilde{\beta}$ such that

$$\exp(E\beta) \int_{|x|<R_1} |u|^2 dx = \exp(-B\beta) \int_{|x|<1} |u|^2 dx.$$

Using such β, we obtain from (4.11) that

$$\begin{aligned}
&\int_{|x|<R_2} |u|^2 dx \\
\leq\ & 2C_9(\log R_2)^6 \exp(E\beta) \int_{|x|<R_1} |u|^2 dx \\
=\ & 2C_9(\log R_2)^6 \left(\int_{|x|<R_1} |u|^2 dx\right)^{\frac{B}{E+B}} \left(\int_{|x|<1} |u|^2 dx\right)^{\frac{E}{E+B}}. \qquad (4.12)
\end{aligned}$$

If $\int_{|x|<R_1} |u|^2 dx = 0$, then letting $\beta \to \infty$ in (4.11) we have $\int_{|x|<R_2} |u|^2 dx = 0$ as well. The three-ball inequality obviously holds.

On the other hand, if

$$\exp(-B\tilde{\beta}) \int_{|x|<1} |u|^2 dx \leq \exp(E\tilde{\beta}) \int_{|x|<R_1} |u|^2 dx,$$

then we have

$$\begin{aligned}
&\int_{|x|<R_2} |u|^2 dx \\
\leq\ & \left(\int_{|x|<1} |u|^2 dx\right)^{\frac{B}{E+B}} \left(\int_{|x|<1} |u|^2 dx\right)^{\frac{E}{E+B}} \\
\leq\ & \exp(B\tilde{\beta}) \left(\int_{|x|<R_1} |u|^2 dx\right)^{\frac{B}{E+B}} \left(\int_{|x|<1} |u|^2 dx\right)^{\frac{E}{E+B}}. \qquad (4.13)
\end{aligned}$$

Putting together (4.12), (4.13), and setting $C_{10} = \max\{2C_9(\log R_2)^6, \exp(\tilde{\beta}\log(5/4))\}$, we arrive at

$$\int_{|x|<R_2} |u|^2 dx \leq C_{10} \left(\int_{|x|<R_1} |u|^2 dx\right)^{\frac{B}{E+B}} \left(\int_{|x|<1} |u|^2 dx\right)^{\frac{E}{E+B}}. \qquad (4.14)$$

It is readily seen that $\frac{B}{E+B} \approx (\log(1/R_1))^{-1}$ when R_1 tends to 0.

Now for the general case, we consider $0 < R_1 < R_2 < R_3 < 1$ with $R_1/R_3 < R_2/R_3 \leq \tilde{R}$, where $\tilde{R}$ is given as above. By scaling, i.e. defining $\widehat{u}(y) := u(R_3 y)$, $\widehat{p}(y) := R_3 p(R_3 y)$ and $\widehat{A}(y) = A(R_3 y)$, (4.14) becomes

$$\int_{|y|<R_2/R_3} |\widehat{u}(y)|^2 dy \leq C_{11}\left(\int_{|y|<R_1/R_3} |\widehat{u}(y)|^2 dy\right)^{\tau}\left(\int_{|y|<1} |\widehat{u}(y)|^2 dy\right)^{1-\tau}, \qquad (4.15)$$

where

$$\tau = B/[E(R_1/R_3, R_2/R_3) + B],$$

$$C_{11} = \max\{2C_9(\log R_2/R_3)^6, \exp(\tilde{\beta}\log(5/4))\}.$$

Note that C_{11} is independent of R_1. Restoring the variable $x = R_3 y$ in (4.15) gives

$$\int_{|x|<R_2} |u|^2 dx \leq C_{11}\left(\int_{|x|<R_1} |u|^2 dx\right)^{\tau}\left(\int_{|x|<R_3} |u|^2 dx\right)^{1-\tau}.$$

The proof of Theorem 1.1 is complete.

We now turn to the proof of Theorem 1.3. We fix R_2, R_3 in Theorem 1.1. By dividing $\int_{|x|<R_2}|u|^2dx$ on the three-ball inequality (1.5), we have that

$$1 \le C\Big(\int_{|x|<R_1}|u|^2dx/\int_{|x|<R_2}|u|^2dx\Big)^{\tau}\Big(\int_{|x|<R_3}|u|^2dx/\int_{|x|<R_2}|u|^2dx\Big)^{1-\tau}. \tag{4.16}$$

Raising both sides by $1/\tau$ yields that

$$\int_{|x|<R_3}|u|^2dx \le \Big(\int_{|x|<R_1}|u|^2dx\Big)\Big(C\int_{|x|<R_3}|u|^2dx/\int_{|x|<R_2}|u|^2dx\Big)^{1/\tau}. \tag{4.17}$$

In view of the formula for τ, we can deduce from (4.17) that

$$\int_{|x|<R_3}|u|^2dx \le \Big(\int_{|x|<R_1}|u|^2dx\Big)(1/R_1)^{\tilde{C}\log(\int_{|x|<R_3}|u|^2dx/\int_{|x|<R_2}|u|^2dx)}, \tag{4.18}$$

where $\tilde{C}$ is a positive constant depending on n and R_2/R_3. Consequently, (4.18) is equivalent to

$$\Big(\int_{|x|<R_3}|u|^2dx\Big)R_1^m \le \int_{|x|<R_1}|u|^2dx$$

for all R_1 sufficiently small, where

$$m = \tilde{C}\log\Big(\frac{\int_{|x|<R_3}|u|^2dx}{\int_{|x|<R_2}|u|^2dx}\Big).$$

We now end the proof of Theorem 1.3.

5. **Proof of Theorem 1.5.**

We prove Theorem 1.5 in this section. Let us first choose $a > \max\{2, \tilde{R}^{-1}\}$, where $\tilde{R}$ is given in Theorem 1.1. By doing so, we can see that if we set $R_2 = ar$ and $R_3 = a^2r$, then $R_2/R_3 < \tilde{R}$ for $r > 0$. Now let $0 < r < 1$ and define R_2, R_3 accordingly. Let $|\tilde{x}| = t$. We pick a sequence of points $0 = x_0, x_1, \cdots, x_N = \tilde{x}$ such that $|x_{j+1} - x_j| \le r$. We shall prove the desired estimate iteratively. To see how the iteration goes, let us assume that $\int_{|x-x_l|<r}|u|^2dx \ge r^{m_l}$ for some $m_l > 0$ since u is nontrivial. By Theorem 1.3 and Remark 1.4, we have that

$$\int_{|x-x_{l+1}|<r}|u|^2dx \ge \int_{|x-x_{l+1}|<R_3}|u|^2dx \cdot r^m, \tag{5.1}$$

where

$$m = \tilde{C}\log\Big(\frac{\int_{|x-x_{l+1}|<R_3}|u|^2dx}{\int_{|x-x_{l+1}|<R_2}|u|^2dx}\Big).$$

Using the boundedness assumption of u (see (1.8)) and $r < 1$, we can deduce that

$$\frac{\int_{|x-x_{l+1}|<R_3}|u|^2dx}{\int_{|x-x_{l+1}|<R_2}|u|^2dx} \le a^{2n}\lambda_2^2 r^{n-m_l} \le r^{-s-m_l} \tag{5.2}$$

for some s depending on λ_2 and n. Note that we can assume $s \le m_l$ by choosing a larger m_l. It follows from (5.2) that

$$r^m \ge r^{\tilde{C}(s+m_l)\log(1/r)} \ge r^{2m_l\tilde{C}\log(1/r)}. \tag{5.3}$$

It is clear that

$$\int_{|x-x_{l+1}|<R_3}|u|^2dx \ge \int_{|x-x_l|<r}|u|^2dx.$$

Thus, combining (5.1) and (5.3) yields that

$$\int_{|x-x_{l+1}|<r} |u|^2 dx \geq r^{m_l[1+2\tilde{C}\log(1/r)]} \geq r^{m_l \zeta}, \tag{5.4}$$

where $\zeta = 1 + 2\tilde{C}\log(1/r)$. Now starting from 0 and iterating N steps with $N \leq [t/r] + 1 \leq t/r + 1$, we obtain that

$$\int_{|x-\tilde{x}|<r} |u|^2 dx \geq r^{m_0 \zeta^N} \geq r^{m_0 \zeta^{(t/r+1)}},$$

where m_0 satisfies

$$\int_{|x|<r} |u|^2 dx \geq r^{m_0}.$$

We now take $c = m_0$, which depends on λ_2, n, and $\int_{|x|<r} |u|^2 dx$. The proof is complete. $\square$

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